AALBORG UNIVERSITY

STUDENT REPORT

Master's Thesis

Flexural Buckling of General Beam Systems

A Method to Determine K-factors using Energy Considerations

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Preface

This Master's thesis "*Flexural Buckling of General Beam System – A Method to Determine K-factors using Energy Considerations*" is made by Jacob S. Mortensen and Mikael Hansen during the 4th semester of the Master's Degree of Structural and Civil Engineering at Aalborg University Esbjerg. The project was completed within the period of 1st of February and 14th of June 2012.

We want to thank Lars Damkilde for his assistance and supervision during the semester. Furthermore we want to thanks ISC A/S for the exchange of expertise and knowledge and for providing office space and supplies.

There will regularly be referred to appendices, these are placed in continuation of the report. Annexes containing material used to produce the report and appendices are placed on the enclosed CD-ROM.

Appendices:

Appendix 1: Examples of Methodical and Experience Based Approaches
Appendix 2: Verification of Cross-section input in Strain Energy Calculations
Appendix 3: Critic of Iterative System Buckling Approach
Appendix 4: Journal Paper (DRAFT)

Annexes:

Annex 1: Structural Drawings
Annex 2: MATLAB script

Enclosed:

One CD-ROM with original documentation for the project:

- Master's Thesis incl. appendices
- Annexes

Abstract

When designing according to the Eurocode, structures, that are not especially sensitive to non-linear behavior, are comprised by an individual member check. This member check includes a reduction factor to account for imperfections. The reduction factor is primarily dependent on the critical load, and this is why the determination of K-factors for individual member becomes important. Traditionally the individual K-factors are determined by methodical approaches such as the isolated subassembly or story methods. These methods do not take the actual system behavior into account, as the linear buckling analysis does. The system buckling approach, which uses the linear buckling analysis, is preferable when the structural behavior is complex, however the method contains the paradox, that compressive members with an relatively small axial force yields excessively large K-factors. In the present thesis a method to circumvent this paradox is proposed. The method distinguishes between members being prone to buckling and members that are not sensitive to additional axial load. Based on energy considerations the load multiplier found by the linear buckling analysis is weighed for each member. The proposed method is verified by 2D examples, where comparisons to known methods are made. It is found that the proposed method provides reasonable K-factors, and it is believed, that future work on the method could lead to an implementation in software, which provides automated code check. Thereby the burdensome manually definition of critical length for each members is avoided.

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1. Introduction

Almost all structures consist of compression members which mean a risk of instability. In the European standards, a stability check often comprise of a check of the individual members. When doing so, a critical length of the compression member is needed as it defines the elastic critical load. The critical length can be determined in a number of ways, either by desk calculation or computer based methods. Most desk calculation approaches are based on a number of assumptions in order to match an analytical solution, therefor the usability of these methods are limited. It also means that these methods are made for specific cases, which for the most part means regular frames. When violating these assumptions, the error can be hard to estimate.

When evaluating structures that cannot be classified as a regular frame by a number of stories and/or bays, the critical length are either determined by experience and standard values or by computer based calculations such as a system buckling approach. The system buckling approach is not limited to specific cases and therefor offers a wide usability. This method does however contain a paradox which means that compression members with a small axial force yields excessively long critical lengths. Therefore a method is needed that overcomes this paradox [1][14].

1.1 Motivation

The stability of structures is becoming harder to evaluate as the complexity of structures increases. The common practice today often relies on methodical approaches and experience to interpret results, where wrongly estimated critical length can lead to disastrous results. In the following pictures, the collapsed practice facility at Dallas Cowboys Stadium is seen. The failure was progressed from buckling in the inner chord of the frame.

Figure 1-1 Failure in slender chord at Cowboys Stadium, Dallas, USA

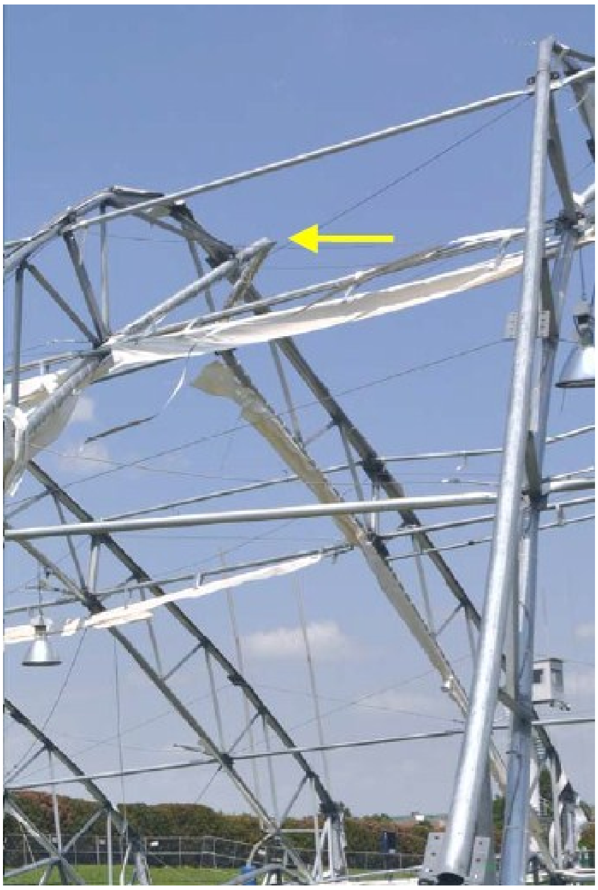


Figure 1-2 Failure in slender chord at Cowboys Stadium, Dallas, USA

In the design aspect instability is a non-linear phenomenon, which in most cases can be handled with linear analysis according to standards as the Eurocode (see section 1.2). The structural integrity is secured by reducing the sectional axial capacity by a factor, which account for the non-linear behavior. This factor have been determined by various experiment of columns with various slenderness and adapted in standards such as the Eurocode.

Figure 1-3 Experiments to determine non-linear behavior of columns

In new designs the critical lengths are usually estimated conservatively (at least when the engineer does not overlook a sway system behavior), and the need for accurate estimates of critical lengths are not that important. However, when performing re-analysis of structures due to changed requirements of the structure, the accuracy of the calculation may become crucial.

In Europe the common practice is to design for stability using the Eurocode. Current software that offers a code-check based on the Eurocode requires the critical length to be set manually. A programmable and reliable method of determining the K-factor, which can be added to the code-check programs, would ease the stability calculations.

1.2 Stability Considerations in Design

The Eurocode defines how the stability analysis should be performed based on the current load level. The load multiplier is found using a linear buckling analysis and it determines how the stability check should be performed.

If the load multiplier is above 10 and 2nd order effects can be neglected, then the stability can be calculated by checking the individual elements. This requires the determination of the critical load or critical length which are interdependent (In the engineering terminology the critical length is normally defined relative to its physical length by the K-factor).

If the load multiplier is below 10 and 2nd order effects should be taking into account, then the method used to check for stability can be chosen according to the sensitivity to 2nd order effects.

The decision procedure is summarized in the flow chart below [6].

