Aspects of Operator Theory in Quantum Mechanics

And a Study of Magnetic Many-body Quantum Systems in the Hartree-Fock Approximation

Master Thesis

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Abstract

This thesis explores aspects of operator theory relevant to quantum mechanics. We primarily focus on three different subjects: Self-adjoint unbounded operators, pseudo-differential calculus, and simple quantum systems of particles in Euclidean space.

First we present some of the theory of self-adjoint unbounded operator covering basic definitions, variational operators, and spectral theory. While introducing the basic definitions we prove standard results such as criteria for self-adjointness and the Kato-Rellich Theorem. For variational operators we also include Friedrich's Extension. As for spectral theory we prove the spectral theorem for bounded and unbounded self-adjoint operators, Stone's Formula, and Helffer-Sjöstrand Formula.

Secondly we study tempered distributions and pseudo-differential operators. The Schwartz space and space of tempered distributions are introduced, and results such as reflexivity of the spaces, Schwartz Kernel Theorem, and the Structure Theorem are proven. Afterwards we deal with quite general quantization schemes for pseudo-differential operators, mostly working with Hörmander classes of smooth symbols with decay controlled by a tempered weight. For these we establish a Calderón-Vaillancourt Theorem, a Moyal product, and for certain quantizations a Beal's Commutator Criterion. To prove all these results we make use of modulated tight Gabor frame, and we characterize the different spaces by their coordinates or matrices in this frame.

Lastly we analyze some one particle systems directly and a many-body particle system in the Hartree-Fock approximation, both under the influence of a regular magnetic field. As a start we omit the magnetic field and give classical results on the free Schrödinger operator and harmonic oscillator. Then we give elementary results on free magnetic Schrödinger operators and find the Landau spectrum. Afterwards we turn our attention toward the Hartree-Fock approximation of a many-body particle system under the influence of a constant magnetic field. Essentially the many-body particle system is approximated by a single particle Schrödinger operator with an added potential representing the particle cloud. This potential satisfies a fix-point equation, which we solve.

Preface

This master thesis was written at the Department of Mathematical Sciences at Aalborg University between the first of September to the 28th of May. It is the culmination of a five year journey studying mathematics at Aalborg University, learning mathematic analysis from the dedicated and knowledgeable people at the department. I am eternally grateful to my teachers and my fellow students for making these years memorable.

A special thanks and appreciation goes to my supervisor of this project: Horia Cornean. H. Cornean has not only suggested the subject of this thesis and supervised it, but he has been a constant inspiration and support of my studies throughout my time in Aalborg. I would not be the same mathematician that i am today if not for him, for better or worse.

The report itself assumes that the reader is comfortable mathematics in general and more specifically, knowledge of introductory functional analysis, measure theory, and distribution theory is assumed. If the reader has acquired the required prerequisites, then the report should be self-contained in its mathematical contents. Moreover, each chapter should for the most part also be self-contained assuming the reader has the necessary prerequisites.

The reader is not expected to know any physics and the report will lean heavily on the mathematics of mathematical physics.

Sources are given at the start of each chapter and otherwise when appropriate. Definitions, propositions, theorems, lemmas, corollaries, and remarks are numbered according to each section and consecutively. Equations are numbered separately. We mark the end of all proofs by a ■. Finally, a complete list of references and an index list can be found on the last pages.

 Beside built-in spell control in the Overleaf LaTeX compiler, the project has not used generative AI.

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Introduction

In the pursuit of understanding the universe, quantum mechanics was developed to explain how the smallest of things the universe have to offer behave. These, such as photons and electrons, were proven to exhibit both particle-like and wavelike behavior through experiments. Having a hard time explaining how e.g. light comes in quanta, but behaves like a wave in an ensemble, lead to this strange and stochastic theory of quantum mechanics.

Quantum mechanics supposes that we work with a Hilbert space as our phase space and associate observables of our quantum system with self-adjoint operators on the Hilbert space. The state of our quantum system is then described by a non-negative, self-adjoint trace class operator with trace one, whom together with the spectral decomposition of an observable gives a probability distribution for what the "realized" value of that observable is. One quite important observable is the energy of the system and the associated operator is called the Schrödinger operator, sometimes the Hamiltonian. The Schrödinger operator governs the time evolution of the quantum system through the Schrödinger equation, which expresses how the state develops in time. See [2, 15, 28] for historical notes or discussion of the "axioms" of quantum mechanics, specifically [15, Chapter 1, 3, and 19] and [28, Chapter 2 and 5].

Dealing with all of the above rigorously in a mathematical sense necessitates the development of new mathematical tools, especially developing the theory of operators to encompass those relevant to quantum mechanics. This is the main interest of this project and we aim to elucidate theory both abstract and concrete, general and specific.

For a start, much of the theory of operators, and even more so unbounded operator, has been created with applications to quantum systems in mind. As seen in e.g. [14, 22, 23], this theory is far from trivial, although well developed at this point. We will in Chapter 2 take a very general approach to the notion of operator and give a short account of self-adjointness and spectral theory for operators on Hilbert spaces. This covers basic material beyond the case of bounded operators with only a few perhaps non-standard topics such as the Helffer-Sjöstrand formula, taken from [10, 16, 28].

Continuing, the question of which operators are associated with which observable begs for an answer. The more intuitive classical mechanics, where one works with systems as deterministic and observables as functions of states giving a specific value, one often has an easier time finding the function associated with a certain observable, see [15, Chapter 2 and 3]. The energy function, henceforth Hamiltonian, is one example. Trying to translate classical mechanics to its quantum counterpart leads to quantization procedures and in cases classes of pseudo-differential operators, see [15, Chapter 13]. Thus this is the subject of Chapter 3, where we present some technical tools in terms of spaces and transformations which enables us to quantize classical observables through general schemes. The approach taken is mainly from the articles [8, 9], where the usefulness of modified Gabor frames to pseudo-differential operators is shown.

Lastly, we would be negligent if we do not delve into some specifics, and so in Chapter 4 several Schrödinger operators for simple quantum systems are studied in detail. A large part is spent on making sure that what should be the Schrödinger operators actually exists in an appropriate sense, i.e. as self-adjoint operators. In some instances we also compute the spectrum of the Schrödinger operator, which is relevant when considering the time evolution of the quantum system. We focus on a range of techniques in large part based off [7, 28]. It should be noted that Schrödinger operators are not always amenable to an analysis, and approximations have to take place. Many-body systems can give raise to such situations and the last topic of Chapter 4 concerns the Hartree-Fock approximation, particularly recovering an approximative Schrödinger operator. This last problem is due to H. Cornean and the proofs are constructed in collaboration, yet to be published.

1.1 Notation and Conventions

Before moving on we introduce some notation and conventions used throughout. While some are common, others are certainly not. Some of the notation clashes, so we count on the viewer to look at the context in which it is used.

We let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0$ the natural numbers plus zero, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. The absolute value of a complex number is denoted $|\cdot|$ and the Euclidean norm in $\mathbb{C}^d$, $d \in \mathbb{N}$, by $\|\cdot\|$. The associated metric is denoted as d, not to be confused with the dimension. On $\mathbb{C}^d$ we also define the Japanese bracket $\langle\cdot\rangle := (1 + \|\cdot\|^2)^{\frac{1}{2}}$ and remind the reader of the following generalization of Peetre's inequality:

$$\langle x\rangle^{-p}\langle y\rangle^{p} \leq 2^{\frac{|p|}{2}}\langle x-y\rangle^{|p|}$$

for $x, y \in \mathbb{C}^d$ and $p \in \mathbb{R}$. At times a $C \in \mathbb{R}$ will denote a positive constant with possible subscripts showing its dependencies. We also let e_j denote the jth canonical basis vector in $\mathbb{C}^d$, $d \geq j$.

The graph of a function f is denoted by $\Gamma(f)$, the domain by $D(f)$, and the range $R(f)$. An indicator function on some subset Ω is denoted by 1_Ω and the identity function by id_Ω. The subscripts are omitted when the context allows it. We also allow the notation $f(\cdot)$ for a function to avoid the cumbersome notation of always having dummy variables.

Given a metric space Ω, we denote the ball at $x \in \Omega$ with radius $r > 0$ by $B_r(x;\Omega)$.

For vector spaces V we denote a translation by $\phi \in V$ as $\tau_\phi := V \ni \psi \mapsto \psi - \phi$ and use the same notation for its pullback, i.e. $\tau_\phi^* = \tau_\phi$. Norms or semi-norms are denoted as $\|\cdot\|_V$ with possibility of added subscripts, and for inner products we use $\langle\cdot,\cdot\rangle_V$. Inner products, and in general sesquilinear forms, are taken antilinear in the first entry and linear in the second. For tensor products we use $\otimes$.

Lebesgue spaces over $\mathbb{R}^d$, $d \in \mathbb{N}$, are denoted as $L^p(\mathbb{R}^d)$ with $p \in [1,\infty]$. Spaces of continuous and continuous differential functions are denoted as $C^n(\mathbb{R}^d)$ with $n \in$

$\mathbb{N} \cup \{\infty\}$, and if these are required to be bounded, then we use $BC^n(\mathbb{R}^d)$. If we take the subspace of functions with compact support, then a c is added as a subscript. All these spaces are equipped with their canonical topology, making them Hilbert, Banach, or Fréchet spaces.

For a linear integral operator T, we denote its, or a, kernel by K_T. We work with

$$\mathcal{F}\colon L^1(\mathbb{R}^d) \ni f \mapsto \left(\mathbb{R}^d \ni \xi \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)dx\right)$$

as the definition of the Fourier transform on $L^1(\mathbb{R}^d)$. The symbol $\mathcal{F}$ will in general denote the Fourier transform in any of its forms.

We make use of multi-index notation: For a multi-index $\alpha \in \mathbb{N}_0^d$, $d \in \mathbb{N}$, we have an absolute value $|\alpha| := \sum_{j=1}^d \alpha_j$, the monomial $x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}$ for $x \in \mathbb{C}^d$, and the differential operator $\partial^\alpha := \prod_{j=1}^d \frac{\partial^{\alpha_j}}{\partial x_j}$ on suitably defined objects.

2

Unbounded Operators, Self-adjointness, and Spectral Theory

In general we mathematicians are quite fond of being precise, but somehow the natural world has not caught up to that fact yet. It seems that one always needs more complex and abstract theory to model different aspects of physics, and quantum mechanics is no exception: We need to have a good grasp on Hilbert spaces and the general notion of an operator on these spaces. This motivates the study of unbounded operators, which will be our goal this chapter.

Our presentation of the subject is based on many well-known sources. Most definitions and common theorems stem from [14, 23, 24, 25, 31], especially the first two sections, which focuses respectively on the general notion of an operator between topological vector spaces and the adjoints for operators between Hilbert spaces. The following section on variational operators is inspired by [14, 22] and delves into the relation between operators and sesquilinear forms. In the last section we present some elementary spectral theory and most importantly the spectral theorems for self-adjoint operators. Much of the spectral theory is stitched together from material in [5, 23, 24, 27] while the Stone's formula is taken form [16, 28] and the Helffer-Sjöstrand formula from [10, 16]. A lot of attention in the second section and onwards is given to exploring self-adjointness, whereas we neglect to study objects such as normal operators.

Note when referring to a vector space we always mean a complex vector space. We also mostly consider Banach and Hilbert spaces in this section though some definitions are stated more generally.

2.1 Unbounded Operators

Let us start with some definitions.

Definition 2.1.1. (Unbounded Operators) A linear map T from a subspace of a topological vector space V_1 into another topological vector space V_2 is called an operator on V_1 into V_2, and the space of such objects denoted by $\mathcal{L}(V_1, V_2)$. Also $\mathcal{L}(V_1) := \mathcal{L}(V_1, V_1)$.

If $T \in \mathcal{L}(V_1, V_2)$ is not bounded, then T is called unbounded. The subspace of $\mathcal{L}(V_1, V_2)$ consisting of bounded operators defined on the entirety of V_1 will be denoted by $\mathcal{B}(V_1, V_2)$ and the bounded dual by $V_1' = \mathcal{B}(V_1, \mathbb{C})$. Spaces of bounded operators are equipped with the strong topology.

We make two notes after this first definition. First of all, for the spaces we consider, boundedness of an operator is equivalent to continuity, see [21, Theorem 4.12] for a general statement of this fact. Secondly, whenever we define operators on topological vector spaces one often defines the domain before the operator, leading to some notational sins.

Definition 2.1.2. (Operations on Operators) Let $T_1, T_2 \in \mathcal{L}(V_1, V_2)$ and $T_3 \in \mathcal{L}(V_2, V_3)$ be operators and $a \in \mathbb{C}$ a scalar.

We define sum of T_1, T_2 on $D(T_1) \cap D(T_2)$ by $(T_1 + T_2)\phi = T_1\phi + T_2\phi$ for $\phi \in D(T_1) \cap D(T_2)$. We define the scalar product of a and T_1 on $D(T_1)$ by $(aT_1)\phi = aT_1\phi$ for $\phi \in D(T_1)$. Lastly, we define the composition of T_1, T_3 on

$$D(T_3T_1) = \{\phi \in D(T_1) | T_1\phi \in D(T_3)\}$$

by $(T_3T_1)\phi = T_3T_1\phi$ for $\phi \in D(T_3T_1)$.

Another important operation on operators invariating some space is the commutator bracket, defined by

$$[T_1, T_2] = T_1T_2 - T_2T_1$$

for $T_1, T_2 \in \mathcal{L}(V)$.

Definition 2.1.3. (Densely Defined) An operator $T \in \mathcal{L}(V_1, V_2)$ is called densely defined if its domain $D(T)$ is dense in V_1.

The operators of interest are mostly densely defined.

Definition 2.1.4. (Extension) For two operators $T_1, T_2 \in \mathcal{L}(V_1, V_2)$, T_2 is called an extension of T_1 if $D(T_1) \subseteq D(T_2)$ and $T_2|_{D(T_1)} = T_1$, and we write $T_1 \subseteq T_2$.

Definition 2.1.5. (Closed, Closable) An operator $T \in \mathcal{L}(V_1, V_2)$ is called closed if its graph $\Gamma(T)$ is closed in the product topology.

If an operator T has a closed extension, then it is called closable. Its smallest closed extension, as measured in the sense of extensions, is then called T's closure and denoted $\overline{T}$.

Note if T is a closable operator, then simply $\Gamma(\overline{T}) = \bigcap_{T \subseteq \tilde{T}, \tilde{T} \text{ is closed}} \Gamma(\tilde{T})$.

A equivalent definition of closedness when dealing with Fréchet spaces is the following: For every sequence $(\phi_n)_{n\in\mathbb{N}}$ in $D(T)$ where $(\phi_n)_{n\in\mathbb{N}}$ is convergent in V_1 and $(T\phi_n)_{n\in\mathbb{N}}$ is convergent in V_2, then $\lim_{n\to\infty} \phi_n \in D(T)$ and $T \lim_{n\to\infty} \phi_n = \lim_{n\to\infty} T\phi_n$. Moreover, if we are dealing with Banach spaces, then $D(\overline{T})$ is the completion of $D(T)$ in the graph norm:

$$\|\cdot\|_{D(T)} \colon D(T) \in \phi \mapsto \|\phi\|_{V_1} + \|T\phi\|_{V_2}$$

2.2 Adjoints

The next step is to define a conjugation for operators between Hilbert spaces.

Definition 2.2.1. (Adjoint Operator) For an densely defined operator $T \in \mathcal{L}(H_1, H_2)$ between Hilbert spaces we define the adjoint operator as follows: On the set

$$D(T^*) = \{\phi \in H_2 | \exists \psi \in H_1 : \langle \phi, T(\cdot) \rangle_{H_2}|_{D(T)} = \langle \psi, \cdot \rangle_{H_1}|_{D(T)}\}$$

define $T^*\phi \in H_1$ for $\phi \in D(T^*)$ by the unique vector such that

$$\langle \phi, T(\cdot) \rangle_{H_2}|_{D(T)} = \langle T^*\phi, \cdot \rangle_{H_1}|_{D(T)}.$$

The uniqueness of the vector $T^*\phi$ comes from the density of $D(T)$ in H_1 and Riesz' representation. We also note the simple identity $\ker(T^*) = R(T)^\perp$ since for $\phi \in R(T)^\perp$ we have $\langle \phi, T(\cdot)\rangle_{H_2}|_{D(T)} \equiv 0$.

Lemma 2.2.2. *The adjoint of any densely defined operator is closed.*

Proof. Let $T \in \mathcal{L}(H_1, H_2)$ be densely defined. Suppose $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $D(T^*)$ such that $(\phi_n)_{n\in\mathbb{N}}$ converges in H_2 and $(T^*\phi_n)_{n\in\mathbb{N}}$ converges in H_1. Then for any $\psi \in D(T)$

$$\langle \phi_n, T\psi\rangle_{H_2} = \langle T^*\phi_n, \psi\rangle_{H_1},$$

and passing to the limit

$$\langle \lim_{n\to\infty} \phi_n, T\psi\rangle_{H_2} = \langle \lim_{n\to\infty} T^*\phi_n, \psi\rangle_{H_1}.$$

By definition $\lim_{n\to\infty} \phi_n \in D(T^*)$ and $T^* \lim_{n\to\infty} \phi_n = \lim_{n\to\infty} T^*\phi_n$. ∎

Proposition 2.2.3. *An densely defined operator $T \in \mathcal{L}(H_1, H_2)$ between Hilbert spaces is closable if and only if T^* is densely defined. In the positive case $\overline{T}^* = T^*$ and $\overline{T} = T^{**}$ hold.*

Proof. If T^* is densely defined, then T^{**} exists and it is a closed extension of T, hence T is closable.

Let us prove the converse. Suppose T is closable. Define the unitary operator $U\colon H_1 \times H_2 \ni (\phi, \psi) \mapsto (-\psi, \phi)$. From the identity defining the adjoint we see that for $\phi \in D(T)$ and $\psi \in D(T^*)$

$$0 = \langle T^*\psi, \phi\rangle_{H_1} - \langle \psi, T\phi\rangle_{H_2} = \langle U^*(\psi, T^*\psi), (\phi, T\phi)\rangle_{H_1\times H_2}.$$

Thus $\Gamma(T)^\perp = U^*\Gamma(T^*)$, so $\Gamma(\overline{T}) = (U^*\Gamma(T^*))^\perp = U^*\Gamma(T^*)^\perp$. Now assume that T^* is not densely defined and find $\phi \in D(T^*)^\perp \setminus \{0\}$. Then $(\phi, 0) \in \Gamma(T^*)^\perp = U\Gamma(\overline{T})$ implying $0 = \overline{T}0 = \phi$, a contradiction. Hence T^* must be densely defined.

Lastly, we prove the two identities. From the above we get $U\Gamma(T^{**}) = \Gamma(T^*)^\perp = U\Gamma(\overline{T})$, showing that $\Gamma(T^{**}) = \Gamma(\overline{T})$ and so $T^{**} = \overline{T}$. Then

$$\overline{T}^* = T^{***} = (T^*)^{**} = \overline{T^*} = T^*,$$

where we use that T^* is closed. ∎

Proposition 2.2.4. *If an densely defined operator $T \in \mathcal{L}(H_1, H_2)$ between Hilbert spaces is closed, injective, and has dense range, then T^* and T^{-1} are both densely defined, closed, and injective with dense range. Furthermore, $(T^*)^{-1} = (T^{-1})^*$ holds.*

Proof. Clearly T^{-1} is densely defined, injective and has dense range. Also, if $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $D(T^{-1})$ where $(\phi_n)_{n\in\mathbb{N}}$ is convergent in H_2 and $(T^{-1}\phi_n)_{n\in\mathbb{N}}$ is convergent in H_1, then by T being closed, $\lim_{n\to\infty} T^{-1}\phi_n \in D(T)$ and $T \lim_{n\to\infty} T^{-1}\phi_n = \lim_{n\to\infty} \phi_n$, showing that T^{-1} is closed.

From Lemma 2.2.2 and Proposition 2.2.3 we have that T^* is densely defined and closed. The dense range of T implies that T^* is injective and using $T^{**} = T$ together with T being injective gives that T^* has dense range.

Let U be defined as in Proposition 2.2.3. Then

$$\Gamma((T^{-1})^*) = U^*\Gamma(T^{-1})^\perp = U^*(U\Gamma(-T))^\perp = \Gamma(-T)^\perp = U^*\Gamma(-T^*) = \Gamma((T^*)^{-1}),$$

so $(T^*)^{-1} = (T^{-1})^*$. ∎

2.2.1 Symmetric and Self-adjoint Operators

Of special interest are operators, which are partly or entirely equal to their adjoint.

Definition 2.2.5. (Symmetric) An densely defined operator $T \in \mathcal{L}(H)$ on a Hilbert space is called symmetric if $T \subseteq T^*$.

For a symmetric operator T we define the lower bound of T by

$$m(T) = \inf\{\langle T\phi, \phi\rangle_H | \phi \in D(T) \cap \partial B_1(0; H)\}$$

and the upper bound by

$$M(T) = \sup\{\langle T\phi, \phi\rangle_H | \phi \in D(T) \cap \partial B_1(0; H)\}.$$

We call T lower bounded if $m(T) > -\infty$ and analogously for upper bounded. If $m(T) \geq 0$ we call T non-negative, if $m(T) > 0$ we call T positive, and analogously for non-positive and negative.

For two symmetric operators $T_1, T_2 \in \mathcal{L}(H)$ we write $T_1 \leq T_2$ if $D(T_1) \subseteq D(T_2)$ and $\langle T_1\phi, \phi\rangle_H \leq \langle T_2\phi, \phi\rangle_H$ for all $\phi \in D(T_1)$.

Note that the above definition of bounds and comparison is well-defined, since $\langle T\phi, \phi\rangle_H \in \mathbb{R}$ for all $\phi \in D(T)$ when T is symmetric. This fact gives the following identity for T and $z \in \mathbb{C}$:

$$\begin{aligned} \|(T-z)\phi\|_H^2 &= \|(T-\mathrm{Re}(z))\phi\|_H^2 - i(\mathrm{Im}(z) - \mathrm{Im}(z))\langle \phi, T - \mathrm{Re}(z)\phi\rangle_H + \mathrm{Im}(z)^2\|\phi\|_H^2 \\ &= \|(T-\mathrm{Re}(z))\phi\|_H^2 + \mathrm{Im}(z)^2\|\phi\|_H^2 \end{aligned} \tag{2.2.1}$$

for all $\phi \in D(T)$.

From the definition we also see that symmetry of T implies that T^{**} exists and so T is closable.

Definition 2.2.6. (Self-adjoint) A densely defined operator $T \in \mathcal{L}(H)$ in a Hilbert space is called self-adjoint if $T = T^*$.

A symmetric operator $T \in \mathcal{L}(H)$ is called essentially self-adjoint if $\overline{T} = T^*$.

Self-adjointness implies T is already closed and symmetric, but the converse is false. Let us give some equivalent statements.

Proposition 2.2.7. *For a densely defined operator $T \in \mathcal{L}(H)$ the following is equivalent to T being self-adjoint:*

(i) T is closed, symmetric and $\ker(T^* \pm i) = \{0\}$

(ii) T is symmetric and $R(T \pm i) = H$

Proof. Suppose T is self-adjoint. Then closedness and symmetry follows as mentioned above. Furthermore, if $\ker(T - a) \neq \{0\}$ for some $a \in \mathbb{C} \setminus \mathbb{R}$, then there exist an eigenvector $\phi \in D(T) \setminus \{0\}$ for T and the eigenvalue a, so

$$a\|\phi\|_H^2 = \langle \phi, T\phi\rangle_H = \langle T\phi, \phi\rangle_H = \overline{a}\|\phi\|_H^2,$$

which is impossible. Hence $\ker(T - a) = \{0\}$ and this is especially true for $a \in \{\pm i\}$. Thus the proof of self-adjointness implying (i) is done.

Now suppose (i) holds. We prove that (ii) holds too by showing that $R(T \pm i)$ is closed and dense in H. For definiteness we prove that $R(T - i)$ is closed and dense, where the same follows for $R(T + i)$ by similar arguments. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in H such that $((T - i)\phi_n)_{n\in\mathbb{N}}$ converges towards $\psi \in H$. Them (2.2.1) implies

$$\|(T - i)\phi\|_H^2 = \|T\phi\|_H^2 + \|\phi\|_H^2$$

for all $\phi \in D(T)$, which shows that since $((T - i)\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then so is $(\phi_n)_{n\in\mathbb{N}}$. Hence $(\phi_n)_{n\in\mathbb{N}}$ has a limit ϕ in H. Now T is closed, hence

$$\psi = \lim_{n\to\infty}(T - i)\phi_n = (T - i)\phi \in R(T - i)$$

and $R(T - i)$ is closed.

Assume then that $R(T - i)$ is not dense. Then there exist $\phi \in R(T - i)^{\perp} \setminus \{0\}$, which means

$$0 = \langle\phi, (T - i)(\cdot)\rangle_H|_{D(T)} = \langle 0, \cdot\rangle_H|_{D(T)}.$$

Thus $(T^* + i)\phi = (T - i)^*\phi = 0$ by definition of the adjoint, which is impossible by (i). Hence $R(T - i)$ is dense in H, and together with being closed we deduce $R(T - i) = H$.

Lastly we shall show that (ii) implies that T is self-adjoint by proving $T^* \subseteq T$. Suppose $\phi \in D(T^*)$. By $R(T - i) = H$ there exists $\psi \in D(T)$ such that $(T - i)\psi = (T^* - i)\phi$, and then using symmetry of T we get $(T^* - i)(\psi - \phi) = 0$. But this implies $\langle\psi - \phi, (T + i)(\cdot)\rangle_H|_{D(T)} = 0$, which when added with $R(T + i) = H$ implies $\psi = \phi$. Hence $D(T^*) \subseteq D(T)$. This and T being symmetric shows $T = T^*$. ■

Similar conditions exists for essential self-adjointness.

Corollary 2.2.8. *For a densely defined operator $T \in \mathcal{L}(H)$ the following is equivalent to T being essentially self-adjoint:*

(i) T is symmetric and $\ker(T^* \pm i) = \{0\}$

(ii) T is symmetric and $R(T \pm i)$ are dense in H

Note that both results still hold if $\pm i$ is switched with $a, \bar{a}$ for $a \in \mathbb{C} \setminus \mathbb{R}$. We now give an equivalent criteria for self-adjointness in case of positivity.

Proposition 2.2.9. *A closed, positive operator $T \in \mathcal{L}(H)$ is self-adjoint if and only if* $\ker(T^*) = \{0\}$ *or equivalently $R(T) = H$.*

Proof. It is clear that $\ker(T^*) = \{0\}$ if T is self-adjoint since T is injective, so suppose $\ker(T^*) = \{0\}$ and let us show that T is self-adjoint.

The first step is to show that $R(T)$ is closed, which we do independent of the hypothesis $\ker(T^*) = \{0\}$. Since T is positive, we have for all $\phi \in D(T)$ that

$$m(T)\|\phi\|_H^2 \leq \langle T\phi, \phi\rangle_H \leq \|T\phi\|_H\|\phi\|_H$$

implying

$$\|T\phi\|_H \geq m(T)\|\phi\|_H.$$

Thus if $(\psi_n)_{n\in\mathbb{N}}$ is a sequence in $R(T)$ converging in H, then the sequence $(\phi_n)_{n\in\mathbb{N}}$ in $D(T)$ satisfying $\psi_n = T\phi_n$ for all $n \in \mathbb{N}$ is a Cauchy sequence, hence convergent in H. By T being closed, this implies that

$$\lim_{n\to\infty}\psi_n = T\lim_{n\to\infty}\phi_n \in R(T),$$

and thus the range of T is closed.

Now the identity $\ker(T^*) = R(T)^\perp$ together with $R(T)$ being closed implies $\ker(T^*)^\perp = R(T)$. Hence for T closed and positive, $\ker(T^*) = \{0\}$ is equivalent to $R(T) = H$.

Let us return to the goal of showing that T is self-adjoint. For $\phi \in D(T^*)$ there exists $\psi \in D(T)$ such that $T^*\phi = T\psi$. Then

$$\langle \psi, T(\cdot)\rangle_H|_{D(T)} = \langle T\psi, \cdot\rangle_H|_{D(T)} = \langle T^*\phi, \cdot\rangle_H|_{D(T)} = \langle \phi, T(\cdot)\rangle_H|_{D(T)}$$

and by $R(T) = H$ we conclude $\psi - \phi \in H^\perp$ or equivalently $\psi = \phi$. Hence $T^* \subseteq T$ and so T is self-adjoint. ■

2.2.2 Relatively Bounded Perturbations

A common situation is having an operator with good qualities and then adding a perturbation. We will show an example of when self-adjointness is preserved.

Definition 2.2.10. (Relatively Bounded) For two densely defined operators $T_1, T_2 \in \mathcal{L}(H)$ in a Hilbert space if $D(T_1) \subseteq D(T_2)$ and there exists $a, b \geq 0$ such that

$$\|T_2\phi\|_H \leq a\|T_1\phi\|_H + b\|\phi\|_H$$

holds for all $\phi \in D(T_1)$, then T_2 is called T_1-bounded. If T_2 is called T_1-bounded, then we call

$$a(T_1, T_2) = \inf\{a \geq 0 | \exists b : \|T_2(\cdot)\|_H|_{D(T_1)} \leq a\|T_1(\cdot)\|_H|_{D(T_1)} + b\|\cdot\|_H|_{D(T_1)}\}$$

the relative T_1-bound of T_2.

Note, when $T_1, T_2 \in \mathcal{L}(H)$ are densely defined and closable, if T_2 is T_1-bounded, then $\overline{T_2}$ is $\overline{T_1}$-bounded.

Theorem 2.2.11. *(Kato-Rellich Theorem) If $T_1, T_2 \in \mathcal{L}(H)$ with T_1 self-adjoint and T_2 symmetric, then T_2 being T_1-bounded with $a(T_1, T_2) < 1$ implies that $T_1 + T_2$ is self-adjoint.*

Additionally, if $V \subseteq D(T_1)$ is a vector space and $T_1|_V$ is essentially self-adjoint, then $T_1|_V + T_2$ is essentially self-adjoint.

Proof. Let $a, b \geq 0$ be constants satisfying the relative boundedness condition for T_1 and T_2 with $a < 1$. Note $T_1 + T_2$ is densely defined and symmetric, thus by Proposition 2.2.7 we only need to prove that $R(T_1 + T_2 \pm ci) = H$ for some $c > 0$.

Fixing $c > 0$ for now, we see

$$\|(T_1 - ic)\phi\|_H^2 = \|T_1\phi\|_H^2 + c^2\|\phi\|_H^2$$

for all $\phi \in D(T_1)$ using (2.2.1). Proposition 2.2.7 shows that $(T_1 - ic)^{-1} \colon H \to D(T_1)$ exists and the above shows that $\|(T_1 - ic)^{-1}\|_{\mathcal{B}(H)} \leq \frac{1}{c}$ and $\|T_1(T_1 - ic)^{-1}\|_{\mathcal{B}(H)} \leq 1$. Using that T_2 is T_1-bounded, we get

$$\|T_2(T_1 - ic)^{-1}\phi\|_H \leq a\|T_1(T_1 - ic)^{-1}\phi\|_H + b\|(T_1 - ic)^{-1}\phi\|_H \leq \left(a + \frac{b}{c}\right)\|\phi\|_H$$

for all $\phi \in H$, hence if $c > \frac{b}{1-a}$, then $T_2(T_1 - ic)^{-1}$ has operator norm less than one. This means that $1 + T_2(T_1 - ic)^{-1}$ is bijective with bounded inverse, seen through using a Neumann series. But then since $R(T_1 - ic) = H$ we have

$$H = R((1 + T_2(T_1 - ic)^{-1})(T_1 - ic)) = R(T_1 + T_2 - ic).$$

Proving $R(T_1 + T_2 + ic) = H$ for $c > \frac{b}{1-a}$ follows along the same lines.

Considering the last part, we know that $T_1|_V + T_2$ has a self-adjoint extension $T_1 + T_2$. Since T_2 is T_1-bounded, the completion of $\Gamma(T_1|_V + T_2)$ and $\Gamma(T_1|_V)$ leads to the same domain for the closure, i.e. $D(\overline{T_1|_V + T_2}) = D(\overline{T_1|_V}) = D(T_1)$, implying $\overline{T_1|_V + T_2} = T_1 + T_2$. ■

2.3 Variational Operators

Variational operators make up an important class of operators, and some of the operators we deal with will partly be of this class.

Definition 2.3.1. (Variational Operator) Let V be an inner product space and H a Hilbert space such that $V \hookrightarrow H$ continuously, densely, and algebraically, and let s be a sesquilinear form on V.

Associated with (V, s) we define the variational operator as follows: On the set

$$D(T) = \{\phi \in V | \exists \psi \in H : s(\phi, \cdot) = \langle \psi, \cdot \rangle_H|_V\}$$

define $T\phi \in H$ for $\phi \in D(T)$ by the unique vector such that

$$s(\phi, \cdot) = \langle T\phi, \cdot \rangle_H|_V.$$

Remark! 2.3.2. Note the adjoint operator of some densely defined operator $T \in \mathcal{L}(H)$ is a variational operator associated with $(D(T), s)$, where

$$s\colon D(T) \times D(T) \ni (\phi, \psi) \mapsto \langle \phi, T\psi \rangle_H.$$

The Lax-Milgram Theorem is important when discussing variational operators. Essentially, when s is bounded and elliptic, and $V = H$, then the variational operator is a homeomorphism of H, see [14, Lemma 12.15].

Theorem 2.3.3. *Let V, H be Hilbert spaces such that $V \hookrightarrow H$ continuously, densely, and algebraically, let s be an elliptic and bounded sesquilinear form on V, and let T be the variational operator associated with (V, s). Then T is closed, bijective onto H with bounded inverse, and $D(T)$ is dense in both V and H.*

Furthermore, the variational operator associated with (V, s^*) is the adjoint of T.

Proof. The operator T being closed essentially follows from the boundedness of s, $V \hookrightarrow H$ continuously, and the arguments of Lemma 2.2.2.

Now on to bijectivity. Let $\tilde{T} \in \mathcal{B}(V)$ be the homeomorphism of V associated with s from the Lax-Milgram Theorem. Since $V \hookrightarrow H$ continuously, densely, and algebraically, the injection $\iota\colon H^* \hookrightarrow V^*$ exists, and it is also continuous and linear, with dense range. Thus for each $\phi \in H$ we have

$$\langle \phi, \cdot \rangle_H|_V = \langle \iota\phi, \cdot \rangle_V = s(\tilde{T}^{-1}\iota\phi, \cdot),$$

and we conclude $T\tilde{T}^{-1}\iota\phi = \phi$, so T is surjective onto H. Also for $\phi \in D(T)$ we have

$$s(\phi, \cdot) = \langle T\phi, \cdot \rangle_H|_V = \langle \iota T\phi, \cdot \rangle_V = s(\tilde{T}^{-1}\iota T\phi, \cdot),$$

implying by ellipticity of s that $\phi = \tilde{T}^{-1}\iota T\phi$. Hence $T^{-1} = \tilde{T}^{-1}\iota$, which is a bounded map.

We move on to proving the density of $D(T)$ in H and V. Suppose $D(T)$ is not dense in H or equivalently that there exists $\phi \in H \setminus \{0\}$ orthogonal to $D(T)$ in H. Then for every $\psi \in D(T)$

$$0 = \langle \phi, \psi \rangle_H = \langle TT^{-1}\phi, \psi \rangle_H = s(T^{-1}\phi, \psi).$$

Choosing $\psi = T^{-1}\phi$ we get by ellipticity of s that $\phi = TT^{-1}\phi = T0 = 0$, a contradiction.

Next assume that there exists $\phi \in V \setminus \{0\}$ orthogonal to $D(T)$ in V. Then for every $\psi \in D(T)$

$$0 = \langle \psi, \phi \rangle_V = \langle \tilde{T}\psi, (\tilde{T}^*)^{-1}\phi \rangle_V = s(\psi, (\tilde{T}^*)^{-1}\phi) = \langle T\psi, (\tilde{T}^*)^{-1}\phi \rangle_H,$$

and since $R(T) = H$, then $\phi = \tilde{T}^*(\tilde{T}^*)^{-1}\phi = 0$, a contradiction.

Last statement to prove is that T^* is the variational operator associated with (V, s^*), which we denote by S. Suppose $\phi \in D(S)$. Then for all $\psi \in D(T)$ we have

$$\langle S\phi, \psi \rangle_H = s^*(\phi, \psi) = \overline{s(\psi, \phi)} = \overline{\langle T\psi, \phi \rangle_H} = \langle \phi, T\psi \rangle_H.$$

This implies that $\phi \in D(T^*)$ and $T^*\phi = S\phi$. Thus T^* is an extension of S.

Now suppose $\phi \in D(T^*)$. Then the same calculation as before shows

$$\langle T^*\phi, \psi \rangle_H = s^*(\phi, \psi),$$

hence $\phi \in D(S)$ and $S\phi = T^*\phi$. They are extensions of each other hence equal. ■

Corollary 2.3.4. *Under the hypothesis of Theorem 2.3.3 except replacing ellipticity of s with $\|\cdot\|_H$-coercivity, then T is closed, $D(T)$ is dense in both V and H, and the variational operator associated with (V, s^*) is the adjoint of T.*

Proof. The $\|\cdot\|_H$-coercivity of s implies that there exists $a, b > 0$ such that

$$\operatorname{Re} s(\phi, \phi) + a\|\phi\|_H \geq b\|\phi\|_V,$$

which means that $\tilde{s} := s + a\langle \cdot, \cdot \rangle_H$ is elliptic and $(V, \tilde{s})$ satisfies the hypothesis in Theorem 2.3.3. Thus the variational operator $\tilde{T}$ for $(V, \tilde{s})$ is closed with domain dense in V and H. Since $D(\tilde{T}) = D(T)$ and $T = \tilde{T} - a$, we see that T is likewise closed with domain dense in V and H.

The same procedure with the same constants works for s^* and from Theorem 2.3.3 we get that $T^* = (\tilde{T} - a)^* = \tilde{T}^* - a$ is the variational operator associated with (V, s^*). ■

Corollary 2.3.5. *Under the hypothesis of Corollary 2.3.4, if s is Hermitian, then T is self-adjoint.*

Proof. The additional assumption that $s = s^*$ implies $T = T^*$ since T and its adjoint T^* are both the variational operator associated with (V, s). ■

2.3.1 Friedrich's Extension

Friedrich's extension is a means of extending a symmetric operator which is lower or upper bounded.

Theorem 2.3.6. *(Friedrich's Extension) Every symmetric and lower bounded operator defined on a Hilbert space has a self-adjoint extension satisfying the same lower bound. An analogous statement holds for upper bounded operators.*

Proof. We reduce the theorem to the case of positive operators: If T is upper bounded, then we consider the operator $-T - m(-T) + 1$, and if T is lower bounded, we consider the operator $T - m(T) + 1$.

So let $T \in \mathcal{L}(H)$ be a positive operator. Define the sesquilinear form

$$s \colon D(T) \times D(T) \ni (\phi, \psi) \mapsto \langle T\phi, \psi \rangle_H + \langle \phi, \psi \rangle_H.$$

This is an inner product on $D(T)$, inducing the graph norm of T, and we denote the completion of $D(T)$ in s by V and the extension of s to V by s again. When $D(T)$ is equipped with s, then $D(T) \hookrightarrow H$ continuously, so it extends to a bounded linear map $\iota \colon V \to H$. If this map is injective, then $V \hookrightarrow H$ continuously, densely, and algebraically. Suppose it is not, i.e. there exists $\phi \in V \setminus \{0\}$ for which $\iota\phi = 0$. We then have a sequence $(\phi_n)_{n \in \mathbb{N}}$ in $D(T)$ such that $\phi_n \overset{n\to\infty}{\to} \phi$ in V and $\phi_n \overset{n\to\infty}{\to} 0$ in H. Then

$$s(\psi, \phi) = \lim_{n\to\infty} s(\psi, \phi_n) = \lim_{n\to\infty} \langle T\psi, \phi_n \rangle_H + \langle \psi, \phi_n \rangle_H = 0$$

for all $\psi \in D(T)$. Since $D(T)$ is dense in V, this implies $\phi = 0$, a contradiction. Thus ι is injective and we can identify V with a subspace of H.

By Corollary 2.3.5, the variational operator $\tilde{T}$ associated with $(V, s - \langle \cdot, \cdot \rangle_H)$ is self-adjoint. It also satisfies

$$\langle \tilde{T}\phi, \phi \rangle_H = s(\phi, \phi) - \langle \phi, \phi \rangle_H \geq m(T) \|\phi\|_H^2$$

for all $\phi \in D(\tilde{T})$, so it is positive with lower bound at least $m(T)$. The fact that $T \subseteq \tilde{T}$ also holds leaves us to conclude $m(\tilde{T}) = m(T)$. ■

Friedrich's extension leads to a simple sufficient condition for essential self-adjointness, building upon Proposition 2.2.9.

Corollary 2.3.7. *A positive operator $T \in \mathcal{L}(H)$ is essentially self-adjoint if and only if* $\ker(T^*) = \{0\}$.

Proof. By Friedrich's Extension 2.3.6, $\overline{T}$ is necessarily positive. Since $\overline{T}^* = T^*$ the rest of the corollary follows from Proposition 2.2.9. ■

2.4 Spectral Theory

Here we focus on the ever so interesting and useful concepts such as spectrum, resolvent, and functional calculus.

Definition 2.4.1. (Resolvent Set, Spectrum) For an operator $T \in \mathcal{L}(V)$ we call

$$\rho(T) = \{z \in \mathbb{C} \,|\, \ker(T - z) = \{0\}, R(T - z) = V\}$$

the resolvent set of T and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ the spectrum of T. The set of eigenvalues we call the point spectrum of T and denote by $\sigma_p(T)$.

In addition, we call

$$\rho(T) \ni z \mapsto (T - z)^{-1} \in \mathcal{L}(V)$$

the resolvent of T.

The resolvent might not seem like much at first glance, but it has nice properties. E.g. For a closed operator $T \in \mathcal{L}(V)$ and $z \in \rho(T)$, the operator $(T-z)^{-1}$ is defined on entirety of V and has a closed graph. Thus if V is a Fréchet space, then by the Closed Graph Theorem $(T-z)^{-1} \in \mathcal{B}(V)$. To find out more, we have to utilize complex analysis.

Theorem 2.4.2. *When $T \in \mathcal{L}(B)$, B a Banach space, is closed and densely defined, then $\rho(T)$ is open and the resolvent is a $\mathcal{B}(B)$-valued holomorphic function.*

Proof. Let $w \in \rho(T)$ be given. Then for every $z \in B_{\|(T-w)^{-1}\|_{\mathcal{B}(B)}}(w;\mathbb{C})$, the power series

$$S_z = \sum_{n=0}^{\infty}(z-w)^n(T-w)^{-n-1}$$

converges absolutely, hence $S_z \in \mathcal{B}(B)$. For such a z we want to confirm that S_z is an inverse to $T-z$. By a short computation

$$\begin{aligned} S_z(T-z) &= \sum_{n=0}^{\infty}(z-w)^n(T-w)^{-n-1}(T-w+w-z) \\ &= \sum_{n=0}^{\infty}(z-w)^n(T-w)^{-n}\mathrm{id}_{D(T)} - \sum_{n=0}^{\infty}(z-w)^{n+1}(T-w)^{-n-1}\mathrm{id}_{D(T)} \\ &= \mathrm{id}_{D(T)} \end{aligned}$$

and similarly, since $R(S_z) \subseteq D(T)$,

$$\begin{aligned} (T-z)S_z &= (T-w+w-z)\sum_{n=0}^{\infty}(z-w)^n(T-w)^{-n-1} \\ &= \sum_{n=0}^{\infty}(z-w)^n(T-w)^{-n} - \sum_{n=0}^{\infty}(z-w)^{n+1}(T-w)^{-n-1} \\ &= \mathrm{id}. \end{aligned}$$

This shows both $\ker(T-z) = \{0\}$ and $R(T-z) = B$, so $z \in \rho(T)$ and we conclude that $\rho(T)$ is open. Also $(T-z)^{-1} = S_z$ for $z \in B_{\|(T-w)^{-1}\|_{\mathcal{B}(B)}}(w;\mathbb{C})$, thus the resolvent is locally expandable into a power series. ■

Theorem 2.4.3. *For $T \in \mathcal{B}(B)$, B a Banach space, then $\sigma(T)$ is non-empty and contained in $\overline{B_{\|T\|_{\mathcal{B}(B)}}(0;\mathbb{C})}$. Moreover,*

$$\sup_{\lambda\in\sigma(T)}|\lambda| = \lim_{n\to\infty}\|T^n\|_{\mathcal{B}(B)}^{\frac{1}{n}}.$$

Proof. Suppose $z \notin \overline{B_{\|T\|_{\mathcal{B}(B)}}(0;\mathbb{C})}$. Then writing $T - z = z(z^{-1}T - \mathrm{id})$ and expanding $z^{-1}T - \mathrm{id}$ in a Neumann series, we see that $z \in \rho(T)$.

If $\sigma(T) = \emptyset$, then the resolvent of T is an entire function by Theorem 2.4.2. From the first paragraph one also gets $(T-z)^{-1} = z^{-1}(z^{-1}T - \mathrm{id})^{-1}$ for $z \in \mathbb{C}\setminus\{0\}$, whence $\|(T-z)^{-1}\|_{\mathcal{B}(B)} \overset{|z|\to\infty}{\to} 0$. Thus Liouville's Theorem tells us that the resolvent of T is constant, which together with the limit tells us that the resolvent is zero everywhere, an impossibility. In conclusion $\sigma(T) \neq \emptyset$.

For the limit result we get a bit creative. Defining $S\colon \{0\} \cup \{z^{-1} | z \in \rho(T) \setminus \{0\}\} \to \mathcal{B}(B)$ by $S(0) = 0$ and otherwise $S(z) = (T - z^{-1})^{-1}$ we get a holomorphic function. In fact for $z \in B_{\|T\|_{\mathcal{B}(B)}}(0;\mathbb{C})$:

$$S(z) = -z \sum_{n=0}^{\infty} (zT)^n$$

The inverse of the radius of convergence for this power series is both $\limsup_{n\to\infty} \|T^n\|_{\mathcal{B}(B)}^{\frac{1}{n}}$ and $\inf_{z\in D(S)^c} |z| = \sup_{\lambda\in\sigma(T)} |\lambda|$, so

$$\sup_{\lambda\in\sigma(T)} |\lambda| = \limsup_{n\to\infty} \|T^n\|_{\mathcal{B}(B)}^{\frac{1}{n}}.$$

Now, $z \in \sigma(T)$ implies $z^n \in \sigma(T^n)$ and by that has been proven we must have $|z|^n \leq \|T^n\|_{\mathcal{B}(B)}$. Hence $|z| \leq \liminf_{n\to\infty} \|T^n\|_{\mathcal{B}(B)}^{\frac{1}{n}}$, so taking the supremum one gets:

$$\sup_{z\in\sigma(T)} |z| \leq \liminf_{n\to\infty} \|T^n\|_{\mathcal{B}(B)}^{\frac{1}{n}},$$

giving us the desired conclusion. ■

The following identities are called the resolvent identities. The one stated in the lemma is called the second, while the one after is called the first.

Lemma 2.4.4. *If $T_1, T_2 \in \mathcal{L}(V)$ have the same domain and $z \in \rho(T_1) \cap \rho(T_2)$, then*

$$(T_1 - z)^{-1} - (T_2 - z)^{-1} = (T_1 - z)^{-1}(T_2 - T_1)(T_2 - z)^{-1}.$$

Proof. Since $D(T_1) = D(T_2)$ it holds that

$$T_1 - z = T_2 - z - (T_2 - T_1),$$

so

$$\mathrm{id}_{D(T_1)} = (T_1 - z)^{-1}(T_2 - z - (T_2 - T_1))$$

and

$$\begin{aligned}(T_2 - z)^{-1} &= (T_1 - z)^{-1}(\mathrm{id} - (T_2 - T_1)(T_2 - z)^{-1}) \\ &= (T_1 - z)^{-1} - (T_1 - z)^{-1}(T_2 - T_1)(T_2 - z)^{-1}.\end{aligned}$$ ■

The difficulty of the proof lies in checking the domains of different compositions and additions of the operators in the above calculation commute in the sense of distributive laws. Here it was enough that the operators in question had equal domain.

For $T \in \mathcal{L}(V)$ and $z, w \in \rho(T)$ the first resolvent identity follows from the first by choosing $T_1 := T$ and $T_2 := T + z - w$ resulting in

$$(T - z)^{-1} - (T - w)^{-1} = (T - z)^{-1}(z - w)(T - w)^{-1}.$$

Remark! 2.4.5. Note that for a self-adjoint operator T on a Hilbert space H, $\mathbb{C} \setminus \mathbb{R} \subseteq \rho(T)$. Also in the case of a self-adjoint operator T we have for $z \in \mathbb{C} \setminus \mathbb{R}$ that

$$\|(T - z)\phi\|_H^2 = \|(T - \mathrm{Re}(z))\phi\|_H^2 + \mathrm{Im}(z)^2\|\phi\|_H^2$$

for all $\phi \in D(T)$, as noted before in (2.2.1), so

$$\|(T - z)^{-1}\|_{\mathcal{B}(H)} \leq |\,\mathrm{Im}(z)|^{-1}.$$

A sharper estimate is presented later in Corollary 2.4.24.

2.4.1 Resolution of the Identity

The spectral theorems covered in this report will be in the form of a resolution of the identity, and we will therefore introduce them here.

Definition 2.4.6. (Resolution of the Identity) A family of orthogonal projections $(E_\lambda)_{\lambda\in\mathbb{R}}$ on a Hilbert space H is called a resolution of the identity if it is non-decreasing, strongly right-continuous,

$$E_\lambda \overset{\lambda\to-\infty}{\to} 0$$

and

$$E_\lambda \overset{\lambda\to\infty}{\to} \mathrm{id}$$

both being strong limits.

We say that a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ is bounded if there exists $\mu, \nu \in \mathbb{R}$ such that $E_\mu = 0$ and $E_\nu = \mathrm{id}$.

For a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ and a measurable function $f\colon \mathbb{R} \to \mathbb{C}$, we want to ascribe an operator to the symbol $\int_{\mathbb{R}} f(\lambda)dE_\lambda$, meant as the limit of Stieltjes-type sums in concept.

Let us first consider f equal to a simple function of the type $\sum_{n=0}^{N} c_n 1_{(a_n,b_n]}$, where the intervals are disjoint. Then for each $\phi \in H$ it would be natural to define

$$\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi := \sum_{n=0}^{N} c_n(E_{b_n} - E_{a_n})\phi$$

and then it would follow that

$$\left\|\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi\right\|_H^2 = \sum_{n=0}^{N} \overline{c_n}c_n\langle(E_{b_n} - E_{a_n})\phi, (E_{b_n} - E_{a_n})\phi\rangle_H = \int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda\phi\|_H^2, \quad (2.4.1)$$

where the integral is taken in the Stieltjes-Lebesgue sense with Stieltjes-Lebesgue measure induced by the non-decreasing, right-continuous function $\mathbb{R} \ni \lambda \mapsto \|E_\lambda\phi\|_H^2$.

Now functions of the above type are dense in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$ for every $\phi \in H$. So if $f \in L^2(\mathbb{R}, d\|E.\phi\|_H^2)$ for some $\phi \in H$, then $\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi$ could be defined as the limit $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(\lambda)dE_\lambda\phi$ in H, where $(f_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of simple functions of the above type converging towards f in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$. This limit is exists and is independent of choice of sequence by (2.4.1).

Henceforth we denote by $BM(\mathbb{R})$ the complex-valued Borel measurable functions and $S(\mathbb{R})$ the simple functions on left-open, right-closed intervals.

Definition 2.4.7. For a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ and $f \in BM(\mathbb{R})$ we define the set

$$D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right) = \left\{\phi \in H \middle| \int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda\phi\|_H^2\right\}$$

and on it the operator $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ by the following procedure: For each $\phi \in D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)$ find any arbitrary sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mathbb{R})$ converging toward f in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$ and then define

$$\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(\lambda)dE_\lambda\phi$$

with convergence in H-norm.

Remark! 2.4.8. From our definition of integration w.r.t. $(E_\lambda)_{\lambda\in\mathbb{R}}$ it follows that the extremities of (2.4.1) holds for all $f \in BM(\mathbb{R})$ and $\phi \in D\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda\right)$.

It is now on us to prove that this definition gives reasonable results: Firstly, for every $\mu \in \mathbb{R}$ we have by dominated convergence that $1_{(\lambda,\mu]} \overset{\lambda\to-\infty}{\to} 1_{(-\infty,\mu]}$ in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$, $\phi \in H$. Thus

$$\int_{\mathbb{R}} 1_{(-\infty,\mu]}(\lambda) dE_\lambda \phi = \lim_{\lambda\to-\infty} (E_\mu - E_\lambda)\phi = E_\mu \phi,$$

where $E_\lambda \overset{\lambda\to-\infty}{\to} 0$ strongly by definition. Furthermore, again by dominated convergence $1_{(-\lambda,\lambda]} \overset{\lambda\to\infty}{\to} 1_{\mathbb{R}}$ in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$ for each $\phi \in H$, showing that

$$\int_{\mathbb{R}} 1_{\mathbb{R}}(\lambda) dE_\lambda \phi = \lim_{\lambda\to-\infty} (E_\lambda - E_{-\lambda})\phi = \phi,$$

using again $E_\lambda \overset{\lambda\to-\infty}{\to} 0$ and also $E_\lambda \overset{\lambda\to\infty}{\to} \text{id}$ strongly.

Now onto a more general result.

Proposition 2.4.9. *Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a resolution of the identity. For every $f \in BM(\mathbb{R})$ the operator $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ is densely defined and closed. Moreover, for $f, g \in BM(\mathbb{R})$ and $a \in \mathbb{C} \setminus \{0\}$*

(i) $\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda\right)^* = \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda$

(ii) $a \int_{\mathbb{R}} f(\lambda) dE_\lambda = \int_{\mathbb{R}} a f(\lambda) dE_\lambda$

(iii) $\int_{\mathbb{R}} f(\lambda) dE_\lambda + \int_{\mathbb{R}} g(\lambda) dE_\lambda \subseteq \int_{\mathbb{R}} (f+g)(\lambda) dE_\lambda$

(iv) $\int_{\mathbb{R}} f(\lambda) dE_\lambda \int_{\mathbb{R}} g(\lambda) dE_\lambda \subseteq \int_{\mathbb{R}} (fg)(\lambda) dE_\lambda$

Proof. The space $S(\mathbb{R})$ is mapped into the space of projections and the algebraic properties listed above holds for $f, g \in S(\mathbb{R})$, which one checks by straightforward calculation.

Next step is looking at $f, g \in L^\infty(\mathbb{R})$. Then quite clearly

$$\int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda \phi\|_H^2 \leq \|f\|_{L^\infty(\mathbb{R})}^2 \|\phi\|_H^2 < \infty$$

for all $\phi \in H$, which shows that $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ is everywhere defined and together with (2.4.1) that $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ is bounded with bound less than or equal to $\|f\|_{L^\infty(\mathbb{R})}$. Similarly for g. The rest of the properties are checked by choosing an approximating sequence from $S(\mathbb{R})$ at every point.

Let $\phi, \psi \in H$ be given and let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be a sequence in $S(\mathbb{R})$ converging to respectively f and g in $L^2(\mathbb{R}, d\|E.\phi\|_H^2) \cap L^2(\mathbb{R}, d\|E.\psi\|_H^2)$. Then $(af_n + g_n)_{n\in\mathbb{N}}$ is in $S(\mathbb{R})$ and converges to $af + g$ in $L^2(\mathbb{R}, d\|E.\phi\|_H^2) \cap L^2(\mathbb{R}, d\|E.\psi\|_H^2)$, and linearity follows

$$\begin{aligned}\int_{\mathbb{R}} (af+g)(\lambda) dE_\lambda \phi &= \lim_{n\to\infty} \int_{\mathbb{R}} (af_n + g_n)(\lambda) dE_\lambda \phi \\ &= a \lim_{n\to\infty} \int_{\mathbb{R}} f_n(\lambda) dE_\lambda \phi + \lim_{n\to\infty} \int_{\mathbb{R}} g_n(\lambda) dE_\lambda \phi \\ &= a \int_{\mathbb{R}} f(\lambda) dE_\lambda \phi + \int_{\mathbb{R}} g(\lambda) dE_\lambda \phi.\end{aligned}$$

For adjoints we again compute

$$\left\langle \int_{\mathbb{R}} f(\lambda) dE_\lambda \phi, \psi \right\rangle_H = \lim_{n\to\infty} \left\langle \int_{\mathbb{R}} f_n(\lambda) dE_\lambda \phi, \psi \right\rangle_H = \lim_{n\to\infty} \left\langle \phi, \int_{\mathbb{R}} \overline{f_n(\lambda)} dE_\lambda \psi \right\rangle_H$$
$$= \left\langle \phi, \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda \psi \right\rangle_H,$$

and from this we get a proof of multiplicativity on $L^\infty(\mathbb{R})$:

$$\begin{aligned}\left\langle \int_{\mathbb{R}} f(\lambda) dE_\lambda \int_{\mathbb{R}} g(\lambda) dE_\lambda \phi, \psi \right\rangle_H &= \left\langle \int_{\mathbb{R}} g(\lambda) dE_\lambda \phi, \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda \psi \right\rangle_H \\ &= \lim_{n\to\infty} \left\langle \int_{\mathbb{R}} g_n(\lambda) dE_\lambda \phi, \int_{\mathbb{R}} \overline{f_n(\lambda)} dE_\lambda \psi \right\rangle_H \\ &= \lim_{n\to\infty} \left\langle \int_{\mathbb{R}} (f_n g_n)(\lambda) dE_\lambda \phi, \psi \right\rangle_H \\ &= \left\langle \int_{\mathbb{R}} (fg)(\lambda) dE_\lambda \phi, \psi \right\rangle_H.\end{aligned}$$

Fix $f \in BM(\mathbb{R})$. Consider $P_n := \int_{\mathbb{R}} 1_{|f|^{-1}([n-1,n))}(\lambda) dE_\lambda$ for some $n \in \mathbb{N}$. Clearly $1_{|f|^{-1}([n-1,n))} \in L^\infty(\mathbb{R})$ and by the above properties P_n is an orthogonal projection. Suppose $\phi \in R(P_n)$. Then

$$\|\phi\|_H^2 = \|P_n \phi\|_H^2 = \int_{|f|^{-1}([n-1,n))} d\|E_\lambda \phi\|_H^2,$$

which implies that $f \in L^\infty(\mathbb{R}, d\|E_\lambda \phi\|_H^2)$ with norm less than or equal to n. Thus $\phi \in D\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda\right)$. Since n was chosen arbitrarily, $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ is defined on each of the space $R(P_n)$, $n \in \mathbb{N}$. Actually $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ invariates each of these spaces and it is bounded with bound less than or equal to n. Let $\phi \in R(P_n)$, $n \in \mathbb{N}$, and choose a sequence $(f_m)_{m\in\mathbb{N}}$ in $S(\mathbb{R})$ converging to f in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$. Then

$$P_n \int_{\mathbb{R}} f(\lambda) dE_\lambda \phi = P_n \lim_{m\to\infty} \int_{\mathbb{R}} f_m(\lambda) dE_\lambda \phi = \lim_{m\to\infty} P_n \int_{\mathbb{R}} f_m(\lambda) dE_\lambda \phi$$
$$= \lim_{m\to\infty} \int_{\mathbb{R}} f_m(\lambda) dE_\lambda P_n \phi = \int_{\mathbb{R}} f(\lambda) dE_\lambda \phi.$$

The bound on $\int_{\mathbb{R}} f(\lambda) dE_\lambda|_{R(P_n)}$ follows from $f \in L^\infty(\mathbb{R}, d\|E.\phi\|_H^2)$, $\phi \in R(P_n)$, with norm less than or equal to n as deduced earlier. These orthogonal projection $(P_n)_{n\in\mathbb{N}}$ are pairwise orthogonal by our functional calculus on $L^\infty(\mathbb{R})$, and by dominated convergence

$$\left\| \int_{\mathbb{R}} 1_{|f|^{-1}([0,n))}(\lambda) dE_\lambda \phi - \phi \right\|_H^2 = \int_{|f|^{-1}([n,\infty))} d\|E_\lambda \phi\|_H^2 \overset{n\to\infty}{\to} 0$$

for every $\phi \in H$, and this has the implication that

$$\sum_{n=1}^\infty P_n = \sum_{n=1}^\infty \int_{\mathbb{R}} 1_{|f|^{-1}([n-1,n))}(\lambda) dE_\lambda = \mathrm{id}$$

with strong convergence. Combining these results we are able to show that $D\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda\right)$ is dense in H: For $\phi \in H$, $\phi = \sum_{n=1}^\infty P_n \phi$ in norm and each finite truncation of the sum is in $D\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda\right)$.

To prove that $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ is closed, let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in H with limit ϕ and such that $(\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi_n)_{n\in\mathbb{N}}$ has limit ψ. Then

$$P_m\psi = \lim_{n\to\infty} P_m \int_{\mathbb{R}} f(\lambda)dE_\lambda\phi_n = \lim_{n\to\infty}\int_{\mathbb{R}} f(\lambda)dE_\lambda P_m\phi_n = \int_{\mathbb{R}} f(\lambda)dE_\lambda P_m\phi$$

and monotone convergence of measures gives

$$\begin{aligned}\int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda\phi\|_H^2 &= \sum_{m=1}^{\infty}\int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda P_m\phi\|_H^2 = \sum_{m=1}^{\infty}\left\|\int_{\mathbb{R}} f(\lambda)dE_\lambda P_m\phi\right\|_H^2 \\ &= \sum_{m=1}^{\infty}\|P_m\psi\|_H^2 = \|\psi\|_H^2 < \infty.\end{aligned}$$

Thus $\phi \in D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)$. Furthermore,

$$\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi = \sum_{m=1}^{\infty} P_m\int_{\mathbb{R}} f(\lambda)dE_\lambda\phi = \sum_{m=1}^{\infty}\int_{\mathbb{R}} f(\lambda)dE_\lambda P_m\phi = \sum_{m=1}^{\infty} P_m\psi = \psi.$$

Hence $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ is closed.

The algebraic properties now follow for $BM(\mathbb{R})$ functions as they did for $L^\infty(\mathbb{R})$ functions, except we only get $\int_{\mathbb{R}} \overline{f(\lambda)}dE_\lambda \subseteq \left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)^*$. Again fix $f \in BM(\mathbb{R})$ and define the projections $(P_n)_{n\in\mathbb{N}}$ as above. Suppose $\phi \in D\left(\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)^*\right)$. Then

$$\left\langle \psi, P_n\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)^*\phi\right\rangle_H = \left\langle \int_{\mathbb{R}} f(\lambda)dE_\lambda\psi, P_n\phi\right\rangle_H = \left\langle \psi, \int_{\mathbb{R}} \overline{f(\lambda)}dE_\lambda P_n\phi\right\rangle_H$$

for all $\psi \in R(P_n)$. Thus $P_n\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)^*\phi = \int_{\mathbb{R}} \overline{f(\lambda)}dE_\lambda P_n\phi$. Now reusing arguments from the last paragraph let us conclude that:

$$\int_{\mathbb{R}} \overline{f(\lambda)}dE_\lambda \subseteq \left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)^* \qquad \blacksquare$$

In the above proof we showed that for $f \in L^\infty(\mathbb{R})$ we had $\int_{\mathbb{R}} f(\lambda)dE_\lambda \in \mathcal{B}(H)$ with norm less than or equal to $\|f\|_{L^\infty(\mathbb{R})}$. This is not necessarily the maximal set of functions mapped into bounded operators by our integral, we will instead need to analyse functions essentially bounded w.r.t. $(E_\lambda)_{\lambda\in\mathbb{R}}$, meaning functions in $\bigcap_{\phi\in H} L^\infty(\mathbb{R}, d\|E.\phi\|_H^2)$ with norms uniformly bounded. Here the idea is that if $\Omega \subseteq \mathbb{R}$ is a Borel measurable set and $\int_{\mathbb{R}} 1_\Omega(\lambda)dE_\lambda = 0$, then it implies that Ω is a null set for each of the measures $d\|E.\phi\|_H^2$.

Definition 2.4.10. For a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ and we call a Borel measurable set $\Omega \subseteq \mathbb{R}$ a $(E_\lambda)_{\lambda\in\mathbb{R}}$-null set if

$$\int_{\mathbb{R}} 1_\Omega(\lambda)dE_\lambda = 0.$$

We say $\Theta \subseteq \mathbb{R}$ $(E_\lambda)_{\lambda\in\mathbb{R}}$-almost everywhere if $\mathbb{R}\setminus\Theta$ is a subset of a $(E_\lambda)_{\lambda\in\mathbb{R}}$-null set.

A function $f \in BM(\mathbb{R})$ is called essentially bounded w.r.t. $(E_\lambda)_{\lambda\in\mathbb{R}}$ if

$$\inf\left\{a \in [0,\infty) \,|\, |f|^{-1}((a,\infty)) \text{ is a } (E_\lambda)_{\lambda\in\mathbb{R}}\text{-null set}\right\} < \infty.$$

The space of functions essentially bounded w.r.t. $(E_\lambda)_{\lambda\in\mathbb{R}}$ will be denoted by $L^\infty(\mathbb{R}, E)$ and for $f \in L^\infty(\mathbb{R}, E)$ the above infimum is denoted by $\|f\|_{L^\infty(\mathbb{R},E)}$.

Note $L^\infty(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}, E)$ continuously.

Lemma 2.4.11. *For a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ we have $f \in L^\infty(\mathbb{R}, E)$ if and only if $f \in \bigcap_{\phi\in H} L^\infty(\mathbb{R}, d\|E.\phi\|_H^2)$ and*

$$\sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\} < \infty.$$

In the positive case, $\|f\|_{L^\infty(\mathbb{R},E)}$ equals the above supremum.

Proof. If $\int_{\mathbb{R}} 1_{|f|^{-1}((a,\infty))}(\lambda)dE_\lambda = 0$ holds for some $f \in BM(\mathbb{R})$ and $a \in [0, \infty)$, then it implies that $|f|^{-1}((a, \infty))$ is a null set for each of the measures $d\|E.\phi\|_H^2$ and hence

$$\sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\} \le a.$$

This shows that $f \in L^\infty(\mathbb{R}, E)$ implies $f \in \bigcap_{\phi\in H} L^\infty(\mathbb{R}, d\|E.\phi\|_H^2)$ and

$$\sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\} \le \|f\|_{L^\infty(\mathbb{R},E)}.$$

Conversely, if $f \in \bigcap_{\phi\in H} L^\infty(\mathbb{R}, d\|E.\phi\|_H^2)$ and

$$a := \sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\} < \infty,$$

then necessarily

$$\int_{\mathbb{R}} 1_{|f|^{-1}((a,\infty))}(\lambda)dE_\lambda = 0$$

by (2.4.1).

We know $\sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\} \le \|f\|_{L^\infty(\mathbb{R},E)}$ for every $f \in L^\infty(\mathbb{R}, E)$, so assume the inequality is strict for some f. Then there would exists an $a \in [0, \infty)$ strictly between $\sup\{\|f\|_{L^\infty(\mathbb{R},d\|E.\phi\|_H^2)} | \phi \in H\}$ and $\|f\|_{L^\infty(\mathbb{R},E)}$ such that $|f|^{-1}((a, \infty))$ is a null set for each of the measures $d\|E.\phi\|_H^2$. Hence by (2.4.1) we must have

$$\int_{\mathbb{R}} 1_{|f|^{-1}((a,\infty))}(\lambda)dE_\lambda = 0,$$

a contradiction. ∎

Proposition 2.4.12. *For a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ the space $L^\infty(\mathbb{R}, E)$ is a C*-algebra when $L^\infty(\mathbb{R}, E)$ is equipped with pointwise multiplication and conjugation, and the map*

$$L^\infty(\mathbb{R}, E) \ni f \mapsto \int_{\mathbb{R}} f(\lambda)dE_\lambda$$

*has image in $\mathcal{B}(H)$ and is a continuous *-algebra-homomorphism into $\mathcal{B}(H)$. Furthermore,*

$$\left\| \int_{\mathbb{R}} f(\lambda)dE_\lambda \right\|_{\mathcal{B}(H)} = \|f\|_{L^\infty(\mathbb{R},E)}.$$

Proof. First we remark that the fact that $L^\infty(\mathbb{R}, E)$ is a C*-algebra follows directly from Lemma 2.4.11. By our definition of $L^\infty(\mathbb{R}, E)$ and (2.4.1), for $f \in L^\infty(\mathbb{R}, E)$ we necessarily have

$$\int_{\mathbb{R}} |f(\lambda)|^2 d\|E_\lambda\phi\|_H^2 \le \|f\|_{L^\infty(\mathbb{R},E)}^2 \|\phi\|_H^2,$$

which first implies $D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right) = H$ and secondly that $\int_{\mathbb{R}} f(\lambda)dE_\lambda \in \mathcal{B}(H)$ with norm less than or equal to $\|f\|_{L^\infty(\mathbb{R},E)}$. Thirdly, we now also have that the map in question is continuous. Note this mimics the calculation for $L^\infty(\mathbb{R})$ functions in the proof of Proposition 2.4.9.

Now the algebraic properties of the map in the proposition simply follows from Proposition 2.4.9 and the fact we only deal with bounded operators. The last identity holds since

$$\int_{\mathbb{R}} dE_\lambda = \mathrm{id}$$

and both $1_{\mathbb{R}}$ and id have norm 1 in the respective spaces. ∎

We also have a version of dominated convergence:

Lemma 2.4.13. *Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a resolution of the identity. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^\infty(\mathbb{R}, E)$ converging pointwise to $f \in L^\infty(\mathbb{R}, E)$ $(E_\lambda)_{\lambda\in\mathbb{R}}$-almost everywhere and being uniformly bounded w.r.t. $(E_\lambda)_{\lambda\in\mathbb{R}}$. Then $\int_{\mathbb{R}} f_n(\lambda)dE_\lambda \overset{n\to\infty}{\to} \int_{\mathbb{R}} f(\lambda)dE_\lambda$ strongly.*

Proof. For each $\phi \in H$ we simply have that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$ by dominated convergence, hence

$$\int_{\mathbb{R}} f_n(\lambda)dE_\lambda\phi \overset{n\to\infty}{\to} \int_{\mathbb{R}} f(\lambda)dE_\lambda\phi$$

in norm by (2.4.1). ∎

Remark! 2.4.14. We have given a strong pointwise definition of integral w.r.t. a resolution of the identity. There also exists a weak variational approach, one we state in this remark.

Fix a resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$. By first verifying the following for functions in $S(\mathbb{R})$ and then by approximation extending it to $BM(\mathbb{R})$ we get

$$\left\langle \int_{\mathbb{R}} f(\lambda)dE_\lambda\phi, \psi \right\rangle_H = \int_{\mathbb{R}} f(\lambda)d\langle E_\lambda\phi, \psi\rangle_H \tag{2.4.2}$$

for $f \in BM(\mathbb{R})$ and $\phi, \psi \in D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)$. The right-hand side of (2.4.2) is well-defined in that $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda\phi, \psi\rangle_H$ is of bounded variation, as seen through the polarization identity. Moreover, it defines a sesquilinear form s on $V := D\left(\int_{\mathbb{R}} f(\lambda)dE_\lambda\right)$. The operator $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ is now easily seen to be the variational operator associated with the space (V, s) by the identity (2.4.2).

2.4.2 Spectral Theorem for Bounded Self-adjoint Operators

It is time to prove the first spectral theorem, here for bounded self-adjoint operators. First, we define the usual continuous functional calculus.

Theorem 2.4.15. *(Continuous Functional Calculus) Let $T \in \mathcal{B}(H)$ be self-adjoint. There exists a unique continuous C*-algebra-homomorphism $\mathcal{T}\colon C(\sigma(T)) \to \mathcal{B}(H)$ such that $\mathcal{T}1 = \mathrm{id}$ and $\mathcal{T}\mathrm{id}_{\sigma(T)} = T$. In addition $\|\mathcal{T}\|_{\mathcal{B}(C(\sigma(T)),\mathcal{B}(H))} = 1$ and if $f \in C(\sigma(T))$ is non-negative, then $\mathcal{T}f$ is non-negative.*

Proof. Let $P(\mathbb{R})$ denote the set of complex-valued polynomials on $\mathbb{R}$ and consider the canonical *-algebra-homomorphism $P(\mathbb{R}) \ni p \mapsto p(T)$. This is the unique *-algebra-homomorphism on $P(\mathbb{R})$ into $\mathcal{B}(H)$ satisfying $\mathcal{T}1 = \mathrm{id}$ and $\mathcal{T}\mathrm{id}_{\sigma(T)} = T$. Fix some $p \in P(\mathbb{R})$. We will prove $p(\sigma(T)) = \sigma(p(T))$, and then secondly $\|p(T)\|_{\mathcal{B}(H)} = \sup_{\lambda\in\sigma(T)} |p(\lambda)|$.

Let $\mu \in \sigma(T)$. Then $p(\lambda) - p(\mu) = (\lambda - \mu)q(\lambda)$ for all $\lambda \in \mathbb{R}$ for some $q \in P(\mathbb{R})$, thus $p(T) - p(\mu) = (T - \mu)q(T)$. Since $T - \mu$ has no inverse we conclude $p(\mu) \in \sigma(p(T))$ and $p(\sigma(T)) \subseteq \sigma(p(T))$. Conversely, if $\mu \in \sigma(p(T))$, then factoring $p - \mu$ into linear factors we get $p(T) - \mu = a \prod_{j=1}^{\deg(p)} (T - \mu_j)$ with $a \neq 0$ and $\mu_i \in \mathbb{C}$ for $j = 1, \ldots, \deg(p)$. If all of the factors $T - \mu_j$ are invertible, then $p(T) - \mu$ would be too, a contradiction. Thus $\mu_j \in \sigma(T)$ for some j, meaning $\mu = p(\mu_j) \in \sigma(p(T))$, so $p(\sigma(T)) = \sigma(p(T))$. Now

$$\|p(T)\|_{\mathcal{B}(H)}^2 = \|p^*(T)p(T)\|_{\mathcal{B}(H)} = \|(\overline{p}p)(T)\|_{\mathcal{B}(H)}.$$

Using that $(\overline{p}p)(T)$ is self-adjoint we know that $\|(\overline{p}p)(T)^2\|_{\mathcal{B}(H)} = \|(\overline{p}p)(T)\|_{\mathcal{B}(H)}^2$, whence Theorem 2.4.3 leads us to

$$\|p(T)\|_{\mathcal{B}(H)}^2 = \sup_{\lambda \in (\overline{p}p)(\sigma(T))} |\lambda| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|^2.$$

Let us move on to the C*-algebra $C(\sigma(T))$. $P(\mathbb{R})$ is dense in $C(\sigma(T))$, so the canonical *-algebra-homomorphism $P(\mathbb{R}) \ni p \mapsto p(T)$ has a unique extension to a bounded operator $\mathcal{T} \colon C(\sigma(T)) \to \mathcal{B}(H)$, also satisfying $\mathcal{T}1 = \mathrm{id}$ and $\mathcal{T}\mathrm{id}_{\sigma(T)} = T$. Furthermore,

$$\|\mathcal{T}f\|_{\mathcal{B}(H)} = \|f\|_{C(\sigma(T))}$$

for all $f \in C(\sigma(T))$. Lastly, if $f \in C(\sigma(T))$ is non-negative, then $\mathcal{T}f = (\mathcal{T}\sqrt{f})^2$ is non-negative. ■

We usually use the notation $f(T) = \mathcal{T}f$ for $f \in C(\sigma(T))$.

Corollary 2.4.16. *Let $T \in \mathcal{B}(H)$ be self-adjoint. If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $C(\sigma(T))$ converging pointwise to $f \in C(\sigma(T))$, then $f_n(T) \overset{n\to\infty}{\to} f(T)$ strongly.*

Proof. Fix $\phi \in H$. The map $C(\sigma(T)) \ni g \mapsto \langle g(T)\phi, \phi\rangle_H$ is a positive linear functional, so from the Riesz–Markov–Kakutani Representation Theorem there exists a positive measure m such that

$$\langle g(T)\phi, \phi\rangle_H = \int_{\sigma(T)} g dm.$$

Hence

$$\|f_n(T)\phi - f(T)\phi\|_H^2 = \langle |f_n - f|^2(T)\phi, \phi\rangle_H = \int_{\sigma(T)} |f_n - f|^2 dm.$$

By dominated convergence we get $\int_{\sigma(T)} |f_n - f|^2 dm \overset{n\to\infty}{\to} 0$, hence $f_n(T)\phi \overset{n\to\infty}{\to} f(T)\phi$ in norm. ■

We are just about ready to construct a resolution of the identity, we will only be needing one more lemma.

Lemma 2.4.17. *Every bounded monotonic sequence of bounded self-adjoint operators has a limit in norm.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a bounded monotonic sequence of bounded self-adjoint operators on H. If $(T_n)_{n\in\mathbb{N}}$ is non-increasing, then we consider $(-T_n)_{n\in\mathbb{N}}$ instead, and furthermore if T_1 is not non-negative, then we consider $(T_n - m(T_1))_{n\in\mathbb{N}}$. Thus we reduce the statement to that of a bounded non-decreasing sequence of bounded, non-negative self-adjoint operators.

Fix $\phi \in H$. Now for $m, n \in \mathbb{N}$, $n < m$, we have $T_m - T_n \geq 0$ and hence

$$\begin{aligned} \|T_m\phi - T_n\phi\|_H^2 &= \langle (T_m - T_n)\phi, (T_m - T_n)\phi\rangle_H \\ &\leq \langle (T_m - T_n)\phi, \phi\rangle_H \langle (T_m - T_n)^2\phi, (T_m - T_n)\phi\rangle_H \\ &\leq (\langle T_m\phi, \phi\rangle_H - \langle T_n\phi, \phi\rangle_H)\|T_m - T_n\|_{\mathcal{B}(H)}^3 \|\phi\|_H^2. \end{aligned}$$

Here we used that $H \times H \ni (\phi, \psi) \mapsto (T\phi, \psi)$ is a semi-inner product whenever T is everywhere defined and non-negative, and so the Cauchy-Schwartz inequality holds. By hypothesis $(\langle T_n\phi, \phi\rangle_H)_{n\in\mathbb{N}}$ is a bounded non-decreasing sequence in $\mathbb{R}$, hence it has a limit. Since also $(T_n)_{n\in\mathbb{N}}$ is bounded, then the above inequalities implies that $(T_n\phi)_{n\in\mathbb{N}}$ is a Cauchy sequence in H. Thus $(T_n)_{n\in\mathbb{N}}$ has a strong limit and it is bounded, which together implies that that $(T_n)_{n\in\mathbb{N}}$ has a limit in norm. ■

Theorem 2.4.18. *(Spectral Theorem for Bounded Self-adjoint Operators) For every bounded self-adjoint operator T there exists a unique resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$, which is bounded, such that*

$$T = \int_{\mathbb{R}} \lambda dE_\lambda.$$

We call $(E_\lambda)_{\lambda\in\mathbb{R}}$ the spectral resolution of T.

Proof. Let some $\mu \in \mathbb{R}$ be given. Find a sequence $(f_n)_{n\in\mathbb{N}}$ of real-valued functions in $BC(\mathbb{R})$ that decrease pointwise to $1_{(-\infty,\mu]}$ on $\sigma(T)$. Then $f_n(T)$ is defined for every $n \in \mathbb{N}$ by the Continuous Functional Calculus 2.4.15 and $(f_n(T))_{n\in\mathbb{N}}$ is a bounded monotonic sequence of bounded self-adjoint operators. Thus $(f_n(T))_{n\in\mathbb{N}}$ has a limit in norm from Lemma 2.4.17 and we define the limit as E_μ. If $(g_n)_{n\in\mathbb{N}}$ is some other sequence of real-valued functions in $BC(\mathbb{R})$ that decrease pointwise to $1_{(-\infty,\mu]}$ on on $\sigma(T)$, then $(g_n - f_n)_{n\in\mathbb{N}}$ goes to zero pointwise on $\sigma(T)$ and are bounded. Hence by Corollary 2.4.16

$$\lim_{n\to\infty} g_n(T) = \lim_{n\to\infty} f_n(T) = E_\mu.$$

Thus E_μ is independent of approximating sequence. Since $(f_n^2)_{n\in\mathbb{N}}$ satisfies this criteria, we must have $E_\mu^2 = E_\mu$, hence E_μ is an orthogonal projection.

Thus we have created a family of orthogonal projections $(E_\lambda)_{\lambda\in\mathbb{R}}$. Let us show that it is a resolution of the identity.

To show that it is non-decreasing, let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ be sequence in $BC(\mathbb{R})$ as above for $\mu, \nu \in \mathbb{R}$ respectively, where $\mu < \nu$. Then $(f_n g_n)_{n\in\mathbb{N}}$ decreasing pointwise to $1_{(-\infty,\mu]}$ on $\sigma(T)$, so

$$E_\mu = \lim_{n\to\infty} f_n(T)g_n(T) = E_\nu E_\mu = E_\mu E_\nu.$$

By this we can conclude $E_\mu \leq E_\nu$.

Next we prove continuity from the right. Let $\mu \in \mathbb{R}$ and construct a sequence of real-valued functions $(f_n)_{n\in\mathbb{N}}$ from $BC(\mathbb{R})$ such that $f_n \geq 1_{(-\infty,\mu+\frac{1}{n}]}$ on $\sigma(T)$ for all $n \in \mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ decrease pointwise to $1_{(-\infty,\mu]}$ on $\sigma(T)$. Then $f_n(T) \geq E_{\mu+\frac{1}{n}}$ and $\lim_{n\to\infty} f_n(T) = E_\mu$, hence $E_\lambda \overset{\lambda\to\mu^+}{\to} E_\mu$ by monotonicity.

To prove the limits at $-\infty$ and ∞ it will be sufficient to prove boundedness: If $\mu < \inf \sigma(T)$, then $(0)_{n\in\mathbb{N}}$ decrease pointwise to $1_{(-\infty,\mu]}$ on $\sigma(T)$, hence $E_\mu = 0$. Similarly, if $\mu > \sup \sigma(T)$, then $(1)_{n\in\mathbb{N}}$ decrease pointwise to $1_{(-\infty,\mu]}$ on $\sigma(T)$, hence $E_\mu = \text{id}$.

So we have proven that $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a resolution of the identity. Left is only to prove the main identity $T = \int_{\mathbb{R}} \lambda dE_\lambda$. For $\mu, \nu \in \mathbb{R}$, $\mu < \nu$, let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ be sequence in $BC(\mathbb{R})$ as above for μ, ν respectively such that $f_n \leq g_n$ and $g_n \equiv 1$ on $(-\infty, \mu]$ for all $n \in \mathbb{N}$. Using these sequences we get

$$\mu(E_\nu - E_\mu) = \lim_{n\to\infty} \mu(g_n(T) - f_n(T)) \leq \lim_{n\to\infty} T(g_n(T) - f_n(T)) = T(E_\nu - E_\mu)$$

by $\mu(g_n - f_n) \leq \mathrm{id}_{\sigma(T)}(g_n - f_n)$ on $\sigma(T)$ for all $n \in \mathbb{N}$ leading to $\mu(g_n(T) - f_n(T)) \leq T(g_n(T) - f_n(T))$, and the norm convergence of the sequences $(f_n(T))_{n\in\mathbb{N}}, (g_n(T))_{n\in\mathbb{N}}$. Similarly $T(E_\nu - E_\mu) \leq \nu(E_\nu - E_\mu)$. From this we see that for every finite, increasing sequence $(\mu_j)_{j=0}^N$, $N \in \mathbb{N}$, in $\mathbb{R}$ that

$$\sum_{j=1}^N \mu_{j-1}(E_{\mu_j} - E_{\mu_{j-1}}) \leq \sum_{j=1}^N T(E_{\mu_j} - E_{\mu_{j-1}}) = T \leq \sum_{j=1}^N \mu_j(E_{\mu_j} - E_{\mu_{j-1}}).$$

We can now choose partitions such that the left and right side converge strongly to $\int_{\mathbb{R}} \lambda dE_\lambda$ by Lemma 2.4.13. But then $T = \int_{\mathbb{R}} \lambda dE_\lambda$ must hold as desired.