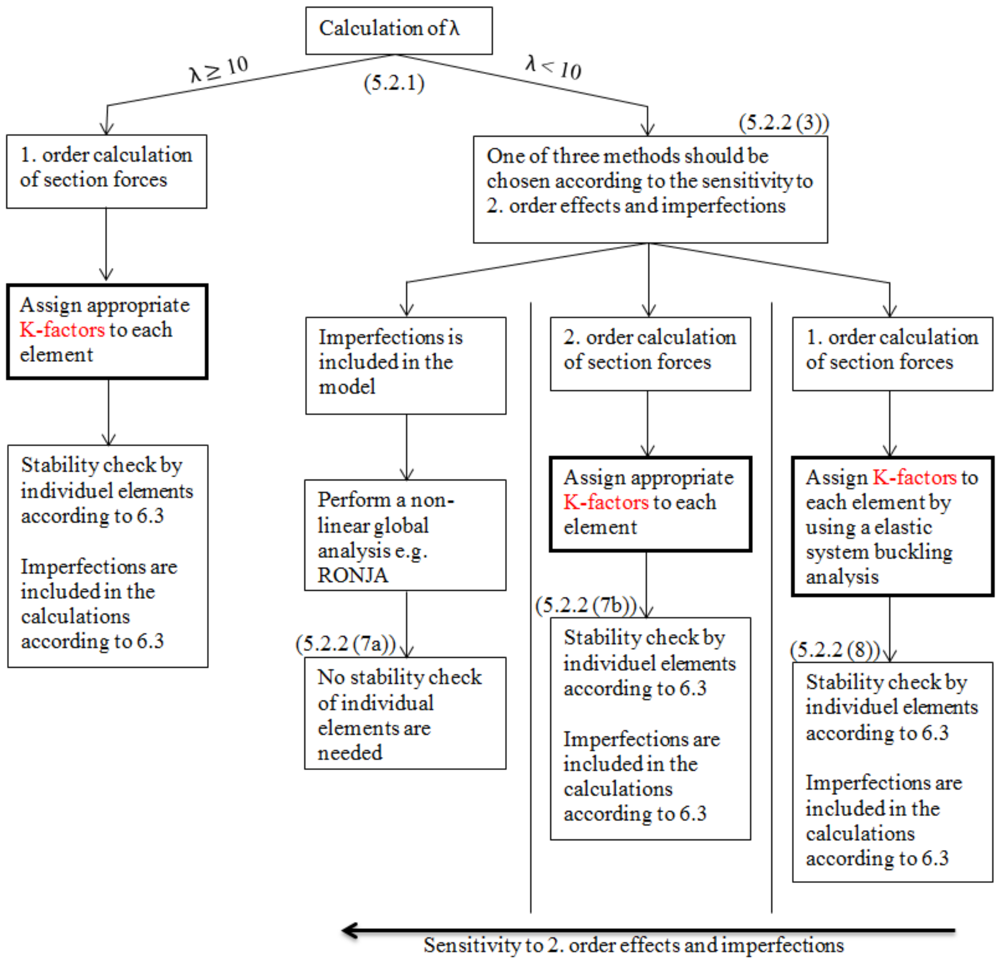


Figure 1-4 Flowchart of the design procedure according to Eurocode

Estimating an appropriate K-factor for individual elements in a system composed of several elements becomes important in three out of four methods given in the flow chart (marked with red).

1.3 Current Approaches for Determining the K-Factor

Besides guessing the K-factors based on experience, there are three main different approaches to calculate the K-factor: The isolated subassembly approach, the story method approach and the system buckling approach.

The isolated subassembly approach is methodical approach which is widely used to determine the K-factor of a single column. This is done by assessing the ratio of stiffness between the column and the adjacent columns and beams. This approach has limiting assumptions. Gantes and Mageirou (2005) has proposed an improved method for calculation the K-factor for a single column in a multi-story sway frame which account for all rotation and translation boundary conditions at the beam ends [1][5][12].

The story method approach is a methodical approach, which is used for regular steel frames and assumes that the shear force can be transferred between columns in a story. This means that the weaker columns are supported against sidesway by the stronger ones. The sidesway buckling resistance is assumed to be equal to the total sidesway resistance of all the columns in that story, which yields a unified K-factor for all members in a story. Two versions of the story method are used and they differ in the way the total resistance is calculated. One method, the story buckling method, uses the isolated subassembly method to determine the critical load for an entire story. The other method, the story stiffness method, uses the first order lateral displacements to include interaction between stories [1].

The system buckling approach uses a linear buckling analysis to determine a load multiplier for the system and thereby a critical load for the entire system. The critical load of each member is calculated by multiplying the axial force in the member with the system load multiplier. When the critical load is calculated a K-factor can be derived. This method's strong point is that it accounts for the entire system behavior, but it also presents the paradox that a small axial compression forces will yield small critical load, and thereby a large K-factor [1].

This paradox is mentioned in other papers and an iterative method has been proposed in order to overcome this problem. The main idea behind the method is that an increase in axial force in some members only has a small effect on the load multiplier. By adding fictitious axial force a higher critical load and thereby a smaller K-factor can be obtained. An iterative procedure has been proposed, where the axial force is increased while calculating the change in K-factor after each step. This is done until the change in K-factor is sufficiently small in all members [1][7][9].

1.4 Project Objective

The main objective of this project is to propose a new method that overcomes the weakness of the system buckling approach when calculating the critical load or critical length for individual members in a system. The method should be programmable in traditional FE-code, and thereby easily implemented into existing software with code-check capabilities. The goal is a method that does not require any user input, besides what is usually given.

In the first part of the project (Chapter 2), the concept of stability of a single member is outlined. Both perfect and imperfect columns are considered, and the design criteria according to the Eurocode are defined.

In the middle part of the project (Chapter 3 and 4), the theory behind the most commonly used methods is presented. Both methodical and numerical method applies for a system of members, however each method has its limitations or disadvantages.

The last part of the project (Chapter 5 and 6) contains the proposition of a new method, referred to as the Energy Ratio Method, for calculating the K-factors. Demonstrative examples showing results of the Energy Ratio Method on simple 2D cases are used to verify the method. Furthermore the method is compared with the other known methods described in Section 1.3.

Scope of this project

- The Eurocode EN1993 (EC3) is used in many countries today and this project will only focus on design according to this code.
- Only steel structures will be considered
- All joints between members are assumed to be rigid.
- Cross sections are assumed to be uniform.
- Only elastic buckling is considered, i.e. no plastic hinges.
- The proposed method is only verified using 2D systems.

2. Stability of Columns – Calculation of Perfect and Real Columns

In this chapter, the stability of a column as a single member using an analytical approach is described.

Firstly, the concept of a columns critical length (also called effective length, free length, reduced length or buckling length) is outlined by considering a perfect column, i.e. no imperfection is considered.

Secondly, the fact that perfect columns do not occur in real structures, means that imperfections have to be taken into account. The design of structures is governed by design codes, and the approach for including imperfections according to design code EN 1993 is explained.

2.1 Derivation of the K-factor using the Differential Equation for a Beam Element

Using the differential equation for bending in a beam-column it is possible to derive the critical axial load for an ideal column, as done by Timoshenko [14].

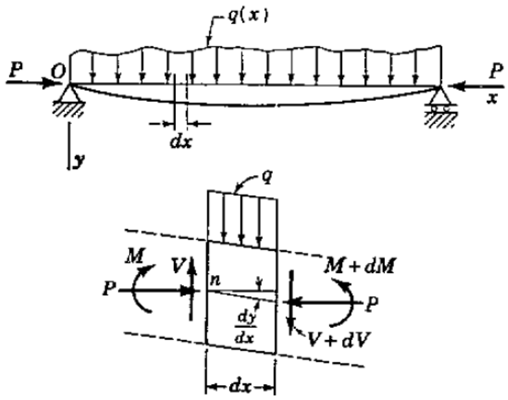


Figure 2-1 Basis for differential equation for a beam column

Based on the beam in Figure 2-1 the basic differential equation for a lateral loaded beam column can be written as:

$$EI \cdot \frac{d^4y}{dx^4} + P \cdot \frac{d^2y}{dx^2} = q \qquad (1)$$

As no lateral loads are present when determining the critical load, and by substituting $k^2 = P/EI$ the equation can be written as:

$$\frac{d^4y}{dx^4} + k^2 \cdot \frac{d^2y}{dx^2} = 0 \qquad (2)$$

The general solution for this equation is:

$$y = A \cdot \sin kx + B \cdot \cos kx + C \cdot x + D \qquad (3)$$

The four integration constants are determined by the geometrical and static end restraints of the beam. These restraints are summarized below:

Fixed end: $y = 0$ and $y' = 0$

Hinge: $y = 0$ and $y'' = 0$

Free end: $M = -EI \cdot y'' = 0$ and $V = (EI \cdot y'')' + P \cdot y' = 0$

For a column with hinged ends the moments and deflection at the ends of the column are zero meaning that:

$$y = 0 \text{ and } y'' = 0 \qquad \text{at } x = 0 \text{ and } x = l$$

When applying these conditions to the general solution (3) the following is obtained:

$$B = C = D = 0$$

As $A \cdot \sin kx$ also must equal zero and $A = 0$ would give a trivial solution, meaning no deflection, it is seen that $\sin kl = 0$. Therefore:

$$kl = n \cdot \pi$$

where $n = 1$ gives the lowest critical load.