Suppose $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ is another resolution of the identity such that $T = \int_{\mathbb{R}} \lambda d\tilde{E}_\lambda$. Then

$$\int_{\mathbb{R}} p(\lambda)dE_\lambda = p(T) = \int_{\mathbb{R}} p(\lambda)d\tilde{E}_\lambda$$

for every $p \in P(\mathbb{R})$. If $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ is unbounded, then we may find $\phi \in H$ and a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$, which is bounded in $L^2(\mathbb{R}, d\|E.\phi\|_H^2)$, but $\|p_n\|_{L^2(\mathbb{R},d\|\tilde{E}.\phi\|_H^2)} \overset{n\to\infty}{\to} \infty$. This contradicts the fact that $\|p_n\|_{L^2(\mathbb{R},d\|E.\phi\|_H^2)} = \|p_n\|_{L^2(\mathbb{R},d\|\tilde{E}.\phi\|_H^2)}$ for all $n \in \mathbb{N}$, thus $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ must be a bounded resolution of the identity.

Since $(E_\lambda)_{\lambda\in\mathbb{R}}$ and $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ are bounded, every $1_{(-\infty,\mu]}$, $\mu \in \mathbb{R}$, is the pointwise limit of a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ $(E_\lambda)_{\lambda\in\mathbb{R}}$- and $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$-almost everywhere, with $(p_n)_{n\in\mathbb{N}}$ bounded in $L^\infty(\mathbb{R}, E) \cap L^\infty(\mathbb{R}, \tilde{E})$. Applying Lemma 2.4.13, we get $E_\mu = \tilde{E}_\mu$. ■

It should not be hard to deduce from the above proof that $f \in BM(\mathbb{R})$ is in $f \in L^\infty(\mathbb{R}, E)$ as long as f is essentially bounded on $\sigma(T)$. Hence $C(\sigma(T)) \hookrightarrow L^\infty(\mathbb{R}, E)$ when extending functions $C(\sigma(T))$ to the entire real line in a continuous, but otherwise arbitrary manner. Now the uniqueness in the Continuous Functional Calculus 2.4.15 shows that

$$\mathcal{T}f = \int_{\mathbb{R}} \tilde{f}(\lambda)dE_\lambda,$$

where $\tilde{f}$ is a continuous extension of f.

For this reason we will also denote the operator $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ by $f(T)$ for each $f \in BM(\mathbb{R})$.

2.4.3 Spectral Theorem for Self-adjoint Operators

Moving on, we go now to the unbounded case and prove the spectral theorem for general self-adjoint operators. But to prove this version, we will need to reduce our problem to the bounded case, which the two following lemmas ensure is possible.

Lemma 2.4.19. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of orthogonal projections on H such that $P_nP_m = 0$ for all $n, m \in \mathbb{N}$ and*

$$\sum_{n=1}^\infty P_n = \mathrm{id},$$

and suppose $(T_n)_{n\in\mathbb{N}}$ *is a sequence of operators such that* $T_n \in \mathcal{B}(R(P_n))$ *and* T_n *is self-adjoint. Then there exists a unique self-adjoint* $T \in \mathcal{L}(H)$ *such that* $TP_n = T_nP_n$ *for all* $n \in \mathbb{N}$*. In fact,*

$$D(T) = \left\{ \phi \in H \middle| \sum_{n=1}^{\infty} \|T_nP_n\phi\|_H^2 < \infty \right\}$$

and for $\phi \in D(T)$

$$T\phi = \sum_{n=1}^{\infty} T_nP_n\phi.$$

Remark! 2.4.20. This lemma and its proof should remind the reader about some of the reasoning in the proof of Proposition 2.4.9. With some adjustments, the above lemma also provides a definition of $\int_{\mathbb{R}} f(\lambda)dE_\lambda$ for general $f \in BM(\mathbb{R})$.

Proof. The set $D(T)$ as defined in the lemma is dense in H since for every $\phi \in H$ we have $\phi = \sum_{n=1}^{\infty} P_n\phi$ in norm and $\sum_{n=1}^{N} P_n\phi \in D(T)$ for each $N \in \mathbb{N}$. Additionally the sum defining T is well-defined on $D(T)$: Actually for any $\phi \in H$, the condition $\sum_{n=1}^{\infty} \|T_nP_n\phi\|_H^2 < \infty$ is equivalent with $\sum_{n=1}^{\infty} T_nP_n\phi$ converging since the sum has orthogonal elements. This follows from:

$$\left\| \sum_{n=N}^{M} T_nP_n\phi \right\|_H^2 = \sum_{n=N}^{M} \langle T_nP_n\phi, T_nP_n\phi\rangle_H = \sum_{n=N}^{M} \|T_nP_n\phi\|_H^2,$$

where $N, M \in \mathbb{N}$.

The identity $TP_n = T_nP_n$ for $n \in \mathbb{N}$ follows from $R(P_n) \subseteq D(T)$ and

$$TP_n\phi = \sum_{m=1}^{\infty} T_mP_mP_n\phi = T_nP_n\phi,$$

which holds for all $\phi \in H$. This leads to a proof of T being symmetric, since for $\phi, \psi \in D(T)$ one then has

$$\langle T\phi, \psi\rangle_H = \sum_{n=1}^{\infty} \langle TP_n\phi, P_n\psi\rangle_H = \sum_{n=1}^{\infty} \langle T_nP_n\phi, P_n\psi\rangle_H = \sum_{n=1}^{\infty} \langle P_n\phi, T_nP_n\psi\rangle_H = \langle \phi, T\psi\rangle_H.$$

So to prove that T is self-adjoint we only have to show $T^* \subseteq T$. Suppose $\phi \in D(T^*)$. Then

$$\langle \psi, P_nT^*\phi\rangle_H = \langle TP_n\psi, \phi\rangle_H = \langle T_n\psi, P_n\phi\rangle_H = \langle \psi, T_nP_n\phi\rangle_H$$

holds for all $\psi \in R(P_n)$ and $n \in \mathbb{N}$, hence $P_nT^*\phi = T_nP_n\phi$, so

$$\sum_{n=1}^{\infty} \|T_nP_n\phi\|_H^2 = \sum_{n=1}^{\infty} \|P_nT^*\phi\|_H^2 = \|T^*\phi\|_H^2 < \infty.$$

Thus $T^* \subseteq T$ as desired.

Assume then that $\tilde{T}$ is a self-adjoint operator on H such that $\tilde{T}P_n = T_nP_n$ for all $n \in \mathbb{N}$. Then we have $\sum_{n=1}^{N} T_nP_n\phi = \tilde{T}\sum_{n=1}^{N} P_n\phi$ for any $\phi \in H$ and $N \in \mathbb{N}$. Since $\tilde{T}$ must be closed, one has that if $\sum_{n=1}^{\infty} T_nP_n\phi$ converges for some $\phi \in H$, then it follows that

$$\tilde{T}\phi = \sum_{n=1}^{\infty} T_nP_n\phi = T\phi.$$

Hence $T \subseteq \tilde{T}$, and because T is self-adjoint it cannot have any strict self-adjoint extensions, so we conclude $T = \tilde{T}$. ■

Lemma 2.4.21. *For a densely defined closed operator $T \in \mathcal{L}(H)$ in a Hilbert space, the operators $(1+T^*T)^{-1}$ and $T(1+T^*T)^{-1}$ are members of $\mathcal{B}(H)$ both with norm less than or equal to 1. Furthermore, $(1+T^*T)^{-1}$ is non-negative and self-adjoint.*

Proof. Let U be defined as in the proof of Proposition 2.2.3 so that $H^2 = \Gamma(T) \oplus U^*\Gamma(T^*)$ and $\Gamma(T) \perp U^*(\Gamma(T^*))$. Thus for each $\phi \in H$, $(\phi, 0)$ has a unique decomposition into the sum of a element of $\Gamma(T)$ and $U^*\Gamma(T^*)$, i.e. there exists unique $\psi \in D(T), \omega \in D(T^*)$ such that

$$\phi = \psi + T^*\omega$$

and

$$0 = T\psi - \omega.$$

Let $S_1, S_2 \in \mathcal{L}(H)$ be the everywhere defined operators such that $\phi = S_1\phi + T^*S_2\phi$ and $0 = TS_1\phi - S_2\phi$. Then $S_2 = TS_1$ and $\text{id} = S_1 + T^*TS_1 = (1+T^*T)S_1$. The operator $1+T^*T$ is positive, hence injective and it has an inverse on $R(1+T^*T)$, which must equal H by $\text{id} = (1+T^*T)S_1$. Hence $S_1 = (1+T^*T)^{-1}$.

The boundedness of S_1, S_2 follows from the definition in that

$$\begin{aligned}\|\phi\|_H^2 = \|(\phi,0)\|_{H^2}^2 &= \|(S_1\phi, TS_1\phi)\|_{H^2}^2 + \|(T^*S_2\phi, -S_2\phi)\|_{H^2}^2 \\ &= \|S_1\phi\|_H^2 + \|TS_1\phi\|_H^2 + \|T^*S_2\phi\|_H^2 + \|S_2\phi\|_H^2\end{aligned}$$

for all $\phi \in H$. For the non-negativity and self-adjointness of S_1 let $\phi, \psi \in H$ be given. Then firstly

$$\langle S_1\phi, \psi\rangle_H = \langle S_1\phi, (1+T^*T)S_1\psi\rangle_H = \langle S_1\phi, S_1\psi\rangle_H + \langle TS_1\phi, TS_1\psi\rangle_H$$

by which we conclude that if $\phi = \psi$, then $\langle S_1\phi, \phi\rangle_H \geq 0$. Continuing the above calculation we get

$$\begin{aligned}\langle S_1\phi, \psi\rangle_H &= \langle S_1\phi, S_1\psi\rangle_H + \langle TS_1\phi, TS_1\psi\rangle_H \\ &= \langle S_1\phi, S_1\psi\rangle_H + \langle T^*TS_1\phi, S_1\psi\rangle_H = \langle \phi, S_1\psi\rangle_H,\end{aligned}$$

showing that S_1 is self-adjoint. ■

Theorem 2.4.22. *(Spectral Theorem for Self-adjoint Operators) For every self-adjoint operator T there exists a unique resolution of the identity $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that*

$$T = \int_{\mathbb{R}} \lambda dE_\lambda.$$

As in the bounded case, $(E_\lambda)_{\lambda\in\mathbb{R}}$ will be called the spectral resolution of T.

Proof. The operator $S_1 = (1+T^2)^{-1}$ is by Lemma 2.4.21 a bounded self-adjoint operator, hence by the Spectral Theorem for Bounded Self-adjoint Operators 2.4.18 there exists a bounded resolution of the identity $(F_\lambda)_{\lambda\in\mathbb{R}}$ such that

$$S_1 = \int_{\mathbb{R}} \lambda dF_\lambda.$$

Since $0 \leq m(S_1)$ and $M(S_1) \leq 1$, then $\sigma(S_1) \subseteq [0,1]$. Thus if we define the sequence $(P_n)_{n\in\mathbb{N}}$ of orthogonal projections by $P_n = F_{\frac{1}{n}} - F_{\frac{1}{n+1}} = 1_{(\frac{1}{n+1},\frac{1}{n}]}(S_1)$ for each $n \in \mathbb{N}$, then $\sum_{n=1}^\infty P_n = \text{id}$ and

$$P_nP_m = (1_{(\frac{1}{n+1},\frac{1}{n}]}1_{(\frac{1}{m+1},\frac{1}{m}]})(S_1) = 0$$

for all $n, m \in \mathbb{N}$, $n \neq m$.

We want to use Lemma 2.4.19 on T and $(P_n)_{n\in\mathbb{N}}$, thus it is essential that T invariates $R(P_n)$ and $T|_{R(P_n)} \in \mathcal{B}(R(P_n))$ for each $n \in \mathbb{N}$. We have

$$S_1 T = S_1 T(1+T^2)S_1 = S_1(1+T^2)TS_1 \subseteq TS_1,$$

and then letting $S_2 := TS_1$ leads to

$$S_1 S_2 = S_1 T S_1 \subseteq T S_1 S_1 = S_2 S_1.$$

Both S_1 and S_2 are bounded from Lemma 2.4.21, so $S_1 S_2 = S_2 S_1$. Since S_2 and S_1 commute, then as a consequence S_2 must also commute with all functions of S_1. For each $n \in \mathbb{N}$ define $f_n \colon \mathbb{R} \ni \lambda \mapsto \frac{1}{\lambda} 1_{(\frac{1}{n+1}, \frac{1}{n}]}$, so that $f_n(S_1)S_1 = P_n$. Then

$$TP_n = TS_1 f_n(S_1) = S_2 f_n(S_1)$$

and

$$P_n T = f_n(S_1)S_1 T \subseteq f_n(S_1)TS_1 = f_n(S_1)S_2$$

hence $P_n T \subseteq TP_n$ for all $n \in \mathbb{N}$. So T invariates $R(P_n)$ and $TP_n \in \mathcal{B}(H)$, implying also that $T|_{R(P_n)} \in \mathcal{B}(R(P_n))$.

To each of the bounded self-adjoint operators $T|_{R(P_n)}$ we have a bounded resolution of the identity $(E_{n,\lambda})_{\lambda\in\mathbb{R}}$ by Theorem 2.4.18. From Lemma 2.4.19 we have for each $\lambda \in \mathbb{R}$ a unique self-adjoint operator E_λ such that $E_\lambda P_n = E_{n,\lambda} P_n$ for all $n \in \mathbb{N}$. Since for $\phi \in H$ we have

$$\sum_{n=1}^{\infty} \|E_{n,\lambda} P_n \phi\|_H^2 \leq \sum_{n=1}^{\infty} \|P_n \phi\|_H^2 = \|\phi\|_H^2,$$

so $D(E_\lambda) = H$. Moreover, it should be clear that $E_\lambda^2 = E_\lambda$ using the pairwise orthogonality of $(P_n)_{n\in\mathbb{N}}$. We conclude that E_λ is an orthogonal projection. Next step is proving that $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a resolution of the identity.

For $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ we have

$$E_\mu E_\nu \phi = \sum_{n=1}^{\infty} E_{n,\mu} P_n \sum_{m=1}^{\infty} E_{m,\nu} P_m \phi = \sum_{n=1}^{\infty} E_{n,\mu} E_{n,\nu} P_n \phi = \sum_{n=1}^{\infty} E_{n,\mu} P_n \phi = E_\mu \phi$$

for all $\phi \in H$ which implies $E_\mu E_\nu = E_\mu$, and by a similar calculation one shows $E_\nu E_\mu = E_\mu$. Hence $(E_\lambda)_{\lambda\in\mathbb{R}}$ is increasing. The properties of right-continuity and limits at $-\infty$ and ∞ follow from dominated convergence: Let us prove that $E_\lambda \overset{\lambda\to\infty}{\to}$ id strongly. For $\phi \in H$ we compute

$$\|(\mathrm{id} - E_\lambda)\phi\|_H^2 = \sum_{n=1}^{\infty} \|(\mathrm{id}_{R(P_n)} - E_{n,\lambda})P_n \phi\|_H^2$$

Because $(E_{n,\lambda})_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity we have $\lim_{\lambda\to\infty} \|\mathrm{id}_{R(P_n)} - E_{n,\lambda}\|_{\mathcal{B}(R(P_n))} = 0$ for all $n \in \mathbb{N}$. Also

$$\sum_{n=1}^{\infty} \|(\mathrm{id}_{R(P_n)} - E_{n,\lambda})P_n \phi\|_H^2 \leq \sum_{n=1}^{\infty} 4\|P_n \phi\|_H^2 = 4\|\phi\|_H^2,$$

hence we may use dominated convergence to conclude that

$$\lim_{\lambda\to\infty} \|(\mathrm{id} - E_\lambda)\phi\|_H^2 = \sum_{n=1}^{\infty} \lim_{\lambda\to\infty} \|(\mathrm{id}_{R(P_n)} - E_{n,\lambda})P_n\phi\|_H^2 = 0.$$

So $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a resolution of the identity, and we are only missing the identity

$$T = \int_{\mathbb{R}} \lambda dE_\lambda.$$

For any $\phi \in H$ the following holds by monotone convergence of measures

$$\sum_{n=1}^{\infty} \left\| \int_{\mathbb{R}} \lambda dE_{n,\lambda} P_n\phi \right\|_H^2 = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \lambda^2 d\|E_{n,\lambda}P_n\phi\|_H^2 = \int_{\mathbb{R}} \lambda^2 d\|E_\lambda\phi\|_H^2.$$

Thus $D(T) = D\left(\int_{\mathbb{R}} \lambda dE_\lambda\right)$. Next, by the pointwise definition of $\int_{\mathbb{R}} \lambda dE_\lambda$ we see that

$$\int_{\mathbb{R}} \lambda dE_\lambda P_n = \int_{\mathbb{R}} \lambda dE_{n,\lambda} P_n = TP_n,$$

by which we conclude $T = \int_{\mathbb{R}} \lambda dE_\lambda$ using Lemma 2.4.19.

For uniqueness, assume $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ is another resolution of the identity such that $T = \int_{\mathbb{R}} \lambda d\tilde{E}_\lambda$. Then each $\tilde{E}_\lambda$ commutes with T and hence commutes with S_1. More significantly each $\tilde{E}_\lambda$ must commute with P_n for each $n \in \mathbb{N}$ since P_n is the strong limit of polynomials in S_1. Then for each $n \in \mathbb{N}$, $(E_\lambda|_{R(P_n)})_{\lambda\in\mathbb{R}} = (E_{n,\lambda})_{\lambda\in\mathbb{R}}$ and $(\tilde{E}_\lambda|_{R(P_n)})_{\lambda\in\mathbb{R}}$ are both spectral resolutions for $T|_{R(P_n)}$. But $T|_{R(P_n)}$ has a unique spectral resolution by the Spectral Theorem for Bounded Self-adjoint Operators 2.4.18. With the help of Lemma 2.4.19 we conclude that $(E_\lambda)_{\lambda\in\mathbb{R}}$ and $(\tilde{E}_\lambda)_{\lambda\in\mathbb{R}}$ must be equal. ■

Like with the bounded version, for any function $f \in BM(\mathbb{R})$ we let $f(T)$ denote $\int_{\mathbb{R}} f(\lambda)dE_\lambda$.

Combining Proposition 2.4.9 and Theorem 2.4.22 we see that there is a one-to-one correspondence between resolution of the identity and self-adjoint operators. Furthermore, bounded resolutions of the identity correspond to bounded self-adjoint operators.

2.4.4 Stone's Formula

As was teased after the bounded version of the spectral theorem, the spectral resolution is "supported" on the spectrum of the associated operator. We look into this connection in this subsection.

Proposition 2.4.23. *Let $T \in \mathcal{L}(H)$ be self-adjoint. Then:*

(i) $\rho(T) = \{\lambda \in \mathbb{R} | \exists \varepsilon > 0 : 1_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) = 0\}$

(ii) $\sigma(T) = \{\lambda \in \mathbb{R} | \forall \varepsilon > 0 : 1_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) \neq 0\}$

(iii) $\sigma_p(T) = \{\lambda \in \mathbb{R} | 1_{\{\lambda\}}(T) \neq 0\}$

Proof. Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral resolution of T.

If $1_{(\mu-\varepsilon,\mu+\varepsilon)}(T) = 0$ for some $\mu \in \mathbb{R}$ and $\varepsilon > 0$, then the function $f\colon \mathbb{R} \ni \lambda \mapsto 1_{(\mu-\varepsilon,\mu+\varepsilon)^c}(\lambda)(\lambda-\mu)^{-1}$ is in $L^\infty(\mathbb{R}, E)$. It follows that $Tf(T) = \mathrm{id}$ and $f(T)T = \mathrm{id}_{D(T)}$ by the calculus of Proposition 2.4.9, whence $\mu \in \rho(T)$.

Conversely, suppose for some $\mu \in \mathbb{R}$, $1_{(\mu-\varepsilon,\mu+\varepsilon)}(T) \neq 0$ for all $\varepsilon > 0$. Then for each $n \in \mathbb{N}$ we may find $\phi_n \in R(1_{(\mu-n^{-1},\mu+n^{-1})}(T)) \cap \partial B_1(0;H)$ and consequently we have a sequence in $\partial B_1(0;H)$ for which

$$\|(T-\mu)\phi_n\|_H^2 = \int_{\mu-n^{-1}}^{\mu+n^{-1}} |\lambda-\mu|^2 d\|E_\lambda\phi_n\|_H^2 \leq n^{-2} \overset{n\to\infty}{\to} 0.$$

So $\mu \in \sigma(T)$.

This finishes (i) and (ii). For (iii), assume first that $\phi \in R(1_{\{\mu\}}(T)) \setminus \{0\}$. Then

$$(T-\mu)\phi = \int_{\{\mu\}} (\lambda-\mu) dE_\lambda \phi = 0,$$

implying that $\mu \in \sigma_p(T)$.

Next, say $\mu \in \sigma_p(T)$ and $\phi \in H \setminus \{0\}$ is a corresponding eigenvector. Define $f_\varepsilon \colon \mathbb{R} \ni \lambda \mapsto 1_{(\mu-\varepsilon,\mu+\varepsilon)^c}(\lambda)(\lambda-\mu)^{-1}$ for each $\varepsilon > 0$. Then

$$1_{(\mu-\varepsilon,\mu+\varepsilon)^c}(T)\phi = f_\varepsilon(T)(T-\mu)\phi = 0,$$

and so $\phi = 1_{(\mu-\varepsilon,\mu+\varepsilon)}(T)\phi \overset{\varepsilon\to 0}{\to} 1_{\{\mu\}}(T)\phi$, as desired. ■

This gives us a sharper estimate for the norm of the resolvent:

Corollary 2.4.24. *Let $T \in \mathcal{L}(H)$ be self-adjoint. Then for each $z \in \rho(T)$ we have*

$$\|(T-z)^{-1}\|_{\mathcal{B}(H)} = \frac{1}{d(z,\sigma(T))}.$$

Proof. From everything we gathered up till this point we get

$$\|(T-z)^{-1}\|_{\mathcal{B}(H)} = \sup_{\lambda\in\sigma(T)} |\lambda-z|^{-1}.$$ ■

Lastly, we present Stone's formula showing that the spectral resolution can be recovered from the resolvent.

Theorem 2.4.25. *(Stone's Formula) For every self-adjoint operator T and $\lambda_1, \lambda_2 \in \mathbb{R}$ it holds that*

$$\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2} \left((T-\lambda-i\varepsilon)^{-1} - (T-\lambda+i\varepsilon)^{-1}\right) d\lambda \overset{\varepsilon\to 0}{\to} \frac{1}{2}\left(1_{[\lambda_1,\lambda_2]}(T) + 1_{(\lambda_1,\lambda_2)}(T)\right)$$

strongly.

Proof. Fix $\phi \in H$. Given $\varepsilon > 0$, the Bochner integral $\int_{\lambda_1}^{\lambda_2} \left((T-\lambda-i\varepsilon)^{-1} - (T-\lambda+i\varepsilon)^{-1}\right) d\lambda$ is well-defined and using the variational approach in Remark 2.4.14 we get:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2} \left((T-\lambda-i\varepsilon)^{-1} - (T-\lambda+i\varepsilon)^{-1}\right) d\lambda\phi \\ &\quad= \int_{\mathbb{R}} \frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2} \left((\mu-\lambda-i\varepsilon)^{-1} - (\mu-\lambda+i\varepsilon)^{-1}\right) d\lambda dE_\mu\phi\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2} \left((\mu-\lambda-i\varepsilon)^{-1} - (\mu-\lambda+i\varepsilon)^{-1}\right) d\lambda &= \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2} d\lambda \\ &\overset{\varepsilon\to 0}{\to} \frac{1}{2}\left(1_{[\lambda_1,\lambda_2]}(\mu) + 1_{(\lambda_1,\lambda_2)}(\mu)\right)\end{aligned}$$

for every $\mu \in \mathbb{R}$ and the functions involved are dominated by 1, the theorem follows from Lemma 2.4.13. ■

2.4.5 Helffer-Sjöstrand Formula

The Helffer-Sjöstrand formula relates the operator $f(T)$ of certain well-behaved functions f to a complex Cauchy-type integral of the resolvent for T, generalization a formula for functions. It is entirely possible to work out a functional calculus for these functions without using the results of the prior subsections as done in [10]. We concentrate our effort on proving the formula for Schwartz functions.

We shall introduce the Schwartz space of functions more fully in the next chapter, but currently a definition will be sufficient. The Schwartz space on $\mathbb{R}$ is:

$$\mathscr{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) | \sup_{\mathbb{R}} \langle \cdot \rangle^m |\partial^n f| < \infty, \forall m, n \in \mathbb{N}_0\}$$

Fix $f \in \mathscr{S}(\mathbb{R})$ and construct $\chi \in C_c^\infty(\mathbb{R})$ such that $1_{[-1,1]} \leq \chi \leq 1_{[-2,2]}$. Then for any $N \in \mathbb{N}$ we define

$$\tilde{f}_N : \mathbb{C} \ni z = x + iy \mapsto \chi\left(\frac{y}{\langle x \rangle}\right) \sum_{n=0}^{N} \frac{i^n}{n!} \partial^n f(x) y^n.$$

No matter the chosen N we call $\tilde{f}_N$ an almost analytic extension of f.

Lemma 2.4.26. *Let $f \in \mathscr{S}(\mathbb{R})$ and $N \in \mathbb{N}$. The almost analytic extension $\tilde{f}_N$ satisfies:*

(i) $\tilde{f}_N|_{\mathbb{R}} = f|_{\mathbb{R}}$

(ii) $|\tilde{f}_N(z)| \leq C_m \langle z \rangle^{-m}$ *for all* $m \geq 0$

(iii) $|\overline{\partial_z} \tilde{f}_N(z)| \leq C_{l,m} |\operatorname{Im}(z)|^l \langle z \rangle^{-m}$ *for all* $l \in \{0, 1, \ldots, N\}$ *and* $m \geq 0$.

Here we note for the uninitiated that $\overline{\partial_z} = \frac{1}{2}(\partial_x + i\partial_y)$.

Proof. (i) is immediate. Pick $m \geq 0$ in (ii). Using the cutoff function χ we get the estimates

$$|y|\chi\left(\frac{y}{\langle x \rangle}\right) \leq 2\langle x \rangle \chi\left(\frac{y}{\langle x \rangle}\right) \leq \sqrt{8}(1 + |x|)\chi\left(\frac{y}{\langle x \rangle}\right)$$

and

$$\begin{aligned} \langle z \rangle \chi\left(\frac{y}{\langle x \rangle}\right) &\leq \sqrt{1 + (|x| + |y|)^2}\chi\left(\frac{y}{\langle x \rangle}\right) \leq \sqrt{1 + (\sqrt{8} + (1 + \sqrt{8})|x|)^2}\chi\left(\frac{y}{\langle x \rangle}\right) \\ &\leq \sqrt{17 + 2(1 + \sqrt{8})^2 |x|^2}\chi\left(\frac{y}{\langle x \rangle}\right) \leq \max\{\sqrt{17}, \sqrt{2}(1 + \sqrt{8})\}\langle x \rangle \chi\left(\frac{y}{\langle x \rangle}\right) \end{aligned}$$

for all $z \in \mathbb{C}$. This especially shows that

$$\langle x \rangle^{-1} \chi\left(\frac{y}{\langle x \rangle}\right) \leq C\langle z \rangle^{-1} \chi\left(\frac{y}{\langle x \rangle}\right) \leq C\langle z \rangle^{-1}.$$

Thus

$$|\tilde{f}_N(z)| \leq \chi\left(\frac{y}{\langle x \rangle}\right) \sum_{n=0}^{N} \frac{1}{n!} |\partial^n f(x)| |y|^n \leq \chi\left(\frac{y}{\langle x \rangle}\right) \sum_{n=0}^{N} \frac{C_{m,n}}{n!} \langle x \rangle^{-n-m} \langle x \rangle^n \leq C_m \langle z \rangle^{-m},$$

where $m \geq 0$ is arbitrary.

Next pick $l \in \{0, 1, \ldots, N\}$ and $m \geq 0$. First we see that

$$\begin{aligned}
2\overline{\partial_z}\tilde{f}_N(z) &= \chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=0}^{N}\frac{i^n}{n!}\partial^{n+1}f(x)y^n - \frac{yx}{\langle x\rangle^3}\partial\chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=0}^{N}\frac{i^n}{n!}\partial^n f(x)y^n \\
&\quad - \chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=1}^{N}\frac{i^{n-1}}{(n-1)!}\partial^n f(x)y^{n-1} + \frac{i}{\langle x\rangle}\partial\chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=0}^{N}\frac{i^n}{n!}\partial^n f(x)y^n \\
&= \chi\left(\frac{y}{\langle x\rangle}\right)\frac{i^N}{N!}\partial^{N+1}f(x)y^N + \left(\frac{yx}{\langle x\rangle^3} - \frac{i}{\langle x\rangle}\right)\partial\chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=0}^{N}\frac{i^n}{n!}\partial^n f(x)y^n.
\end{aligned}$$

Dealing with one term at the time we get

$$\chi\left(\frac{y}{\langle x\rangle}\right)|\partial^{N+1}f(x)y^N| \leq C_{l,m}\chi\left(\frac{y}{\langle x\rangle}\right)|y|^l\langle x\rangle^{-m} \leq C_{l,m}|y|^l\langle z\rangle^{-m},$$

and with the extra estimates

$$\langle x\rangle\partial\chi\left(\frac{y}{\langle x\rangle}\right) \leq |y|\partial\chi\left(\frac{y}{\langle x\rangle}\right) \leq 2\langle x\rangle\partial\chi\left(\frac{y}{\langle x\rangle}\right),$$

we also get

$$\begin{aligned}
\left|\left(\frac{yx}{\langle x\rangle^3} - \frac{i}{\langle x\rangle}\right)\partial\chi\left(\frac{y}{\langle x\rangle}\right)\sum_{n=0}^{N}\frac{i^n}{n!}\partial^n f(x)y^n\right| &\leq C_{l,m}\left|\partial\chi\left(\frac{y}{\langle x\rangle}\right)\right|\sum_{n=0}^{N}\langle x\rangle^{-m+l-m}|y|^n \\
&\leq C_{l,m}|y|^l\langle x\rangle^{-m}.
\end{aligned}$$

Thus we are done. ■

It will not matter which N we choose in the following, or indeed which $C^1(\mathbb{C})$ extension of f we choose, as long as we have the properties (ii) and (iii). Actually weaker conditions will work, see [16, Theorem 1.17].

Theorem 2.4.27. *(Helffer-Sjöstrand Formula) Let $T \in \mathcal{L}(H)$ be self-adjoint and $f \in \mathscr{S}(\mathbb{R})$. Then*

$$f(T) = -\frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial_z}\tilde{f}_N(z)(T-z)^{-1}dxdy$$

for any $N \in \mathbb{N}$ with convergence of the integral in $\mathcal{B}(H)$-norm.

Proof. By Lemma 2.4.26 and $\|(T-z)^{-1}\|_{\mathcal{B}(H)} \leq |\operatorname{Im}(z)|^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$ we see that

$$\|\overline{\partial_z}\tilde{f}_N(z)(T-z)^{-1}\|_{\mathcal{B}(H)} \leq C\langle z\rangle^{-3}.$$

This decay at infinity together with the continuity and boundedness of $\mathbb{C}\setminus\mathbb{R} \in z \to \overline{\partial_z}\tilde{f}_N(z)(T-z)^{-1}$ gives us uniform continuity and also importantly:

$$\begin{aligned}
\sup_{\mathbb{C}\setminus\mathbb{R}}\Big\|&\overline{\partial_z}\tilde{f}_N(z)(T-z)^{-1} \\
&- \sum_{w\in[-n,n]^2\cap n^{-1}(2^{-1}+i2^{-1}+\mathbb{Z}^2)}\overline{\partial_z}\tilde{f}_N(w)(T-w)^{-1}1_{w+[-(2n)^{-1},(2n)^{-1}]^2}(z)\Big\|_{\mathcal{B}(H)} \\
&\overset{n\to\infty}{\to} 0
\end{aligned}$$

This implies that the Bochner integral in the theorem exists and defined by

$$\int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z)(T-z)^{-1} dxdy = \lim_{n\to\infty} \frac{1}{n^2} \sum_{w\in[-n,n]^2\cap n^{-1}(2^{-1}+i2^{-1}+\mathbb{Z}^2)} \overline{\partial_z} \tilde{f}_N(w)(T-w)^{-1}$$

Let now $(E_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral resolution of T. Then by $\rho(T)$ being open, $\mathbb{R} \ni \lambda \mapsto (\lambda - z)^{-1}$ is in $L^\infty(\mathbb{R}, E)$ for each $z \in \rho(T)$, and one can check that $(T-z)^{-1} = \int_{\mathbb{R}} (\lambda - z)^{-1} dE_\lambda$. Furthermore, the following identity holds:

$$f(w) = \tilde{f}_N(w) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z)(z-w)^{-1} dxdy$$

for all $w \in \mathbb{R}$. Hence, using the variational form of $(T-z)^{-1}$, see Remark 2.4.14, we get for $\phi, \psi \in D(f(T))$ that

$$\begin{aligned} \langle f(T)\phi, \psi\rangle_H &= \int_{\mathbb{R}} f(\lambda) d\langle E_\lambda \phi, \psi\rangle_H = -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z)(z-\lambda)^{-1} dxdyd\langle E_\lambda\phi, \psi\rangle_H \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z) \int_{\mathbb{R}} (z-\lambda)^{-1} d\langle E_\lambda \phi, \psi\rangle_H dxdy \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z) \langle (T-z)^{-1}\phi, \psi\rangle_H dxdy. \end{aligned}$$

We needed to use Fubini's Theorem, but

$$\int_{\mathbb{C}} |\overline{\partial_z} \tilde{f}_N(z)(z-\lambda)^{-1}| dxdy < \infty$$

and the measure $d\langle E_\cdot \phi, \psi\rangle_H$ has finite total variation, so that is no problem. Using the norm convergence of the integral we see

$$\begin{aligned} \langle f(T)\phi, \psi\rangle_H &= -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z) \langle (T-z)^{-1}\phi, \psi\rangle_H dxdy \\ &= -\frac{1}{\pi} \left\langle \int_{\mathbb{C}} \overline{\partial_z} \tilde{f}_N(z)(T-z)^{-1} dxdy\phi, \psi \right\rangle_H \end{aligned}$$

for all $\phi, \psi \in D(f(T))$, giving the desired result. ■

3

Tempered Distributions and Pseudo-differential Calculus

The ordinary or classical calculus has long been replaced by its weak or distributional counterpart, giving more structure to differentiation and new tools. We will also have need of this apparatus, so in this chapter we introduce tempered distributions and calculus on them as well as a pseudo-differential calculus dealing with quantization of classical Hamiltonians.

To start with we introduce a phase-modified tight Gabor frame, a generalization of the magnetic tight Gabor frame from [8, 9]. Basic results from frame theory will be used, see [4]. Next is our presentation of Schwartz space and tempered distribution, largely inspired by the various sources [1, 12, 13, 14, 21, 29, 30, 31, 32], but some original proofs are provided. Lastly, we explore a quite general definition of pseudo-differential operators and deduce a calculus for these operators. The main results follow [8, 9, 30], while [20, 26, 33] has been used for background material.

The setting is Euclidean space and we fix dimensions $d, d_1, d_2 \in \mathbb{N}$ throughout the chapter.

3.1 A Modulated Tight Gabor Frame

We will work out most of the theory in this chapter using a modified tight Gabor frame, i.e. a modification of the standard combination of a quadratic partition of unity and Fourier series. Thus in this short section we make a proper introduction of this tool.

Definition 3.1.1. (Phase Function) A continuous, uni-modular function $\vartheta\colon \mathbb{R}^d \times \mathbb{R}^d \to \partial B_1(0;\mathbb{C})$ will be called a phase function.

Note by uni-modular we mean that $|\vartheta| \equiv 1$.

Let ϑ be a phase function. Construct $u \in C_c^\infty(\mathbb{R})$ satisfying $\operatorname{supp}(u) \subseteq (-1,1)$ and $\sum_{\alpha\in\mathbb{Z}}(\tau_\alpha u)^2 \equiv 1$. Let $E_{\alpha'}\colon \mathbb{R} \ni x \mapsto e^{i\alpha' x}$ for $\alpha' \in \mathbb{Z}$. Then the elements of our modulated tight Gabor frame are:

$$\mathcal{G}_{\tilde{\alpha},\vartheta} := \vartheta(\cdot,\alpha) \otimes_{j=1}^{d} (2\pi)^{-\frac{1}{2}} \tau_{\alpha_j}(u E_{\alpha'_j}),$$

where $\tilde{\alpha} = (\alpha, \alpha') \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$.

Lemma 3.1.2. *The family $(\mathcal{G}_{\tilde{\alpha},\vartheta})_{\tilde{\alpha}\in\mathbb{Z}^{2d}}$ constitutes a Parseval frame in $L^2(\mathbb{R}^d)$. Importantly:*

$$\phi = \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$$

for all $\phi \in L^2(\mathbb{R}^d)$ with unconditional convergence in $L^2(\mathbb{R}^d)$-norm.

Remark! 3.1.3. In this context, unconditional convergence means that the convergence is not conditional upon the way one takes the limit in the index of the sum.

Proof. For $\phi, \psi \in L^2(\mathbb{R}^2)$ we have by dominated convergence:

$$\langle \phi, \psi\rangle_{L^2(\mathbb{R}^d)} = \sum_{\alpha\in\mathbb{Z}^d} \langle \vartheta(\cdot,\alpha)(\otimes_{j=1}^d \tau_{\alpha_j} u)\phi, \vartheta(\cdot,\alpha)(\otimes_{j=1}^d \tau_{\alpha_j} u)\psi\rangle_{L^2(\alpha+(-\pi,\pi)^d)}$$

The space $L^2(\alpha + (-\pi,\pi)^d)$ has as an orthonormal basis $(\otimes_{j=1}^d (2\pi)^{-\frac{1}{2}}\tau_{\alpha_j}(E_{\alpha'_j}))_{\alpha'\in\mathbb{Z}^d}$, whence

$$\begin{aligned}\langle \phi, \psi\rangle_{L^2(\mathbb{R}^d)} &= \sum_{\alpha\in\mathbb{Z}^d}\sum_{\alpha'\in\mathbb{Z}^d} \langle \phi, \mathcal{G}_{\tilde{\alpha},\vartheta}\rangle_{L^2(\alpha+(-\pi,\pi)^d)} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi\rangle_{L^2(\alpha+(-\pi,\pi)^d)} \\ &= \sum_{\alpha\in\mathbb{Z}^d}\sum_{\alpha'\in\mathbb{Z}^d} \langle \phi, \mathcal{G}_{\tilde{\alpha},\vartheta}\rangle_{L^2(\mathbb{R}^d)} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi\rangle_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{3.1.1}$$

This implies that

$$\|\phi\|^2_{L^2(\mathbb{R}^d)} = \sum_{\alpha\in\mathbb{Z}^d}\sum_{\alpha'\in\mathbb{Z}^d} |\langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)}|^2, \tag{3.1.2}$$

hence $(\mathcal{G}_{\tilde{\alpha},\vartheta})_{\tilde{\alpha}\in\mathbb{Z}^{2d}}$ is a Parseval frame.

Now according to [4, Corollary 5.1.7], the convergence statement follows from $(\mathcal{G}_{\tilde{\alpha},\vartheta})_{\tilde{\alpha}\in\mathbb{Z}^{2d}}$ being a Parseval frame. However, we will give a simple direct proof of this fact using a technique from [9]. Fix $\phi \in L^2(\mathbb{R}^d)$ and enumerate $\mathbb{Z}^{2d}$ arbitrarily to get a sequence $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$. Then for $N, M \in \mathbb{N}$ with $N \leq M$

$$\begin{aligned}&\left\| \sum_{n=1}^N \langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha}_n,\vartheta} - \sum_{n=1}^M \langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha}_n,\vartheta}\right\|_{L^2(\mathbb{R}^d)} \\ &\qquad = \sup_{\psi\in\partial B_1(0;L^2(\mathbb{R}^d))} \left| \left\langle \psi, \sum_{n=N+1}^M \langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha}_n,\vartheta} \right\rangle_{L^2(\mathbb{R}^d)} \right| \\ &\qquad \leq \sup_{\psi\in\partial B_1(0;L^2(\mathbb{R}^d))} \sum_{n=N+1}^M |\langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \langle \psi, \mathcal{G}_{\tilde{\alpha}_n,\vartheta}\rangle_{L^2(\mathbb{R}^d)}| \\ &\qquad \leq \sum_{n=N+1}^M |\langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)}|^2,\end{aligned}$$

where we used the Cauchy-Schwartz inequality, (3.1.2), and $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$. When $N, M \to \infty$ the above goes to zero by (3.1.2), whence $(\sum_{n=1}^N \langle \mathcal{G}_{\tilde{\alpha}_n,\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha}_n,\vartheta})_{N\in\mathbb{N}}$ is a Cauchy sequence. The limit must then be ϕ by (3.1.1), and so the lemma has been proven. ∎

In each of the preceding sections we require extra conditions on the phase function, which are stated when needed.

3.2 Schwartz Space and Tempered Distributions

The spaces of Schwartz functions and tempered distributions are hugely important spaces by the fact that they together extend in what ways we are able to use derivatives and the Fourier transform. The apparently crucial conditions are faster decay than any polynomial and polynomial growth, conditions implied by classical results on the Fourier transform on $L^1(\mathbb{R}^d)$.

We make a note that every polynomial p in $\mathbb{R}^d$ has growth controlled by some $\langle\cdot\rangle^n$, $n \in \mathbb{N}$, i.e. $\sup_{\mathbb{R}^d} |p|\langle\cdot\rangle^{-n} < \infty$ and vice-versa in that $\langle\cdot\rangle^n \leq \langle\cdot\rangle^{2n}$, where $\langle\cdot\rangle^{2n}$ is a polynomial. Here $\langle\cdot\rangle$ denotes the Japanese bracket, see Section 1.1.

Definition 3.2.1. (Schwartz Space) The space of Schwartz functions $\mathscr{S}(\mathbb{R}^d)$ consists of smooth functions $\phi \in C^\infty(\mathbb{R}^d)$ such that for each $n \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}_0^d$:

$$\sup_{\mathbb{R}^d} \langle\cdot\rangle^n |\partial^\gamma \phi| < \infty$$

We equip $\mathscr{S}(\mathbb{R}^d)$ with the topology induced by the semi-norms $\|\cdot\|_{\mathscr{S}(\mathbb{R}^d),n,m}$, $n, m \in \mathbb{N}_0$, defined by

$$\|\phi\|_{\mathscr{S}(\mathbb{R}^d),n,m} = \sum_{\substack{\gamma\in\mathbb{N}_0^d \\ |\gamma|\leq m}} \sup_{\mathbb{R}^d} \langle\cdot\rangle^n |\partial^\gamma \phi|$$

for all $\phi \in \mathscr{S}(\mathbb{R}^d)$.

One often calls the decay property of Schwartz functions for rapid decay.

Proposition 3.2.2. *$\mathscr{S}(\mathbb{R}^d)$ is a Fréchet space with the Heine-Borel property.*

Proof. The semi-norm $\|\cdot\|_{\mathscr{S}(\mathbb{R}^d),0,0}$ is just the $L^\infty(\mathbb{R}^d)$-norm restricted to $\mathscr{S}(\mathbb{R}^d)$, so the family of semi-norms on $\mathscr{S}(\mathbb{R}^d)$ is separating. The topology is also metrizable by an invariant metric since it is induced by countably many semi-norms. Thus we have only to prove that $\mathscr{S}(\mathbb{R}^d)$ is complete and the Heine-Borel property.

Suppose $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{S}(\mathbb{R}^d)$. Then the sequence and the sequence of all derivatives converge uniformly to some functions in $BC(\mathbb{R}^d)$ by $BC(\mathbb{R}^d)$ being a Banach space in the norm $\|\cdot\|_{BC(\mathbb{R}^d)}$. Hence $(\phi_n)_{n\in\mathbb{N}}$ converges uniformly to some smooth function $\phi \in BC^\infty(\mathbb{R}^d)$. Next for $m \in \mathbb{N}_0$, $r > 0$, and $\gamma \in \mathbb{N}_0^d$ we have

$$\begin{aligned}\sup_{B_r(0;\mathbb{R}^d)} \langle\cdot\rangle^m |\partial^\gamma \phi| &\leq \sup_{\mathbb{R}^d} \langle\cdot\rangle^m |\partial^\gamma \phi_n| + \sup_{B_r(0;\mathbb{R}^d)} \langle\cdot\rangle^m |\partial^\gamma(\phi - \phi_n)| \\ &\leq \sup_{j\in\mathbb{N}} \sup_{\mathbb{R}^d} \langle\cdot\rangle^m |\partial^\gamma \phi_j| + \langle r\rangle^m \sup_{B_r(0;\mathbb{R}^d)} |\partial^\gamma(\phi - \phi_n)| \overset{n\to\infty}{\to} \sup_{j\in\mathbb{N}} \sup_{\mathbb{R}^d} \langle\cdot\rangle^m |\partial^\gamma \phi_j|.\end{aligned}$$

Taking $r \to \infty$ we see that $\phi \in \mathscr{S}(\mathbb{R}^d)$. Moreover, for $m, j \in \mathbb{N}_0$

$$\begin{aligned}\|\phi - \phi_n\|_{\mathscr{S}(\mathbb{R}^d),m,j} &\leq \sum_{\substack{\gamma\in\mathbb{N}_0^d \\ |\gamma|\leq j}} \sup_{B_r(0;\mathbb{R}^d)} \langle\cdot\rangle^m |\partial^\gamma(\phi - \phi_n)| + \sup_{\mathbb{R}^d\setminus B_r(0;\mathbb{R}^d)} \langle\cdot\rangle^m |\partial^\gamma(\phi - \phi_n)| \\ &\leq \sum_{\substack{\gamma\in\mathbb{N}_0^d \\ |\gamma|\leq j}} \langle r\rangle^m \sup_{B_r(0;\mathbb{R}^d)} |\partial^\gamma(\phi - \phi_n)| + \frac{2}{\langle r\rangle} \sup_{k\in\mathbb{N}} \sup_{\mathbb{R}^d\setminus B_r(0;\mathbb{R}^d)} \langle\cdot\rangle^{m+1} |\partial^\gamma \phi_k| \\ &\overset{n\to\infty}{\to} \frac{2}{\langle r\rangle} \sup_{k\in\mathbb{N}} \sup_{\mathbb{R}^d} \langle\cdot\rangle^{m+1} |\partial^\gamma \phi_k|,\end{aligned}$$

where $r > 0$ is arbitrary. Taking $r \to \infty$ again, we see that $(\phi_n)_{n\in\mathbb{N}}$ converge to ϕ in $\mathscr{S}(\mathbb{R}^d)$.

Since $\mathscr{S}(\mathbb{R}^d)$ is metrizable, the Heine-Borel property follows from showing that every bounded sequence has a convergent subsequence. But this follows from a standard diagonalization argument, the Arzelà–Ascoli Theorem, and the reasoning above. ■

In the following we shall make use of the spaces $\langle\cdot\rangle^n L^p(\mathbb{R}^d)$, $n \in \mathbb{N}_0, p \in [1,\infty]$, meaning the space of elements of $L^p(\mathbb{R}^d)$ multiplied by $\langle\cdot\rangle^n$ pointwise. These spaces are Banach spaces when equipped with the norms

$$\|\cdot\|_{\langle\cdot\rangle^n L^p(\mathbb{R}^d)} \colon \langle\cdot\rangle^n L^p(\mathbb{R}^d) \ni \phi \mapsto \|\langle\cdot\rangle^{-n}\phi\|_{L^p(\mathbb{R}^d)},$$

respectively. We use a similar notation in case of the space $BC^m(\mathbb{R}^d)$, $m \in \mathbb{N}_0$.

Proposition 3.2.3. *$\mathscr{S}(\mathbb{R}^d)$ is continuously injected into the spaces $\langle\cdot\rangle^n L^p(\mathbb{R}^d)$, $n \in \mathbb{N}_0, p \in [1,\infty]$, and $\langle\cdot\rangle^n BC^m(\mathbb{R}^d)$, $n \in \mathbb{N}_0, m \in \mathbb{N}_0$.*

Proof. It will be enough to pick $n = 0$ since $L^p(\mathbb{R}^d) \hookrightarrow \langle\cdot\rangle^n L^p(\mathbb{R}^d)$ and $BC^m(\mathbb{R}^d) \hookrightarrow \langle\cdot\rangle^n BC^m(\mathbb{R}^d)$.

For any $p \in [1,\infty)$ and $\phi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |\phi|^p \leq \int_{\mathbb{R}^d} \langle\cdot\rangle^{-d-1}\langle\cdot\rangle^{p(d+1)}|\phi|^p \leq \left(\sup_{\mathbb{R}^d}\langle\cdot\rangle^{d+1}|\phi|\right)^p \int_{\mathbb{R}^d}\langle\cdot\rangle^{-d-1},$$

which clearly shows that $\phi \in L^p(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$. It is even easier to see $\mathscr{S}(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ since the norm on $BC^m(\mathbb{R}^d)$ corresponds to the semi-norm $\|\phi\|_{\mathscr{S}(\mathbb{R}^d),0,m}$ for $\phi \in \mathscr{S}(\mathbb{R}^d)$, and this argument also works for $L^\infty(\mathbb{R}^d)$, so we are done. ■

As a last basic result in developing the Schwartz space, we define several continuous operations.

Definition 3.2.4. (Slowly Increasing Functions) The space $\mathscr{O}_m(\mathbb{R}^d)$ consists of smooth functions $\phi \in C^\infty(\mathbb{R}^d)$ such that for each $\gamma \in \mathbb{N}_0^d$ there exists $n_\gamma \in \mathbb{N}_0$ for which

$$\sup_{\mathbb{R}^d}\langle\cdot\rangle^{-n_\gamma}|\partial^\gamma\phi| < \infty.$$

Proposition 3.2.5. *The following are bounded operators on $\mathscr{S}(\mathbb{R}^d)$:*

(i) ∂^γ, $\gamma \in \mathbb{N}_0^d$.

(ii) $\mathcal{F}$ *and* $\mathcal{F}^{-1}$.

(iii) Pointwise multiplication by a fixed $\phi \in \mathscr{O}_m(\mathbb{R}^d)$.

(iv) Convolution with a fixed $\psi \in \mathscr{S}(\mathbb{R}^d)$.

(v) Affine change of coordinates.

Proof. (i) and (iii) are trivial by definitions. Also, (v) becomes obvious by Peetre's inequality.

Since $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$, then both the Fourier transform and its inverse are well-defined on $\mathscr{S}(\mathbb{R}^d)$. Classical results shows that rapid decay of $\omega \in L^1(\mathbb{R}^d)$ implies $\mathcal{F}\omega \in BC^\infty(\mathbb{R}^d)$ and smoothness of ω implies rapid decay of $\mathcal{F}\omega$. We may use these results to conclude that $\mathcal{F}$ and $\mathcal{F}^{-1}$ invariates $\mathscr{S}(\mathbb{R}^d)$. Furthermore,

$$\|\mathcal{F}\omega\|_{\mathscr{S}(\mathbb{R}^d),n,m} \leq C_{n,m} \sup_{\mathbb{R}^d} \langle\cdot\rangle^m |(1-\Delta)^n \omega|$$

for all $\omega \in \mathscr{S}(\mathbb{R}^d)$ and $n, m \in \mathbb{N}_0$, by which continuity follows.

Let us now consider the convolution in (iv). Fix $\omega \in \mathscr{S}(\mathbb{R}^d)$. The rapid decay shows that $\psi * \omega \in C^\infty(\mathbb{R}^d)$ with the known identities

$$\partial^\gamma(\psi * \omega) = \partial^\gamma \psi * \omega = \psi * \partial^\gamma \omega$$

holding for all $\gamma \in \mathbb{N}_0^d$. Furthermore, for any such $\gamma \in \mathbb{N}_0^d$ and a $n \in \mathbb{N}_0$ we have

$$\langle\cdot\rangle^n |\partial^\gamma(\psi * \omega)|(\cdot) \leq C_{n,\gamma} \|\langle\cdot\rangle^n \psi\|_{L^1(\mathbb{R}^d)} \sup_{\mathbb{R}^d} \langle\cdot\rangle^n |\partial^\gamma \omega|$$

using Peetre's inequality. By this inequality we claim $\psi * \omega \in \mathscr{S}(\mathbb{R}^d)$ and continuity of convolution with a fixed element. ■

As bonus information, the inversion formula implies that $\mathcal{F}$ is an isomorphism of $\mathscr{S}(\mathbb{R}^d)$. Moreover, the proof of continuity for convolution actually shows that it is a bounded bilinear map $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$.

We now have a rather nice space $\mathscr{S}(\mathbb{R}^d)$. Using variational principles we want to extend these pleasant properties to its dual $\mathscr{S}'(\mathbb{R}^d)$.

Definition 3.2.6. (Tempered Distribution) The elements of the dual $\mathscr{S}'(\mathbb{R}^d)$ are called tempered distributions and $\mathscr{S}'(\mathbb{R}^d)$ is called the space of tempered distributions.

Remark! 3.2.7. It is uncommon to use function notation for tempered distributions. Instead, one makes use of the duality bracket:

$$\langle \cdot, \cdot \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \colon \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \ni (\phi, \psi) \mapsto \phi(\psi)$$

Remark! 3.2.8. Recall that we in Definition 2.1.1 equipped bounded duals with the strong topology. Explicitly, the strong topology on $\mathscr{S}'(\mathbb{R}^d)$ is induced by the semi-norms

$$\|\cdot\|_{\mathscr{S}'(\mathbb{R}^d),\Omega} \colon \mathscr{S}'(\mathbb{R}^d) \ni \phi \mapsto \sup_{\Omega} |\phi|,$$

where $\Omega \subseteq \mathscr{S}(\mathbb{R}^d)$ varies over bounded sets of Schwartz functions.

There will be given three results for tempered distribution before moving on to coordinate representations of Schwartz functions and tempered distribution. Thereafter we shall give more information on $\mathscr{S}'(\mathbb{R}^d)$.

The first is an instance of a broader proposition: The dual of a Fréchet space is complete.

Proposition 3.2.9. *$\mathscr{S}'(\mathbb{R}^d)$ is complete.*

Proof. Suppose $(\phi_j)_{j\in I}$ is a Cauchy net in $\mathscr{S}'(\mathbb{R}^d)$. Then $(\langle \phi_j, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)})_{j\in I}$ is a Cauchy net in $\mathbb{C}$, hence convergent. Thus we can define a linear functional ϕ on $\mathscr{S}(\mathbb{R}^d)$ by pointwise limit of $(\phi_j)_{j\in I}$.

Suppose now that $\Omega \subseteq \mathscr{S}(\mathbb{R}^d)$ is a bounded subset and let $\varepsilon > 0$ be given. Then there exists $J \in I$ such that

$$\sup_{\psi\in\Omega} |\langle \phi_j - \phi_k, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq \varepsilon$$

for all $j, k \geq J$. Thus for $\psi \in \Omega$, taking the limit in k we get

$$|\langle \phi_j - \phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq \varepsilon$$

for all $j \geq J$. Now, this implies

$$\sup_{\psi\in\Omega} |\langle \phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq \sup_{\psi\in\Omega} |\langle \phi - \phi_J, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| + \sup_{\psi\in\Omega} |\langle \phi_J, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| < \infty,$$

so ϕ is bounded, hence $\phi \in \mathscr{S}'(\mathbb{R}^d)$. Furthermore, we get

$$\|\phi_j - \phi\|_{\mathscr{S}'(\mathbb{R}^d),\Omega} \leq \varepsilon$$

for $j \geq J$ showing convergence of $(\phi_j)_{j\in I}$ to ϕ in $\mathscr{S}'(\mathbb{R}^d)$. ∎

Next we show a form for converse of Proposition 3.2.3.

Proposition 3.2.10. *Let B be one of the Banach spaces $\langle\cdot\rangle^n L^p(\mathbb{R}^d)$, $n \in \mathbb{N}_0, p \in [1,\infty]$, or $\langle\cdot\rangle^n BC^m(\mathbb{R}^d)$, $n \in \mathbb{N}_0, m \in \mathbb{N}_0$. Then the map sending $\phi \in B$ into the integral with density ϕ, i.e.*

$$\mathscr{S}(\mathbb{R}^d) \ni \psi \mapsto \int_{\mathbb{R}^d} \psi\phi,$$

defines a continuously injected of B into the space $\mathscr{S}'(\mathbb{R}^d)$.

Note this implies that $\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$.

Proof. Let us first consider $B = \langle\cdot\rangle^n L^p(\mathbb{R}^d)$. By Hölder's inequality

$$\int_{\mathbb{R}^d} |\psi\phi| = \int_{\mathbb{R}^d} \langle\cdot\rangle^n |\psi| \langle\cdot\rangle^{-n} |\phi| \leq \|\langle\cdot\rangle^n \psi\|_{L^q(\mathbb{R}^d)} \|\phi\|_{\langle\cdot\rangle^n L^p(\mathbb{R}^d)}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $\phi \in B$ with $q = (1 - p^{-1})^{-1}$ if $p > 1$ and otherwise $q = \infty$. So not only is the map $\mathscr{S}(\mathbb{R}^d) \ni \psi \mapsto \int_{\mathbb{R}^d} \psi\phi$ a well-defined linear functional, but since multiplication by $\langle\cdot\rangle^n$ is continuous on $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ continuously, $\|\langle\cdot\rangle^n \psi\|_{L^q(\mathbb{R}^d)}$ is bounded by a constant times a semi-norm of $\mathscr{S}(\mathbb{R}^d)$. This firstly gives that $\mathscr{S}(\mathbb{R}^d) \ni \psi \mapsto \int_{\mathbb{R}^d} \psi\phi$ is continuous for each $\phi \in B$, and secondly that the so defined injection $B \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$ is continuous.

For the spaces $\langle\cdot\rangle^n BC^m(\mathbb{R}^d)$ the proposition follows from $\langle\cdot\rangle^n BC^m(\mathbb{R}^d) \hookrightarrow \langle\cdot\rangle^n L^\infty(\mathbb{R}^d)$ continuously. ∎

Thus we have a wealth of "normal" functions in $\mathscr{S}'(\mathbb{R}^d)$. The strength of these injections is that the operations in Proposition 3.2.5 can be defined for all these functions and in general on tempered distributions through duality. We have here need of the transpose identity for these operations: For a map $T\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ we call $S\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ the formal transpose of T if

$$\int_{\mathbb{R}^d} (T\phi)\psi = \int_{\mathbb{R}^d} \phi(S\psi)$$

for all $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$. If S is continuous and linear, then we have a real transpose of S defined as $S^t\colon \mathscr{S}'(\mathbb{R}^d) \ni \phi \mapsto \phi \circ S$, which would be weak*-continuous. But then if both T and S are continuous and linear, we would have

$$\langle S^t\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \phi, S\psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle T\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

for all $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$. Identifying now S^t with T on $\mathscr{S}'(\mathbb{R}^d)$ we get an extension of T to $\mathscr{S}'(\mathbb{R}^d)$.

Using this methodology on the operations in Proposition 3.2.5, whom all have formal transpose operators, gives:

Definition 3.2.11. We define the following operators on $\mathscr{S}'(\mathbb{R}^d)$:

(i) $\partial^\gamma := ((-1)^{|\gamma|}\partial^\gamma|_{\mathscr{S}(\mathbb{R}^d)})^t$, $\gamma \in \mathbb{N}_0^d$.

(ii) $\mathcal{F} := \mathcal{F}|^t_{\mathscr{S}(\mathbb{R}^d)}$ and $\mathcal{F}^{-1} := \mathcal{F}^{-1}|^t_{\mathscr{S}(\mathbb{R}^d)}$.

(iii) Pointwise multiplication by a fixed $\phi \in \mathscr{O}_m(\mathbb{R}^d)$ as the transpose of pointwise multiplication by ϕ on $\mathscr{S}(\mathbb{R}^d)$.

(iv) Convolution with a fixed $\psi \in \mathscr{S}(\mathbb{R}^d)$ as the transpose of convolution by $(-\mathrm{id})^*\psi$ on $\mathscr{S}(\mathbb{R}^d)$.

(v) Affine change of coordinates by the transpose of the inverse affine change of coordinates multiplied by the determinant of the inverse.

Remark! 3.2.12. With regards to convolution we know $\phi * \psi = \psi * \phi$ when both elements are Schwartz functions, which implies that extending left or right convolution to $\mathscr{S}'(\mathbb{R}^d)$ gives the same operator and we shall write $\phi * \psi = \psi * \phi$ even if one element is a tempered distribution.

Remark! 3.2.13. By definition relations like $\tau_y \mathcal{F}\phi = \mathcal{F}(\otimes_{j=1}^d E_{y_j}\phi)$ for $\phi \in \mathscr{S}(\mathbb{R}^d)$ extend to tempered distributions.

Sometimes we will add "distributional" before the operations in Definition 3.2.11 to differentiate them from classical operations, e.g. distributional derivatives.

Proposition 3.2.14. *All the operators in Definition 3.2.11 are continuous on* $\mathscr{S}'(\mathbb{R}^d)$.

Proof. Note standard results gives that the operators are linear and weak*-continuous, but since they all are transpose operators of bounded operators, then they are also continuous on the strong topology on $\mathscr{S}'(\mathbb{R}^d)$. ■

We end this introduction by noting that $\mathscr{S}'(\mathbb{R}^d)$ contains many "abnormal" objects. One example is the Dirac delta distribution δ defined by

$$\langle \delta, \phi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \phi(0).$$

No tempered distribution arising from a representation like the one in Proposition 3.2.10 is equal to δ. Even though the Dirac delta δ can be seen as a representation of the point measure at 0 in $\mathscr{S}(\mathbb{R}^d)$, the derivative $\partial^\gamma \delta$, $\gamma \in \mathbb{N}_0^d \setminus \{0\}$, has no representation as an integral.

The Dirac delta δ is often single out as an extremely useful tempered distribution. One example is through the concept of a fundamental solution. Consider a linear partial differential operator $T := \sum_{\substack{\gamma \in \mathbb{N}_0^d \\ |\gamma| \leq N}} \phi_\gamma \partial^\gamma$ with $N \in \mathbb{N}_0$ and $\phi_\gamma \in \mathscr{O}_m(\mathbb{R}^d)$ for $\gamma \in \mathbb{N}_0^d, |\gamma| \leq N$. Looking at this operator distributionally, we may ask ourselves the question of the existence of some $\psi \in \mathscr{S}'(\mathbb{R}^d)$ such that $T\psi = \delta$. Such a distribution is called a fundamental solution of the operator. This is especially useful if all the coefficients are constants, since then for every $\omega \in \mathscr{S}(\mathbb{R}^d)$ we would have:

$$T(\omega * \psi) = \omega * T\psi = \omega * \delta = \omega.$$

3.2.1 Coordinate Representation of Schwartz Functions and Tempered Distributions

For the advanced results on the Schwartz space and space of tempered distributions we need a coordinate representation of both in our modulated tight Gabor frame.

Lemma 3.2.15. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a phase function.*

If $(a_{\tilde{\alpha}})_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \in \mathbb{C}^{\mathbb{Z}^{2d}}$ satisfies

$$\sup_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \alpha \rangle^k \langle \alpha' \rangle^l |a_{\tilde{\alpha}}| < \infty$$

for all $k, l \in \mathbb{N}_0$, then $\sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} a_{\tilde{\alpha}} \mathcal{G}_{\tilde{\alpha},\vartheta}$ converges absolutely in $\mathscr{S}(\mathbb{R}^d)$.

Additionally, for $\phi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\sup_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \alpha \rangle^k \langle \alpha' \rangle^l |\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)}| \leq C_{\vartheta,k,l} \|\phi\|_{\mathscr{S}(\mathbb{R}^d), \tilde{k}_{k,\vartheta}, 2l}$$

for all $k, l \in \mathbb{N}_0$ and some $\tilde{k}_{k,\vartheta} \in \mathbb{N}_0$, and $\phi = \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$.

Remark! 3.2.16. By absolute convergence we mean that for any of the semi-norms $\| \cdot \|_{\mathscr{S}(\mathbb{R}^d),n,m}$ on $\mathscr{S}(\mathbb{R}^d)$ one gets:

$$\sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} |a_{\tilde{\alpha}}| \|\mathcal{G}_{\tilde{\alpha},\vartheta}\|_{\mathscr{S}(\mathbb{R}^d),n,m} < \infty$$

Since $\mathscr{S}(\mathbb{R}^d)$ is complete, this implies that the sum converges to some element of $\mathscr{S}(\mathbb{R}^d)$ no matter the truncations.

Proof. For $n \in \mathbb{N}$ and $\gamma \in \mathbb{N}^d$ we have by Leibniz's rule and Peetre's inequality:

$$\begin{aligned} \sup_{\mathbb{R}^d} \langle \cdot \rangle^n |\partial^\gamma \mathcal{G}_{\tilde{\alpha},\vartheta}| &\leq C_{n,|\gamma|} \langle \alpha \rangle^n \sup_{\mathbb{R}^d} \langle \tau_\alpha(\cdot) \rangle^n \sum_{\substack{\delta \in \mathbb{N}^d \\ \delta \leq \gamma}} |\partial^\delta (\vartheta(\cdot, \alpha) \tau_\alpha(\otimes^d u))| \langle \alpha' \rangle^{|\gamma - \delta|} \\ &\leq C_{n,|\gamma|} \langle \alpha \rangle^{\tilde{n}_{n,\vartheta}} \langle \alpha' \rangle^{|\gamma|} \end{aligned} \tag{3.2.1}$$

with $\tilde{n}_{n,\vartheta} \in \mathbb{N}_0$ possibly larger than n. Taking into account the criteria on the coefficients stated in the lemma, we see that the sum $\sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} a_{\tilde{\alpha}} \mathcal{G}_{\tilde{\alpha},\vartheta}$ must converge absolutely in $\mathscr{S}(\mathbb{R}^d)$.

As for the second half of the lemma, we already know that

$$\phi = \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$$

holds in $L^2(\mathbb{R}^d)$ by Lemma 3.1.2, so if we can prove a uniform estimate on

$$\langle \alpha \rangle^k \langle \alpha' \rangle^l |\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}|$$

for arbitrary $k, l \in \mathbb{N}$, then we are done by the first part of the lemma. Luckily, such an estimate follows from integration by parts, properties of the exponential function, and Peetre's inequality:

$$\begin{aligned}\langle \alpha \rangle^k \langle \alpha' \rangle^l |\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| &= (2\pi)^{-\frac{d}{2}} \langle \alpha \rangle^k \left| \int_{\mathbb{R}^d} \phi \vartheta(\cdot, \alpha) \tau_\alpha \left(\otimes^d u (1-\Delta)^l (\otimes_{j=1}^d E_{-\alpha_j'}) \right) \right| \\ &\leq (2\pi)^{-\frac{d}{2}} \langle \alpha \rangle^k \int_{\mathbb{R}^d} \left| (1-\Delta)^l \left(\phi \vartheta(\cdot, \alpha) \tau_\alpha (\otimes^d u) \right) \right| \\ &\leq C_{\vartheta,k,l} \|\phi\|_{\mathscr{S}(\mathbb{R}^d),\tilde{k}_{k,\vartheta},2l}\end{aligned} \tag{3.2.2}$$

■

Lemma 3.2.17. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a phase function.*
If $(a_{\tilde{\alpha}})_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \in \mathbb{C}^{\mathbb{Z}^{2d}}$ satisfies

$$\sup_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \langle \alpha \rangle^{-k} \langle \alpha' \rangle^{-l} |a_{\tilde{\alpha}}| < \infty$$

for some $k, l \in \mathbb{N}_0$, then $\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} a_{\tilde{\alpha}} \mathcal{G}_{\tilde{\alpha},\vartheta}$ converges absolutely in $\mathscr{S}'(\mathbb{R}^d)$.
Moreover, for $\phi \in \mathscr{S}'(\mathbb{R}^d)$ there exists $k, l \in \mathbb{N}_0$ such that

$$\sup_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \langle \alpha \rangle^{-k} \langle \alpha' \rangle^{-l} |\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| < \infty$$

and $\phi = \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$.

Proof. Suppose $(a_{\tilde{\alpha}})_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \in \mathbb{C}^{\mathbb{Z}^{2d}}$ satisfies the criteria in the lemma. Then by Lemma 3.2.15, for every $\psi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} |a_{\tilde{\alpha}} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq C \|\psi\|_{\mathscr{S}(\mathbb{R}^d),\tilde{k}_{k,\vartheta},2l}$$

for some $k, l \in \mathbb{N}_0$ not dependent on ψ. This shows that

$$\mathscr{S}(\mathbb{R}^d) \ni \psi \mapsto \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} a_{\tilde{\alpha}} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

defines a continuous linear functional on $\mathscr{S}(\mathbb{R}^d)$. Furthermore, for a bounded set $\Omega \subset \mathscr{S}(\mathbb{R}^d)$, the estimates used above ensures that

$$\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \sup_{\psi\in\Omega} |a_{\tilde{\alpha}} \langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| < \infty,$$

whence the sum $\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} a_{\tilde{\alpha}} \mathcal{G}_{\tilde{\alpha},\vartheta}$ converges absolutely in $\mathscr{S}'(\mathbb{R}^d)$.

Given $\phi \in \mathscr{S}'(\mathbb{R}^d)$ and using the estimate (3.2.1) we get

$$|\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq C_\phi \|\mathcal{G}_{\tilde{\alpha},\vartheta}\|_{S(\mathbb{R}^d),k,l} \leq C_\phi \langle \alpha \rangle^{\tilde{k}_{k,\vartheta}} \langle \alpha' \rangle^l$$

for some $k, l \in \mathbb{N}$. Hence $\sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$ exists as a tempered distribution. Lemma 3.2.15 and duality now shows that

$$\left\langle \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi \right\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$
$$= \left\langle \phi, \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \psi, \mathcal{G}_{\tilde{\alpha},\vartheta} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \right\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \phi, \psi \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

for every $\psi \in \mathscr{S}(\mathbb{R}^d)$, finishing the proof. ■

These results also gives us a frame version of Poissons summation formula for Schwartz functions. For any $\phi \in \mathscr{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{Z}^d$, then either by using Lemma 3.2.17 and $\tau_\alpha \delta$, or Lemma 3.2.15, the easier choice, we get

$$\phi(\alpha) = \sum_{\alpha' \in \mathbb{Z}^d} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} (2\pi)^{-\frac{d}{2}} \vartheta(\alpha, \alpha).$$

Thus

$$\sum_{\alpha \in \mathbb{Z}^d} (2\pi)^{\frac{d}{2}} \overline{\vartheta(\alpha, \alpha)} \phi(\alpha) = \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

with both sums absolutely convergent by the rapid decay of the elements. Of course ϑ is a phase function in $\mathscr{O}_m(\mathbb{R}^{2d})$.

3.2.2 Fundamental Theorems for Schwartz Functions and Tempered Distributions

With the basics of $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ understood and having the important tool of our frame, it has become time to prove some advanced results.

Corollary 3.2.18. *$\otimes^d \mathscr{S}(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$.*

Proof. Choosing $\vartheta \equiv 1$, the corollary follows from Lemma 3.2.15 and Lemma 3.2.17. ■

Theorem 3.2.19. *$\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ are reflexive.*

Proof. Clearly $\mathscr{S}(\mathbb{R}^d) \hookrightarrow (\mathscr{S}'(\mathbb{R}^d))'$ continuously by the map

$$\mathscr{S}(\mathbb{R}^d) \ni \phi \mapsto \langle \cdot, \phi \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)},$$

i.e. the evaluation maps.

Suppose ϕ is a continuous linear functional on $\mathscr{S}'(\mathbb{R}^d)$ and let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a phase function. Then

$$\phi(\psi) = \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \phi(\mathcal{G}_{\tilde{\alpha},\vartheta}) \langle \psi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

for any $\psi \in \mathscr{S}'(\mathbb{R}^d)$ by Lemma 3.2.17, so $\phi = \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \phi(\mathcal{G}_{\tilde{\alpha},\vartheta})\overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}$ with weak*-convergence in $(\mathscr{S}'(\mathbb{R}^d))'$. But using Lemma 3.2.15

$$\begin{aligned}|\phi(\mathcal{G}_{\tilde{\alpha},\vartheta})| &\leq C_\phi \|\mathcal{G}_{\tilde{\alpha},\vartheta}\|_{\mathscr{S}'(\mathbb{R}^d),\Omega} = C_\phi \sup_{\psi\in\Omega} |\langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \psi\rangle|_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &\leq C_{\phi,\Omega,k,l}\langle\alpha\rangle^{-k}\langle\alpha'\rangle^{-l},\end{aligned}$$

where Ω is some bounded subset of $\mathscr{S}(\mathbb{R}^d)$ dependent on ϕ and $k, l \in \mathbb{N}_0$ are arbitrary. So $\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \phi(\mathcal{G}_{\tilde{\alpha},\vartheta})\overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}$ converges absolutely in $\mathscr{S}(\mathbb{R}^d)$ and by continuity of $\mathscr{S}(\mathbb{R}^d) \hookrightarrow (\mathscr{S}'(\mathbb{R}^d))'$ we must have $\phi \in \mathscr{S}(\mathbb{R}^d)$, finishing the proof. ∎

Now comes a big one: The Schwartz Kernel Theorem. Before we prove it, we need to define the notion of a Schwartz kernel.

Definition 3.2.20. (Schwartz Kernel) Fix a map $T\colon \mathscr{S}(\mathbb{R}^{d_1}) \to \mathscr{S}'(\mathbb{R}^{d_2})$. We call $K_T \in \mathscr{S}'(\mathbb{R}^{d_1+d_2})$ a Schwartz kernel of T if

$$\langle T\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} = \langle K_T, \psi\otimes\phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}$$

for all $\phi \in \mathscr{S}(\mathbb{R}^{d_1}), \psi \in \mathscr{S}(\mathbb{R}^{d_2})$.

Theorem 3.2.21. *(The Schwartz Kernel Theorem) For each $T \in \mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2}))$ there exists a unique Schwartz kernel $K_T \in \mathscr{S}'(\mathbb{R}^{d_1+d_2})$, and the map*

$$\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2})) \ni T \mapsto K_T$$

defines a topological vector space isomorphism of $\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2}))$ and $\mathscr{S}'(\mathbb{R}^{d_1+d_2})$.

Proof. Uniqueness of Schwartz kernels is trivial by Corollary 3.2.18. Let us show existence. Fix two phase functions $\vartheta_1 \in \mathscr{O}_m(\mathbb{R}^{2d_1}), \vartheta_2 \in \mathscr{O}_m(\mathbb{R}^{2d_2})$. Applying Lemma 3.2.15 twice, we get the following identity

$$\begin{aligned}&\langle T\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \\ &\quad= \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2},\tilde{\beta}\in\mathbb{Z}^{2d_1}} \langle T\mathcal{G}_{\tilde{\beta},\vartheta_1}, \mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \langle \overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}}, \psi\otimes\phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})},\end{aligned} \tag{3.2.3}$$

which holds for all $\phi \in \mathscr{S}(\mathbb{R}^{d_1}), \psi \in \mathscr{S}(\mathbb{R}^{d_2})$. This leads us to make the ansatz that K_T should be defined by the expansion:

$$\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2},\tilde{\beta}\in\mathbb{Z}^{2d_1}} \langle T\mathcal{G}_{\tilde{\beta},\vartheta_1}, \mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}}.$$

Indeed, the coefficients satisfy

$$\begin{aligned}&|\langle T\mathcal{G}_{\tilde{\beta},\vartheta_1}, \mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}| \\ &\qquad\leq C_T \|\mathcal{G}_{\tilde{\beta},\vartheta_1}\|_{\mathscr{S}(\mathbb{R}^{d_1}),n,m} \|\mathcal{G}_{\tilde{\alpha},\vartheta_2}\|_{\mathscr{S}(\mathbb{R}^{d_2}),n,m} \leq C_T(\langle\beta\rangle\langle\beta'\rangle\langle\alpha\rangle\langle\alpha'\rangle)^l\end{aligned}$$

using the Banach-Steinhaus Theorem and (3.2.1) with $n, m \in \mathbb{N}_0$ determined by T and $l \in \mathbb{N}_0$ determined by both T and ϑ_1, ϑ_2. Thus Lemma 3.2.17 tells us that

$$K_T := \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2},\tilde{\beta}\in\mathbb{Z}^{2d_1}} \langle T\mathcal{G}_{\tilde{\beta},\vartheta_1}, \mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}}$$

is a well-defined tempered distribution. From (3.2.3) and the absolute convergence of the involved sums, we conclude that T has Schwartz kernel K_T. The mapping of T into K_T is clearly linear by the explicit formula for K_T.

Second step is to show that the map $\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2})) \ni T \mapsto K_T$ is surjective. Let $K \in \mathscr{S}'(\mathbb{R}^{d_1+d_2})$. Then the map $T_K \colon \mathscr{S}(\mathbb{R}^{d_1}) \to \mathscr{S}'(\mathbb{R}^{d_2})$ defined by

$$T_K\phi = \langle K, (\cdot) \otimes \phi \rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}), \mathscr{S}(\mathbb{R}^{d_1+d_2})}$$

for $\phi \in \mathscr{S}(\mathbb{R}^{d_1})$ is a continuous operator with Schwartz kernel K if the tensor product $\otimes \colon \mathscr{S}(\mathbb{R}^{d_2}) \times \mathscr{S}(\mathbb{R}^{d_1}) \to \mathscr{S}(\mathbb{R}^{d_1+d_2})$ is continuous. But this follows from the elementary estimate

$$\|\psi \otimes \phi\|_{\mathscr{S}(\mathbb{R}^{d_1+d_2}),n,m} \leq C_n \|\psi\|_{\mathscr{S}(\mathbb{R}^{d_2}),n,m} \|\phi\|_{\mathscr{S}(\mathbb{R}^{d_1}),n,m}$$

for all $\phi \in \mathscr{S}(\mathbb{R}^{d_1}), \psi \in \mathscr{S}(\mathbb{R}^{d_2})$ and $n, m \in \mathbb{N}_0$. Whence we get surjectivity.

In the third and final step we show continuity. Here we note that the strong topology on $\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2}))$ is induced by the semi-norms

$$\|\cdot\|_{\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2})),\Omega,\Theta} \colon \mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2})) \ni T \mapsto \sup_{\phi\in\Omega,\psi\in\Theta} |\langle T\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}|,$$

where Ω ranges over bounded subsets of $\mathscr{S}(\mathbb{R}^{d_1})$ and Θ over bounded subsets of $\mathscr{S}(\mathbb{R}^{d_2})$.

Assume $(K_j)_{j\in I}$ is a convergent net of Schwartz kernels in $\mathscr{S}'(\mathbb{R}^{d_1+d_2})$ with limit K, and let $(T_{K_j})_{j\in I}, T_K$ be the corresponding operators. Then for bounded sets $\Omega \subseteq \mathscr{S}(\mathbb{R}^{d_1})$ and $\Theta \subseteq \mathscr{S}(\mathbb{R}^{d_2})$, the set

$$\mathrm{Y} := \{\phi \otimes \psi | \phi \in \mathscr{S}(\mathbb{R}^{d_1}), \psi \in \mathscr{S}(\mathbb{R}^{d_2})\}$$

is bounded in $\mathscr{S}(\mathbb{R}^{d_1+d_2})$, implying by hypothesis that:

$$\|T_K - T_{K_j}\|_{\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}),\mathscr{S}'(\mathbb{R}^{d_2})),\Omega,\Theta} = \|K - K_j\|_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathrm{Y}} \xrightarrow{j} 0$$

Now assume that $(T)_{j\in I}$ is a net in $\mathcal{B}(\mathscr{S}(\mathbb{R}^{d_1}), \mathscr{S}'(\mathbb{R}^{d_2}))$ converging to T, and let $(K_{T_j})_{j\in I}, K_T$ be the corresponding Schwartz kernels. Then for $\Phi \in \mathscr{S}(\mathbb{R}^{d_1+d_2})$ we have

$$\begin{aligned}&\langle K_T - K_{T_j}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})} \\ &= \sum_{\tilde\alpha\in\mathbb{Z}^{2d_2},\tilde\beta\in\mathbb{Z}^{2d_1}} \langle (T - T_j)\mathcal{G}_{\tilde\beta,\vartheta_1}, \mathcal{G}_{\tilde\alpha,\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \langle \overline{\mathcal{G}_{\tilde\alpha,\vartheta_2} \otimes \mathcal{G}_{\tilde\beta,\vartheta_1}}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})},\end{aligned}$$

for every $j \in I$. We will then use dominated convergence to take the limit in j under the sum. The sets

$$\Omega := \{|\langle \overline{\mathcal{G}_{\tilde\alpha,\vartheta_2} \otimes \mathcal{G}_{\tilde\beta,\vartheta_1}}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}|^{\frac{1}{3}} \mathcal{G}_{\tilde\beta,\vartheta_1} | \tilde\alpha \in \mathbb{Z}^{2d_2}, \tilde\beta \in \mathbb{Z}^{2d_1}\}$$

and

$$\Theta := \{|\langle \overline{\mathcal{G}_{\tilde\alpha,\vartheta_2} \otimes \mathcal{G}_{\tilde\beta,\vartheta_1}}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}|^{\frac{1}{3}} \mathcal{G}_{\tilde\alpha,\vartheta_2} | \tilde\alpha \in \mathbb{Z}^{2d_2}, \tilde\beta \in \mathbb{Z}^{2d_1}\}$$

are bounded sets in $\mathscr{S}(\mathbb{R}^{d_1})$ and $\mathscr{S}(\mathbb{R}^{d_2})$ respectively by the rapid decay of the coefficients $\langle \overline{\mathcal{G}_{\tilde\alpha,\vartheta_2} \otimes \mathcal{G}_{\tilde\beta,\vartheta_1}}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}$. Hence

$$\sup_{\tilde\alpha\in\mathbb{Z}^{2d_2},\tilde\beta\in\mathbb{Z}^{2d_1}} |\langle \overline{\mathcal{G}_{\tilde\alpha,\vartheta_2} \otimes \mathcal{G}_{\tilde\beta,\vartheta_1}}, \Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}|^{\frac{2}{3}} |\langle (T - T_j)\mathcal{G}_{\tilde\beta,\vartheta_1}, \mathcal{G}_{\tilde\alpha,\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}|$$

$$\xrightarrow{j} 0.$$

Furthermore,

$$\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2},\tilde{\beta}\in\mathbb{Z}^{2d_1}} |\langle \overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}},\Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}|^{\frac{1}{3}} < \infty.$$

Thus using dominated convergence we see that

$$\begin{aligned} &|\langle K_T - K_{T_j},\Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}| \\ &\quad\le \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2},\tilde{\beta}\in\mathbb{Z}^{2d_1}} |\langle (T-T_j)\mathcal{G}_{\tilde{\beta},\vartheta_1},\mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}||\langle \overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}},\Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}| \\ &\quad\xrightarrow{j} 0. \end{aligned}$$

Lemma 3.2.15 shows that the necessary estimates can be made uniformly for Φ in a bounded set, so by the above we conclude that $K_j \xrightarrow{j} K$ in $\mathscr{S}'(\mathbb{R}^{d_1+d_2})$. ■

As a corollary of the above we get that the tensor product of tempered distributions has an extension as a tempered distribution on a larger Schwartz space. But more than that, we get a distributional Fubini's Theorem. Note we make use of some formal variables in order to state the theorem.

Theorem 3.2.22. *(Fubini's Theorem) Fix any two distributions $\phi \in \mathscr{S}'(\mathbb{R}^{d_1}), \psi \in \mathscr{S}'(\mathbb{R}^{d_2})$. Then $\phi\otimes\psi$ has a unique extension in $\mathscr{S}'(\mathbb{R}^{d_1+d_2})$.*

Moreover, for every $\Phi \in \mathscr{S}(\mathbb{R}^{d_1+d_2})$ both

$$\mathbb{R}^{d_1} \ni y \mapsto \langle \psi, \Phi(\cdot,y)\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}$$

and

$$\mathbb{R}^{d_2} \ni x \mapsto \langle \phi, \Phi(x,\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^{d_1}),\mathscr{S}(\mathbb{R}^{d_1})}$$

defines Schwartz functions, and

$$\begin{aligned} \langle \phi\otimes\psi,\Phi\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})} &= \langle \phi_y, \langle \psi,\Phi(\cdot,y)\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}\rangle_{\mathscr{S}'(\mathbb{R}^{d_1}),\mathscr{S}(\mathbb{R}^{d_1})} \\ &= \langle \psi_x, \langle \phi,\Phi(x,\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^{d_1}),\mathscr{S}(\mathbb{R}^{d_1})}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}. \end{aligned}$$

Proof. The tensor product $\phi\otimes\psi$ defines a separately continuous bilinear form on $\mathscr{S}(\mathbb{R}^{d_1})\times\mathscr{S}(\mathbb{R}^{d_2})$, which then can be subjected to a use of the Schartz Kernel Theorem 3.2.21, showing the existence of a unique $K\in\mathscr{S}'(\mathbb{R}^{d_1+d_2})$ such that

$$\langle \phi,\omega_1\rangle_{\mathscr{S}'(\mathbb{R}^{d_1}),\mathscr{S}(\mathbb{R}^{d_1})}\langle \psi,\omega_2\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} = \langle K,\omega_1\otimes\omega_2\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}$$

for all $\omega_1\in\mathscr{S}(\mathbb{R}^{d_1}), \omega_2\in\mathscr{S}(\mathbb{R}^{d_2})$. We identify K with $\phi\otimes\psi$, as already done in the theorem.