By inserting $k^2 = P/EI$ and $n = 1$ the critical load is found:

$$\sqrt{\frac{P}{EI}} \cdot l = 1 \cdot \pi \Rightarrow P_{cr} = \frac{\pi^2 \cdot EI}{(1.0 \cdot l)^2}$$

In this case the critical column length is 1.0 times the actual column length. This is also called the K-factor.

If we look at a column with one end fixed and the other end hinged, also called a propped-end column, then the restraints become:

$$y = 0 \text{ and } y'' = 0 \; at \; x = 0$$

$$y = 0 \text{ and } y' = 0 \; at \; x = l$$

By inserting these restraints into the general solution the four restraints becomes:

$$B + D = 0$$

$$-B \cdot k^2 = 0$$

$$A \cdot \sin kl + B \cdot \cos kl + C \cdot l + D = 0$$

$$A \cdot k \cdot \cos kl - B \cdot k \cdot \sin kl + C = 0$$

By combining these four equations it can be shown that

$$\tan kl = kl$$

The smallest root to this equation is $kl = 4.4934$ which gives the follow critical load:

$$\sqrt{\frac{P}{EI}} \cdot l = 4.4934 \Rightarrow P_{cr} = \frac{\pi^2 \cdot EI}{(0.699 \cdot l)^2}$$

Hence a propped-end column has a critical length of 0.699 times the actual column length.

Based on the preceding examples it is clear that the critical load generally can be written in the form also known as the Euler load:

$$P_{cr} = \frac{\pi^2 \cdot EI}{{l_s}^2} \tag{4}$$

where $l_s = K \cdot l$ is the critical column length. Hereby the K-factor for a column is defined, and it is noted that the critical load and the K-factor are interdependent. If the critical load is known the K-factor can be determined as:

$$K = \sqrt{\frac{\pi^2 \cdot EI}{P_{cr} \cdot l^2}} \tag{5}$$

The covered examples and other basic cases can be seen on Figure 2-2.

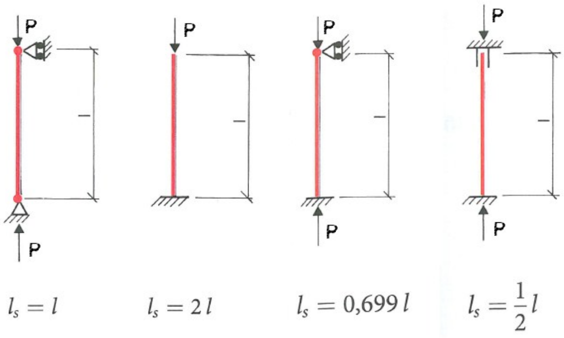


Figure 2-2 Critical lengths for basic cases

This covers the well-defined end restraints. In the case of flexible end restraints which give an infinity number of solutions, no general solution can be obtained before the flexibility is defined. An example is a column whose ends cannot not move but are restrained against rotation by springs such as seen on Figure 2-3a.

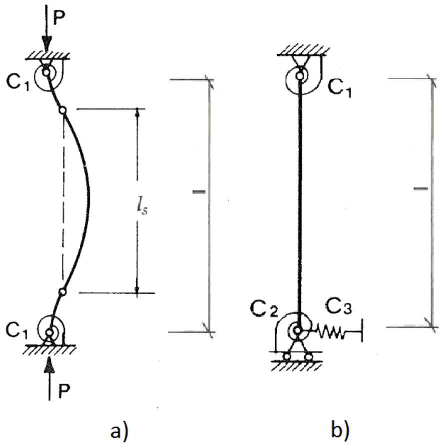


Figure 2-3 Flexible end restraints

Infinity large spring stiffness at both ends will give the same result as a column with fixed ends, meaning a critical length of $0.5 \cdot l$. The other limit state is an infinity small spring stiffness at both ends which corresponds to hinges at both ends and a K-factor of 1. Hence the K-factor for such end restraints is in the interval $0.5 \leq K \leq 1.0$.

The most general case, illustrated in Figure 2-3b, is obtained by applying a rotation spring (C1) and no movement at one end, and a rotation spring (C2) plus a spring against movement (C3) at the other end. Such a column has a K-factor in the interval $0.5 \geq K \geq \infty$. The infinity occurs when bottom springs (C2 and C3) are equal to zero, meaning that the column acts as a cantilever. This combined with a non-existing resistance against rotation in the last spring (C1) means that the column becomes a mechanism [2].

The critical length or K-factor is a concept derived from perfect columns. Nevertheless the K-factor is often used as a key parameter in calculations of the strength of compressive elements in real structures.

2.2 Calculation of Columns According to EN 1993 – Including Imperfections

In real structures, a column will fail before the theoretical critical load is reached. This is due to various imperfections such as:

- Geometrical imperfections: column out-of-straightness, profile tolerances etc.

- Material imperfections: residual stresses from production or non-linear material behavior.

The geometrical imperfections causes amplification to the transverse deflection of the column which is most pronounced for slender columns.

Non-linear material behavior in steel is experienced when the stress level is larger than approximately 80 % of the yield stress, also known as the proportional limit. After this point the modulus of elasticity is reduced and the transverse deflection increases more rapidly. This effect is not experienced in slender columns were the critical load is reached at a lower stress level.

Residual stresses become important when the sum of the residual stress and applied stress exceeds the proportional limit. This can be seen in a welded I-section that buckles about the minor-axis because there are residual compressive stresses at the free ends of the flanges.

In EN 1993 imperfections are accounted for by introducing a generalized imperfection factor that covers both geometrical and material imperfections. Ayrton-Perrys formula is an analytical way of including imperfections and forms the basis for the method used in EN 1993, which is explained in the next sections.

2.2.1 Column with an imperfection – Ayrton-Perrys formula

Ayrton-Perrys formula is the basis for the column strength given in EN 1993 [3]. We consider a simply supported beam with an imperfection, which is assumed to be a sinus shaped initial deflection u_0. The model of the imperfect column is seen in Figure 2-4.

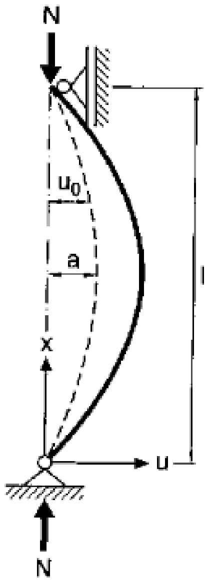


Figure 2-4 Imperfect column

If the maximum deflection is a, then u_0 can be written as:

$$u_0 = a \sin\left(\frac{\pi x}{l}\right) \tag{6}$$

The beam is subjected to an axial force N, therefor the bending moment in the beam is:

$$M = Nu = Nu_0 + N(u - u_0) \tag{7}$$

where u is the total deflection.

By inserting in the differential equation for the beam deflection we get:

$$Nu = -EI\frac{d^2(u-u_0)}{dx^2} \tag{8}$$

where E is the modulus of elasticity and I is the second moment of inertia.

The differential equation can be solved using the particular solution $u = C\sin\left(\frac{\pi x}{l}\right)$ and the result is:

$$u = \frac{N_{cr}\,u_0}{N_{cr} - N} \tag{9}$$

The limit is found when the stress at the edge reaches the yield stress f_y. Using Naviers formula we get:

$$\frac{N}{A} + \frac{Nu}{W} = f_y \tag{10}$$

where A and W is the section area and section modulus respectively. The maximal normal stress is given by:

$$\sigma_b = \frac{N}{A} \tag{11}$$

and combining formula (9), (10) and (11) we get:

$$\sigma_b + \sigma_b\frac{A}{W}\frac{N_{cr}}{N_{cr} - N}u_0 = f_y \tag{12}$$

The critical stress (Euler stress) is defined as:

$$\sigma_{cr} = \frac{N_{cr}}{A} \tag{13}$$

and formula (12) can be written as the Ayrton-Perry formula:

$$(\sigma_{cr} - \sigma_b)(f_y - \sigma_b) = \sigma_b\sigma_{cr}\eta \tag{14}$$

where $\eta = \frac{u_0 A}{W}$ represent the column straightness.

The link between the Ayrton-Perry formula and the column design formula in EN 1993 follows in the next section.

2.2.2 Column formula in EN 1993

In the Ayrton-Perry formula the maximum normal stress σ_b is given implicitly. For convenience the formula is rewritten introducing the column strength reduction factor:

$$\chi = \frac{\sigma_b}{f_y} \tag{15}$$

and the non-dimensional slenderness:

$$\bar{\lambda} = \sqrt{\frac{f_y}{\sigma_{cr}}} \tag{16}$$

By dividing formula (14) with $\frac{\sigma_{cr}}{f_y}$ we get:

$$(1-\chi)(1-\bar{\lambda}^2\chi) = \eta\chi \tag{17}$$

or

$$\bar{\lambda}^2\chi^2 - \chi(\bar{\lambda}^2 + \eta + 1) + 1 = 0 \tag{18}$$

The smallest value of the roots in this quadratic equation is the relevant value of χ given by:

$$\chi = \frac{1+\eta+\bar{\lambda}^2 - \left((1+\eta+\bar{\lambda}^2)^2 - 4\bar{\lambda}^2\right)^{1/2}}{2\bar{\lambda}^2} \tag{19}$$

Letting the column straightness η cover both geometrical and material imperfection by the expression:

$$\eta = \alpha(\bar{\lambda} - \bar{\lambda}_0) \tag{20}$$

and introducing:

$$\phi = \frac{1}{2}\left(1 + \alpha(\bar{\lambda} - \bar{\lambda}_0) + \bar{\lambda}^2\right) \tag{21}$$

formula (19) can be written as

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}} \tag{22}$$

The last two equations are identical to the ones used in EN 1993-1-1 when calculating columns, where $\bar{\lambda}_0 = 0.2$ and the imperfection factor α depends on the cross section type, dimensions and yield stress. The imperfection factor and dependence of cross section can be seen in Figure 2-6 and Figure 2-7. The reduction factor χ can be seen in Figure 2-8. The figures are taken from EN 1991.

The reduction factor χ is used in the utilization ratio given by:

$$UR_{axial} = \frac{N_E}{\chi \cdot A \cdot f_y} \tag{23}$$

where A is the cross sectional area, f_y is the yield stress and N_E is the exposed load.

When members are imposed to both axial and bending stress, the axial utilization ratio is included as part of the total utilization ratio. The utilization ratio in equation (23) is thereby the first of three terms in the design equations for members subjected to both bending and compression which is seen in Figure 2-5. The two equations express bending about y- and z-axis respectively.

$$\frac{N_{Ed}}{\frac{\chi_y N_{Rk}}{\gamma_{M1}}} + k_{yy}\frac{M_{y,Ed} + \Delta M_{y,Ed}}{\chi_{LT}\frac{M_{y,Rk}}{\gamma_{M1}}} + k_{yz}\frac{M_{z,Ed} + \Delta M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}} \le 1 \qquad (6.61)$$

$$\frac{N_{Ed}}{\frac{\chi_z N_{Rk}}{\gamma_{M1}}} + k_{zy}\frac{M_{y,Ed} + \Delta M_{y,Ed}}{\chi_{LT}\frac{M_{y,Rk}}{\gamma_{M1}}} + k_{zz}\frac{M_{z,Ed} + \Delta M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}} \le 1 \qquad (6.62)$$

Figure 2-5 Design equations for members subjected to axial load and bending

Table 6.2: Selection of buckling curve for a cross-section

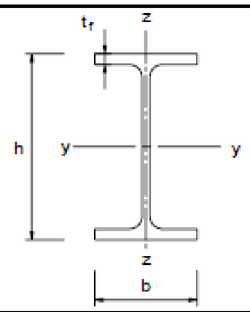
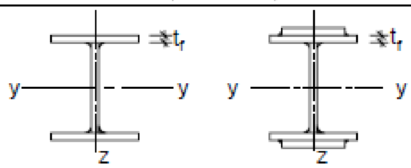
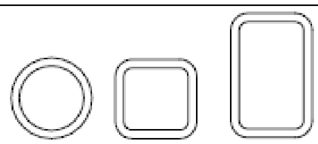
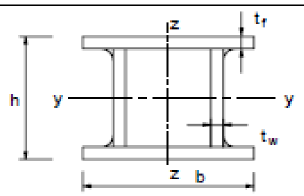
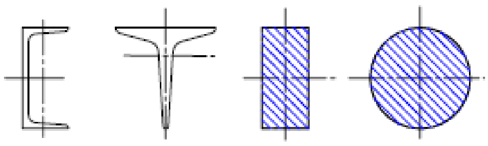
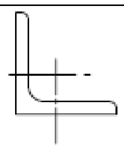
	Cross section	Limits		Buckling about axis	Buckling curve S 235, S 275, S 355, S 420	Buckling curve S 460
Rolled sections		$h/b > 1{,}2$	$t_f \le 40$ mm	y – y z – z	a b	a_0 a_0
			40 mm < $t_f \le 100$	y – y z – z	b c	a a
		$h/b \le 1{,}2$	$t_f \le 100$ mm	y – y z – z	b c	a a
			$t_f > 100$ mm	y – y z – z	d d	c c
Welded I-sections		$t_f \le 40$ mm		y – y z – z	b c	b c
		$t_f > 40$ mm		y – y z – z	c d	c d
Hollow sections		hot finished		any	a	a_0
		cold formed		any	c	c
Welded box sections		generally (except as below)		any	b	b
		thick welds: $a > 0{,}5t_f$ $b/t_f < 30$ $h/t_w < 30$		any	c	c
U-, T- and solid sections				any	c	c
L-sections				any	b	b