As for the second statement, fix some phase functions $\vartheta_1\in\mathscr{O}_m(\mathbb{R}^{2d_1}), \vartheta_2\in\mathscr{O}_m(\mathbb{R}^{2d_2})$. By Lemma 3.2.15 we have

$$\begin{aligned} &\langle \psi,\Phi(\cdot,y)\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \\ &\quad= \sum_{\tilde{\beta}\in\mathbb{Z}^{2d_1}}\left(\sum_{\tilde{\alpha}\in\mathbb{Z}^{2d_2}} \langle \Phi,\overline{\mathcal{G}_{\tilde{\alpha},\vartheta_2}\otimes\mathcal{G}_{\tilde{\beta},\vartheta_1}}\rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})}\langle \psi,\mathcal{G}_{\tilde{\alpha},\vartheta_2}\rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})}\right)\mathcal{G}_{\tilde{\beta},\vartheta_1}(y) \end{aligned}$$

for all $y \in \mathbb{R}^{d_1}$. Combining Lemma 3.2.15 and Lemma 3.2.17, we see that the above coefficients satisfy the sufficient criteria in Lemma 3.2.15 for the above sum to converge to a Schwartz function. It is now trivial by the expansions to show that

$$\langle \phi \otimes \psi, \Phi \rangle_{\mathscr{S}'(\mathbb{R}^{d_1+d_2}),\mathscr{S}(\mathbb{R}^{d_1+d_2})} = \langle \phi_y, \langle \psi, \Phi(\cdot, y) \rangle_{\mathscr{S}'(\mathbb{R}^{d_2}),\mathscr{S}(\mathbb{R}^{d_2})} \rangle_{\mathscr{S}'(\mathbb{R}^{d_1}),\mathscr{S}(\mathbb{R}^{d_1})}.$$

The statements with ϕ and ψ interchanged follows by similar arguments. ■

Our last theorem gives a sort of minimality of the extension of calculus to tempered distributions.

Theorem 3.2.23. *(Structure Theorem) Each $\phi \in \mathscr{S}'(\mathbb{R}^d)$ is the distributional derivative of a continuous function with polynomial growth, i.e. there exists an $n \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^d$ and $\psi \in \langle \cdot \rangle^n BC(\mathbb{R}^d)$ such that $\partial^\gamma \psi = \phi$ in terms of distributions.*

Proof. Our strategy is to take the convolution of our frame elements with a suitably rotated fundamental solution and use the expansion in Lemma 3.2.17. Fix some phase function $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$.

Define

$$H_N \colon \mathbb{R} \ni x \mapsto \frac{x^{N-1}}{(N-1)!} 1_{\mathbb{R}_{\geq 0}}(x)$$

for $N \in \mathbb{N} \setminus \{0\}$. Then $\partial^N H_N = \delta$ with δ being the Dirac-delta distribution, i.e. H_N is a fundamental solution for the differential operator ∂^N.

Now we make the ansatz that for $N \in \mathbb{N}$ large enough, the following will be the continuous function we are looking for:

$$\psi := \sum_{\tilde{\alpha} \in \mathbb{Z}^{2d}} \langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta} * (\otimes_{j=1}^d \operatorname{sign}(\alpha_j)^N (\operatorname{sign}(\alpha_j)\mathrm{id})^* H_N) \tag{3.2.4}$$

Here we let $\operatorname{sign}(0) = 1$. The steps to be taken are now that the sum converges to a continuous function with polynomial growth, that the distributional derivative can be taken under the sum, and finally use that

$$\partial^{(N,\dots,N)} \mathcal{G}_{\tilde{\alpha},\vartheta} * (\otimes_{j=1}^d \operatorname{sign}(\alpha_j)^N (\operatorname{sign}(\alpha_j)\mathrm{id})^* H_N) = \mathcal{G}_{\tilde{\alpha},\vartheta} \tag{3.2.5}$$

and Lemma 3.2.17.

For the convergence part, we want the sum (3.2.4) to converge absolutely in $\langle \cdot \rangle^M BC(\mathbb{R}^d)$ for some $M \in \mathbb{N}$. From Lemma 3.2.17 we know for some $k, l \in \mathbb{N}$ that

$$|\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq C_\phi \langle \alpha \rangle^k \langle \alpha' \rangle^l.$$

Furthermore, for $x \in \mathbb{R}^d$ and if N is large enough we have

$$\begin{aligned}
&\langle \alpha \rangle^{k+d+1} \langle \alpha' \rangle^{l+d+1} |\mathcal{G}_{\tilde{\alpha},\vartheta} * (\otimes_{j=1}^d \operatorname{sign}(\alpha_j)^N (\operatorname{sign}(\alpha_j)\mathrm{id})^* H_N)(x)| \\
&= \langle \alpha \rangle^{k+d+1} \bigg| \int_{\mathbb{R}^d} \tau_{x-\alpha}(-\mathrm{id})^* (\otimes_{j=1}^d \operatorname{sign}(\alpha_j)^N (\operatorname{sign}(\alpha_j)\mathrm{id})^* H_N) \\
&\qquad (\vartheta(\cdot + \alpha, \alpha) \otimes^d u)(1-\Delta)^{l+d+1} (\otimes_{j=1}^d E_{\alpha'_j}) \bigg| \\
&\leq \langle \alpha \rangle^{k+d+1} \int_{\mathbb{R}^d} \Big| (1-\Delta)^{l+d+1} \big(\tau_{x-\alpha}(-\mathrm{id})^* (\otimes_{j=1}^d \operatorname{sign}(\alpha_j)^N (\operatorname{sign}(\alpha_j)\mathrm{id})^* H_N)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad (\vartheta(\cdot+\alpha,\alpha)\otimes^d u))\Big| \\
&\le C_l\langle\alpha\rangle^{k+d+1}\sum_{\substack{\delta\in\mathbb{N}^d\\|\delta|\le 2l}}\sum_{\substack{\sigma\in\mathbb{N}^d\\|\sigma|\le 2l+2d+2}}\int_{\mathbb{R}^d}\Big|\tau_{x-\alpha}(-\mathrm{id})^*(\partial^\delta(\otimes_{j=1}^d(\mathrm{sign}(\alpha_j)\mathrm{id})^*H_N))\Big| \\
&\qquad\qquad \Big|\partial^\sigma(\vartheta(\cdot+\alpha,\alpha)\otimes^d u)\Big|.
\end{aligned}
$$

Now $\mathrm{supp}(\otimes^d u)\subset(-1,1)^d$ and $\mathrm{supp}(\otimes_{j=1}^d(\mathrm{sign}(\alpha_j)\mathrm{id})^*H_N) = \prod_{j=1}^d \mathrm{sign}(\alpha_j)\mathbb{R}_{\ge 0}$, so the above is only non-zero when $1+\mathrm{sign}(\alpha_j)x_j \ge |\alpha_j|$ for all $j=1,\dots,d$. This implies that

$$\langle\alpha\rangle \le C\langle x\rangle$$

on the support of the $\tilde{\alpha}$-term, hence

$$\langle\alpha\rangle^{k+d+1}\langle\alpha'\rangle^{l+d+1}|\mathcal{G}_{\tilde{\alpha}}*(\otimes_{j=1}^d\mathrm{sign}(\alpha_j)^N(\mathrm{sign}(\alpha_j)\mathrm{id})^*H_N)(x)|\le C_{l,k,N}\langle x\rangle^M$$

for M large enough. Thus each term in the sum (3.2.4) is contained in $\langle\cdot\rangle^M BC(\mathbb{R}^d)$ and the sum converges absolutely in this space.

Since $\langle\cdot\rangle^M BC(\mathbb{R}^d)\hookrightarrow\mathscr{S}'(\mathbb{R}^d)$ continuously, see Proposition 3.2.10, the sum converges in $\mathscr{S}'(\mathbb{R}^d)$, so we may take the distributional derivative term by term. Factoring in (3.2.5) we are done. ■

3.3 Pseudo-differential Calculus

Pseudo-differential operators generalize a large class of operators including some multiplication operators and linear partial differential operators. Simply put we "sandwich" a function between its Fourier transform and its inverse, translating something that is easy to interpret into a perhaps obscure operator. The function may be a classical observable, which we quantize into a pseudo-differential operator, so it becomes a quantum observable.

We start by defining the Weyl transform:

Definition 3.3.1. (Weyl Transform) For $t\in\mathbb{R}$ and a phase function $\vartheta\in\mathscr{O}_m(\mathbb{R}^{2d})$ we define the (t,ϑ)-Weyl transform of $\Phi\in\mathscr{S}(\mathbb{R}^{2d})$ by

$$\mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi(x,y)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{i\xi\cdot(x-y)}\vartheta(x,y)\Phi(tx+(1-t)y,\xi)d\xi$$

for $x,y\in\mathbb{R}^d$.

The Weyl transform is a continuous linear operator on $\mathscr{S}(\mathbb{R}^{2d})$: It is the composition of a partial Fourier transform, a linear change of coordinates, and a multiplication by a $\mathscr{O}_m(\mathbb{R}^{2d})$-function. It also has an inverse, the (t,ϑ)-Wigner transform, which for $\Phi\in\mathscr{S}(\mathbb{R}^{2d})$ is given by

$$\mathcal{T}^{t,\vartheta}_{\mathrm{Wigner}}\Phi(x,\xi)=\int_{\mathbb{R}^d}e^{iy\cdot\xi}\overline{\vartheta(x-(1-t)y,x+ty)}\Phi(x-(1-t)y,x+ty)dy$$

for all $x,\xi\in\mathbb{R}^d$.

Similarly to other operators on Schwartz space, we can extend the Weyl transform to tempered distributions using the transpose identity. Here the formal transpose of $\mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}$ is given by

$$\begin{aligned}(2\pi)^{-d}(\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*)\mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}}\Phi(x,\xi) \\ = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\vartheta(x-(1-t)y, x+ty)\Phi(x-(1-t)y, x+ty)dy\end{aligned}$$

for $\Phi \in \mathscr{S}(\mathbb{R}^{2d})$ and $x, \xi \in \mathbb{R}^d$. Thus it follows that a suitable definition of the (t,ϑ)-Weyl transform on $\mathscr{S}'(\mathbb{R}^{2d})$ is the real transpose of $(2\pi)^{-d}(\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*)\mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}}$. This results in the Weyl transform being a linear homeomorphism of $\mathscr{S}'(\mathbb{R}^{2d})$. Furthermore, the Wigner transform can be extended in the same manner and becomes the inverse of the Weyl transform on $\mathscr{S}'(\mathbb{R}^{2d})$.

Definition 3.3.2. (Pseudo-differential Operator) Fix $t \in \mathbb{R}$ and a phase function $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$. Then we define the (t,ϑ)-quantization

$$\mathfrak{Op}_{t,\vartheta}\colon \mathscr{S}'(\mathbb{R}^{2d}) \to \mathcal{B}(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$$

in the following manner: For $\Phi \in \mathscr{S}'(\mathbb{R}^{2d})$, $\mathfrak{Op}_{t,\vartheta}(\Phi)\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is the unique continuous linear operator with Schwartz kernel $\mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi$.

The operator $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is called a pseudo-differential operator and we call Φ the symbol of $\mathfrak{Op}_{t,\vartheta}(\Phi)$.

Remark! 3.3.3. The choices $t = 1$ or $t = \frac{1}{2}$ are the most popular.

The case $t = 1$ is interesting since the symbol only impacts the outer position variable, making it easier to deal with multiplication operators. When $t = 1$ we call $\mathfrak{Op}_{1,\vartheta}$ the ϑ-standard quantization.

For $t = \frac{1}{2}$ our interest comes from symmetry reasons, which we will elaborate on shortly. We shall also mostly be interested in this case, and we omit t in the notation and call $\mathfrak{Op}_\vartheta$ the ϑ-Weyl-quantization.

The space of tempered distributions $\mathscr{S}'(\mathbb{R}^{2d})$ is vast and we only study the quantization of some. Two particularly interesting and simple sets of operators are the position and momentum operators. These will have a role to play beyond this chapter.

Fix $j \in \{1, \dots, d\}$. The jth position operator $X_{\vartheta,j}$ is defined as the (t,ϑ)-quantization of the symbol $\mathbb{R}^{2d} \ni (x,\xi) \mapsto x_j$, while the jth momentum operator $\Pi_{\vartheta,j}$ is defined by the symbol $\mathbb{R}^{2d} \ni (x,\xi) \mapsto \xi_j$. Thus they both depend on the phase function chosen for the quantization. A short calculation gives

$$X_{\vartheta,j}\phi(x) = \vartheta(x,x)x_j\phi(x)$$

and

$$\Pi_{\vartheta,j}\phi(x) = -i\left(\vartheta(x,x)\partial^{e_j} + \partial^{e_{d+j}}\vartheta(x,x)\right)\phi(x)$$

for $\phi \in \mathscr{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Both have trivial continuous extensions to $\mathscr{S}'(\mathbb{R}^d)$.

3.3.1 Hörmander Symbols

The symbols for which we develop a pseudo-differential calculus are called Hörmander symbols, which are smooth functions of tempered growth.

Definition 3.3.4. (Tempered Weight) A tempered weight is a positive continuous function $M\colon \mathbb{R}^{2d} \to (0,\infty)$ such that there exists $s_M, C_M > 0$ for which

$$M(x+y,\xi+\zeta) \leq C_M M(x,\xi)\langle(y,\zeta)\rangle^{s_M}$$

holds for all $x,y,\xi,\zeta \in \mathbb{R}^d$. We define the standard tempered weight by $M_0\colon \mathbb{R}^{2d} \ni (x,\xi) \mapsto \langle\xi\rangle$.

The space of tempered weights is denoted by $\mathcal{M}(\mathbb{R}^{2d})$, and on $\mathcal{M}(\mathbb{R}^{2d})$ we define the relation $\leq_{\mathcal{M}(\mathbb{R}^{2d})}$ by $M_1 \leq_{\mathcal{M}(\mathbb{R}^{2d})} M_2$ if and only if $M_1 \leq CM_2$ pointwise for $M_1, M_2 \in \mathcal{M}(\mathbb{R}^{2d})$

For a tempered weight $M \in \mathcal{M}(\mathbb{R}^{2d})$ we have a generalized Peetre's inequality:

$$M(x,\xi)^p M(y,\zeta)^{-p} \leq C_M^{|p|}\langle(x-y,\xi-\zeta)\rangle^{s_M|p|} \tag{3.3.1}$$

for $p \in \mathbb{R}, x,y,\xi,\zeta \in \mathbb{R}^d$. Furthermore

$$C_M^{-1}M(0,0)\langle(x,\xi)\rangle^{-s_M} \leq M(x,\xi) \leq C_M M(0,0)\langle(x,\xi)\rangle^{s_M}$$

for $x,\xi \in \mathbb{R}^d$. Also $\mathcal{M}(\mathbb{R}^{2d})$ is closed under multiplication and taking real powers, and these operations act as expected w.r.t. the order $\leq_{\mathcal{M}(\mathbb{R}^{2d})}$.

Definition 3.3.5. (Hörmander Class) The Hörmander class with tempered weight $M \in \mathcal{M}(\mathbb{R}^{2d})$ is defined as

$$S_M(\mathbb{R}^{2d}) := \left\{ \Phi \in C^\infty(\mathbb{R}^{2d}) \,\middle|\, \sup_{x,\xi\in\mathbb{R}^d} M(x,\xi)^{-1}|\partial^\gamma\Phi(x,\xi)| < \infty, \forall\gamma \in \mathbb{N}_0^{2d} \right\}.$$

On $S_M(\mathbb{R}^{2d})$ we define the semi-norms:

$$\|\cdot\|_{S_M(\mathbb{R}^{2d}),n}\colon S_M(\mathbb{R}^{2d}) \ni \Phi \mapsto \sum_{\substack{\gamma\in\mathbb{N}_0^d\\|\gamma|\leq n}} \sup_{x,\xi\in\mathbb{R}^d} M(x,\xi)^{-1}|\partial^\gamma\Phi(x,\xi)|$$

for all $n \in \mathbb{N}_0$.

Moreover, we define $S_\infty(\mathbb{R}^{2d}) := \bigcup_{M\in\mathcal{M}(\mathbb{R}^{2d})} S_M(\mathbb{R}^{2d})$.

Proposition 3.3.6. *$S_M(\mathbb{R}^{2d})$ is a Fréchet space with the Heine-Borel property.*

Proof. We skip many of the details since these mirror Proposition 3.2.2. See also [30, Proposition 3.1.6.].

The family of semi-norms we defined on $S_M(\mathbb{R}^{2d})$ are clearly separating and there is countably many of them, hence the induced topology is metrizable. If we consider some Cauchy sequence in $S_M(\mathbb{R}^{2d})$, then the sequence and the sequences of derivatives converge uniformly on compact sets to some smooth function and its derivatives respectively. Then by the arguments in the proof of Proposition 3.2.2 we find that the limit is in $S_M(\mathbb{R}^{2d})$ and so is the convergence. The Heine-Borel property is then a consequence of Arzelá-Ascoli Theorem and a diagonalization argument. ∎

Proposition 3.3.7. *$\mathscr{S}(\mathbb{R}^{2d}) \hookrightarrow S_{M_1}(\mathbb{R}^{2d}) \hookrightarrow S_{M_2}(\mathbb{R}^{2d}) \hookrightarrow \mathscr{S}'(\mathbb{R}^{2d})$ continuously for any $M_1, M_2 \in \mathcal{M}(\mathbb{R}^{2d})$ with $M_1 \leq_{\mathcal{M}(\mathbb{R}^{2d})} M_2$.*

Proof. The continuous injections $\mathscr{S}(\mathbb{R}^{2d}) \hookrightarrow S_{M_1}(\mathbb{R}^{2d})$ and $S_{M_2}(\mathbb{R}^{2d}) \hookrightarrow \mathscr{S}'(\mathbb{R}^{2d})$ are simple extensions of the proofs of Proposition 3.2.3 and Proposition 3.2.10, and $S_{M_1}(\mathbb{R}^{2d}) \hookrightarrow S_{M_2}(\mathbb{R}^{2d})$ is trivial. ∎

Later on in Lemma 3.3.8 we essentially prove that $\mathscr{S}(\mathbb{R}^{2d})$ is dense in $S_M(\mathbb{R}^{2d})$.

3.3.2 Coordinate Representation of Pseudo-differential Operators

When dealing with Schwartz functions and tempered distributions we had a lot of success applying coordinate representations. One such application is the expansion

$$T\phi = \sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} \langle T\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \phi, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta}$$

for any $T \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$, $\phi \in \mathscr{S}(\mathbb{R}^d)$, and phase function $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$, with the sum converging absolutely in $\mathscr{S}'(\mathbb{R}^d)$. Then a reasonable hypothesis is that if the matrix elements $\langle T\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$ have some nice properties then so must the map T and conversely. Note that these matrix elements are really the coordinates of the Schwartz kernel of T.

Specifically, we consider the Hörmander symbols. In order to exploit the tempered weights governing the Hörmander classes and make sharp estimate on the matrix elements, we need to make some assumptions on the phase function $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$.

The first assumption is that ϑ is Hermitian, meaning that $\vartheta(x,y) = \overline{\vartheta(y,x)}$ for all $x, y \in \mathbb{R}^d$. The second is that ϑ has the following triangle property: The function

$$\eta\colon \mathbb{R}^{3d} \ni (x,y,z) \mapsto \vartheta(x,y)\vartheta(y,z)\vartheta(z,x)$$

satisfies that for every $\gamma \in \mathbb{N}_0^{3d}$ there exists some $n_\gamma \in \mathbb{N}_0$ such that

$$\sup_{x,y,z\in\mathbb{R}^d} (\langle x-y\rangle\langle y-z\rangle)^{-n_\gamma} |\partial^\gamma \eta(x,y,z)| < \infty.$$

Note η is cyclic:

$$\eta(x,y,z) = \eta(z,x,y) = \eta(y,z,x)$$

for all $x, y, z \in \mathbb{R}^d$.

Lemma 3.3.8. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a Hermitian phase function satisfying the triangle property and fix $t \in \mathbb{R}$, $M \in \mathcal{M}(\mathbb{R}^{2d})$.*

If $(A_{\tilde{\alpha},\tilde{\beta}})_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}}$ satisfies

$$\sup_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} \langle \alpha-\beta\rangle^k \langle \alpha'-\beta'\rangle^l M(t\alpha+(1-t)\beta, (1-t)\alpha'+t\beta')^{-1} |A_{\tilde{\alpha},\tilde{\beta}}| < \infty$$

for arbitrary $k, l \in \mathbb{N}_0$, then $\sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} A_{\tilde{\alpha},\tilde{\beta}} \mathcal{T}^{t,\vartheta}_{\mathrm{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})$ converges unconditionally in $S_M(\mathbb{R}^{2d})$.

Conversely, for $\Phi \in S_M(\mathbb{R}^{2d})$ we have

$$\begin{aligned}\sup_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} &\langle \alpha-\beta\rangle^k \langle \alpha'-\beta'\rangle^l M(t\alpha+(1-t)\beta, (1-t)\alpha'+t\beta')^{-1} \\ &\qquad |\langle \mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \otimes \mathcal{G}_{\tilde{\beta},\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}| \\ &\qquad \le C_{M,t,\vartheta,k,l} \|\Phi\|_{S_M(\mathbb{R}^{2d}),n}\end{aligned}$$

for arbitrary $k, l \in \mathbb{N}_0$ and some $n \in \mathbb{N}_0$ determined by k, l, and

$$\Phi = \sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} \langle \mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \otimes \mathcal{G}_{\tilde{\beta},\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})} \mathcal{T}^{t,\vartheta}_{\mathrm{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}).$$

Remark! 3.3.9. The interpretation of the lemma is that the operators in $\mathfrak{Op}_{t,\vartheta}(S_M(\mathbb{R}^{2d}))$ are exactly the operators having matrix elements satisfying the stated decay conditions: The Schwartz kernel of $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is $\mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi$ and

$$\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \otimes \mathcal{G}_{\tilde{\beta},\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}.$$

Proof. Define η as above.

Let us begin by studying the Schwartz functions $\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})$. For $x, \xi \in \mathbb{R}^d$ we have

$$\begin{aligned}
&\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})(x,\xi) \\
&\quad= \int_{\mathbb{R}^d} e^{iy\cdot\xi}\overline{\vartheta(x-(1-t)y, x+ty)}\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}(x-(1-t)y, x+ty)dy \\
&\quad= \frac{\vartheta(\beta,\alpha)e^{i(\beta'\cdot\beta-\alpha'\cdot\alpha)}}{(2\pi)^d}\int_{(-2,2)^d} e^{i(z+\beta-\alpha)\cdot(\xi-(1-t)\alpha'-t\beta')}e^{i(\alpha'-\beta')\cdot x} \\
&\qquad\quad \eta(\alpha, x+t(z+\beta-\alpha), x-(1-t)(z+\beta-\alpha))\eta(\alpha,\beta, x+t(z+\beta-\alpha)) \\
&\qquad\quad \otimes^d u(x-(1-t)z-t\alpha-(1-t)\beta) \otimes^d u(x+tz-t\alpha-(1-t)\beta)dz,
\end{aligned} \tag{3.3.2}$$

where we used the translation $z := y-\beta+\alpha$, the Hermitian property of the phase function, and the fact that $\operatorname{supp}(\otimes^d u) \subseteq (-1,1)$ implies that the above integrand has support in $(-2,2)^d$ by

$$z = (x+tz-t\alpha-(1-t)\beta) - (x-(1-t)z-t\alpha-(1-t)\beta).$$

Moreover, the support of the u's tells us that (3.3.2) is zero when $x \notin t\alpha + (1-t)\beta + (-1,1)^d$.

If we take the γ-derivative, $\gamma \in \mathbb{N}_0^{2d}$, of $\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})$, then by (3.3.2) we get polynomials in $z+\beta-\alpha$ and $\alpha'-\beta'$, and derivatives of η and u. Furthermore, using integration by parts we can get arbitrary high powers of $\langle \xi-(1-t)\alpha'+t\beta'\rangle^{-1}$ at the cost of additional derivatives of η and u by using the exponential factor $e^{iz\cdot(\xi-(1-t)\alpha'-t\beta')}$. We do this to get a factor of at least $\langle \xi-(1-t)\alpha'+t\beta'\rangle^{-d-1-s_M}$.

The polynomials in z and derivatives of u pose no problem since the integrand has bounded support in z and u is a $C_c^\infty(\mathbb{R})$-function. Recalling the triangle property of ϑ and Peetre's inequality we may bound the derivatives of the η's uniformly by

$$C_{M,t,\vartheta,\gamma}\langle z\rangle^{n_1}\langle \alpha-\beta\rangle^{n_2}\langle x+tz-t\alpha-(1-t)\beta\rangle^{n_3}\langle x-(1-t)z-t\alpha-(1-t)\beta\rangle^{n_4}$$

for some $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$ dependent on ϑ, M, γ. Note again the factor of $\langle z\rangle^{n_1}$ gives no issues and the factor $\langle x+tz-t\alpha-(1-t)\beta\rangle^{n_3}\langle x-(1-t)z-t\alpha-(1-t)\beta\rangle^{n_4}$ is bounded by the support of the derivatives of the u's.

Keeping tabs, we conclude that

$$\begin{aligned}
&|\partial^\gamma \mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})(x,\xi)| \\
&\quad \le C_{M,t,\vartheta,\gamma}\langle \alpha-\beta\rangle^{m_1}\langle \alpha'-\beta'\rangle^{m_2}\langle \xi-(1-t)\alpha'+t\beta'\rangle^{-d-1-s_M}1_{t\alpha+(1-t)\beta+(-1,1)^d}(x)
\end{aligned}$$

for all $x, \xi \in \mathbb{R}^d$ and some $m_1, m_2 \in \mathbb{N}_0$ dependent on ϑ, M, γ. Thus together with the hypothesis on the coefficients $(A_{\tilde{\alpha},\tilde{\beta}})_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}}$ and Peetre's inequality (3.3.1) we get

$$\sup_{x,\xi\in\mathbb{R}^d} \sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} M(x,\xi)^{-1}|A_{\tilde{\alpha},\tilde{\beta}}||\partial^\gamma \mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde{\alpha},\vartheta} \otimes \overline{\mathcal{G}_{\tilde{\beta},\vartheta}})(x,\xi)| \le C_{M,t,\vartheta,\gamma,A},$$

whence the sum $\sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}} A_{\tilde\alpha,\tilde\beta}\partial^\gamma\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}})$ converges unconditionally and uniformly on $\mathbb{R}^{2d}$ when multiplied with a factor of M^{-1}. Consequently, one can interchange the sum and derivative, and so it is clear now that $\sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}} A_{\tilde\alpha,\tilde\beta}\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}})$ converges unconditionally in $S_M(\mathbb{R}^{2d})$.

For the last part of the lemma we recall from Lemma 3.2.17 that

$$\mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi = \sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}}\langle\mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi,\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}}$$

holds in $\mathscr{S}'(\mathbb{R}^{2d})$, and using $\mathcal{T}^{t,\vartheta}_{\text{Wigner}}$ on both sides we get

$$\Phi = \sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}}\langle\mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi,\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}\mathcal{T}^{t,\vartheta}_{\text{Wigner}}(\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}}).$$

Hence if we can prove the stated estimate on the coefficients, then the convergence of the last sum also holds unconditionally in $S_M(\mathbb{R}^{2d})$ by the first part.

The coefficients are more explicitly given by

$$\begin{aligned}
&\langle\mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi,\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}\\
&\qquad= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,\xi)e^{-iy\cdot\xi}\vartheta(x-(1-t)y,x+ty)\\
&\qquad\quad\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}(x-(1-t)y)\mathcal{G}_{\tilde\beta,\vartheta}(x+ty)dydxd\xi\\
&\qquad= \frac{\vartheta(\beta,\alpha)e^{i(\alpha-\beta)\cdot((1-t)\alpha'+t\beta')}}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{iz\cdot(\beta'-\alpha')}e^{-iw\cdot\zeta}e^{i\zeta\cdot(\alpha-\beta)}\\
&\qquad\quad\Phi(z+t\alpha+(1-t)\beta,\zeta+(1-t)\alpha'+t\beta')\\
&\qquad\quad\eta(\alpha,z-(1-t)w+\alpha,z+tw+\beta)\eta(\alpha,\beta,z+tw+\beta)\\
&\qquad\quad\otimes^d u(z-(1-t)w)\otimes^d u(z+tw)dwdzd\zeta
\end{aligned}\tag{3.3.3}$$

where we used the translation

$$(z,w,\zeta) := (x-t\alpha-(1-t)\beta,y+\alpha-\beta,\xi-(1-t)\alpha'-t\beta')$$

and that the phase function is Hermitian. We also note that the integrand has support in $(-1,1)^d$ for the z variable and support in $(-2,2)^d$ for the w variable.

Let $k,l\in\mathbb{N}_0$ be given. Being mindful of the order in which we integrate, we may use integration by parts in (3.3.3), first in w, then z and lastly ζ to introduce arbitrary powers of $\langle\zeta\rangle^{-1}$, $\langle\alpha'-\beta'\rangle^{-1}$, and $\langle\alpha-\beta\rangle^{-1}$, respectively, using the exponential factors to do so. For the integration by parts in w we need at least a factor of $\langle\zeta\rangle^{-d-1-s_M}$, which introduces derivatives of η and u. The second integration by parts, which is in z, we make a factor of $\langle\alpha'-\beta'\rangle^{-l}$ appear, and again we get derivatives of η,u,Φ. The last integration by parts, in ζ this time, we do not know yet to which power we want $\langle\alpha-\beta\rangle^{-1}$, but we remark that the only problematic side-effect is the introduction of additional derivatives of Φ.

We end up with a sum of functions in the integrand. The derivatives of u are not problematic, and the derivatives of the η's, since there are finitely many, can be uniformly bounded by

$$C_{M,t,\vartheta,l,t}\langle\alpha-\beta\rangle^{n_1}\langle z\rangle^{n_2}\langle w\rangle^{n_3}$$

for some $n_1, n_2, n_3 \in \mathbb{N}_0$ dependent on ϑ, M, l. Thus we want to create a factor of $\langle \alpha - \beta \rangle^{-n_1-k}$ with the last integration by parts. As for the derivatives of Φ, we note that if $\gamma \in \mathbb{N}_0^{2d}$, then

$$\begin{aligned}
&\sup_{z,\zeta\in\mathbb{R}^d} \langle z\rangle^{-s_M}\langle\zeta\rangle^{-s_M}|\partial^\gamma\Phi(z+t\alpha+(1-t)\beta,\zeta+(1-t)\alpha'+t\beta')| \\
\leq C_M &\sup_{z,\zeta\in\mathbb{R}^d} M(t\alpha+(1-t)\beta,(1-t)\alpha'+t\beta')M(z+t\alpha+(1-t)\beta,\zeta+(1-t)\alpha'+t\beta')^{-1} \\
&\qquad |\partial^\gamma\Phi(z+t\alpha+(1-t)\beta,\zeta+(1-t)\alpha'+t\beta')| \\
&\qquad \leq C_{\Phi,M} M(t\alpha+(1-t)\beta,(1-t)\alpha'+t\beta')
\end{aligned}$$

by use of Peetre's inequality.

Thus all in all we estimate (3.3.3) by:

$$\begin{aligned}
&\langle \mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi, \overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})} \\
&\quad = C_{M,t,\vartheta,k,l}\langle\alpha-\beta\rangle^{-k}\langle\alpha'-\beta'\rangle^{-l}M(t\alpha+(1-t)\beta,(1-t)\alpha'+t\beta')\|\Phi\|_{S_M(\mathbb{R}^{2d}),n}
\end{aligned}$$

with $k, l \in \mathbb{N}_0$ arbitrary and $n \in \mathbb{N}_0$ dependent on k, l, ϑ, M. ■

On a technical note we remark that t is of no importance. By the required arbitrary decay in the differences $\alpha - \beta$ and $\alpha' - \beta'$ and Peetre's inequality, coefficients $(A_{\tilde\alpha,\tilde\beta})_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}}$ satisfying the condition in the lemma for one $t \in \mathbb{R}$ will satisfy it for all.

From this fact we deduce that the operator space $\mathfrak{Op}_{t,\vartheta}(S_M(\mathbb{R}^{2d}))$ is independent of t. Furthermore, the change of $S_M(\mathbb{R}^{2d})$-symbol is a homeomorphism of $S_M(\mathbb{R}^{2d})$.

Corollary 3.3.10. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be Hermitian phase function satisfying the triangle property and fix $t_1, t_2 \in \mathbb{R}$.*

The change of symbol map

$$\mathfrak{Op}_{t_2,\vartheta}^{-1}\circ\mathfrak{Op}_{t_1,\vartheta} = \mathcal{T}^{t_2,\vartheta}_{\mathrm{Wigner}}\circ\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}$$

restricts to a homeomorphism of $S_M(\mathbb{R}^{2d})$ for every $M \in \mathcal{M}(\mathbb{R}^{2d})$.

Proof. Since the inverse of $\mathfrak{Op}_{t_2,\vartheta}^{-1}\circ\mathfrak{Op}_{t_1,\vartheta}$ is $\mathfrak{Op}_{t_1,\vartheta}^{-1}\circ\mathfrak{Op}_{t_2,\vartheta}$ we only need to prove that $\mathfrak{Op}_{t_2,\vartheta}^{-1}\circ\mathfrak{Op}_{t_1,\vartheta}|_{S_M(\mathbb{R}^{2d})}$ has range in $S_M(\mathbb{R}^{2d})$ and is continuous as a map $S_M(\mathbb{R}^{2d}) \to S_M(\mathbb{R}^{2d})$, where the rest follows by symmetry.

Lemma 3.3.8 tells us that

$$\begin{aligned}
\mathcal{T}^{t_2,\vartheta}_{\mathrm{Wigner}}\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}\Phi &= \sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}}\langle\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}\Phi,\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}\mathcal{T}^{t_2,\vartheta}_{\mathrm{Wigner}}\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}\mathcal{T}^{t_1,\vartheta}_{\mathrm{Wigner}}(\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}}) \\
&= \sum_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}}\langle\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}\Phi,\overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\otimes\mathcal{G}_{\tilde\beta,\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}\mathcal{T}^{t_2,\vartheta}_{\mathrm{Wigner}}(\mathcal{G}_{\tilde\alpha,\vartheta}\otimes\overline{\mathcal{G}_{\tilde\beta,\vartheta}})
\end{aligned}$$

converges to some function in $S_M(\mathbb{R}^{2d})$ for every $\Phi \in S_M(\mathbb{R}^{2d})$. Moreover, using this expansion and the proof of Lemma 3.3.8 we get that

$$\|\mathcal{T}^{t_2,\vartheta}_{\mathrm{Wigner}}\mathcal{T}^{t_1,\vartheta}_{\mathrm{Weyl}}\Phi\|_{S_M(\mathbb{R}^{2d}),n} \leq C_{M,t_1,t_2,\vartheta,n}\|\Phi\|_{S_M(\mathbb{R}^{2d}),m_n}$$

for all $n \in \mathbb{N}_0$ and some $m_n \in \mathbb{N}_0$, showing continuity. ■

3.3.3 Boundedness Properties of Pseudo-differential Operators

We will immediately put this coordinate characterization to good use:

Proposition 3.3.11. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a Hermitian phase function satisfying the triangle property and fix $t \in \mathbb{R}, M \in \mathcal{M}(\mathbb{R}^{2d})$.*

For $\Phi \in S_M(\mathbb{R}^{2d})$ we have $\mathfrak{Op}_{t,\vartheta}(\Phi) \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$ and the map

$$\mathfrak{Op}_{t,\vartheta}|_{S_M(\mathbb{R}^{2d})} \colon S_M(\mathbb{R}^{2d}) \to \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$$

is continuous.

Proof. Our goal is to use Lemma 3.2.15 on the expansion

$$\mathfrak{Op}_{t,\vartheta}(\Phi)\phi = \sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \phi, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \mathcal{G}_{\tilde{\alpha},\vartheta} \tag{3.3.4}$$

holding in $\mathscr{S}'(\mathbb{R}^d)$ for $\phi \in \mathscr{S}(\mathbb{R}^d)$. Firstly, note that

$$\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \mathcal{T}^{t,\vartheta}_{\text{Weyl}}\Phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}} \otimes \mathcal{G}_{\tilde{\beta},\vartheta}\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})}.$$

The estimate in Lemma 3.3.8 can, with the use of Peetre's inequality, be reformulated into saying

$$\begin{aligned}\sup_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} &\langle\alpha\rangle^k\langle\alpha'\rangle^l\langle\beta\rangle^{-m}\langle\beta'\rangle^{-m}|\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \\ &\leq C_{M,t,\vartheta,k,l}\|\Phi\|_{S_M(\mathbb{R}^{2d}),n}\end{aligned}$$

for arbitrary $k, l \in \mathbb{N}_0$ and some $n, m \in \mathbb{N}_0$ determined by k, l. Thus this estimate used in conjunction with Lemma 3.2.15 tells us that the sum in (3.3.4) defines a Schwartz function, hence $R(\mathfrak{Op}_{t,\vartheta}(\Phi)) \subseteq \mathscr{S}(\mathbb{R}^d)$.

Tracing the necessary estimates and using the absolute convergence of the sum in (3.3.4) we get the bound

$$\|\mathfrak{Op}_{t,\vartheta}(\Phi)\phi\|_{\mathscr{S}(\mathbb{R}^d),n,m} \leq C_{M,t,\vartheta,n,m}\|\Phi\|_{S_M(\mathbb{R}^{2d}),k}\|\phi\|_{\mathscr{S}(\mathbb{R}^d),\tilde{n},\tilde{m}}$$

for arbitrary $n, m \in \mathbb{N}_0$ and $k, \tilde{n}, \tilde{m} \in \mathbb{N}_0$ dependent on n, m. This shows that $\mathfrak{Op}_{t,\vartheta}(\Phi) \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$ and that the (t,ϑ)-quantization is continuous from $S_M(\mathbb{R}^{2d})$ into $\mathcal{B}(\mathscr{S}(\mathbb{R}^d))$. ■

This proposition itself has consequences. For one, by evaluating $\mathfrak{Op}_{t,\vartheta}(\Phi)\phi$ variationally one gets the identity:

$$\mathfrak{Op}_{t,\vartheta}(\Phi)\phi(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\xi\cdot(x-y)}\vartheta(x,y)\Phi(tx+(1-t)y,\xi)\phi(y)dyd\xi$$

for all $x \in \mathbb{R}^d, \phi \in \mathscr{S}(\mathbb{R}^d), \Phi \in S_\infty(\mathbb{R}^d)$ and t, ϑ satisfying the assumption in Proposition 3.3.11. We observe that the order of integration is non-trivial and cannot be interchanged without further assumptions.

A second consequence is that we obtain a transpose identity for the pseudo-differential operators

$$\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \phi, S\psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

for all $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$ with $S\colon \mathscr{S}(\mathbb{R}^d) \ni \psi \mapsto \overline{\mathfrak{Op}_{1-t,\vartheta}(\overline{\Phi})\overline{\psi}}$. This allows us to extend $\mathfrak{Op}_{t,\vartheta}(\Phi)$ to a continuous linear operator on $\mathscr{S}'(\mathbb{R}^d)$.

The third consequence, and last for now, is that $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is a densely defined operator on $L^2(\mathbb{R}^d)$ and using the expression found above we see that

$$\langle \phi, \mathfrak{Op}_{t,\vartheta}(\Phi)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle \mathfrak{Op}_{1-t,\vartheta}(\overline{\Phi})\phi, \psi\rangle_{L^2(\mathbb{R}^d)}$$

holds for all $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$. Hence $\mathscr{S}(\mathbb{R}^d) \subseteq D(\mathfrak{Op}_{t,\vartheta}(\Phi)^*)$ and $\mathfrak{Op}_{t,\vartheta}(\Phi)^*|_{\mathscr{S}(\mathbb{R}^d)} = \mathfrak{Op}_{1-t,\vartheta}(\overline{\Phi})$. Now invoking Proposition 2.2.3 we reach the result:

Corollary 3.3.12. *Under the assumptions of Proposition 3.3.11 the operator $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is closable and its $L^2(\mathbb{R}^d)$-adjoint is given by $\mathfrak{Op}_{1-t,\vartheta}(\overline{\Phi})$ on $\mathscr{S}(\mathbb{R}^d)$.*

Remark! 3.3.13. Importantly, since $L^2(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$ continuously and $\mathscr{S}(\mathbb{R}^d)$ is dense in both spaces, the $L^2(\mathbb{R}^d)$-closure of $\mathfrak{Op}_{t,\vartheta}(\Phi)$ correspond to the value of the $\mathscr{S}'(\mathbb{R}^d)$-extension of $\mathfrak{Op}_{t,\vartheta}(\Phi)$ on their common domain.

Recalling Remark 3.3.3, we are now able to argue for the importance of the Weyl-quantization, i.e. $t = \frac{1}{2}$. It is exactly in this case that the condition $\Phi = \overline{\Phi}$ is equivalent with the statement that $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is symmetric.

Being closable is nice, but it would be even better to have bounded operators. This happens for certain Hörmander symbols as the next theorem states:

Theorem 3.3.14. *(Calderón-Vaillancourt Theorem) Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a Hermitian phase function satisfying the triangle property and fix $t \in \mathbb{R}$.*

For $\Phi \in S_1(\mathbb{R}^{2d})$ the operator $\mathfrak{Op}_{t,\vartheta}(\Phi)$ is extendable to a bounded operator on $L^2(\mathbb{R}^d)$ and the map

$$\mathfrak{Op}_{t,\vartheta}|_{S_1(\mathbb{R}^{2d})}\colon S_1(\mathbb{R}^{2d}) \to \mathcal{B}(L^2(\mathbb{R}^d))$$

is continuous.

Note we use the tempered weight $M \equiv 1$.