Figure 2-6 Buckling curve dependence of cross section

Table 6.1: Imperfection factors for buckling curves

Buckling curve	a_0	a	b	c	d
Imperfection factor α	0,13	0,21	0,34	0,49	0,76

Figure 2-7 Imperfection factor dependence of buckling curve

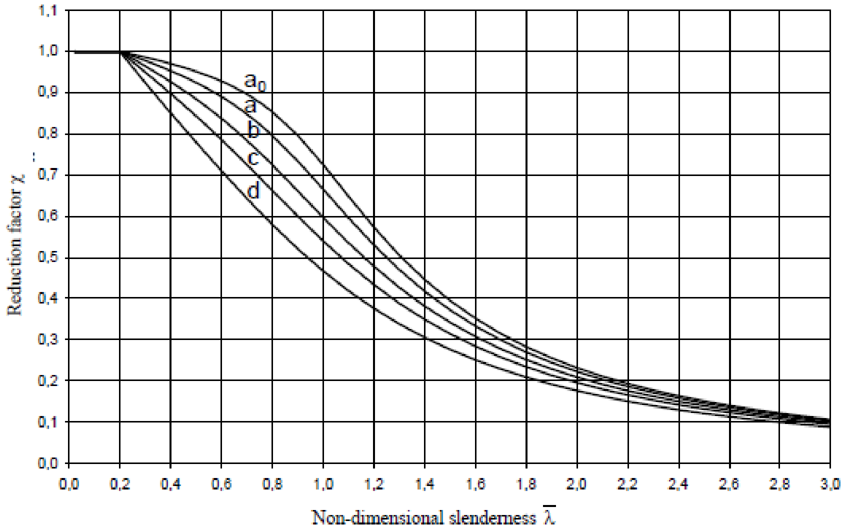


Figure 6.4: Buckling curves

Figure 2-8 Buckling curves

In Figure 2-8 the reduction factor χ is depends on the non-dimensional slenderness $\bar{\lambda}$, in which the K-factor is included by:

$$\bar{\lambda} = \sqrt{\frac{f_y}{\sigma_{cr}}} = \sqrt{\frac{A \cdot f_y \cdot (K \cdot L)^2}{\pi^2 \cdot EI}} \tag{24}$$

2.3 Summary

To summarize we now know that the reduction factor is a function the critical load, so even if the critical value is not reached in real structures, it is still important in the calculations of the individual member design. Hence, the more precise you can determine the critical length of a column, the more economical your design can be.

If it is possible to establish the critical load of a column, back calculations can be used to determine the K-factor by:

$$P_{cr} = \frac{\pi^2 \cdot EI}{(K \cdot l)^2} \quad => \quad K = \sqrt{\frac{\pi^2 \cdot EI}{P_{cr} \cdot l^2}} \tag{25}$$

The K-factor is determinative for the reduction factor χ and thereby an important parameter when calculating the utilization ratio of a compression member by:

$$UR_{axial} = \frac{N_E}{\chi \cdot A \cdot f_y} \tag{26}$$

3. Methodical Approach to Stability of Beam-column Systems

It is seldom seen that a column acts as an independent column. Often columns are a part of a larger system of beams which means that the end restraints are flexible. Therefore columns in such systems are not well-defined as the basic cases, however the basic cases may give the engineer an idea of in which interval the K-factor is expected to be. The connections between the beams are often assumed to be rigid, which for the most part is an adequate assumption, wherefore joint flexibility is not included in the scope of this thesis.

A widely used method for calculating K-factors is the isolated subassembly approach (also known as the nomograph/alignment chart method), which is presented in pr-EN 1993 amongst many others. The method requires a categorization of the system into either sway or non-sway, which is illustrated in Section 3.1. The method is an analytical approach which gives a correct critical length if several assumptions are fulfilled as explained in Section 3.2.

Due to the fact that these assumptions seldom are fulfilled, other methods have been developed for typical structures such as rectangular frames or story frames. These methods are known as story methods and stability calculations concerning both single frames and multi-bay/multi-story frames are well documented in acknowledged literature [1]. In asymmetric cases, the story buckling method tends to give better estimates of the K-factors than the isolated subassembly approach; however in some cases there is significant buckling interaction between stories, which means that a numerical approach such as the system buckling approach (see Chapter 4) would be more appropriate. A discussion of these different approached can be found in (ASCE, 1997) [1]. The two most common story methods are The Story Buckling Method and The Story Stiffness Method which is outlined in Section 3.3.

3.1 Categorization into sway and non-sway systems

Systems are divided into two different categories: Sway or Non-sway. A sway frame is allowed to sway to either side, whereas a non-sway frame is restricted from doing so. This is illustrated on the following figures.

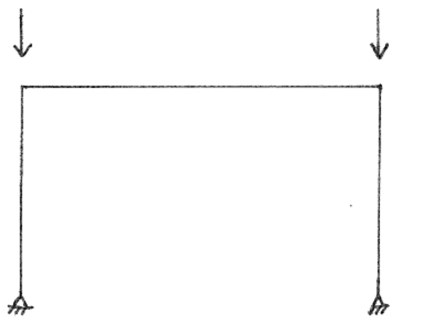


Figure 3-1 Boundary conditions for a sway frame

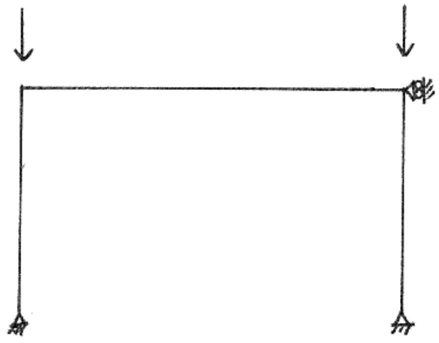


Figure 3-2 Boundary conditions for a non-sway frame

If a linear buckling analysis of the frames is performed, the distorted configurations, also called mode shapes, are established.