Proof. Fix $\phi \in \mathscr{S}(\mathbb{R}^2)$. Using the expansion (3.3.4) twice we get

$$\|\mathfrak{Op}_{t,\vartheta}(\Phi)\phi\|^2_{L^2(\mathbb{R}^2)} = \sum_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\in\mathbb{Z}^{2d}} \overline{\langle \phi, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\alpha},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \phi, \overline{\mathcal{G}_{\tilde{\gamma},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \tag{3.3.5}$$

with absolute convergence of the sum. Lemma 3.3.8 implies that

$$\sup_{\tilde{\alpha}\in\mathbb{Z}^{2d}} \sum_{\tilde{\beta},\tilde{\gamma}\in\mathbb{Z}^{2d}} |\overline{\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\alpha},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq C_{t,\vartheta}\|\Phi\|^2_{S_1(\mathbb{R}^d),n_d}$$

and similarly result is obtained by switching the roles of $\tilde{\alpha}$ and $\tilde{\beta}$:

$$\sup_{\tilde{\beta}\in\mathbb{Z}^{2d}} \sum_{\tilde{\alpha},\tilde{\gamma}\in\mathbb{Z}^{2d}} |\overline{\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\alpha},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta}, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| \leq \tilde{C}_{t,\vartheta}\|\Phi\|^2_{S_1(\mathbb{R}^d),n_d}$$

Here $n_d \in \mathbb{N}_0$ depends solely on the dimension d. Thus we can invoke Schur's test in $l^2(\mathbb{Z}^{2d})$ on the expression (3.3.5) and conclude that

$$\|\mathfrak{Op}_{t,\vartheta}(\Phi)\phi\|^2_{L^2(\mathbb{R}^2)} \leq C_{t,\vartheta}\|\Phi\|^2_{S_1(\mathbb{R}^d),n_d} \sum_{\tilde{\alpha}\in\mathbb{Z}^{2d}} |\langle \mathcal{G}_{\tilde{\alpha},\vartheta}, \phi\rangle_{L^2(\mathbb{R}^d)}|^2 = C_{d,t,\vartheta}\|\Phi\|^2_{S_1(\mathbb{R}^d),n_d}\|\phi\|^2_{L^2(\mathbb{R}^d)}.$$

The obtained estimate implies that $\overline{\mathfrak{Op}_{t,\vartheta}(\Phi)} \in \mathcal{B}(L^2(\mathbb{R}^d))$ and

$$\|\overline{\mathfrak{Op}_{t,\vartheta}(\Phi)}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq C_{t,\vartheta}\|\Phi\|_{S_1(\mathbb{R}^d),n_d},$$

so the (t,ϑ)-quantization is continuous from $S_1(\mathbb{R}^{2d})$ into $\mathcal{B}(L^2(\mathbb{R}^d))$. ■

3.3.4 Algebra of Pseudo-differential Operators

We saw in Proposition 3.3.11 that $\mathfrak{Op}_{t,\vartheta}(S_\infty(\mathbb{R}^{2d}))$ is a subset of the algebra $\mathcal{B}(\mathscr{S}(\mathbb{R}^d))$, so it makes sense to question whether or not $\mathfrak{Op}_{t,\vartheta}(S_\infty(\mathbb{R}^{2d}))$ is itself an algebra with the operator product. The answer is in the affirmative.

Theorem 3.3.15. *Let $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ be a Hermitian phase function satisfying the triangle property and fix $t \in \mathbb{R}$.*

The product

$$\#_{t,\vartheta}\colon S_\infty(\mathbb{R}^{2d}) \times S_\infty(\mathbb{R}^{2d}) \ni (\Phi,\Psi) \mapsto \mathfrak{Op}^{-1}_{t,\vartheta}(\mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi))$$

has image in $S_\infty(\mathbb{R}^{2d})$ and $S_\infty(\mathbb{R}^{2d})$ becomes a filtered algebra when endowed with $\#_{t,\vartheta}$: For $M_1, M_2 \in \mathcal{M}(\mathbb{R}^{2d})$ and $\Phi \in S_{M_1}(\mathbb{R}^{2d})$ and $\Psi \in S_{M_2}(\mathbb{R}^{2d})$ we have $\Phi \#_{t,\vartheta} \Psi \in S_{M_1M_2}(\mathbb{R}^{2d})$ and

$$\#_{t,\vartheta}|_{S_{M_1}(\mathbb{R}^{2d})\times S_{M_2}(\mathbb{R}^{2d})}\colon S_{M_1}(\mathbb{R}^{2d}) \times S_{M_2}(\mathbb{R}^{2d}) \to S_{M_1M_2}(\mathbb{R}^{2d})$$

is a continuous map.

Remark! 3.3.16. An alternative filtered product on $S_\infty(\mathbb{R}^{2d})$ is given by pointwise multiplication which is proven by a straightforward calculation. Although we will not have need of it, often one finds, such as in the case of semiclassical analysis, situations where the difference $\Phi \#_{t,\vartheta} \Psi - \Phi\Psi$ is "small".

The product is called the (t,ϑ)-Moyal product. Note $\bigcap_{M\in\mathcal{M}(\mathbb{R}^{2d})} S_M(\mathbb{R}^{2d}) = \mathscr{S}(\mathbb{R}^{2d})$ becomes a two-sided ideal in this product.

Proof. Note $\#_{t,\vartheta}$ is well-defined by Proposition 3.3.11. Fix $\Phi \in S_{M_1}(\mathbb{R}^{2d})$ and $\Psi \in S_{M_2}(\mathbb{R}^{2d})$ with $M_1, M_2 \in \mathcal{M}(\mathbb{R}^{2d})$. First, an expansion á la (3.3.4), but for the product $\mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)$, is

$$\begin{aligned}\mathfrak{Op}_{t,\vartheta}(\Phi \#_{t,\vartheta} \Psi)\phi &= \mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)\phi \\ &= \sum_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}\in\mathbb{Z}^{2d}} \langle \mathfrak{Op}_{t,\vartheta}(\Psi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\gamma},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &\quad \langle \phi, \overline{\mathcal{G}_{\tilde{\beta},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}\mathcal{G}_{\tilde{\alpha},\vartheta}\end{aligned}$$

holding in $\mathscr{S}(\mathbb{R}^d)$ for $\phi \in \mathscr{S}(\mathbb{R}^d)$, where we used

$$\begin{aligned}&\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &\quad = \sum_{\tilde{\gamma}\in\mathbb{Z}^{2d}} \langle \mathfrak{Op}_{t,\vartheta}(\Psi)\mathcal{G}_{\tilde{\beta},\vartheta}, \overline{\mathcal{G}_{\tilde{\gamma},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta}, \overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}.\end{aligned}$$

Secondly, by Lemma 3.3.8 we get

$$\begin{aligned}
&\sup_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} \langle \alpha-\beta\rangle^k \langle \alpha'-\beta'\rangle^l (M_1M_2)(t\alpha+(1-t)\beta,(1-t)\alpha'+t\beta')^{-1} \\
&\qquad \left| \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)\mathcal{G}_{\tilde{\beta},\vartheta},\overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \right| \\
&\qquad \leq \sup_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}^{2d}} C_{M_1,M_2,t} \sum_{\tilde{\gamma}\in\mathbb{Z}^{2d}} \langle \alpha-\gamma\rangle^k \langle \alpha'-\beta'\rangle^l (M_1M_2)(t\alpha+(1-t)\beta,(1-t)\alpha'+t\beta')^{-1} \\
&\qquad\quad \frac{\langle \alpha-\gamma\rangle^{k+d+1}\langle \alpha'-\gamma'\rangle^{l+d+1} M_1(t\alpha+(1-t)\gamma,(1-t)\alpha'+t\gamma')^{-1}}{\langle \alpha-\gamma\rangle^{d+1}\langle \alpha'-\gamma'\rangle^{d+1}} \\
&\qquad\quad\qquad \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde{\gamma},\vartheta},\overline{\mathcal{G}_{\tilde{\alpha},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\
&\qquad\quad \frac{\langle \gamma-\beta\rangle^{k+d+1}\langle \gamma'-\beta'\rangle^{l+d+1} M_2(t\gamma+(1-t)\beta,(1-t)\gamma'+t\beta')^{-1}}{\langle \gamma-\beta\rangle^{d+1}\langle \gamma'-\beta'\rangle^{d+1}} \\
&\qquad\quad\qquad \langle \mathfrak{Op}_{t,\vartheta}(\Psi)\mathcal{G}_{\tilde{\beta},\vartheta},\overline{\mathcal{G}_{\tilde{\gamma},\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\
&\qquad \leq C_{M_1,M_2,t,\vartheta,k,l}\|\Phi\|_{S_{M_1}(\mathbb{R}^{2d}),n}\|\Psi\|_{S_{M_2}(\mathbb{R}^{2d}),n}
\end{aligned}$$

for arbitrary $k,l\in\mathbb{N}_0$ and some $n\in\mathbb{N}_0$ determined by k,l. Thus the matrix elements for the product $\mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)$ satisfies the hypothesis of Lemma 3.3.8. So thirdly, one can conclude that the symbol in the (t,ϑ)-quantization of $\mathfrak{Op}_{t,\vartheta}(\Phi)\mathfrak{Op}_{t,\vartheta}(\Psi)$ is in $S_{M_1M_2}(\mathbb{R}^{2d})$, and by the proof of Lemma 3.3.8 we get an estimate of the kind:

$$\|\Phi \#_{t,\vartheta} \Psi\|_{S_{M_1M_2}(\mathbb{R}^{2d}),n} \leq C_{M_1,M_2,t,\vartheta,k,l}\|\Phi\|_{S_{M_1}(\mathbb{R}^{2d}),m_n}\|\Psi\|_{S_{M_2}(\mathbb{R}^{2d}),m_n}$$

for every $n\in\mathbb{N}_0$ with $m_n\in\mathbb{N}_0$. This last estimate also proves the continuity claim, so we are done. ∎

If we fix a tempered weight M, then we may create the one-parameter group of tempered weights $(M^p)_{p\in\mathbb{R}}$. For the associated Hörmander classes $(S_{M^p}(\mathbb{R}^{2d}))_{p\in\mathbb{R}}$ and the Moyal product we get

$$S_{M^p}(\mathbb{R}^{2d}) \#_{t,\vartheta} S_{M^q}(\mathbb{R}^{2d}) \subseteq S_{M^{p+q}}(\mathbb{R}^{2d})$$

for $p,q\in\mathbb{R}$, and so the Moyal product makes $\bigcup_{p\in\mathbb{R}} S_{M^p}(\mathbb{R}^{2d})$ into a filtered algebra in the parameter p.

3.3.5 Phase Functions Induced by Antisymmetric Forms and Beal's

We now restrict the phase functions we consider even further.

Lemma 3.3.17. *Let $A\in\mathbb{R}^{d\times d}$ be antisymmetric and define*

$$\vartheta\colon \mathbb{R}^{2d}\ni(x,y)\mapsto e^{ix\cdot Ay}.$$

Then ϑ is a Hermitian phase function in $\mathscr{O}_m(\mathbb{R}^{2d})$ satisfying the triangle inequality.

Proof. The only non-trivial part is the triangle inequality. If we differentiate

$$\eta\colon \mathbb{R}^{3d}\ni(x,y,z)\mapsto \vartheta(x,y)\vartheta(y,z)\vartheta(z,x)$$

in the first d variables, then it equals η multiplied by a polynomial in $A(y-z)$. If we differentiate in the next d variables we gain a polynomial in $A(z-x)$ and if differentiation is done in the last d variables we gain a polynomial in $A(x-y)$. Thus for every $\gamma \in \mathbb{N}_0^{3d}$ there exists $n_\gamma \in \mathbb{N}_0$ such that

$$\sup_{x,y,z \in \mathbb{R}^d} (\langle y-z\rangle \langle z-x\rangle \langle x-y\rangle)^{-n_\gamma} |\partial^\gamma \eta(x,y,z)| < \infty.$$

Now one use of Peetre's inequality finishes the proof. ■

A phase function such as the one in Lemma 3.3.17 is said to be induced by the antisymmetric matrix A.

For any phase function $\vartheta \in \mathscr{O}_m(\mathbb{R}^{2d})$ we may create the phase translations

$$\tau_{\vartheta,y} = \vartheta(\cdot, y)\tau_y$$

for $y \in \mathbb{R}^d$ as continuous linear maps on $\mathscr{S}'(\mathbb{R}^d)$, which restrict to continuous maps on $\mathscr{S}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. The intuition is that, if a pseudo-differential operator in the (t,ϑ)-quantization acts homogeneously throughout space, then it will commute with all phase translations. We see that this property has an appealing characterization for the phase functions induced by an antisymmetric matrix:

Lemma 3.3.18. *Let $A \in \mathbb{R}^{d\times d}$ be antisymmetric and define*

$$\vartheta \colon \mathbb{R}^{2d} \ni (x,y) \mapsto e^{ix\cdot Ay}.$$

Also fix $t \in \mathbb{R}$.

For $\Phi \in \mathscr{S}'(\mathbb{R}^{2d})$ and $z \in \mathbb{R}^d$ we have

$$\tau_{\vartheta,z}\mathfrak{Op}_{t,\vartheta}(\Phi)\tau_{\vartheta,-z} = \mathfrak{Op}_{t,\vartheta}(\tau_{(z,0)}\Phi).$$

Thus $[\mathfrak{Op}_{t,\vartheta}(\Phi), \tau_{\vartheta,z}] = 0$ if and only if $\tau_{(z,0)}\Phi = \Phi$.

Similarly to the above, one could consider invariance under other families of operators such as multiplication by a phase factor.

Proof. We will prove the equivalent statement:

$$\tau_{\vartheta,z}\mathfrak{Op}_{t,\vartheta}(\tau_{(-z,0)}\Phi)\tau_{\vartheta,-z} = \mathfrak{Op}_{t,\vartheta}(\Phi).$$

The identity $\tau_{\vartheta,z}\mathfrak{Op}_{t,\vartheta}(\tau_{(-z,0)}\Phi)\tau_{\vartheta,-z} = \mathfrak{Op}_{t,\vartheta}(\Phi)$ is equivalent with

$$\begin{aligned}\mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\Phi &= \mathcal{T}^{t,\vartheta}_{\mathrm{Weyl}}\tau_{(-z,0)}\Phi \circ (\tau_{\overline{\vartheta},-z} \otimes \tau_{\vartheta,-z}) \\ &= (2\pi)^{-d}\Phi \circ \tau_{(z,0)} \circ (\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*)\mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}} \circ (\tau_{\overline{\vartheta},-z} \otimes \tau_{\vartheta,-z}),\end{aligned}$$

which in turn is equivalent with

$$(\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*)\mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}} = \tau_{(z,0)} \circ (\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*)\mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}} \circ (\tau_{\overline{\vartheta},-z} \otimes \tau_{\vartheta,-z})$$

holding on $\mathscr{S}(\mathbb{R}^{2d})$. This finishes the abstract nonsense and gives us the condition we will prove.

For every $\Psi \in \mathscr{S}(\mathbb{R}^{2d})$ we have

$$\begin{aligned}
&\tau_{(z,0)} \circ (\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*) \mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}} \circ (\tau_{\overline{\vartheta},-z} \otimes \tau_{\vartheta,-z}) \Psi(x,\xi) \\
&\quad = \int_{\mathbb{R}^d} e^{-iy\cdot\xi} e^{i(x-z)\cdot Ay} e^{i(x-(1-t)y)\cdot Az} e^{-i(x+ty)\cdot Az} \Psi(x-(1-t)y, x+ty) dy \\
&\quad = \int_{\mathbb{R}^2} e^{-i\xi\cdot y} e^{ix\cdot Ay} \Psi(x-(1-t)y, x+ty) dy \\
&\quad = (\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)} \otimes (-\mathrm{id}_{\mathbb{R}^d})^*) \mathcal{T}^{t,\overline{\vartheta}}_{\mathrm{Wigner}} \Psi(x,\xi),
\end{aligned}$$

for all $x, \xi \in \mathbb{R}^d$, and we are done. ■

There is one last big and helpful result we need, Beal's Commutator Criterion, which holds for certain phase functions. We prove it for phase functions induced by an antisymmetric matrix, but note that it is a subcase of the Beal's Commutator Criterion proven in [9, Theorem 3.8].

Theorem 3.3.19. *(Beal's Commutator Criterion) Let $A \in \mathbb{R}^{d\times d}$ be antisymmetric and define*

$$\vartheta \colon \mathbb{R}^{2d} \ni (x,y) \mapsto e^{ix\cdot Ay}.$$

Also fix $t \in \mathbb{R}$.

If $\Phi \in S_1(\mathbb{R}^{2d})$, then all commutators of the form

$$[T_1[T_2, \dots [T_n, \mathfrak{Op}_{t,\vartheta}(\Phi)] \dots]] \colon , \tag{3.3.6}$$

where $T_j \in \{X_{\vartheta,1}, \dots, X_{\vartheta,d}, \Pi_{\vartheta,1}, \dots, \Pi_{\vartheta,d}\}$ for $j = 1, \dots, n$, $n \in \mathbb{N}_0$, are extendable to operators in $\mathcal{B}(L^2(\mathbb{R}^d))$.

Conversely, suppose that $T \colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is linear and extends to a bounded operator in $L^2(\mathbb{R}^d)$, and all the commutators of the form (3.3.6) *are also extendable to operators in $\mathcal{B}(L^2(\mathbb{R}^d))$. Then T is continuous and $\mathfrak{Op}^{-1}_{t,\vartheta}(T) \in S_1(\mathbb{R}^{2d})$.*

Proof. For the first part, we will show that $[T, \mathfrak{Op}_{t,\vartheta}(\Phi)] \in \mathfrak{Op}_{t,\vartheta}(S_1(\mathbb{R}^{2d}))$ for all $T \in \{X_{\vartheta,1}, \dots, X_{\vartheta,d}, \Pi_{\vartheta,1}, \dots, \Pi_{\vartheta,d}\}$, whence the rest then follows by the Calderón-Vaillancourt Theorem 3.3.14 and applying induction. We first note that the definition of ϑ implies that it equals one on the diagonal, i.e. $\vartheta(x,x) = 1$ for all $x \in \mathbb{R}^d$.

Fix $j \in \{1, \dots, d\}$, $\phi \in \mathscr{S}(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$. Then, using dominated convergence for limits and integration by parts in ξ,

$$\begin{aligned}
&[X_{\vartheta,j}, \mathfrak{Op}_{t,\vartheta}(\Phi)]\phi(x) \\
&\quad = (2\pi)^{-d} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} (x_j - y_j) e^{i\xi\cdot(x-y)} \vartheta(x,y) \Phi(tx + (1-t)y, \xi) \phi(y) dy d\xi \\
&\quad = (2\pi)^{-d} \lim_{\varepsilon\to 0} \Bigg(\int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y) i\partial^{e_{d+j}} \Phi(tx + (1-t)y, \xi) \phi(y) dy d\xi \\
&\qquad - 2\varepsilon \int_{\mathbb{R}^2} \xi_j e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y) \Phi(tx + (1-t)y, \xi) \phi(y) dy d\xi \Bigg) \\
&\quad = \mathfrak{Op}_{t,\vartheta}(i\partial^{e_{d+j}}\Phi)\phi(x).
\end{aligned}$$

For $\Pi_{\vartheta,j}$, we use again dominated convergence, integration by parts in both y and ξ, and also differentiation under the integral in x,

$$[\Pi_{\vartheta,j}, \mathfrak{Op}_{t,\vartheta}(\Phi)]\phi(x)$$

$$= (2\pi)^{-d} \lim_{\varepsilon\to 0} \Bigg(\int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y)$$

$$\left(\xi_j + (A(y-x))_j - it\partial^{e_j}\right) \Phi(tx+(1-t)y,\xi)\phi(y)dyd\xi$$

$$- \int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y)\Phi(tx+(1-t)y,\xi)\Pi_{\vartheta,j}\phi(y)dyd\xi \Bigg)$$

$$= (2\pi)^{-d} \lim_{\varepsilon\to 0} \Bigg(\int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y)$$

$$\left(2(A(y-x))_j - i\partial^{e_j}\right) \Phi(tx+(1-t)y,\xi)\phi(y)dyd\xi \Bigg)$$

$$= (2\pi)^{-d} \lim_{\varepsilon\to 0} \Bigg(\int_{\mathbb{R}^d} e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y)$$

$$\left(-i\partial^{e_j} - 2i\sum_{k=1}^{d} A_{jk}\partial^{e_k}\right) \Phi(tx+(1-t)y,\xi)\phi(y)dyd\xi$$

$$+ 4i\varepsilon \sum_{k=1}^{d} A_{jk} \int_{\mathbb{R}^d} \xi_k e^{-\varepsilon\|\xi\|^2} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \vartheta(x,y)\Phi(tx+(1-t)y,\xi)\phi(y)dyd\xi \Bigg)$$

$$= \mathfrak{Op}\left(-i\left(\partial^{e_j} + 2\sum_{k=1}^{d} A_{jk}\right)\Phi\right)\phi(x).$$

The derivatives of Φ are clearly in $S_1(\mathbb{R}^{2d})$, so we are done with the first statement.

For the second part of the theorem we also use an iterative argument. Two things have to be shown: T is continuous $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ and its matrix elements satisfies Lemma 3.3.8.

Continuity as a map $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is rather simple: T bounded in $L^2(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$ with both injections continuous.

As for the condition on T's matrix elements, we aim to show the equivalent condition that for any $\gamma \in \mathbb{N}_0^{2d}$

$$\sup_{\tilde\alpha,\tilde\beta\in\mathbb{Z}^{2d}} |(\tilde\alpha - \tilde\beta)^\gamma \langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde\beta,\vartheta}, \overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}| < \infty. \tag{3.3.7}$$

The above is essentially making sure that multiplying $\langle \mathfrak{Op}_{t,\vartheta}(\Phi)\mathcal{G}_{\tilde\beta,\vartheta}, \overline{\mathcal{G}_{\tilde\alpha,\vartheta}}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$ repeatedly by a factor $\alpha_j - \beta_j$ or $\alpha'_j - \beta'_j$ for $j = 1,\dots,d$ gives bounded terms in $(\tilde\alpha, \tilde\beta)$. Our goal is to deduce some general identities and estimates, which, when applied iteratively, shows (3.3.7).

Fix $j \in \{1,\dots,d\}$. For $\phi \in \mathscr{S}(\mathbb{R}^d)$ define

$$S_{\tilde\alpha}\phi \colon \mathbb{R}^d \ni x \mapsto (2\pi)^{-\frac{d}{2}} \vartheta(x,\alpha)\phi(x-\alpha)e^{i\alpha'\cdot(x-\alpha)}.$$

This definition is such that $S_{\tilde\alpha} \otimes^d u = \mathcal{G}_{\tilde\alpha,\vartheta}$ and we have the important commutators $[S_{\tilde\alpha}, X_{\vartheta,j}] = -\alpha_j S_{\tilde\alpha}$ and $[S_{\tilde\alpha}, \Pi_{\vartheta,j}] = -\alpha'_j S_{\tilde\alpha}$. Now for every choice of $\tilde T \in \mathcal{B}(L^2(\mathbb{R}^d))$ and $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$, we have

$$(\alpha_j - \beta_j)\langle \tilde T S_{\tilde\beta}\phi, \overline{S_{\tilde\alpha}\psi}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

$$= -\langle \tilde{T} S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} X_{\vartheta,j} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} + \langle [X_{\vartheta,j}, \tilde{T}] S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$
$$+ \langle \tilde{T} S_{\tilde{\beta}} X_{\vartheta,j} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)},$$

and

$$(\alpha_j' - \beta_j') \langle \tilde{T} S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$
$$= -\langle \tilde{T} S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} \Pi_{\vartheta,j} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} + \langle [\Pi_{\vartheta,j}, \tilde{T}] S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$
$$+ \langle \tilde{T} S_{\tilde{\beta}} \Pi_{\vartheta,j} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)},$$

and finally

$$\left| \langle \tilde{T} S_{\tilde{\beta}} \phi, \overline{S_{\tilde{\alpha}} \psi} \rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \right| \leq (2\pi)^{-d} \|\tilde{T}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \|\phi\|_{L^2(\mathbb{R}^d)} \|\psi\|_{L^2(\mathbb{R}^d)}.$$

Now, $T, \otimes^d u$ satisfies the assumptions on $\tilde{T}, \phi, \psi$ and furthermore, T satisfies the commutator criterion in the theorem by hypothesis. Thus we can make repeated use of the above facts to obtain (3.3.7). ■

Remark! 3.3.20. Let us note what was important in the above proof. The first part dependent upon differentiation of the phase, where every time we differentiated the phase function we got a polynomial in x, y. If the phase function only produces a polynomial in x, y plus a $BC^\infty(\mathbb{R}^{2d})$ function, then the proof will go through with no problem.

In the second part we look towards the commutators $[S_{\tilde{\alpha}}, X_{\vartheta,j}] = -\alpha_j S_{\tilde{\alpha}}$ and $[S_{\tilde{\alpha}}, \Pi_{\vartheta,j}] = -\alpha_j' S_{\tilde{\alpha}}$. More general phase functions might perturb these commutators slightly, but for a reasonable perturbation the proof still holds.

The proof of Beal's Commutator Criterion 3.3.19 gives us a ways of giving bounds on the matrix elements as in Lemma 3.3.8 without working with the symbol directly. This will become a strong tool in the next chapter.

4

Schrödinger Operators for Particles in Euclidean Space

We will finally be analyzing some quantum systems varying from simple to complex. All these systems are of particles in Euclidean space, meaning we focus on the Schrödinger operator of particles roaming around in some Euclidean space, affected by its surroundings.

In the beginning we deal with realizations of differential operators, which are important for modeling kinetic energy. This is largerly inspired by [14], but also [22, 23]. Afterwards, we study the free Schrödinger operator and harmonic oscillator in larger details, based on [16, 22] and [15, 22, 23, 28] respectively. Next we complicate matters an introduce a magnetic field, but with a focus on constant magnetic fields and the Landau operator. For this matter we use the materials [8, 9, 19, 22] and additionally [6, 7, 17] for the Landau operator. As a finale, we consider the Hartree-Fock approximation of the Schrödinger operator in a magnetic many-body system. The potential from the cumulative effect of the many particles is given by a formal self-consistent equation, which is then formalized and solved. The interpretation of the physics leading to the self-consistent equation has been done by H. Cornean with inspiration from [3, Chapter 8] and it has also been solved in collaboration with H. Cornean.

We ignore most parameters and constants stemming from the physics of the models, since our motivation is the study of the mathematics underlying the models. W.r.t the underlying space we fix a dimension $d \in \mathbb{N}$ for the chapter.

4.1 Schrödinger Operator of a Particle with Outer Potential

Say we deal with the quantum system of one particle, which perhaps is under the influence of some outer potential, e.g. an electron being attracted by a proton, but otherwise roaming space as it pleases (while obeying the laws of physics). This we model by using the Hilbert space $L^2(\mathbb{R}^d)$ with d representing the number of parameters for our particle such as physical coordinates. For its Schrödinger operator $\mathcal{H}$ we have to consider the kinetic energy of the particle and the potential energy stemming other sources, so $\mathcal{H}$ is effectively the sum of two operators.

In classical mechanics such a system would have Hamiltonian $H\colon \mathbb{R}^{2d} \ni (x, \xi) \mapsto \|\xi\|^2 + V(x)$ with the first term being the kinetic energy and the second the potential energy. For regular enough V and using the standard quantization procedure from the previous chapter, we get the Schrödinger operator as

$$\mathcal{H} = \mathfrak{Op}_{1,1}(H) = \mathfrak{Op}_{1,1}\left(\mathbb{R}^{2d} \ni (x,\xi) \mapsto \|\xi\|^2 + V(x)\right) = -\Delta + V$$

with the operator V being pointwise multiplication with V. Again, if V is regular enough, then $\mathcal{H}$ is a closable operator, see Corollary 3.3.12. But this is not quite enough, we need $\mathcal{H}$ to be a self-adjoint operator. Thus it is up to us to show that operators of the above type can be interpreted as observables, i.e. are self-adjoint. We first look at multiplication operators as V and secondly differential operators as $-\Delta$, then at two specific cases of Schrödinger operators.

Lemma 4.1.1. *Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be measurable. Then the operator $V \in \mathcal{L}(L^2(\mathbb{R}^d))$ defined by $V\colon \phi \mapsto V\phi$ on*

$$D(V) = \{\phi \in L^2(\mathbb{R}^d) | V\phi \in L^2(\mathbb{R}^d)\}$$

is self-adjoint.

Proof. As defined, V is clearly symmetric. It is also densely defined since for any $\phi \in L^2(\mathbb{R}^d)$ we have $1_{V^{-1}([-n,n])}\phi \overset{n\to\infty}{\to} \phi$ in $L^2(\mathbb{R}^d)$ with the $(1_{V^{-1}([-n,n])}\phi)_{n\in\mathbb{N}}$ residing in $D(V)$. Thus by Proposition 2.2.7 (ii) it is enough to show that $R(T \pm i) = L^2(\mathbb{R}^d)$. But $\psi = (V \pm i)^{-1}\phi \in D(V)$ for every $\phi \in L^2(\mathbb{R}^d)$ and $(V \pm i)\psi = \phi$, implying that $R(T \pm i) = L^2(\mathbb{R}^d)$, so V is self-adjoint. ■

Definition 4.1.2. (Realizations of Differential Operators) Let $T := \sum_{\gamma\in\mathbb{N}_0^d, |\gamma|\le N} \phi_\gamma \partial^\gamma$ be a linear partial differential operator with $\mathscr{O}_m(\mathbb{R}^d)$-coefficients defined on $\mathscr{S}'(\mathbb{R}^d)$.

The minimal realization of T on $L^2(\mathbb{R}^d)$ is defined as

$$T_{\min} := T|_{\mathscr{S}(\mathbb{R}^d)},$$

and the maximal realization as

$$T_{\max} := T|_{\{\phi\in L^2(\mathbb{R}^d) | T\phi \in L^2(\mathbb{R}^d)\}}.$$

Every operator $\tilde{T} \in \mathcal{L}(L^2(\mathbb{R}^d))$ which satisfies $T_{\min} \subseteq \tilde{T} \subseteq T_{\max}$ is called a realization of T.

Lemma 4.1.3. *Let T be a partial differential operator with constant coefficients defined on $\mathscr{S}'(\mathbb{R}^d)$ such that $T = \mathcal{F}^{-1}p\mathcal{F}$, where p is a polynomial on $\mathbb{R}^d$ with real-valued coefficients. If $T_{\min}$ is symmetric, then it is essentially self-adjoint and*

$$\overline{T_{\min}} = T_{\max}.$$

Proof. We prove the lemma in two steps: First we show that $T_{\max}$ is self-adjoint, and then that $T_{\max} = T_{\min}^*$.

Since $\mathscr{S}(\mathbb{R}^d) \subseteq D(T_{\max})$, $T_{\max}$ is densely defined. It is also symmetric:

$$\langle T_{\max}\phi, \psi\rangle_{L^2(\mathbb{R}^d)} = \langle p\mathcal{F}\phi, \mathcal{F}\psi\rangle_{L^2(\mathbb{R}^d)} = \langle \mathcal{F}\phi, p\mathcal{F}\psi\rangle_{L^2(\mathbb{R}^d)} = \langle \phi, T_{\max}\psi\rangle_{L^2(\mathbb{R}^d)}$$

for all $\phi, \psi \in D(T_{\max})$. Now, like Lemma 4.1.1, we use Proposition 2.2.7 (ii). For $\phi \in L^2(\mathbb{R}^d)$ we have $\psi = \mathcal{F}^{-1}(p \pm i)^{-1}\mathcal{F}\phi \in D(T_{\max})$ and $(T \pm i)\psi = \phi$, hence we conclude that $T_{\max}$ is self-adjoint.

As for the second step, if $\phi \in D(T_{\min}^*)$, then

$$\langle T_{\min}^*\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \overline{\psi}, T_{\min}^*\phi\rangle_{L^2(\mathbb{R}^d)} = \langle T_{\min}\overline{\psi}, \phi\rangle_{L^2(\mathbb{R}^d)} = \langle T\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^d)$, so $T\phi = T_{\min}^*\phi \in L^2(\mathbb{R}^d)$ meaning $\phi \in D(T_{\max})$ and $T_{\min}^* \subseteq T_{\max}$. Oppositely, if $\phi \in D(T_{\max})$, then

$$\langle T_{\min}\psi, \phi\rangle_{L^2(\mathbb{R}^d)} = \langle T\phi, \overline{\psi}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle \psi, T_{\max}\phi\rangle_{L^2(\mathbb{R}^d)}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^d)$, so $\phi \in D(T_{\min}^*)$ and $T_{\max} \subseteq T_{\min}^*$. Thus $T_{\max} = T_{\min}^*$.

Thus $T_{\min}^*$ is densely defined and by Proposition 2.2.3 $\overline{T_{\min}} = T_{\max}^* = T_{\max}$, which implies that $T_{\min}$ is self-adjoint. ■

Note that the first lemma includes the position operators $X_{1,j}$, $j \in \{1, \ldots, d\}$, and second lemma includes both the Laplacian Δ and momentum operators $\Pi_{1,j}$, $j \in \{1, \ldots, d\}$.

For the interested reader, general criteria for the sum $-\Delta + V$ to be self-adjoint are presented in [22, Theorem X.16, Theorem X.20, Theorem X.28, and Theorem X.29]. We note that trivially, or using the Kato-Rellich Theorem 2.2.11, when $V \in L^\infty(\mathbb{R}^d)$, then $-\Delta + V$ is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^d)$.

4.1.1 The Free Schrödinger Operator

The Schrödinger operator with pure kinetic energy $-\Delta_{\max}$ models a particle free of outside forces and is therefore called the free Schrödinger operator. For this quantum system one might expect the possible energies to be every non-negative number, corresponding to the kinetic energy of the particle. This is indeed correct.

Theorem 4.1.4. *The operator $\mathcal{H} = -\Delta_{\max} \in \mathcal{L}(L^2(\mathbb{R}^d))$ has no eigenvalues and its spectrum is*

$$\sigma(\mathcal{H}) = [0, \infty).$$

Proof. Let us deal with the statement about eigenvalues first. Suppose λ is an eigenvalue of $\mathcal{H}$. Then there exists $\phi \in D(\mathcal{H}) \setminus \{0\}$ such that

$$0 = (\mathcal{H} - \lambda)\phi = \mathcal{F}^{-1}(\|\cdot\|^2 - \lambda)\mathcal{F}\phi,$$

so $(\|\cdot\|^2 - \lambda)\mathcal{F}\phi = 0$ leading to $\phi = 0$, a contradiction. Thus $\mathcal{H}$ must have no eigenvalues.

Now for the spectrum. If $z \in \mathbb{C} \setminus [0, \infty)$, then $T_z := \mathcal{F}^{-1}(\|\cdot\|^2 - z)^{-1}\mathcal{F}$ is an operator in $\mathcal{B}(L^2(\mathbb{R}^d))$, and $R(T_z) \subseteq D(\mathcal{H})$ since

$$\mathcal{F}^{-1}\|\cdot\|^2\mathcal{F}T_z\phi = \mathcal{F}^{-1}\frac{\|\cdot\|^2}{\|\cdot\|^2 - z}\mathcal{F}\phi \in L^2(\mathbb{R}^d)$$

for all $\phi \in L^2(\mathbb{R}^d)$. Moreover, $T_z(\mathcal{H} - z)\phi = \phi$ for $\phi \in D(\mathcal{H})$ and $(\mathcal{H} - z)T_z\phi = \phi$ for $\phi \in L^2(\mathbb{R}^d)$, whence $T_z = (\mathcal{H} - z)^{-1}$ and $z \in \rho(\mathcal{H})$.

This gives $\sigma(\mathcal{H}) \subseteq [0, \infty)$. To prove that we have an equality, we use Stone's Formula 2.4.25 and Proposition 2.4.23. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be given and note that the absence of eigenvalues means that $1_{[\lambda_1,\lambda_2]}(\mathcal{H}) = 1_{(\lambda_1,\lambda_2)}(\mathcal{H})$, whence

$$\begin{aligned} 1_{(\lambda_1,\lambda_2)}(\mathcal{H})\phi &= \mathcal{F}^{-1}\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2}\left((\|\cdot\|^2 - \lambda - i\varepsilon)^{-1} - (\|\cdot\|^2 - \lambda + i\varepsilon)^{-1}\right)d\lambda\mathcal{F}\phi \\ &= \mathcal{F}^{-1}1_{(\lambda_1,\lambda_2)}(\|\cdot\|^2)\mathcal{F}\phi \end{aligned}$$

for all $\phi \in L^2(\mathbb{R}^d)$. If $z \in [0, \infty)$ and $\varepsilon > 0$, then $1_{(z-\varepsilon, z+\varepsilon)}(\|\cdot\|^2) \neq 0$ as a multiplication operator. By Proposition 2.4.23 we conclude that $\sigma(\mathcal{H}) = [0, \infty)$. ■

4.1.2 The Harmonic Oscillator

A favorite example of introductory classical physics and ordinary differential equations is that of the harmonic oscillator. Is quantum counterpart is formally given by the sum of the momentum and position squared, i.e. $-\Delta + X^2 = \mathfrak{Op}_{1,1}\left(\mathbb{R}^{2d} \ni (x, \xi) \mapsto \|(x, \xi)\|^2\right)$, and similarly to the classical harmonic oscillator, the quantum harmonic oscillator has a nice solution in terms of spectrum. It is also a teaser for the next section, where we work with magnetic Schrödinger operators whom share some likeness with the quantum harmonic oscillator.

Theorem 4.1.5. *The operator $(-\Delta + X^2)_{\min} \in \mathcal{L}(L^2(\mathbb{R}^d))$ is essentially self-adjoint and $\mathcal{H} = \overline{(-\Delta + X^2)_{\min}}$ has spectrum:*

$$\sigma(\mathcal{H}) = \sigma_p(\mathcal{H}) = \{d + 2n | n \in \mathbb{N}_0\}$$

Proof. For this spectrum we use algebraic methods. The steps are to find the eigenfunctions of $(-\Delta + X^2)_{\min}$, show that they constitute an orthogonal basis in $L^2(\mathbb{R}^d)$, and this will lead to the desired conclusion. We also reduce the statement to the case of the Harmonic oscillator on $L^2(\mathbb{R})$ since if this operator has an orthogonal basis of eigenfunctions with a certain behavior, then taking the tensor product of this basis with itself, we get an orthogonal basis of eigenfunctions for the Harmonic oscillator in any dimension.

Consider $T = (-\Delta + X^2)_{\min} \in \mathcal{L}(L^2(\mathbb{R}))$. Note T is symmetric and non-negative. On $\mathscr{S}(\mathbb{R})$ we decompose T as follows

$$T = A^\dagger A + 1,$$

where $A^\dagger = -\partial + X$ and $A = \partial + X$ satisfies $A_{\min} \subseteq (A^\dagger_{\min})^*$ and conversely. The strength of this decomposition is that we can work with the first order differential operators $A, A^\dagger$ instead of the second order differential operator of T.

First of all, the equation $A\phi = 0$ has the non-zero solution $\phi_0 \colon \mathbb{R} \ni x \mapsto e^{-\frac{x^2}{2}}$, getting us our first eigenfunction of T with eigenvalue 1. In opposition, $A^\dagger\phi = 0$ has no non-zero solution in $\mathscr{S}(\mathbb{R})$.

Second of all, to obtain more eigenfunctions, we note that $[T, A^\dagger] = 2A^\dagger$, so if $T\phi = z\phi$ for $z \in \mathbb{C}, \phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$, then

$$TA^\dagger\phi = A^\dagger T\phi + [T, A^\dagger]\phi = (z+2)A^\dagger\phi,$$

i.e. $A^\dagger\phi$ is a new eigenfunction with a different eigenvalue. Importantly $A^\dagger\phi \neq 0$ if $\phi \neq 0$.

Essentially, this means that $\psi_n := \frac{(A^\dagger)^n\phi_0}{\|(A^\dagger)^n\phi_0\|_{L^2(\mathbb{R}^d)}}$ for $n \in \mathbb{N}_0$ make up a sequence of normalized eigenfunctions of T. The sequence $(\psi_n)_{n\in\mathbb{N}_0}$ is also orthogonal since each element corresponds to a different eigenvalue, that is

$$2(n-m)\langle\psi_n, \psi_m\rangle_{L^2(\mathbb{R})} = \langle T\psi_n, \psi_m\rangle_{L^2(\mathbb{R})} - \langle\psi_n, T\psi_m\rangle_{L^2(\mathbb{R})} = 0$$

for $n, m \in \mathbb{N}_0$, hence if $n \neq m$, then $\langle\psi_n, \psi_m\rangle_{L^2(\mathbb{R})} = 0$.

The last step of our analysis is then a study of the span of $(\psi_n)_{n\in\mathbb{N}_0}$. A short investigation shows that each ψ_n is the product of ϕ_0 and a polynomial of degree n. Hence every product of ϕ_0 with a polynomial on $\mathbb{R}$ is in $\overline{\operatorname{span}((\psi_n)_{n\in\mathbb{N}_0})}$. Taking this one step further, this implies that $\omega_\xi \colon \mathbb{R} \ni x \mapsto e^{i\xi x}\phi_0(x)$ is in $\overline{\operatorname{span}((\psi_n)_{n\in\mathbb{N}_0})}$ for all $\xi \in \mathbb{R}$. Thus if $\phi \in L^2(\mathbb{R})$ is orthogonal to all ψ_n, $n \in \mathbb{N}_0$, then

$$\mathcal{F}(\phi\phi_0)(\xi) = \frac{1}{\sqrt{2\pi}}\langle \omega_\xi, \phi\rangle_{L^2(\mathbb{R})} = 0$$

for all $\xi \in \mathbb{R}$, meaning $\phi\phi_0 \equiv 0$. Since ϕ_0 is zero nowhere, $\phi = 0$. We conclude that $\overline{\operatorname{span}((\psi_n)_{n\in\mathbb{N}_0})} = L^2(\mathbb{R})$, and $(\psi_n)_{n\in\mathbb{N}_0}$ is an orthogonal basis for $L^2(\mathbb{R})$.

Finishing the proof is now a walk in the park. The operator T is non-negative and has dense range, so by Corollary 2.3.7, it is essentially self-adjoint. The closure $\mathcal{H} = \overline{T}$ has $(\psi_n)_{n\in\mathbb{N}_0}$ as an orthogonal basis of eigenvectors, and for every $z \notin \sigma_p(\mathcal{H}) = \{1 + 2n | n \in \mathbb{N}_0\}$:

$$(\mathcal{H} - z)^{-1} = \sum_{n\in\mathbb{N}_0} \frac{1}{1 + 2n - z}\langle \psi_n, \cdot\rangle_{L^2(\mathbb{R})}\psi_n,$$

meaning $\sigma(\mathcal{H}) = \sigma_p(\mathcal{H})$. ■

The reader may have recognized the eigenfunctions in the above as the Hermite functions. See [11, Section 6.4] for a study of these in detail.

Also, the self-adjoint extension of $(-\Delta + X^2)_{\text{min}}$ is not necessarily $(-\Delta + X^2)_{\text{max}}$, but maybe a smaller realization. This will also be the case for magnetic Schrödinger operators.

4.2 Schrödinger Operator of a Particle in a Magnetic Field

To complicate matters we now deal with a magnetic field in our quantum system. The magnetic fields and potentials will be quite regular, but more general situations can be considered, see [22, Theorem X.22 and Theorem X.34].

A d-dimensional regular magnetic field B, henceforth just called a magnetic field, is a closed 2-form which can be represented by $BC^\infty(\mathbb{R}^d)$ coefficients, that is the magnetic field itself can be thought of as a map $B \in BC^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ such that $B_{jk} = -B_{kj}$ and $\partial^{e_j}B_{kl} + \partial^{e_k}B_{lj} + \partial^{e_l}B_{jk} \equiv 0$.

Associated with any magnetic field we have a magnetic potential, which is a 1-form A with smooth coefficients such that $dA = B$, where dA is the exterior derivative of A. Like the magnetic field, we may think of A as an element of $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$. As a canonical choice of magnetic potential we take

$$A_B \colon \mathbb{R}^d \ni x \mapsto \left(\sum_{k=1}^{d} \int_0^1 s x_k B_{kj}(sx)ds\right)_{j=1}^{d}.$$

This is the magnetic potential in the transversal gauge. Note A_B has components in $\mathscr{O}_m(\mathbb{R}^d)$.

We will now construct a magnetic phase, and from this we shall derive the magnetic Schrödinger operator.

Lemma 4.2.1. *Given a magnetic field B, the map*

$$\vartheta_B \colon \mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto e^{-i\int_{[x,y]} A_B}$$

is a Hermitian phase function in $\mathscr{O}_m(\mathbb{R}^{2d})$ obeying the triangle property.

Proof. It is clearly a Hermitian phase function. The other properties of ϑ_B will follow from analyzing the integral $\int_{[x,y]} A_B$. For $x, y \in \mathbb{R}^d$, we see

$$\int_{[x,y]} A_B = \int_0^1 (y-x)\cdot A_B(x+t(y-x))dt = \int_0^1\int_0^1 sx\cdot B(sty+s(1-t)x)ydsdt$$

using that we can impose antisymmetry on the coefficients of B. Since we can choose coefficients in $BC^\infty(\mathbb{R}^d)$, the above expression is in $\mathscr{O}_m(\mathbb{R}^{2d})$ when considering (x,y) as variables, which implies $\vartheta_B \in \mathscr{O}_m(\mathbb{R}^{2d})$.

As for the triangle property, a quick use of Stokes Theorem lets us conclude that

$$\begin{aligned}\int_{[x,y]} A_B + \int_{[y,z]} A_B + \int_{[z,x]} A_B &= \int_{<x,y,z>} B \\ &= \int_0^1\int_0^1 t(y-x)\cdot B(stz+t(1-s)y+(1-t)x)(z-y)dsdt \\ &\quad - \int_0^1\int_0^1 t(z-y)\cdot B(stz+t(1-s)y+(1-t)x)(y-x)dsdt\end{aligned}$$

for any $x, y, z \in \mathbb{R}^d$. Noting that the coefficients of B can be chosen in $BC^\infty(\mathbb{R}^d)$ we get the triangle property. ■

Now we may use the ϑ_B-quantization from Section 3.3 to get the magnetic position and momentum operators. The position operator does not change from the non-magnetic case to the magnetic case, but the momentum operator ends up becoming $\Pi_{B,j} = -i\partial^{e_j} - (A_B)_j$ and hence the kinetic energy in the presence of a magnetic field should formally be

$$(-i\nabla - A_B)^2 = \sum_{j=1}^d \Pi_{B,j}^2 = \mathfrak{Op}_B(\mathbb{R}^{2d} \ni (x,\xi) \mapsto \|\xi\|^2)$$

This also corresponds to the effect of a magnetic field in classical physics. Note we replaced the phase used in the notation of Section 3.3 with the magnetic field B. This will be a convention from here on out, and we will also use the prefix magnetic instead of phase.

Again, it is up to us to make sense of these operators as self-adjoint operators on $L^2(\mathbb{R}^d)$:

Proposition 4.2.2. *For any magnetic field B, $(-i\nabla - A_B)^2_{\min} \in \mathcal{L}(L^2(\mathbb{R}^d))$ is a non-negative, essentially self-adjoint operator. Moreover, for every $j \in \{1,\dots,d\}$, $\Pi_{B,j}|_{\mathscr{S}(\mathbb{R}^d)}$ is essentially self-adjoint and $(-i\nabla - A_B)^2_{\min}$-bounded.*

Proof. It is routine to check that the operators in question are symmetric using integration by parts. Using this once for $(-i\nabla - A_B)^2_{\min}$ we get:

$$\langle (-i\nabla - A_B)^2_{\min}\phi, \phi\rangle_{L^2(\mathbb{R}^d)} = \sum_{j=1}^d \|\Pi_{B,j}\phi\|^2_{L^2(\mathbb{R}^d)} \geq 0$$

for all $\phi \in \mathscr{S}(\mathbb{R}^d)$, hence we conclude that $(-i\nabla - A_B)^2_{\min}$ is non-negative. Furthermore, by the Cauchy-Schwartz inequality:

$$\begin{aligned}\sum_{j=1}^{d} \|\Pi_{B,j}\phi\|^2_{L^2(\mathbb{R}^d)} &\leq \|(-i\nabla - A_B)^2_{\min}\phi\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)} \\ &\leq 2^{-1}\|(-i\nabla - A_B)^2_{\min}\phi\|^2_{L^2(\mathbb{R}^d)} + 2^{-1}\|\phi\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

so

$$\|\Pi_{B,j}\phi\|_{L^2(\mathbb{R}^d)} \leq 2^{-\frac{1}{2}}\|(-i\nabla - A_B)^2_{\min}\phi\|_{L^2(\mathbb{R}^d)} + 2^{-\frac{1}{2}}\|\phi\|_{L^2(\mathbb{R}^d)}$$

for every $j \in \{1, \dots, d\}$, which shows that the magnetic momentum operators are $(-i\nabla - A_B)^2_{\min}$-bounded.

All that is left are the statements about essential self-adjointness. We use a similar technique on all the operators, and so only do the proof of $T_{\min} := (-i\nabla - A_B)^2_{\min}$ being essentially self-adjoint. By Corollary 2.2.8 (i) and the associated comments, if $T_{\min}$ is not essentially self-adjoint, then for every $c \in \mathbb{R} \setminus \{0\}$, either $\ker(T^*_{\min} + ci) \neq \{0\}$ or $\ker(T^*_{\min} - ci) \neq \{0\}$. We shall prove that this is impossible for $|c|$ large enough, implying that $T_{\min}$ must then be essentially self-adjoint.

First we note that if $u \in C^\infty_c(\mathbb{R}^d)$ satisfies $\mathrm{supp}(u) \subseteq (-1,1)^d$ and $\sum_{\alpha\in\mathbb{Z}^d}(\tau_\alpha u)^2 \equiv 1$, then $(\tau_{B,\alpha}u)_{\alpha\in\mathbb{Z}^d}$ constitutes a Parseval frame in $L^2(\mathbb{R}^d)$. This follows from using dominated convergence:

$$\|\phi\|^2_{L^2(\mathbb{R}^d)} = \sum_{\alpha\in\mathbb{Z}^d} \|\tau_{B,\alpha}u\phi\|^2_{L^2(\mathbb{R}^d)},$$

which holds for $\phi \in L^2(\mathbb{R}^d)$. Note the similarities with Lemma 3.1.2. We will use this Parseval frame to construct something close to the resolvent of $T_{\min}$.

Assume $\phi_c \in \partial B_1(0; L^2(\mathbb{R}^d)) \cap \ker(T^*_{\min} - ci)$ for $c \in \mathbb{R} \setminus \{0\}$. Thus

$$\langle (T_{\min} + ci)\psi, \phi_c\rangle_{L^2(\mathbb{R}^d)} = 0$$

for all $\psi \in \mathscr{S}(\mathbb{R}^d)$. Fix $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ for now. Then denote

$$\psi_N = \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} \tau_{B,\alpha}u(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\psi) \in \mathscr{S}(\mathbb{R}^d)$$

and see

$$\begin{aligned}(T_{\min} + ci)\psi_N = &\sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (\tau_\alpha u)^2\psi + \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (T_{\min}\tau_{B,\alpha}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\psi) \\ &- 2i\sum_{j=1}^{d}\sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (\Pi_{B,j}\tau_{B,\alpha}u)\partial^{e_j}(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\psi).\end{aligned}$$

The first sum on the right converges to ψ as $N \to \infty$ in $L^2(\mathbb{R}^d)$, which is nice, but we still have to manage the remainder. We define the remainder as the map $S_{c,N}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, so

$$\begin{aligned}S_{c,N}\omega = &\sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (T_{\min}\tau_{B,\alpha}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\omega) \\ &- 2i\sum_{j=1}^{d}\sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (\Pi_{B,j}\tau_{B,\alpha}u)\partial^{e_j}(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\omega)\end{aligned}$$

for $\omega \in L^2(\mathbb{R}^d)$. It is a sum of sums, each with terms having a similar structure. We shall study the components of these terms next.

Here

$$\|(-\Delta_{\max} + ci)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq |c|^{-1},$$

and $\partial^{e_j}_{\min}$ is $-\Delta_{\min}$-bounded and

$$\|\partial^{e_j}_{\max}(-\Delta_{\max} + ci)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq |c|^{-\frac{1}{2}}\sqrt{|c|^{-1}+1}$$

since by the help of integration by parts

$$\begin{aligned}\sum_{j=1}^{d} &\|\partial^{e_j}_{\max}(-\Delta_{\max} + ci)^{-1}\omega\|^2_{L^2(\mathbb{R}^d)} \\ &= \langle(-\Delta_{\max} + ci)^{-1}\omega, (\mathrm{id}_{L^2(\mathbb{R}^d)} - ic(-\Delta_{\max} + ci)^{-1})\omega\rangle_{L^2(\mathbb{R}^d)} \leq 2|c|^{-1}\|\omega\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for all $\omega \in L^2(\mathbb{R}^d)$. Furthermore, for $j \in \{1, \ldots, d\}, \alpha \in \mathbb{Z}^d$ we have on $\mathscr{S}(\mathbb{R}^d)$ the identities

$$\Pi_{B,j}\tau_{B,\alpha} = -i\partial^{e_j}\vartheta_B(\cdot,\alpha)\tau_\alpha - i\tau_{B,\alpha}\partial^{e_j} - (A_B)_j\tau_{B,\alpha} = C_{B,j,\alpha}\tau_{B,\alpha} - i\tau_{B,\alpha}\partial^{e_j}$$

and

$$\Pi^2_{B,j}\tau_{B,\alpha} = -i\partial^{e_j}C_{B,j,\alpha}\tau_{B,\alpha} + C^2_{B,j,\alpha}\tau_{B,\alpha} - 2iC_{B,j,\alpha}\tau_{B,\alpha}\partial^{e_j} - \tau_{B,\alpha}\partial^{2e_j},$$

where

$$C_{B,j,\alpha} \colon \mathbb{R}^d \ni x \mapsto \sum_{k=1}^{d}\int_0^1 (1-t)(\alpha_k - x_k)B_{kj}((1-t)x + t\alpha)dt$$

is a $\mathscr{O}_m(\mathbb{R}^d)$-function, and it has, along with all of its derivatives, growth controlled by a power of $\langle \cdot - \alpha\rangle$. This implies that

$$\{T_{\min}\tau_{B,\alpha}u | \alpha \in \mathbb{Z}^d\} \bigcup \{\Pi_{B,j}\tau_{B,\alpha}u | j \in \{1, \ldots, j\}, \alpha \in \mathbb{Z}^d\}$$

is a bounded set in $L^\infty(\mathbb{R}^d)$.

Using this and the support of u, one gets:

$$\begin{aligned}&\left\| \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} (T_{\min}\tau_{B,\alpha}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\omega)\right\|^2_{L^2(\mathbb{R}^d)} \\ &\quad= \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} \sum_{\substack{\beta\in\mathbb{Z}^d \\ \alpha-\beta\in[-1,1]^d}} \Bigg| \Bigg\langle (T_{\min}\tau_{B,\alpha}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\omega), \\ &\qquad\qquad (T_{\min}\tau_{B,\beta}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\beta}u}\omega) \Bigg\rangle_{L^2(\mathbb{R}^d)} \Bigg| \\ &\quad\leq \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\leq N}} \frac{3^d}{2}(3^d - 1)\|(T_{\min}\tau_{B,\alpha}u)(-\Delta_{\max} + ci)^{-1}(\overline{\tau_{B,\alpha}u}\omega)\|^2_{L^2(\mathbb{R}^d)} \\ &\quad\leq |c|^{-1}\frac{3^d}{2}(3^d - 1) \sup_{\beta\in\mathbb{Z}^d} \|T_{\min}\tau_{B,\beta}u\|^2_{L^\infty(\mathbb{R}^d)} \sum_{\alpha\in\mathbb{Z}^d} \|\overline{\tau_{B,\alpha}u}\omega\|^2_{L^2(\mathbb{R}^d)} \\ &\quad= |c|^{-2}C_{B,u}\|\omega\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for $\omega \in L^2(\mathbb{R}^d)$. Similar treatment of the other d sums defining $S_{c,N}$ shows that

$$\sup_{N\in\mathbb{N}} \|S_{c,N}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq (|c|^{-1} + |c|^{-\frac{1}{2}})C_{B,u}.$$

Moreover, $(S_{c,N})_{N\in\mathbb{N}}$ converges pointwise, thus using the Banach-Steinhaus Theorem we conclude that $(S_{c,N})_{N\in\mathbb{N}}$ converges in $\mathcal{B}(L^2(\mathbb{R}^d))$ to some S_c and

$$\|S_c\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq (|c|^{-1} + |c|^{-\frac{1}{2}})C_{B,u}.$$

Recall ϕ_c was assume to be in $\partial B_1(0; L^2(\mathbb{R}^d)) \cap \ker(T^*_{\min} - ci)$. Find $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathbb{R}^d)$ such that $\psi_n \overset{n\to\infty}{\to} \phi_c$ in $L^2(\mathbb{R}^d)$, see Lemma 3.1.2. Then

$$\begin{aligned} 0 &= \langle (T^*_{\min} + ci)(\psi_n)_N, \phi_c\rangle_{L^2(\mathbb{R}^d)} \overset{N\to\infty}{\to} \langle \psi_n, \phi_c\rangle_{L^2(\mathbb{R}^d)} + \langle S_c\psi_n, \phi_c\rangle_{L^2(\mathbb{R}^d)} \\ &\overset{n\to\infty}{\to} \langle \phi_c, \phi_c\rangle_{L^2(\mathbb{R}^d)} + \langle S_c\phi_c, \phi_c\rangle_{L^2(\mathbb{R}^d)} = 1 + \langle S_c\phi_c, \phi_c\rangle_{L^2(\mathbb{R}^d)} \end{aligned} \tag{4.2.1}$$

with the last term obeying the estimate:

$$|\langle S_c\phi_c, \phi_c\rangle_{L^2(\mathbb{R}^d)}| \leq (|c|^{-1} + |c|^{-\frac{1}{2}})C_{B,u}$$

If $|c|$ is large enough, then $|\langle S_c\phi_c, \phi_c\rangle_{L^2(\mathbb{R}^d)}| \leq 2^{-1}$, which gives a contradiction with (4.2.1). This implies that taking $|c|$ large enough, the operators $T^*_{\min} \pm ci$ have trivial kernels, hence by Corollary 2.2.8, $T_{\min} = (-i\nabla - A_B)^2_{\min}$ is essentially self-adjoint. ■

We will not study when one can add an outer potential V and still obtain a self-adjoint operator, but remark that the trivial case $V \in L^\infty(\mathbb{R}^d)$ still holds.

In the following we shall only deal with the case of a constant magnetic field, i.e. it can be represented by constant coefficients. For constant magnetic fields B, the magnetic potential A_B is just a linear map with an antisymmetric matrix, and every antisymmetric matrix M defines a constant magnetic field, for which A_B has matrix $-2^{-1}M$. Furthermore, the associated magnetic phase ϑ_B is then induced by the matrix of A_B:

$$-\int_{[x,y]} A_B = x \cdot A_B y = -\frac{1}{2} x \cdot By$$

for all $x, y \in \mathbb{R}^d$.

One significant fact for constant magnetic field is the following:

Lemma 4.2.3. *For a constant magnetic field B, $\mathcal{H}_B = \overline{(-i\nabla - A_B)^2_{\min}}$ commutes with all magnetic translations.*

Proof. That $(-i\nabla - A_B)^2_{\min}$ commutes with $\tau_{B,y}$ for all $y \in \mathbb{R}^d$ follows from a direct calculation using classical calculus, or alternative note that $(-i\nabla - A_B)^2_{\min} = \sum_{j=1}^d \Pi^2_{B,j}|_{\mathscr{S}(\mathbb{R}^d)}$ and that the magnetic momenta commute with $\tau_{B,y}$ by Lemma 3.3.18.

Now for $\phi \in D(\mathcal{H}_B)$ we can find a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathbb{R}^d)$ such that $\phi_n \overset{n\to\infty}{\to} \phi$ in the $D(\mathcal{H}_B)$-norm, the graph norm of $\mathcal{H}_B$. But then $(\tau_{B,y}\phi_n)_{n\in\mathbb{N}}$ is Cauchy in the $D(\mathcal{H}_B)$-norm, so

$$\tau_{B,y}\mathcal{H}_B\phi = \lim_{n\to\infty} \mathcal{H}_B\tau_{B,y}\phi_n = \mathcal{H}_B \lim_{n\to\infty} \tau_{B,y}\phi_n = \mathcal{H}_B\tau_{B,y}\phi.$$ ■

4.2.1 The Landau Operator

The free magnetic Schrödinger operator with constant magnetic field in two-dimensions is special among magnetic Schrödinger operators, since in this case much more is known about the operator. It is called the Landau operator and we shall find its spectrum, called the Landau spectrum.

Theorem 4.2.4. *For a constant magnetic field $B = b dx_1 \wedge dx_2$, $b > 0$, in $\mathbb{R}^2$, the operator $\mathcal{H}_B = \overline{(-i\nabla - A_B)^2_{\min}}$ has spectrum:*

$$\sigma(\mathcal{H}_B) = \sigma_p(\mathcal{H}_B) = \{b(1+2n) | n \in \mathbb{N}_0\}$$

Proof. This time, we focus on solving differential equations in a distributional sense, so given $z \in \mathbb{C}$ we want to find $\phi \in \mathscr{S}'(\mathbb{R}^2)$ such that $\left((-i\nabla - A_B)^2 - z\right)\phi = 0$ and $\phi \in D(\mathcal{H}_B)$ or $\left((-i\nabla - A_B)^2 - z\right)\phi = \delta$. Here the first corresponds to a point of the point spectrum and the second to a point of the resolvent set.

Define $C_B \colon \mathbb{R}^2 \ni x \mapsto \frac{b}{4}\|x\|^2$. Due to the form of the differential operator, we make the ansatz that

$$\phi = e^{-C_B}(\psi \circ C_B)$$

for some function ψ. Applying the operator we get:

$$\left((-i\nabla - A_B)^2 - z\right)\phi = e^{-C_B}(-bC_B\partial^2\psi + b(2C_B - 1)\partial\psi + (b-z)\psi) \circ C_B$$

Setting this equal to zero and multiplying by $-\frac{e^{C_B}}{b}$, the above shows that ψ should satisfy

$$x\partial^2\psi(x) + (1-2x)\partial\psi(x) + \left(\frac{z}{b} - 1\right)\psi(x) = 0$$

for all $x \in \mathbb{R}_+$. Scaling $x \mapsto 2x =: y$, we get

$$y\partial^2\psi(y) + (1-y)\partial\psi(y) + \frac{1}{2}\left(\frac{z}{b} - 1\right)\psi(y) = 0 \tag{4.2.2}$$

for all $y \in \mathbb{R}_+$, which is the differential equation for confluent hypergeometric functions with parameters 1 and $\frac{1}{2}\left(1 - \frac{z}{b}\right)$.

According to [18, Chapter 13] on confluent hypergeometric functions, when $\frac{1}{2}\left(1 - \frac{z}{b}\right) \in \mathbb{N}_0$, or equivalently $z \in b(1 + 2\mathbb{N}_0)$, we may find a polynomial ψ on $\mathbb{R}$ solving (4.2.2). Thus

$$\phi \colon \mathbb{R}^2 \ni x \mapsto e^{-\frac{b}{4}\|x\|^2}\psi\left(\frac{b}{2}\|x\|^2\right)$$

should be an eigenfunction for $\mathcal{H}_B$. Indeed $\phi \in \mathscr{S}(\mathbb{R}^2) \subseteq D(\mathcal{H}_B)$, and the above analysis shows that $(\mathcal{H}_B - z)\phi = 0$.

If $z \notin b(1 + 2\mathbb{N}_0)$, then we may find an analytic function ψ on $\mathbb{R}_+$ solving (4.2.2), see again [18, Chapter 13]. This solution have just the right asymptotics such that

$$\phi \colon \mathbb{R}^2 \ni x \mapsto e^{-\frac{b}{4}\|x\|^2}\psi\left(\frac{b}{2}\|x\|^2\right)1_{\mathbb{R}^2\setminus\{0\}}(x)$$

is a tempered distribution and $\left((-i\nabla - A_B)^2 - z\right)\phi = \delta$ when ψ is scaled properly. Additionally, ϕ has exponential decay when $\|x\| \to \infty$.

Now to find an inverse to $\mathcal{H}_B - z$ in this case, we take inspiration from the theory of partial differential operators with constant coefficients, where convolution with a fundamental solution provides an inverse. Actually, in our situation we only have to add the magnetic phase, i.e. we consider the integral operator T on $L^2(\mathbb{R}^2)$ with kernel

$$K_T(x,y) = \vartheta_B(x,y)\phi(x-y)$$

for $x, y \in \mathbb{R}^2$. By properties of ϕ, $T \in \mathcal{B}(L^2(\mathbb{R}^2))$. If $\omega \in \mathscr{S}(\mathbb{R}^2)$, then for $x \in \mathbb{R}^2$ and using distribution theory we get

$$\begin{aligned} T(\mathcal{H}_B - z)\omega(x) &= \langle \vartheta_B(x,\cdot)\phi(x-\cdot), (\mathcal{H}_B - z)\omega\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)} \\ &= \langle \tau_x\delta, \omega\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)} = \omega(x). \end{aligned}$$

Furthermore,

$$T\omega(x) = \int_{\mathbb{R}^2} \vartheta_B(x,-y)\phi(y)\omega(x-y)dy$$

can be differentiated under the integral, and so

$$\begin{aligned} (\mathcal{H}_B - z)T\omega(x) &= \langle \vartheta_B(x,-\cdot)\phi, (\mathcal{H}_B - z)\omega(x-\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)} \\ &= \langle \delta, \omega(x-\cdot)\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)} = \omega(x). \end{aligned}$$

Finally, the density of $\mathscr{S}(\mathbb{R}^2)$ in both $D(\mathcal{H}_B)$ and $L^2(\mathbb{R}^2)$ lets us conclude that $T = (\mathcal{H}_B - z)^{-1}$, so $z \in \rho(\mathcal{H}_B)$ if $z \notin b(1 + 2\mathbb{N}_0)$. ■

4.3 Hartree-Fock Approximation of the Schrödinger Operator of a Particle in a Magnetic Many-body System

Up till this point we have worked with one particle quantum systems. If one theoretically wants to study many-body systems, then for N d-dimensional particles one would work in a subspace of $L^2(\mathbb{R}^{Nd})$ and a collective Schrödinger operator for the system. We instead take the approach of the Hartree-Fock approximation, so that when our quantum system fulfills certain assumptions, each particles self-energy can be approximated by a one-particle Schrödinger operator with added potential from the mean-field of the particle cloud. This added potential has to satisfy a self-consistent equation, which we translate into mathematics and solve rigorously. We remark that the particles considered in this section are fermionic.

To set the stage, we fix a constant magnetic field B and a one-particle free magnetic Schrödinger operator

$$\mathcal{H}_B = \overline{(-i\nabla - A_B)^2_{\min}}.$$

For particle-particle interactions we take a potential satisfying a symmetry condition and a rapid decay condition:

Definition 4.3.1. The space of even, integrable functions with rapid decay $L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$ consists of all real-valued integrable functions $v \in L^1(\mathbb{R}^d, \mathbb{R})$ such that $\langle\cdot\rangle^n v \in L^1(\mathbb{R}^d)$ for all $n \in \mathbb{N}$ and $v(\cdot) = v(-\cdot)$.

The rapid decay symbolizes that particles far away from each other have less impact on one another, and on that note, a physically interesting v is often singular at zero. The symmetry should be interpreted as the potential not caring about two particles switching places.

Lastly, we need a quantum distribution function:

Definition 4.3.2. The space of smooth functions on $\mathbb{R}$ with rapid decay towards $+\infty$ consists of real-valued functions $f \in C^\infty(\mathbb{R})$ satisfying

$$\sup_{[0,\infty)} \langle\cdot\rangle^n |\partial^j f| < \infty$$

for all $n, j \in \mathbb{N}_0$ and is denoted by $\mathscr{S}_+(\mathbb{R}, \mathbb{R})$.

The quantum distribution function $f \in \mathscr{S}_+(\mathbb{R}, \mathbb{R})$ will control the mean-field and the rapid decay restriction is defined such that f can cope with particles of large positive energies. Note $\mathcal{H}_B$ is lower bounded. For a probabilistic interpretation, one should additionally require $\lim_{\lambda\to-\infty} f(\lambda) = 1$.

Coming back to the mean-field potential, the self-consistent equation for the potential $\mathcal{W}_\lambda$ with coupling $\lambda \geq 0$ is given by

$$K_{\mathcal{W}_\lambda}(x,y) = \lambda\delta(x-y)\int_{\mathbb{R}^d} v(x-z)K_{f(\mathcal{H}_B+\mathcal{W}_\lambda)}(z,z)dz - \lambda v(x-y)K_{f(\mathcal{H}_B+\mathcal{W}_\lambda)}(x,y) \quad (4.3.1)$$

for $x, y \in \mathbb{R}^d$, where δ is the Dirac delta distribution. We wish to define $\mathcal{W}_\lambda$ through a rigorous version of (4.3.1) and add it to the one-particle Schrödinger operator $\mathcal{H}_B$. The Equation (4.3.1) involves kernels and so before we work on it we prove a lemma on integral operators:

Lemma 4.3.3. *Given a constant magnetic field B, a densely defined integral operator $T \in \mathcal{L}(L^2(\mathbb{R}^d))$ commutes with all magnetic translations if and only if*

$$K_T(x,y) = \vartheta_B(x,y)K_T(x-y,0) \quad (4.3.2)$$

for almost all $x, y \in \mathbb{R}^d$.

Moreover, in the positive case K_T is constant on the diagonal $\{(x,x)|x \in \mathbb{R}^d\}$ and T is then symmetric if and only if $K_T(\cdot,0) = \overline{K_T(-\cdot,0)}$.

Of course these equalities have to be interpreted correctly. Also, the first part of the lemma essentially follows from Lemma 3.3.18, but we provide another proof.

Proof. For $\phi \in D(T)$ and $x, y \in \mathbb{R}^d$ we have

$$\tau_{B,y}T\phi(x) = \int_{\mathbb{R}^d} \vartheta_B(x,y)K_T(x-y,z)\phi(z)dz = \int_{\mathbb{R}^d} \vartheta_B(x,y)K_T(x-y,z-y)\phi(z-y)dz$$

and

$$T\tau_{B,y}\phi(x) = \int_{\mathbb{R}^d} K_T(x,z)\vartheta_B(z,y)\phi(z-y)dz$$

for $x, y \in \mathbb{R}^d$. Thus T commutes with all magnetic translations if and only if

$$\vartheta_B(x,y)K_T(x-y,z-y) = \vartheta_B(z,y)K_T(x,z) \quad (4.3.3)$$

for $x, y, z \in \mathbb{R}^d$. Setting $z = y$ in (4.3.3) we get (4.3.2). Conversely, assuming (4.3.2) holds, we get

$$\vartheta_B(x,y)K_T(x-y,z-y) = \vartheta_B(x,y)\vartheta_B(x-y,z-y)K_T(x-z,0)$$

and

$$\vartheta_B(z,y)K_T(x,z) = \vartheta_B(z,y)\vartheta_B(x,z)K_T(x-z,0)$$

for $x, y, z \in \mathbb{R}^d$. Since

$$x \cdot A_B y + (x-y) \cdot A_B(z-y) = x \cdot A_B z + z \cdot A_B y,$$

we conclude that (4.3.2) implies (4.3.3).

Setting $y = x$ in (4.3.2) we get $K_T(0,0) = K_T(x,x)$, hence in the positive case, K_T would be constant on the diagonal. Next, an integral operator is symmetric if and only if its kernel is Hermitian. Since ϑ_B is Hermitian, T is symmetric if and only if

$$\overline{K_T(x-y,0)} = K_T(y-x,0)$$

for $x, y \in \mathbb{R}^d$. This condition is equivalent with $K_T(\cdot,0) = \overline{K_T(-\cdot,0)}$, so we are done. ■

Since the mean-field potential $\mathcal{W}_\lambda$ is the same for all particles in the particle cloud, it makes sense to impose the commutative property of Lemma 4.3.3 on $\mathcal{W}_\lambda$. A reasonable implication is that $f(\mathcal{H}_B + \mathcal{W}_\lambda)$ also possesses this commutative property, and so using Lemma 4.3.3, the Equation (4.3.1) would be equivalent with

$$K_{\mathcal{W}_\lambda}(x-y,0) = \lambda K_{f(\mathcal{H}_B+\mathcal{W}_\lambda)}(0,0) \int_{\mathbb{R}^d} v - \lambda v(x-y) K_{f(\mathcal{H}_B+\mathcal{W}_\lambda)}(x-y,0)$$

for $x, y \in \mathbb{R}^d$. Setting $F := K_{f(\mathcal{H}_B+\mathcal{W}_\lambda)}(\cdot,0)$ we get

$$K_{\mathcal{W}_\lambda}(\cdot,0) = F(0) \int_{\mathbb{R}^d} v - \lambda v(\cdot)F(\cdot),$$

and so defining $Z_{B,v,F}$ as the integral operator commuting with all magnetic translations with kernel $K_{Z_{B,v,F}}(\cdot,0) = \lambda v(\cdot)F(\cdot)$ we conclude

$$\mathcal{W}_\lambda = \lambda F(0) \int_{\mathbb{R}^d} v - \lambda Z_{B,v,F}.$$

Plugging this expression for the mean-field potential $\mathcal{W}_\lambda$ into the definition of F gives a self-consistent equation for F:

$$F = K_{f(\mathcal{H}_B + \lambda F(0) \int_{\mathbb{R}^d} v - \lambda Z_{B,v,F})}(\cdot,0) \tag{4.3.4}$$

This is the real fix point equation we will solve, and then recover $\mathcal{W}_\lambda$ as $\lambda W_{B,v,F} = \lambda(F(0) \int_{\mathbb{R}^d} v - Z_{B,v,F})$.

We have a lot of analysis to do. For a start we will look for a suitable definition for and properties of the operator $Z_{B,v,F}$, and next in line the operator $W_{B,v,F}$. Then it is time for the perturbed Schrödinger operator $\mathcal{H}_B + \mathcal{W}_\lambda$ and its quantum distribution $f(\mathcal{H}_B + \mathcal{W}_\lambda)$. Lastly, we solve (4.3.4) in a certain sense using Banach's Fix Point Theorem.

4.3.1 Integral Potential

Given a constant magnetic field B and $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$ we want a map $F \mapsto Z_{B,v,F}$ with good properties. Recall $Z_{B,v,F}$ was loosely defined as the integral operator commuting with all magnetic translations and with kernel satisfying $K_{Z_{B,v,F}}(\cdot,0) = v(\cdot)F(\cdot)$, i.e.

$$K_{Z_{B,v,F}}(x,y) = \vartheta_B(x,y) v(x-y) F(x-y)$$

for $x, y \in \mathbb{R}^d$.

One requirement we impose is $Z_{B,v,F} \in \mathcal{B}(L^2(\mathbb{R}^d))$ for all F, and by Schur's test a sufficient criteria is $vF \in L^1(\mathbb{R}^d)$ and it that case

$$\|Z_{B,v,F}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq \sqrt{\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K_{Z_{B,v,F}}(x,y)|dy \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K_{Z_{B,v,F}}(x,y)|dx} \leq \|vF\|_{L^1(\mathbb{R}^d)}.$$

A wide class of functions F satisfies $vF \in L^1(\mathbb{R}^d)$, but a simple and direct criteria is to take $F \in L^1(\mathbb{R}^d)' = L^\infty(\mathbb{R}^d)$. For this choice we get:

Lemma 4.3.4. *Given a constant magnetic field B and $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$, the map*

$$Z_{B,v,(\cdot)} \colon L^\infty(\mathbb{R}^d) \ni F \mapsto Z_{B,v,F}$$

enjoys the following properties:

(i) $Z_{B,v,(\cdot)}$ *has image in $\mathcal{B}(L^2(\mathbb{R}^d))$ and is bounded as an operator $L^\infty(\mathbb{R}^d) \to \mathcal{B}(L^2(\mathbb{R}^d))$ with operator norm less than or equal to $\|v\|_{L^1(\mathbb{R}^d)}$. The integral operator $Z_{B,v,F}$ commutes with all magnetic translations, and it is self-adjoint if $F(\cdot) = \overline{F(-\cdot)}$.*

(ii) $Z_{B,v,(\cdot)}$ *also has image in $\mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$ and is bounded as an operator $L^\infty(\mathbb{R}^d) \to \mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$ with operator norm less than or equal to $\|v\|_{L^2(\mathbb{R}^d)}$.*

(iii) For any $F \in L^\infty(\mathbb{R}^d)$ and $T_j \in \{X_{B,1}, \dots, X_{B,d}, \Pi_{B,1}, \dots, \Pi_{B,d}\}$, $j = 1, \dots, n$, $n \in \mathbb{N}_0$, the repeated commutator $[T_1, [T_2, \dots [T_n, Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)}] \dots]]$ extends to an operator in $\mathcal{B}(L^2(\mathbb{R}^d)) \cap \mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$. Consequently, $Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$.

Moreover, the linear maps

$$L^\infty(\mathbb{R}^d) \ni F \mapsto Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$$

and

$$L^\infty(\mathbb{R}^d) \ni F \mapsto \mathfrak{Op}_B^{-1}(Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)}) \in S_1(\mathbb{R}^{2d})$$

are continuous.

Proof. By Schur's test

$$\|Z_{B,v,F}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq \|vF\|_{L^1(\mathbb{R}^d)} \leq \|v\|_{L^1(\mathbb{R}^d)} \|F\|_{L^\infty(\mathbb{R}^d)}$$

for all $F \in L^\infty(\mathbb{R}^d)$, which deals with most of (i). The latter statements about commutating with all magnetic translations and the sufficient condition for self-adjointness follows from the explicit kernel of $Z_{B,v,F}$, Lemma 4.3.3, and the fact that v is real-valued.

(ii) is also not much of a problem to prove. For $\phi \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$|Z_{B,v,F}\phi(x)| \leq \int_{\mathbb{R}^d} |(vF)(x-y)||\phi(y)|dy \leq \|v\|_{L^2(\mathbb{R}^d)} \|F\|_{L^\infty(\mathbb{R}^d)} \|\phi\|_{L^2(\mathbb{R}^d)},$$

whence

$$\|Z_{B,v,F}\phi\|_{L^\infty(\mathbb{R}^d)} \leq \|v\|_{L^2(\mathbb{R}^d)} \|F\|_{L^\infty(\mathbb{R}^d)} \|\phi\|_{L^2(\mathbb{R}^d)}.$$

The operator norm estimate follows easily.

The last point, (iii), of the lemma boils down to proving that the commutators are integral operators with kernel of the same type as $Z_{B,v,F}$ and then prove that the maps in the latter half are continuous. Essentially, the particle-particle potential v has enough decay so that we may neglect whatever unwanted factors are introduced.

Fix $\phi \in \mathscr{S}(\mathbb{R}^d)$ and $j \in \{1, \dots, d\}$. For $x \in \mathbb{R}^d$ we have

$$[X_{B,j}, Z_{B,v,F}]\phi(x) = \int_{\mathbb{R}^d} \vartheta_B(x,y)(x_j - y_j)v(x-y)F(x-y)\phi(y)dy,$$

where the kernel on the right hand side gives rise to a map in both $\mathcal{B}(L^2(\mathbb{R}^d))$ and $\mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$ by the decay of v and the reasoning in the proof of (i) and (ii), which also gives the estimates

$$\|[X_{B,j}, Z_{B,v,F}]\phi\|_{L^2(\mathbb{R}^d)} \leq \|F\|_{L^\infty(\mathbb{R}^d)}\|X_{B,j}v\|_{L^1(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}$$

and

$$\|[X_{B,j}, Z_{B,v,F}]\phi\|_{L^\infty(\mathbb{R}^d)} \leq \|F\|_{L^\infty(\mathbb{R}^d)}\|X_{B,j}v\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}.$$

Concerning the momentum operators we show that $Z_{B,v,F}\phi$ is classically differentiable. A translation under the integral gives

$$Z_{B,v,F}\phi(x) = \int_{\mathbb{R}^d} \vartheta_B(x,-y)v(y)F(y)\phi(x-y)dy.$$

The integrand on the right hand side is differentiable in x and integrable in y uniformly in x, hence $Z_{B,v,F}\phi$ is differentiable by standard results of integration theory. Explicitly,

$$\begin{aligned}\partial^{e_j} Z_{B,v,F}\phi(x) = &-i\int_{\mathbb{R}^d}(A_B)_j(x-y)\vartheta_B(x,y)v(x-y)F(x-y)\phi(y)dy \\ &+ \int_{\mathbb{R}^d} \vartheta_B(x,y)v(x-y)F(x-y)\partial^{e_j}\phi(y)dy,\end{aligned}$$

implying that

$$[\Pi_{B,j}, Z_{B,v,F}]\phi(x) = -2i\int_{\mathbb{R}^d}(A_B)_j(x-y)\vartheta_B(x,y)v(x-y)F(x-y)\phi(y)dy,$$

and so the commutator $[\Pi_{B,j}, Z_{B,v,F}]$ also gives rise to a map in both $\mathcal{B}(L^2(\mathbb{R}^d))$ and $\mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$ with similar estimates on the norms.

Now it is trivially to show by a proper induction argument that repeated commutators of the kind considered in (iii) have nice kernels and that they are extendable to maps in $\mathcal{B}(L^2(\mathbb{R}^d)) \cap \mathcal{B}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$. Using Beal's Commutator Criterion 3.3.19, we conclude that $Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathfrak{Op}_B(S_1(\mathbb{R}^{2d})) \subseteq \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$. Furthermore, for all $n \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^d$

$$\begin{aligned}\sup_{\mathbb{R}^d}\langle\cdot\rangle^n|\partial^\gamma Z_{B,v,F}\phi| &\leq C_B \sum_{\substack{\delta\in\mathbb{N}_0^d \\ \delta\leq\gamma}} \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\langle x-y\rangle^n|X_{B,j}^{\gamma-\delta}v(x-y)F(x-y)|\langle y\rangle^n|\partial^\delta\phi(y)|dy \\ &\leq C_B\|F\|_{L^\infty(\mathbb{R}^d)}\|\langle\cdot\rangle^{n+|\gamma|}v\|_{L^1(\mathbb{R}^d)}\sum_{\substack{\delta\in\mathbb{N}_0^d \\ \delta\leq\gamma}}\sup_{\mathbb{R}^d}\langle\cdot\rangle^{|\gamma|}|\partial^\delta\phi|,\end{aligned}$$

and so by n, γ being arbitrary,

$$L^\infty(\mathbb{R}^d) \ni F \mapsto Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$$

is shown to be continuous.

For the map

$$L^\infty(\mathbb{R}^d) \ni F \mapsto \mathfrak{Op}_B^{-1}(Z_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)}) \in S_1(\mathbb{R}^{2d})$$

we note that by Parseval's identity one sees that $\Phi_{v,F}\colon \mathbb{R}^{2d} \ni (x,\xi) \mapsto \mathcal{F}(vF)(\xi)$ is the symbol of $Z_{B,v,F}$, whence

$$\sup_{\mathbb{R}^{2d}} |\partial^\gamma \Phi_{v,F}| \leq C\|F\|_{L^\infty(\mathbb{R}^d)} \|\langle\cdot\rangle^{|\gamma|} v\|_{L^1(\mathbb{R}^d)}$$

for all $\gamma \in \mathbb{N}_0^{2d}$, showing continuity. ■

Lemma 4.3.4 vindicates our choice of $L^\infty(\mathbb{R}^d)$ as domain for $Z_{B,v,(\cdot)}$. Note the significance of Lemma 4.3.4 (iii) lies in the fact that we are able to use the pseudo-differential calculus of Section 3.3 in the following.

Many of the qualities of $Z_{B,v,(\cdot)}$ should transfer to $W_{B,v,(\cdot)}$. One problem with $L^\infty(\mathbb{R}^d)$ as domain for $W_{B,v,(\cdot)}$ is that $W_{B,v,F}$ is defined using the value $F(0)$. A quick, and sufficient, fix is to restrict the domain of $W_{B,v,(\cdot)}$ to $BC(\mathbb{R}^d)$.

Corollary 4.3.5. *Given a constant magnetic field B and $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$, the map*

$$W_{B,v,(\cdot)}\colon BC(\mathbb{R}^d) \ni F \mapsto W_{B,v,F}$$

enjoys the following properties:

(i) $W_{B,v,(\cdot)}$ has image in $\mathcal{B}(L^2(\mathbb{R}^d))$ and is bounded as an operator $BC(\mathbb{R}^d) \to \mathcal{B}(L^2(\mathbb{R}^d))$ with operator norm less than or equal to $2\|v\|_{L^1(\mathbb{R}^d)}$. The operator $W_{B,v,F}$ commutes with all magnetic translations, and it is self-adjoint if $F(\cdot) = \overline{F(-\cdot)}$.