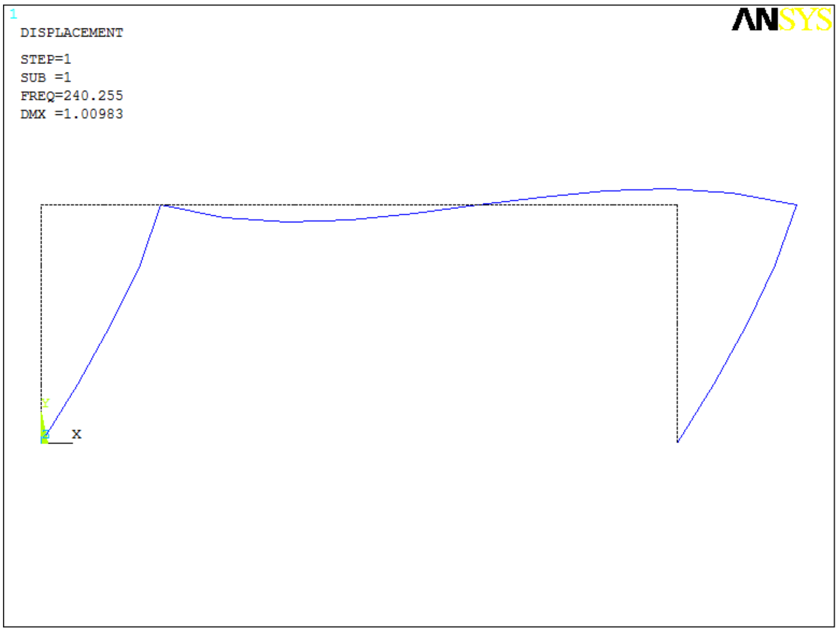


Figure 3-3 First mode shape for sway frame

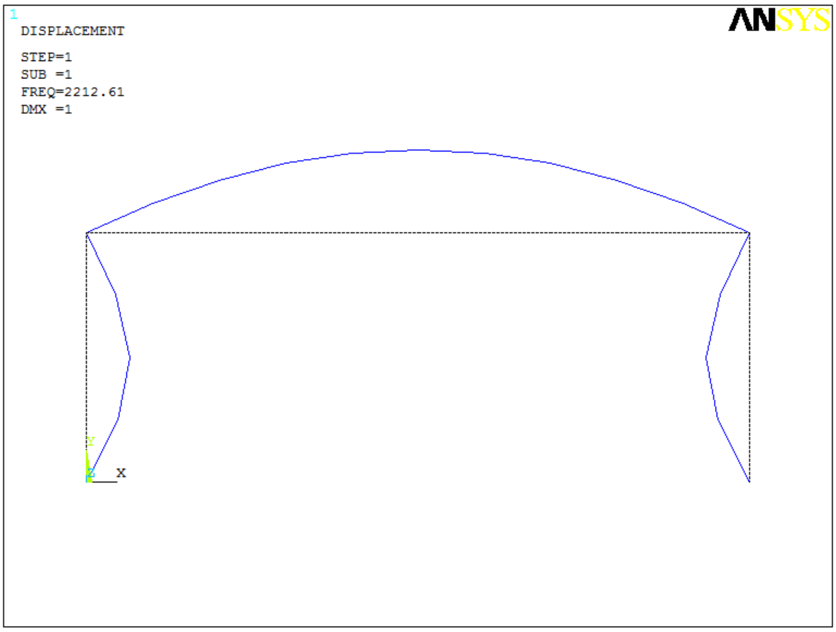


Figure 3-4 First modeshape for non-sway frame

By comparing the load multiplier (FREQ) in the distorted configurations on Figure 3-3 and Figure 3-4, it is seen that the general behavior of the system has a large effect on the critical length of the individual columns. As the load case for the two frames are exactly the same it is possible to compare the load multiplier directly in order to determining the buckling resistance of the two systems. The sway frame has a load multiplier of 240 and the non-sway frame a load multiplier of 2212. This means that the resistance against buckling for the non-sway frame is about 9 times larger in the given case.

If a standard critical length with the value of the actual member length is assigned for every member, the risk of overlooking a general system behavior as the one seen on Figure 3-3 arises. Overlooking this means that an individual member check will show a higher buckling resistance than the system as a whole.

3.2 Stability of Frame Systems Based on the Isolated Subassembly Approach

This section contains a description of stability of frames based on annex E of the pr-EN1993 (pre-standard to EC3). The annex provides a mean of establishing the critical length of a column based on the stiffness of the adjacent beams and the general behavior of the system (sway or non-sway).

This method is widely used and adopted by both the European and American standards, even though they are based on different but similar work by Wood (1974) and Julian and Lawrence (1959) respectively [13][15]. When used on a reference frame with pinned ends base they provide similar results.

The method described in the European standards offer a higher degree of customization than the American to overcome violation of some of the assumptions.

The method is based on the following assumptions [1][4][5].

1. All members have constant cross section.
2. All joints are rigid.
3. Behavior is purely elastic.
4. No significant axial compression force exists in the girders.
5. The column stiffness parameter $l\sqrt{\frac{P}{I}}$ must be identical for all columns.
6. Rotations at the far ends of the restraining members are fixed.
7. Joint restraint is distributed to the column above and below the joint in proportion to I/l of the two columns.
8. The frame is subjected to vertical loads applied only at the joints.
9. All columns in the frame become unstable simultaneously, and no shear force is transferred to the subassembly from other portions of the structure.

By determining stiffness distribution factors for the end joints of the column in question, a K-factor is obtained either by using Figure 3-6 (non-sway) and Figure 3-7 (sway), or by using the conservatively fitted formulas (29) and (30) based on these figures.

By using the following formulas, the distribution factors η_1 and η_2 are obtained:

$$\eta_1 = \frac{K_C + K_1}{K_C + K_1 + K_{11} + K_{12}} \tag{27}$$

$$\eta_2 = \frac{K_C + K_2}{K_C + K_2 + K_{21} + K_{22}} \tag{28}$$

where

K_C is the column stiffness coefficient I/l

K_i are the stiffness coefficients for the adjacent columns

K_{ij} are the effective beam stiffness coefficients

The stiffness coefficients K_C , K_i , K_{ij} can be seen on Figure 3-5.

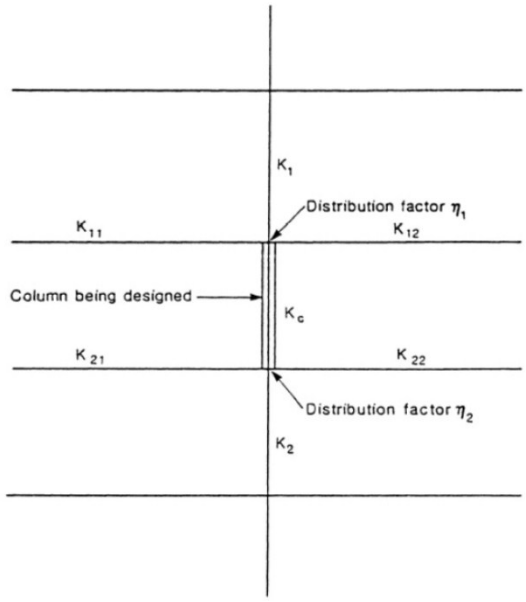


Figure 3-5 Stiffness designations for surrounding columns

As the method is based on fixed ends of the adjacent beams, the effective beam stiffness K_{ij} can be used in order to simulate other boundary conditions.

The effective beam stiffness for beams without axial force can be taken from Table 3-1.

Conditions of rotational restraint at far end of beam	$\boldsymbol{K_{ij}}$ provided that the beam remains elastic
Fixed at far end	1,0 I / l
Pinned at far end	0,75 I / l
Rotation as at near end (double curvature)	1,5 I / l
Rotation equal and opposite to that at near end (single curvature)	0,5 I / l
General case. Rotation $\boldsymbol{\theta_a}$ at near end $\boldsymbol{\theta_b}$ at far end	$(1 + 0{,}5 \cdot \theta_b / \theta_a)\ I\ /\ l$

Table 3-1 Effective beam stiffness coefficient for a beam without axial force

When the distribution factors are known the critical column length can be found either by reading Figure 3-6 (non-sway) and Figure 3-7 (sway) or by using the following formulas.

Non-sway mode: (29)

$$\frac{l_s}{l} = K = \left[\frac{1 + 0{,}145\ (\eta_1 + \eta_2) - 0{,}265\ \eta_1\ \eta_2}{2 - 0{,}364\ (\eta_1 + \eta_2) - 0{,}247\ \eta_1\ \eta_2}\right]$$

Sway mode: (30)

$$\frac{l_s}{l} = K = \left[\frac{1 - 0{,}2\ (\eta_1 + \eta_2) - 0{,}12\ \eta_1\ \eta_2}{1 - 0{,}8\ (\eta_1 + \eta_2) + 0{,}6\ \eta_1\ \eta_2}\right]^{0{,}5}$$