(ii) For any $F \in BC(\mathbb{R}^d)$ and $T_j \in \{X_{B,1}, \dots, X_{B,d}, \Pi_{B,1}, \dots, \Pi_{B,d}\}$, $j = 1, \dots, n$, $n \in \mathbb{N}_0$, the repeated commutator $[T_1, [T_2, \dots [T_n, W_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)}] \dots]]$ extends to an operator in $\mathcal{B}(L^2(\mathbb{R}^d))$. Consequently, $W_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$.

Moreover, the linear maps

$$BC(\mathbb{R}^d) \ni F \mapsto W_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)} \in \mathcal{B}(\mathscr{S}(\mathbb{R}^d))$$

and

$$BC(\mathbb{R}^d) \ni F \mapsto \mathfrak{Op}_B^{-1}(W_{B,v,F}|_{\mathscr{S}(\mathbb{R}^d)}) \in S_1(\mathbb{R}^{2d})$$

are continuous.

Proof. We utilize that $W_{B,v,F}$ is the sum of two operators in $\mathcal{B}(L^2(\mathbb{R}^d))$, the constant $F(0)\int_{\mathbb{R}^d} v$ and $Z_{B,v,F}$. Clearly $F(0)\int_{\mathbb{R}^d} v$ commutes with all operators and $F(\cdot) = \overline{F(-\cdot)}$ is sufficient for the operator $F(0)\int_{\mathbb{R}^d} v$ to be self-adjoint since v is real-valued. It does then not take much more effort to prove (ii) for $F(0)\int_{\mathbb{R}^d} v$ in place of $W_{B,v,F}$.

This means that we are essentially finished since $Z_{B,v,F}$ has the required properties by Lemma 4.3.4 and the corollary is then quickly verified for the sum $W_{B,v,F} = F(0)\int_{\mathbb{R}^d} v - Z_{B,v,F}$. ■

4.3.2 Symbol of the Perturbed Magnetic Schrödinger Operator

Having a good deal of information on the operators $W_{B,v,F}$, we turn to the study of the operator $\mathcal{H}_{B,v,\lambda,F} := \mathcal{H}_B + \lambda W_{B,v,F}$. A sufficient condition for $\mathcal{H}_{B,v,\lambda,F}$ to be self-adjoint and essentially self-adjoint when restricted to $\mathscr{S}(\mathbb{R}^d)$ is that $F(\cdot) = \overline{F(-\cdot)}$, see Corollary 4.3.5 (i). Thus we restrict ourselves to that case and define the space:

$$BC_H(\mathbb{R}^d) = \{F \in BC(\mathbb{R}^d) | F(\cdot) = \overline{F(-\cdot)}\}$$

As an example, $BC_H(\mathbb{R}^d)$ contains all real-valued even functions. We endow $BC_H(\mathbb{R}^d)$ with the subspace topology of $BC(\mathbb{R}^d)$.

Now for each $F \in BC_H(\mathbb{R}^d)$, we have $\mathcal{H}_{B,v,\lambda,F} \in \mathfrak{Op}_B(S_{M_0^2}(\mathbb{R}^{2d}))$, so one might suspect from the pseudo-differential calculus that the resolvent of $\mathcal{H}_{B,v,\lambda,F}$ takes values in $\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$. This is indeed the case, which we will prove in two steps: First our goal will be to use Beal's Commutator Criterion 3.3.19 to show that the resolvent is in $\mathfrak{Op}_B(S_1(\mathbb{R}^{2d}))$, and then use the explicit form of $\mathcal{H}_B$ and the definition of the Weyl quantizations to show that it is really in $\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$.

Proposition 4.3.6. *Given a constant magnetic field B, $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$, $F \in BC_H(\mathbb{R}^d)$, and $\lambda \geq 0$, the resolvent of $\mathcal{H}_{B,v,\lambda,F}$ takes values in $\mathfrak{Op}_B(S_1(\mathbb{R}^{2d}))$.*

Furthermore, for $T_j \in \{X_{B,1}, \dots, X_{B,d}, \Pi_{B,1}, \dots, \Pi_{B,d}\}$, $j = 1, \dots, n$, $n \in \mathbb{N}_0$, we have the following estimate on the operator norm of the repeated commutator:

$$\begin{aligned} &\|[T_1, [T_2, \dots [T_n, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}] \dots]]\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\ &\quad \leq C_B d(z, \sigma(\mathcal{H}_{B,v,\lambda,F}))^{-1} \langle d(z, \sigma(\mathcal{H}_{B,v,\lambda,F}))\rangle^{-n} \langle z \rangle^{\frac{n}{2}} \langle \lambda \| \langle \cdot \rangle^n v\|_{L^1(\mathbb{R}^d)} \|F\|_{L^\infty(\mathbb{R}^d)}\rangle^n \end{aligned} \tag{4.3.5}$$

for $z \in \rho(\mathcal{H}_{B,v,\lambda,F})$.

Proof. Fix $z \in \rho(\mathcal{H}_{B,v,\lambda,F})$ throughout. To use Beal's Commutator Criterion 3.3.19 we need to handle the repeated commutators advertised in the proposition. We will start with computing the commutators of $X_{B,j}, \Pi_{B,j}, (-i\nabla - A_B)^2$, $j \in \{1, \dots, d\}$, on $\mathscr{S}'(\mathbb{R}^d)$. These can then be shown to correspond with the commutators of $X_{B,j}, \Pi_{B,j}, \mathcal{H}_B$ on $L^2(\mathbb{R}^d)$ by use of duality. Then, using regularization and variational techniques, we deal with commutators of $X_{B,j}, \Pi_{B,j}, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}$.

We have

$$\begin{aligned} [X_{B,j}, X_{B,k}] &= [X_{B,j}, \Pi_{B,k}] = 0, \\ [X_{B,j}, \Pi_{B,j}] &= -iX_{B,j}\partial^{e_j} + i(1 + X_{B,j}\partial^{e_j}) = i, \\ [\Pi_{B,j}, \Pi_{B,k}] &= i[\partial^{e_j}, (A_B)_k] + i[(A_B)_j, \partial^{e_k}] = \frac{iB_{jk}}{2}[\partial^{e_j}, X_{B,j}] + \frac{iB_{kj}}{2}[X_{B,k}, \partial^{e_k}] = B_{jk}, \end{aligned} \tag{4.3.6}$$

for $j, k \in \{1, \dots, d\}, j \neq k$, and recalling that $(-i\nabla - A_B)^2 = \sum_{j=1}^d \Pi_{B,j}^2$ we get

$$[X_{B,j}, (-i\nabla - A_B)^2] = [X_{B,j}, \Pi_{B,j}]\Pi_{B,j} + \Pi_{B,j}[X_{B,j}, \Pi_{B,j}] = 2i\Pi_{B,j} \tag{4.3.7}$$

and

$$[\Pi_{B,j}, (-i\nabla - A_B)^2] = \sum_{k=1}^d [\Pi_{B,j}, \Pi_{B,k}]\Pi_{B,k} + \Pi_{B,k}[\Pi_{B,j}, \Pi_{B,k}] = \sum_{k=1}^d 2B_{jk}\Pi_{B,k} \tag{4.3.8}$$

for $j \in \{1, \dots, d\}$.

Moving on, we show that $\Pi_{B,j}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}$ is a bounded operator. It is defined on the entirety of $L^2(\mathbb{R}^d)$ since $\overline{\Pi_{B,j}|_{\mathscr{S}(\mathbb{R}^d)}}$ is $\mathcal{H}_B$-bounded, see Proposition 4.2.2. An estimate of the bound is then, for $\phi \in L^2(\mathbb{R}^d)$, computed by

$$\begin{aligned}\sum_{k=1}^{d} \|\Pi_{B,k}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi\|^2_{L^2(\mathbb{R}^d)} &= \langle(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, \mathcal{H}_B(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi\rangle_{L^2(\mathbb{R}^d)} \\ &\leq \|\mathrm{id}_{L^2(\mathbb{R}^d)} + (z - \lambda W_{B,v,F})(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\ &\qquad \|(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^d))}\|\phi\|^2_{L^2(\mathbb{R}^d)} \\ &= \left(1 + \frac{|z| + 2\lambda\|v\|_{L^1(\mathbb{R}^d)}\|F\|_{L^\infty(\mathbb{R}^d)}}{d(z, \sigma(\mathcal{H}_{B,v,\lambda,F}))}\right)\frac{1}{d(z, \sigma(\mathcal{H}_{B,v,\lambda,F}))}\|\phi\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

where we used integration by parts and the norm estimates in Corollary 2.4.24 and Corollary 4.3.5.

Finally, we study the commutators of $X_{B,j}, \Pi_{B,j}$ with $(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}$. For $X_{B,j}$, we shall first make use of the regularization $X_{B,j,\varepsilon} := X_{B,j}e^{-\varepsilon\langle\cdot\rangle} \in \mathcal{B}(L^2(\mathbb{R}^d))$, $\varepsilon > 0$. As operators on Schwartz functions or tempered distributions

$$X_{B,j,\varepsilon} \overset{\varepsilon\to 0}{\to} X_{B,j}$$

and if $\phi \in L^2(\mathbb{R}^d)$ with $X_{B,j}\phi \in L^2(\mathbb{R}^d)$, then also

$$X_{B,j,\varepsilon}\phi \overset{\varepsilon\to 0}{\to} X_{B,j}\phi$$

in $L^2(\mathbb{R}^d)$-norm. Furthermore, on $D(\mathcal{H}_B)$ we have pointwise convergence

$$[X_{B,j,\varepsilon}, \mathcal{H}_B] \overset{\varepsilon\to 0}{\to} 2i\Pi_{B,j}$$

in $L^2(\mathbb{R}^d)$-norm. Using these convergence properties in unison we get

$$\begin{aligned}&\langle[X_{B,j}, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}]\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \lim_{\varepsilon\to 0}\langle[X_{B,j,\varepsilon}, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}]\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &= -\lim_{\varepsilon\to 0}\langle(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}[X_{B,j,\varepsilon}, \mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &= -\langle(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}(2i\Pi_{B,j} + \lambda[X_{B,j}, W_{B,v,F}])(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}\end{aligned}$$

for all $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$, and so $[X_{B,j}, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}]$ equals, on $\mathscr{S}(\mathbb{R}^d)$, a $L^2(\mathbb{R}^d)$-bounded operator by $\Pi_{B,j}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^d))$ and Corollary 4.3.5.

For $\Pi_{B,j}$ we use the $\mathcal{H}_B$-boundedness again. Given $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$ we get

$$\begin{aligned}&\langle[\Pi_{B,j}, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}]\phi, \psi\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle\Pi_{B,j}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, \mathcal{H}_{B,v,\lambda,F}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\psi\rangle_{L^2(\mathbb{R}^d)} \\ &\quad - \langle\mathcal{H}_{B,v,\lambda,F}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, \Pi_{B,j}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\psi\rangle_{L^2(\mathbb{R}^d)} \\ &= s((\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\psi) + s^*((\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\phi, (\mathcal{H}_{B,v,\lambda,F} - z)^{-1}\psi),\end{aligned}$$

where s is the sesquilinear form:

$$s\colon D(\mathcal{H}_B) \times D(\mathcal{H}_B) \ni (\omega_1, \omega_2) \mapsto \langle\Pi_{B,j}\omega_1, \mathcal{H}_{B,v,\lambda,F}\omega_2\rangle_{L^2(\mathbb{R}^d)},$$

Both s and its adjoint are bounded in $D(\mathcal{H}_B)$, and so we may find sequences $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathbb{R}^d)$ converging to $(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\phi, (\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\psi$ respectively in $D(\mathcal{H}_B)$, and compute:

$$\begin{aligned}\langle[\Pi_{B,j},(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}]\phi,\psi\rangle_{L^2(\mathbb{R}^d)} &= \lim_{n\to\infty} s(\phi_n,\psi_n)+s^*(\phi_n,\psi_n)\\ &= -\lim_{n\to\infty}\langle[\Pi_{B,j},\mathcal{H}_{B,v,\lambda,F}]\phi_n,\psi_n\rangle_{L^2(\mathbb{R}^d)}\\ &= -\langle(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}[\Pi_{B,j},\mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\phi,\psi\rangle_{L^2(\mathbb{R}^d)},\end{aligned}$$

where the limit in the last equality stems from (4.3.8) and Corollary 4.3.5. This then shows that $[\Pi_{B,j},(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}]$ equals a $L^2(\mathbb{R}^d)$-bounded operator on $\mathscr{S}(\mathbb{R}^d)$.

Thus the commutators of $X_{B,j},\Pi_{B,j}$ with $(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}$ extend to bounded operators on $L^2(\mathbb{R}^d)$. These arguments can be extended by induction to prove that all the repeated commutators considered in the proposition extend to operators in $\mathcal{B}(L^2(\mathbb{R}^d))$, and additionally, by our method of switching commutators in $(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}$ with ones in $\mathcal{H}_{B,v,\lambda,F}$ and Leibniz's rule, one obtains the formula:

$$\begin{aligned}&[T_1,[T_2,\cdots[T_n,(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}]\cdots]]\\ &\quad=(-1)^n\sum_{p\in S_n}(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\prod_{l=1}^n\Big([T_{p_l},\mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\Big)\\ &\qquad+(-1)^{n-1}\sum_{p\in S_n,p_1<p_2}(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}[T_{p_1},[T_{p_2},\mathcal{H}_{B,v,\lambda,F}]](\mathcal{H}_b-z)^{-1}\\ &\qquad\qquad\prod_{l=3}^n\Big([T_{p_l},\mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\Big)\\ &\qquad+(-1)^{n-1}\sum_{p\in S_n,p_2<p_3}(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}[T_{p_1},\mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\\ &\qquad\qquad[T_{p_2},[T_{p_3},\mathcal{H}_{B,v,\lambda,F}]](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\prod_{l=4}^n\Big([T_{p_l},\mathcal{H}_{B,v,\lambda,F}](\mathcal{H}_{B,v,\lambda,F}-z)^{-1}\Big)\\ &\qquad+\cdots\\ &\qquad-(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}[T_1,[T_2,\cdots[T_n,\mathcal{H}_{B,v,\lambda,F}]\cdots]](\mathcal{H}_{B,v,\lambda,F}-z)^{-1},\end{aligned}$$

where S_n is the symmetric group of n-elements, or otherwise known as the set of all permutations of n-elements. Note also, that by the commutators (4.3.6), (4.3.7), and (4.3.8), the triple, or more, repeated commutator of operators in $\{X_{B,1},\dots,X_{B,d},\Pi_{B,1},\dots,\Pi_{B,d}\}$ with $(-i\nabla-A_B)^2$ is zero, which affects the above formula. We use this expansion of the repeated commutators to get the estimate of its operator norm, as stated in the proposition. ■

Proposition 4.3.7. *Given a constant magnetic field B, $v\in L^1_{0,E}(\mathbb{R}^d,\mathbb{R})$, $F\in BC_H(\mathbb{R}^d)$, and $\lambda\geq 0$, the resolvent of $\mathcal{H}_{B,v,\lambda,F}$ takes values in $\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$.*

Proof. A considerable amount of work went into showing that $(\mathcal{H}_{B,v,\lambda,F}-z)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}\in\mathfrak{Op}_B(S_1(\mathbb{R}^{2d}))$. Luckily this proof is simpler and somewhat direct. We prove that $(\mathcal{H}_B-i)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}\in\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$, whence the second resolvent identity and the Moyal product, see Lemma 2.4.4 and Theorem 3.3.15, implies that $(\mathcal{H}_{B,v,\lambda,F}-i)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}\in\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$ from which the statement follows from the first resolvent identity and the Moyal product.

Let

$$\Phi := \mathfrak{Op}_B^{-1}\left((\mathcal{H}_B - i)^{-1}|_{\mathscr{S}(\mathbb{R}^d)}\right).$$

For $\phi, \psi \in \mathscr{S}(\mathbb{R}^d)$, this means that

$$\begin{aligned}\langle \phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} &= \langle (\mathcal{H}_B - i)\mathfrak{Op}_B(\Phi)\phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} \\ &= \left\langle \mathcal{T}_{\mathrm{Weyl}}^{\frac{1}{2},B}\Phi, \left((i\nabla - A_B)^2 - i\right)\psi\otimes\phi\right\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})} \\ &= \lim_{\varepsilon\to 0}\left\langle \mathcal{T}_{\mathrm{Weyl}}^{\frac{1}{2},B}\Phi_\varepsilon, \left((i\nabla - A_B)^2 - i\right)\psi\otimes\phi\right\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})},\end{aligned}$$

where $\Phi_\varepsilon \colon \mathbb{R}^{2d} \ni (x,\xi) \mapsto e^{-\varepsilon\langle\xi\rangle}\Phi(\xi)$ for $\varepsilon > 0$ and the last equality holds by dominated convergence. For each $\varepsilon > 0$ we can, by using Fubini's Theorem, integration by parts, and the exponential factors, get

$$\begin{aligned}&\left\langle \mathcal{T}_{\mathrm{Weyl}}^{\frac{1}{2},B}\Phi_\varepsilon, \left((i\nabla - A_B)^2 - i\right)\psi\otimes\phi\right\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})} \\ &= \left\langle \mathcal{T}_{\mathrm{Weyl}}^{\frac{1}{2},B}\left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi_\varepsilon, \psi\otimes\phi\right\rangle_{\mathscr{S}'(\mathbb{R}^{2d}),\mathscr{S}(\mathbb{R}^{2d})} \\ &= \left\langle \mathfrak{Op}_B\left(\left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi_\varepsilon\right)\phi, \psi\right\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}.\end{aligned}$$

Here, when viewing Φ_ε as a function of $(x,\xi) \in \mathbb{R}^{2d}$, we use $\Xi \colon \mathbb{R}^{2d} \ni (x,\xi) \mapsto \xi$ and $\nabla_\xi = (\partial^{e_{d+j}})_{j=1}^d$.

Since

$$\left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi_\varepsilon \overset{\varepsilon\to\infty}{\to} \left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi$$

holds in $\mathscr{S}'(\mathbb{R}^{2d})$ by continuity of the map $(-iA_B(\nabla_\xi) + \Xi)^2 - i$, we get

$$\langle \phi, \psi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \left\langle \mathfrak{Op}_B\left(\left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi\right)\phi, \psi\right\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}.$$

The identity function $\mathrm{id}_{\mathscr{S}(\mathbb{R}^d)}$ has symbol $1 \in S_1(\mathbb{R}^{2d})$, so we get by uniqueness of symbols that

$$1 = \left((-iA_B(\nabla_\xi) + \Xi)^2 - i\right)\Phi.$$

It follows that

$$\Phi = \frac{1}{\|\Xi\|^2 - i}\left(1 - 2i\Xi\cdot A_B(\nabla_\xi) - A_B(\nabla_\xi)\cdot A_B(\nabla_\xi)\Phi\right) \in S_{M_0^{-1}}(\mathbb{R}^{2d})$$

with the initially dominating term being $\frac{1}{\|\Xi\|^2 - i}\Xi\cdot A_B(\nabla_\xi)$, and thus secondly it follows still that $\Phi \in S_{M_0^{-2}}(\mathbb{R}^{2d})$ with the dominating term being $\frac{1}{\|\Xi\|^2 - i} \in S_{M_0^{-2}}(\mathbb{R}^{2d})$. ■

4.3.3 Quantum Distribution of the Perturbed Magnetic Schrödinger Operator

We have gathered most of results we need to prove that (4.3.4) has a fix point. Essential in this task, and the last bit of preparation, is to examine the operator $f(\mathcal{H}_{B,v,\lambda,F})$.

Theorem 4.3.8. *Let B be a constant magnetic field, $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$, $F \in BC_H(\mathbb{R}^d)$, $f \in \mathscr{S}_+(\mathbb{R}, \mathbb{R})$, and $\lambda \geq 0$.*

The operator $f(\mathcal{H}_{B,v,\lambda,F})$ is a self-adjoint, bounded operator that commutes with all magnetic translations. It has a kernel for which $K_{f(\mathcal{H}_{B,v,\lambda,F})}(\cdot, 0) \in \mathscr{S}(\mathbb{R}^d)$.

Proof. The self-adjointness and boundedness follows from our spectral theory, see Proposition 2.4.9 and Proposition 2.4.12. For the rest we will use the Helffer-Sjöstrand Formula 2.4.27.

We need a suitable almost analytic extension of f. Noting that $\mathcal{H}_{B,v,\lambda,F}$ is lower bounded, we can construct an increasing function $g \in BC^\infty(\mathbb{R})$ such that $g(x) = 1$ for $x \geq m(\mathcal{H}_{B,v,\lambda,F}) - 1$ and $g(x) = 0$ for $x \leq m(\mathcal{H}_{B,v,\lambda,F}) - 2$. Then $f(\mathcal{H}_{B,v,\lambda,F}) = (gf)(\mathcal{H}_{B,v,\lambda,F})$ and $gf \in \mathscr{S}(\mathbb{R})$. By the construction in Subsection 2.4.5 we have a family of almost analytic extensions of $\widetilde{(gf)}_N$ indexed by $N \in \mathbb{N}$. Fix one for now.

Then by the Helffer-Sjöstrand Formula 2.4.27:

$$f(\mathcal{H}_{B,v,\lambda,F}) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \widetilde{(gf)}_N(z) (\mathcal{H}_{B,v,\lambda,F} - z)^{-1} dxdy$$

Here we may then use the first resolvent identity and integration by parts any number of times to get:

$$\begin{aligned}
f(\mathcal{H}_{B,v,\lambda,F}) &= -\frac{1}{\pi} \sum_{j=0}^{n-1} \int_{\mathbb{C}} \overline{\partial_z} \widetilde{(gf)}_N(z) (z-i)^j dxdy (\mathcal{H}_{B,v,\lambda,F} - i)^{-j-1} \\
&\quad - \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \widetilde{(gf)}_N(z) (z-i)^n (\mathcal{H}_{B,v,\lambda,F} - z)^{-1} dxdy (\mathcal{H}_{B,v,\lambda,F} - i)^{-n} \\
&= -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \widetilde{(gf)}_N(z) (z-i)^n (\mathcal{H}_{B,v,\lambda,F} - z)^{-1} dxdy (\mathcal{H}_{B,v,\lambda,F} - i)^{-n}
\end{aligned}$$

We take at least $n > \frac{d}{2} - 1$. Applying this to a Schwartz function $\phi \in \mathscr{S}(\mathbb{R}^d)$ we compute

$$f(\mathcal{H}_{B,v,\lambda,F})\phi = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial_z} \widetilde{(gf)}_N(z) (z-i)^n \mathfrak{Op}_B(\Phi_z) \phi dxdy,$$

where we used simple results about the commutativity of bounded operators with Bochner integrals to get ϕ inside the integral and

$$\Phi_z := \mathfrak{Op}_B^{-1} \left((\mathcal{H}_{B,v,\lambda,F} - z)^{-1} (\mathcal{H}_{B,v,\lambda,F} - i)^{-n} |_{\mathscr{S}(\mathbb{R}^d)} \right) \in S_{M_0^{-2n-2}}(\mathbb{R}^{2d})$$

having Proposition 4.3.7 and Theorem 3.3.15 in mind.

It is now time to interchange the integral with the magnetic Weyl-quantization. Much as in the proof of Theorem 2.4.27 we see that:

$$\begin{aligned}
f(\mathcal{H}_{B,v,\lambda,F})\phi &= -\frac{1}{\pi} \lim_{k\to\infty} \frac{1}{k^2} \sum_{w \in [-k,k]^2 \cap k^{-1}(2^{-1} + i2^{-1} + \mathbb{Z}^2)} \overline{\partial_z} \widetilde{(gf)}_N(w) (w-i)^n \mathfrak{Op}_B(\Phi_z) \phi \\
&= -\frac{1}{\pi} \lim_{k\to\infty} \mathfrak{Op}_B \left(\frac{1}{k^2} \sum_{w \in [-k,k]^2 \cap k^{-1}(2^{-1} + i2^{-1} + \mathbb{Z}^2)} \overline{\partial_z} \widetilde{(gf)}_N(w) (w-i)^n \Phi_w \right) \phi
\end{aligned}$$

Hence if the sequence

$$\left(\frac{1}{k^2} \sum_{w \in [-k,k]^2 \cap k^{-1}(2^{-1} + i2^{-1} + \mathbb{Z}^2)} \overline{\partial_z} \widetilde{(gf)}_N(w) (w-i)^n \Phi_w \right)_{k \in \mathbb{N}}$$

converges in $\mathscr{S}'(\mathbb{R}^{2d})$, then we may do the wanted interchange by the Schwartz Kernel Theorem 3.2.21. To show the convergence we look for estimates on $\|\Phi_z\|_{S_{M_0^{-2n-2}}(\mathbb{R}^{2d}),0}$.

From the proof of Lemma 3.3.8 and Theorem 3.3.15, we see that it is enough to have control of the matrix elements of $(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}$ in the magnetic Gabor frame $(\mathcal{G}_{\tilde{\alpha},B})_{\tilde{\alpha}\in\mathbb{Z}^{2d}}$ for z in a dense set. Beal's Commutator Criterion 3.3.19 then provides a way for us to do this by controlling the operator norms of repeated commutators of $\{X_{B,1},\dots,X_{B,d},\Pi_{B,1},\dots,\Pi_{B,d}\}$ with the map $(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}$, and an estimation of these operator norms was done in Proposition 4.3.6, see (4.3.5). We found that the z-dependence of these acted like $d(z,\sigma(\mathcal{H}_{B,v,\lambda,F}))^{-1}\langle d(z,\sigma(\mathcal{H}_{B,v,\lambda,F}))\rangle^{-l}\langle z\rangle^{\frac{l}{2}}$, for some $l\in\mathbb{N}$, when $z\in\rho(\mathcal{H}_{B,v,\lambda,F})$. To deal with this behavior, we only have to chose N large enough, see Lemma 2.4.26. Hence

$$\sup_{\xi\in\mathbb{R}^d}|\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\langle\xi\rangle^{2n+2}\Phi_z(\xi)| \le c_{N,n,\gamma}\langle z\rangle^{-d-1}$$

for all $z\in\rho(\mathcal{H}_{B,v,\lambda,F})$ if N is chosen appropriately large. This implies that uniformly in ξ we have

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k^2}&\sum_{w\in[-k,k]^2\cap k^{-1}(2^{-1}+i2^{-1}+\mathbb{Z}^2)}\overline{\partial_z}\widetilde{(gf)}_N(w)(w-i)^n\langle\xi\rangle^{2n+2}\Phi_w(\xi)\\&=\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\langle\xi\rangle^{2n+2}\Phi_z(\xi)dxdy\end{aligned}$$

with

$$\sup_{\xi\in\mathbb{R}^d}\left|\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\langle\xi\rangle^{2n+2}\Phi_z(\xi)dxdy\right|<\infty.$$

These results also hold if we consider $\partial^\gamma\Phi_z$, $\gamma\in\mathbb{N}_0^d$, in-place of Φ_z so long as N is chosen larger.

This is a stronger result than just convergence in $\mathscr{S}'(\mathbb{R}^{2d})$, so we are indeed able to conclude that

$$f(\mathcal{H}_{B,v,\lambda,F})\phi=\mathfrak{Op}_B\left(-\frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\Phi_z dxdy\right)\phi.$$

Now the form of the symbol and decay in the variable ξ gives us

$$\begin{aligned}&\langle\psi,f(\mathcal{H}_{B,v,\lambda,F})\phi\rangle_{L^2(\mathbb{R}^d)}\\&=\left\langle\mathfrak{Op}_B\left(-\frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\Phi_z(\xi)dxdy\right)\phi,\overline{\psi}\right\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}\\&=\int_{\mathbb{R}^d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\xi\cdot(u-\zeta)}\vartheta_B(u,\zeta)\left(-\frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\Phi_z(\xi)dxdy\right)d\xi\\&\qquad\phi(\zeta)d\zeta\overline{\psi}(u)du\end{aligned}$$

for all $\psi\in\mathscr{S}(\mathbb{R}^d)$, hence

$$\begin{aligned}&f(\mathcal{H}_{B,v,\lambda,F})\phi(u)\\&=(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\xi\cdot(u-\zeta)}\vartheta_B(u,\zeta)\left(-\frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n\Phi_z(\xi)dxdy\right)d\xi\phi(\zeta)d\zeta\end{aligned}$$

for almost all $u \in \mathbb{R}^d$. By our arguments up to this point,

$$\mathbb{R}^d \ni \xi \mapsto \frac{-1}{2\pi^d} \int_{\mathbb{C}} \overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n \Phi_z(\xi) dxdy \tag{4.3.9}$$

is a function with smoothness regulated by N and decay like $\langle \xi \rangle^{-2n-2}$. For n large enough, this means that its Fourier transform is well-defined, its in $C^{n+1}(\mathbb{R}^d)$, and has decay of $\langle \xi \rangle^{-k}$ for $k \in \mathbb{N}_0$ under some threshold decided by N. Let h be the Fourier transform, so

$$f(\mathcal{H}_{B,v,\lambda,F})\phi(u) = \int_{\mathbb{R}^d} \vartheta_B(u,\zeta) h(\zeta - u)\phi(\zeta) d\zeta \tag{4.3.10}$$

for almost all $u \in \mathbb{R}^d$. Note h is dependent on the N and n chosen, but we are able to choose these arbitrarily large. Furthermore, (4.3.10) holds for all large enough N, n and every $\phi \in \mathscr{S}(\mathbb{R}^d)$, whence we must conclude that h is smooth and has decay of $\langle \xi \rangle^{-k}$ for any $k \geq 0$. Choosing appropriate N, n and analyzing the derivative of h, with h as the Fourier transform of (4.3.9), we see that $h \in \mathscr{S}(\mathbb{R}^d)$. Since $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, (4.3.10) holds for all $\phi \in L^2(\mathbb{R}^d)$ and it follows that $f(\mathcal{H}_{B,v,\lambda,F})$ has a kernel with the property $K_{f(\mathcal{H}_{B,v,\lambda,F})}(\cdot, 0) = h \in \mathscr{S}(\mathbb{R}^d)$.

All that is left is the commutativity with all magnetic translation. We can use Lemma 4.3.3 and (4.3.10), or more simply the Hellfer-Sjöstrand Formula 2.4.27 directly and the fact that the resolvent of $\mathcal{H}_{B,v,\lambda,F}$ commutes with all magnetic translations. ■

4.3.4 Existence of the Hartree-Fock Approximation of the Schrödinger Operator

At last, we prove the existence of solutions to (4.3.4) given λ small enough:

Theorem 4.3.9. *Let B be a constant magnetic field, $v \in L^1_{0,E}(\mathbb{R}^d, \mathbb{R})$, and $f \in \mathscr{S}_+(\mathbb{R}, \mathbb{R})$. Define the map*

$$\mathcal{T}_\lambda \colon BC_H(\mathbb{R}^d) \ni F \mapsto K_{f(\mathcal{H}_{B,v,\lambda,F})}(\cdot, 0) \in BC_H(\mathbb{R}^d) \cap \mathscr{S}(\mathbb{R}^d)$$

for every $\lambda \geq 0$.

Then for every $M > \|\mathcal{T}_0(0)\|_{L^\infty(\mathbb{R}^d)}$ there exists $\lambda_0 > 0$ such that $\mathcal{T}_\lambda$ has a unique fix point in $\overline{B_M(0; BC_H(\mathbb{R}^d))} \cap \mathscr{S}(\mathbb{R}^d)$ for all $\lambda \in [0, \lambda_0]$.

Proof. As a start, note that Theorem 4.3.8 tells us that $\mathcal{T}_\lambda$ is a well-defined map with image in $\mathscr{S}(\mathbb{R}^d) \subseteq BC(\mathbb{R}^d)$ for any $\lambda \geq 0$. Furthermore, the theorem stated that $f(H_{B,v,\lambda,F})$ is self-adjoint, which implies by Lemma 4.3.3 that $\mathcal{T}_\lambda(F)$ has the required symmetry to confirm $R(\mathcal{T}_\lambda) \subseteq BC_H(\mathbb{R}^d) \cap \mathscr{S}(\mathbb{R}^d)$.

Let $M > \|\mathcal{T}_0(0)\|_{L^\infty(\mathbb{R}^d)}$ be given. The plan is to make estimates on the difference $\|\mathcal{T}_\lambda(F) - \mathcal{T}_\lambda(G)\|_{L^\infty(\mathbb{R}^d)}$, $F, G \in \overline{B_M(0; BC_H(\mathbb{R}^d))}$, for $\lambda \in [0, \lambda_0]$ where λ_0 is yet to be determined. Then, as already alluded to earlier, we invoke Banach's Fix Point Theorem to get a fix point for $\mathcal{T}_\lambda$. We will throughout the proof choose λ_0 smaller when needed and always think of λ as some number in $[0, \lambda_0]$.

To estimate the difference $\|\mathcal{T}_\lambda(F) - \mathcal{T}_\lambda(G)\|_{L^\infty(\mathbb{R}^d)}$, we apply the methodology used in the proof of Theorem 4.3.8, i.e. the Helffer-Sjöstrand Formula 2.4.27, resolvent identities, and Propositions 4.3.6 and 4.3.7, to find a good expression for the kernel of $\mathcal{T}_\lambda(F) -$

$\mathcal{T}_\lambda(G)$. In that proof we used a cutoff function, which we now choose uniform: For every $F \in \overline{B_M(0; BC_H(\mathbb{R}^d))}$ we have

$$\|W_{B,v,F}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq 2M\|v\|_{L^1(\mathbb{R}^d)}$$

by Corollary 4.3.5, so for $\lambda_0 < (2M\|v\|_{L^1(\mathbb{R}^d)})^{-1}$, $\|\lambda W_{B,v,F}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} < 1$ and

$$m(\mathcal{H}_{B,v,\lambda,F}) > -1.$$

Hence, if we construct $g \in BC^\infty(\mathbb{R})$ such that g is increasing, $g(x) = 1$ for $x > -1$ and $g(x) = 0$ for $x < -2$, then $gf \in \mathscr{S}(\mathbb{R})$ and

$$gf(\mathcal{H}_{B,v,\lambda,F}) = f(\mathcal{H}_{B,v,\lambda,F})$$

for all $F \in \overline{B_M(0; BC_H(\mathbb{R}^d))}$.

Thus for such a g and using the Helffer-Sjöstrand Formula 2.4.27, resolvent identities, and integration by parts, we get

$$\begin{aligned}
&f(\mathcal{H}_{B,v,\lambda,F}) - f(\mathcal{H}_{B,v,\lambda,G}) \\
&\quad= \frac{\lambda}{\pi}\int_{\mathbb{C}} \overline{\partial_z}\widetilde{(gf)}_N(z)(\mathcal{H}_{B,v,\lambda,G} - z)^{-1}W_{B,v,F-G}(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}dxdy \\
&\quad= \frac{\lambda}{\pi}\int_{\mathbb{C}} \overline{\partial_z}\widetilde{(gf)}_N(z)(z-i)^n(\mathcal{H}_{B,v,\lambda,G} - z)^{-1}(\mathcal{H}_{B,v,\lambda,G} - i)^{-n}W_{B,v,F-G} \\
&\qquad\left(\sum_{j=0}^{n-1}(z-i)^j(\mathcal{H}_{B,v,\lambda,F} - i)^{-j-1}\right. \\
&\qquad\quad\left. + (z-i)^n(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}(\mathcal{H}_{B,v,\lambda,F} - i)^{-n}\right)dxdy \\
&\quad+ \frac{\lambda}{\pi}\int_{\mathbb{C}} \overline{\partial_z}\widetilde{(gf)}_N(z)\left(\sum_{j=0}^{n-1}(z-i)^j(\mathcal{H}_{B,v,\lambda,G} - i)^{-j-1}\right. \\
&\qquad\quad\left. + (z-i)^n(\mathcal{H}_{B,v,\lambda,G} - z)^{-1}(\mathcal{H}_{B,v,\lambda,G} - i)^{-n}\right) \\
&\qquad W_{B,v,F-G}(z-i)^n(\mathcal{H}_{B,v,\lambda,F} - z)^{-1}(\mathcal{H}_{B,v,\lambda,F} - i)^{-n}dxdy
\end{aligned}$$

for all $F, G \in \overline{B_M(0; BC_H(\mathbb{R}^d))}$, an $n \in \mathbb{N}$, and a large enough N. We will need at least $n > \frac{d}{2} - 2$ in the following. Theorem 3.3.15, Proposition 4.3.7, and Corollary 4.3.5 shows that each operator product in the above sum is a magnetic pseudo-differential operator with symbol in $S_{M_0^{-2n-4}}(\mathbb{R}^{2d})$. Gauging the z-dependence of these symbols using (4.3.5), we see that for n, N large enough, the integral in $\mathbb{C}$ and the magnetic Weyl-quantization can be interchanged, and we get

$$f(\mathcal{H}_{B,v,\lambda,F}) - f(\mathcal{H}_{B,v,\lambda,G}) = \mathfrak{Op}_B(\Phi_{F,G,z})$$

for some $\Phi_{F,G,z} \in BC(\mathbb{R}^{2d})$ with limited decay and smoothness. Note $\Phi_{F,G,z}$ is only dependent on the last d coordinates, see Lemma 3.3.18. If again n, N are chosen large enough, then, as argued in the proof of Theorem 4.3.8, we obtain a function $h_{F,G}$ with limited decay and smoothness such that

$$f(\mathcal{H}_{B,v,\lambda,F})\phi(u) - f(\mathcal{H}_{B,v,\lambda,G})\phi(u) = \int_{\mathbb{R}^d} \vartheta_B(u,\zeta)h_{F,G}(\zeta - u)\phi(\zeta)d\zeta$$

holds for all $\phi \in L^2(\mathbb{R}^d)$. This implies that

$$h_{F,G} = \mathcal{T}_\lambda(F) - \mathcal{T}_\lambda(G),$$

so to get an estimate on the difference $\|\mathcal{T}_\lambda(F) - \mathcal{T}_\lambda(G)\|_{L^\infty(\mathbb{R}^d)}$ we can look at $h_{F,G}$. The advantage is that $h_{F,G}$ has norm controlled by, among other things, operator norms of the repeated commutators in Corollary 4.3.5 (ii), which involves the factor $\|F - G\|_{L^\infty(\mathbb{R}^d)}$ in our situation. Let us be more concrete:

To estimate the norm $\|h_{F,G}\|_{L^\infty(\mathbb{R}^d)}$ we can use the proof of Beal's Commutator Criterion 3.3.19, Corollary 4.3.5, and Proposition 4.3.6. This gives an estimate of the type

$$\|h_{F-G}\|_{L^\infty(\mathbb{R}^d)} \leq \lambda C_{B,f,v,n,N} \|F - G\|_{L^\infty(\mathbb{R}^d)} \langle \lambda M \rangle^{m_{n,N}},$$

where $m_{n,N} \in \mathbb{N}$. Fixing n, N large enough, we can choose λ_0 small enough resulting in

$$\|\mathcal{T}_\lambda(F) - \mathcal{T}_\lambda(G)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{2} \|F - G\|_{L^\infty(\mathbb{R}^d)}.$$

Moreover, choosing λ_0 perhaps even smaller, we can conclude that

$$\|\mathcal{T}_\lambda(F)\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathcal{T}_0(0)\|_{L^\infty(\mathbb{R}^d)} + \|\mathcal{T}_\lambda(F) - \mathcal{T}_0(0)\|_{L^\infty(\mathbb{R}^d)} \leq M.$$

Note $\mathcal{T}_0(0) = \mathcal{T}_\lambda(0)$. Summarizing, when λ_0 is small enough, $\mathcal{T}_\lambda$ invariates and becomes a contraction on $\overline{B_M(0; BC_H(\mathbb{R}^d))}$. This implies by Banach's Fix Point Theorem that $\mathcal{T}_\lambda$ indeed has a fix point, concluding the proof. ■

Remark! 4.3.10. Let us take some time to explain how the different objects affect our bounds in the proof of Theorem 4.3.9. We concentrate on the magnetic field B, interaction potential v, and quantum distribution function f.

The main contributions for the magnetic field is in the operator norms of the repeated commutators in Corollary 4.3.5 (ii). Here the coefficients of B give polynomial growth in the estimates. This is also the case for v, where factors of $\|\langle \cdot \rangle^n v\|_{L^1(\mathbb{R}^d)}$, $n \in \mathbb{N}_0$, are introduced. Lastly, f contributes through the semi-norms $\sup_{[-1,\infty)} \langle \cdot \rangle^n |\partial^j f|$ with $n, j \in \mathbb{N}_0$.

All three affect the spectrum of $\mathcal{H}_{B,v,\lambda,F}$ and hence the distances in Proposition 4.3.6. These can however be ignored and instead one can only consider $z \in \mathbb{C} \setminus \mathbb{R}$ throughout and replace the distances with $|\operatorname{Im}(z)|$.

Remark! 4.3.11. One might wonder if the results proven here hold for the magnetic Schrödinger operator with added outer potential, i.e. considering $\mathcal{H}_B + V$ for some measurable function $V\colon \mathbb{R}^d \to \mathbb{R}$ instead of merely the free operator $\mathcal{H}_B$.

The problem occurs in the locality introduced: $\mathcal{H}_B$ acts homogeneously throughout space in the sense of Lemma 4.2.3, and for interesting V, this property is lost with $\mathcal{H}_B + V$. The mean-field potential $\mathcal{W}_\lambda$ would also change, having to take into account V.

It would be interesting to study the problem for rather regular V. Note, if $V \in BC^\infty(\mathbb{R}^d)$, then $V \in \mathfrak{Op}_B(S_1(\mathbb{R}^{2d}))$, as an operator, and so $\mathcal{H}_B + V \in \mathfrak{Op}_B(S_{M_0^2}(\mathbb{R}^{2d}))$ with its resolvent in $\mathfrak{Op}_B(S_{M_0^{-2}}(\mathbb{R}^{2d}))$. But this will have to wait for another time.

Another extension is working with non-constant magnetic fields. Again some problems with commutating with magnetic translations occur, and so like with adding a non-zero V, we would have to analyze Equation (4.3.1) in another way.

Remark! 4.3.12. The concept of quantum distribution function as used in this text is quite vague, but it is inspired by the Fermi-Dirac distribution:

$$f_{\mathrm{FD}} \colon \mathbb{R} \ni x \mapsto (1 + e^{\beta(x-\mu)})^{-1},$$

where β is the inverse temperature and μ is the chemical potential. This function appear in quantum statistical mechanics and is the relevant "quantum distribution function" for our problem in a physical sense.

Clearly $f_{\mathrm{FD}} \in \mathscr{S}_{+}(\mathbb{R}, \mathbb{R})$ for any choice of parameters. Note the parameters regulate the behavior of f_{FD} w.r.t. decay at $+\infty$.

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