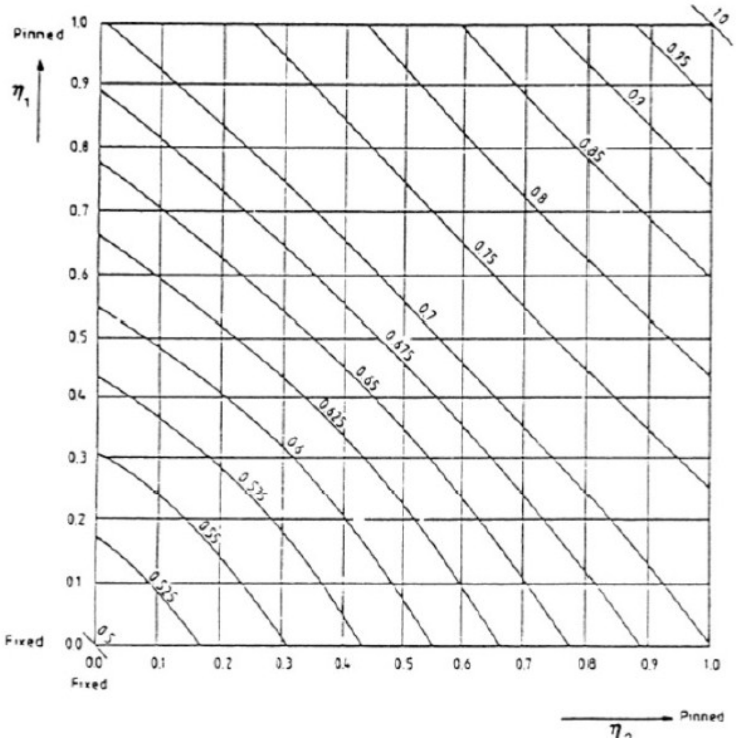


Figure 3-6 Critical length ratio l/L for a column in a non- sway mode

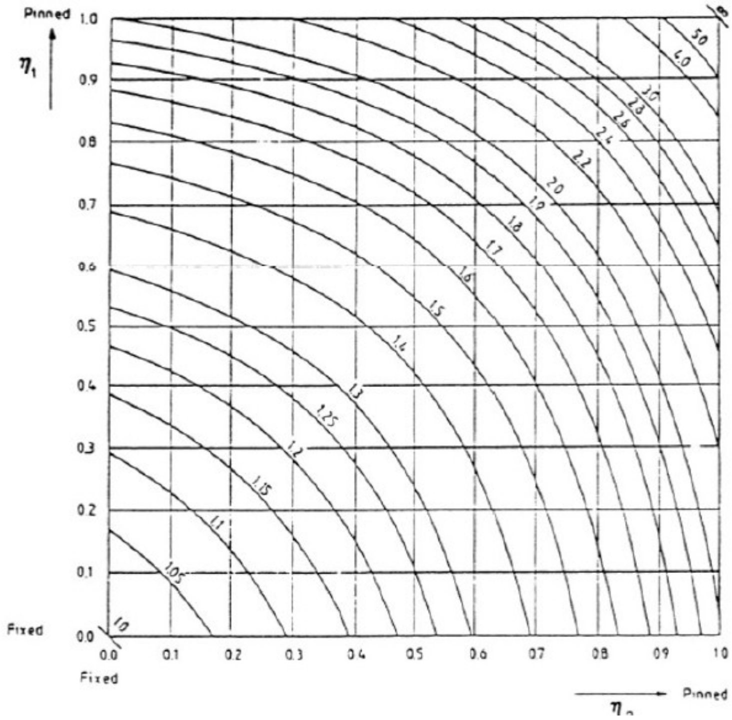


Figure 3-7 Critical length ratio l/L for a column in sway mode

By looking at Figure 3-6 and Figure 3-7 it becomes apparent that the most conservative values is obtained by using high values of the distribution factors. A low value of the effective beam stiffness K_{ij} is equal to a single curvature deflection of the adjacent beam.

The method is based on a large number of assumptions, which means that the validity becomes limited on real life structures. Especially assumption number 5 which requires the same stiffness parameter for each column is often not fulfilled in real life situations. This means that the columns and beams do not become unstable simultaneously. As a result hereof, the isolated subassembly method will in some cases give conservatively large K-factors, which had led to the development of the story methods.

3.3 Stability of Frame Systems Based on the Story Method Approach

When designing regular frames, the story method approach is widely used as it offers a reasonably amount of accuracy with relatively simple calculations, that for the most case can be done by hand.

The story buckling approach differs from the isolated subassembly approach in the fact that it can account for the transfer of shear forces between columns. This means that if a column in any given story is stronger than some of the other columns, then this column will provide the weaker column with some of its strength. So in cases where there is significant buckling interaction between columns, the story buckling approach is the better choice when compared to the isolated subassembly approach [1].

The following is based on frames with equal column lengths, however methods that account for different column lengths does exist. It is assumed that the stronger columns in a story will brace the weaker columns until an overall story buckling load is reached. Furthermore it is assumed, that at the story buckling load, the story as a whole buckles in a sidesway mode.

All story based procedures can be formulated bases on the following equation:

$$\lambda_{story} \cdot \sum_{all} P_u = \sum_{non-leaner} P_{cr(story)} \tag{31}$$

where λ_{story} is the load multiplier that the axial loads P_u in the respective story must be scaled by to achieve story sidesway buckling, and $P_{cr(story)}$ represents the contribution from each column to the story sidesway buckling resistance. Leaning columns are pinned-pinned, and does not contribute to the sideway stiffness of the story. As this project is limited to rigid joints no leaning column will be present.

From this equation the K-factor for a member in the story can be derived:

$$K_{story} = \sqrt{\frac{1}{P_u} \cdot \frac{\pi^2 EI}{L^2} \cdot \frac{\sum_{all} P_u}{\sum_{non-leaner} P_{cr(story)}}} \tag{32}$$

The different story methods differs primarily in the way $P_{cr(story)}$ is calculated.

The Story Buckling Method, also known as just story buckling, assumes that the buckling capacity of the story is equal to the sum the column buckling loads computed using a K-factor based on isolated assembly method. This however has the same limitation and must fulfill the same assumptions as stated in Section 3.2.

Another method is known as the Story Stiffness Method or Practical Story Based Effective Length Factor. This method is more ideal when there is significant violation of the assumptions on which the isolated subassembly approach is based.

Here the K-factor is derived as:

$$K_{story} = \sqrt{\frac{1}{P_u} \cdot \frac{\pi^2 EI}{L^2} \cdot \frac{\Delta_{oh} \cdot \sum_{all} P_u}{(0.85 + 0.15 \cdot R_L) \cdot \sum_{non-leaner} HL}} \tag{33}$$

Where H is the lateral displacement forces, L is the length of the column and Δ_{oh} is the 1. order lateral displacement of the story, and R_L is defined by:

$$R_L = \frac{\sum_{Leaner} P_u}{\sum_{all} P_u} \tag{34}$$

which means, that $R_L = 0$ in the case where no leaning columns are present.

3.4 Summary

Stability of frames is well documented for a handful of reference cases. When doing desk calculations and using predetermined formulas given in the isolated subassembly approach, one of the most important tasks for the engineer is to use the correct state of sway. An overlooked state of sway may result in large errors in the K-factor meaning an overestimated buckling resistance. It is possible to determine K-factors for beams in a larger system, but the assumptions behind the isolated subassembly approach limit the usability and require careful attention and understanding of the consequence by overruling those assumptions.

The story method approach is another analytical approach suited for regular frames, which takes the transfer of shear between columns into account. The approach is usually split into two different methods, the story buckling method, which uses the isolated subassembly approach to calculate the critical load, and the story stiffness method which uses a 1. order deflection analysis to calculate the critical load. The advantage of using the story stiffness method is that the assumptions behind the isolated subassembly approach do not apply.

In this chapter the common analytical ways of estimating K-factors has been outlined. In practice, the engineer sometimes choose to guess the K-factors based on experience, and in Appendix 1 two real life examples of complex structures is analyzed. The first example illustrates the advantage and justification of using experience by applying standard values of K-factor to certain types of members. The second example demonstrates the use of the subassembly method and emphasizes the importance of determining the correct state of sway.

4. Numerical Approach to Stability of Beam-column Systems

As explained in Chapter 3, the usability of methodical approaches is limited, and when the complexity of the system increases an alternative approach is needed. An often used method is to perform a linear buckling analysis, where it is possible to obtain a critical load factor based on the current load case. However this method only defines the system stability and not the critical load of the individual members. An accepted way of determining K-factor of individual members is the system buckling approach. In the flowchart on Figure 4-1 the process of determining K-factors using numerical software is outlined.

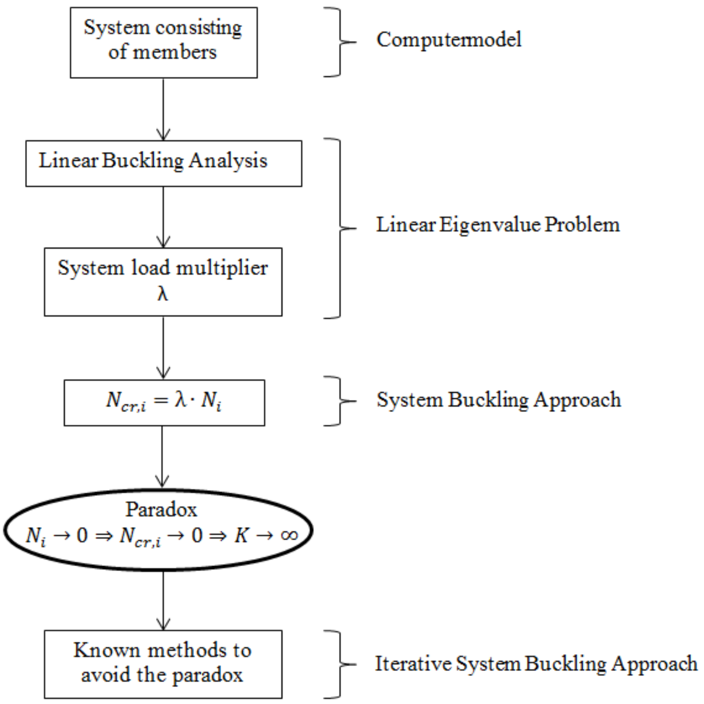


Figure 4-1 Flowchart of numerical procedure

In this chapter the linear eigenvalue problem and the load multiplication factors is explained. The element type and number of member divisions needed is also discussed to set a basis for the further use of FEM in this thesis.

Furthermore, the system buckling approach which uses the linear buckling analysis is described. Here a multiplication factor for the system is obtained and a critical load is found for each element by multiplying the load multiplier with the axial force in the element. The method automatically includes the system behavior, however it means that elements with a small axial force also have a small critical load and thereby a large K-factor. This paradox is discussed further in this chapter.

It is verified, that the large K-factors does not give a correct picture of the members individual sensitivity to an increase in compressive load, and a newly proposed method by Choi and Yoo (2008) to avoid the K-factor paradox is described [7].

4.1 The Linear Eigenvalue Problem – Load Multiplication Factor

In complex structures consisting of several beams and columns, the effective length of each element is an indistinct concept. Instead the stability is assessed by calculating the critical load level for each given load case. The amount of time used to calculate the critical load level by desk calculations (e.g. using the slope deflection method [11]) increases rapidly with the number of elements. Therefore most structures require the use of FEM.

An eigenvalue problem can be defined by applying the principal of virtual work to two situations of the structure with the same load level. The load level is defined as a load multiplication factor λ_{cr} multiplied a known load case, which can consist of one or more applied loads. Situation 1 is where the structures load response is equal to the linear static solution, and Situation 2 is where the structures load response is equal to bifurcation. Now an expression for the difference in the structures displacements in the two situations is formulated, and the value of the load multiplication factor, for which solutions of the additional displacements exist, defines the critical load level. The governing equation can be expressed as:

$$(K + \lambda_{cr} \cdot K_g)S = 0 \tag{35}$$

Formula (35) forms a linear eigenvalue problem, where K is the global elastic stiffness matrix, K_g is the global geometrical stiffness matrix, λ_{cr} is the load multiplier and S is vector containing the displacements.

The local elastic and geometrical stiffness matrices can be expressed as:

$$\boldsymbol{k} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

$$\boldsymbol{k_g} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6P}{5L} & \frac{P}{10} & 0 & -\frac{6P}{5L} & \frac{P}{10} \\ 0 & \frac{P}{10} & \frac{2PL}{15} & 0 & -\frac{P}{10} & -\frac{P}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6P}{5L} & -\frac{P}{10} & 0 & \frac{6P}{5L} & -\frac{P}{10} \\ 0 & \frac{P}{10} & -\frac{P}{30} & 0 & -\frac{P}{10} & \frac{2PL}{15} \end{bmatrix}$$

The results of the eigenvalue problem are both the load multiplication factor λ_{cr} (eigenvalue) and the corresponding displacements or mode shape (eigenvector). The magnitude of the node displacements are correct relative to each other, but will have to be scaled if used as the structures initial geometrical imperfection in a second order analysis. The modeshape corresponding to the lowest value of the load multiplication factor is the relevant modeshape to consider when applying initial geometrical imperfections, and it can be shown that it is the most unfavorable shape of imperfection.

This method is known as the Linear Buckling Analysis, and it is implicitly assumed that the structure have a linear behavior until the critical load is reached. The derivation of the eigenvalue problem can be found the literature [10].

4.1.1 Member division

The accuracy of the solution in the linear buckling analysis depends on the number of elements, into which the members are divided. If a too crude member division is used, it is not possible to describe the modeshape correctly and the corresponding load multiplication factor will be incorrect.

If the modeshape is a local deformation of a single member, the result can be rather sensitive to the member division. If a buckling analysis is performed where the members are not subdivided, one might get a result were the first modeshape is global, meaning the structure deforms as a whole, but where the actual lowest modeshape should be a local modeshape of a single member with a lower multiplication factor.

The sensitivity of the member division is performed for a simply supported column and a column fixed in both ends. The columns have the length $1\ m$ and therefore the analytical value of the effective lengths are $1\ m$ and $0.5\ m$ respectively. The crosssection is chosen to be quadratic with side length of $10\ mm$. The results of the buckling analysis can be seen in Table 4-1 and Table 4-2; the corresponding modeshapes can be seen in Figure 4-2 and Figure 4-3.

Number of elements	Analytical Euler load	Numerical buckling analysis	Deviation
1		2100 N	21.6 %
2		1740 N	0.8 %
4	1727 N	1728 N	0.1 %
8		1727 N	0 %
16		1727 N	0 %

Table 4-1 Buckling analysis for simply supported column

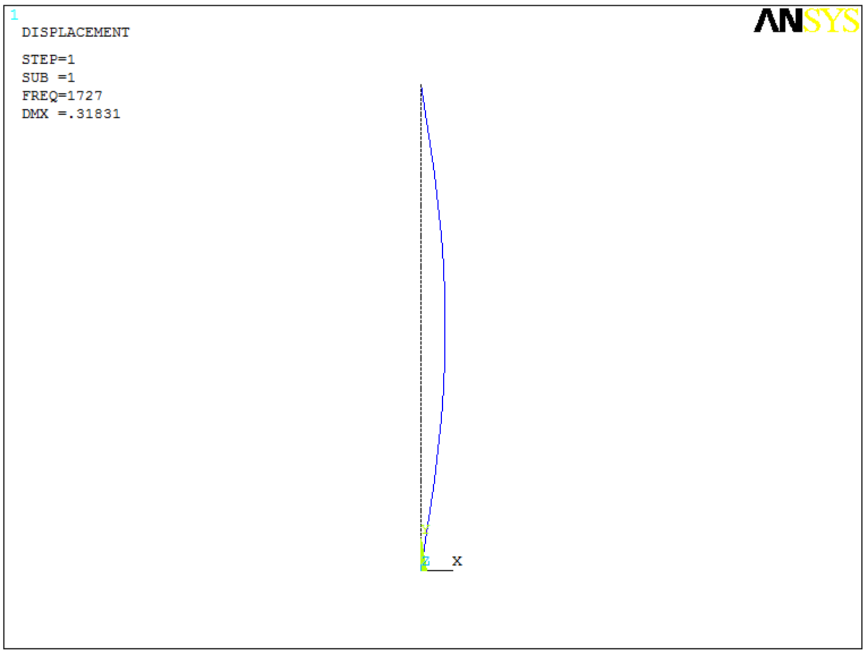


Figure 4-2 Buckling modeshape of simply supported column

Number of elements	Analytical Euler load	Numerical buckling analysis	Deviation
2		$7000\ N$	1.3 %
4		$6961\ N$	0.8 %
8	$6909\ N$	$6912\ N$	0.04 %
16		$6909\ N$	0 %

Table 4-2: Buckling analysis for a column with fixed support at the ends

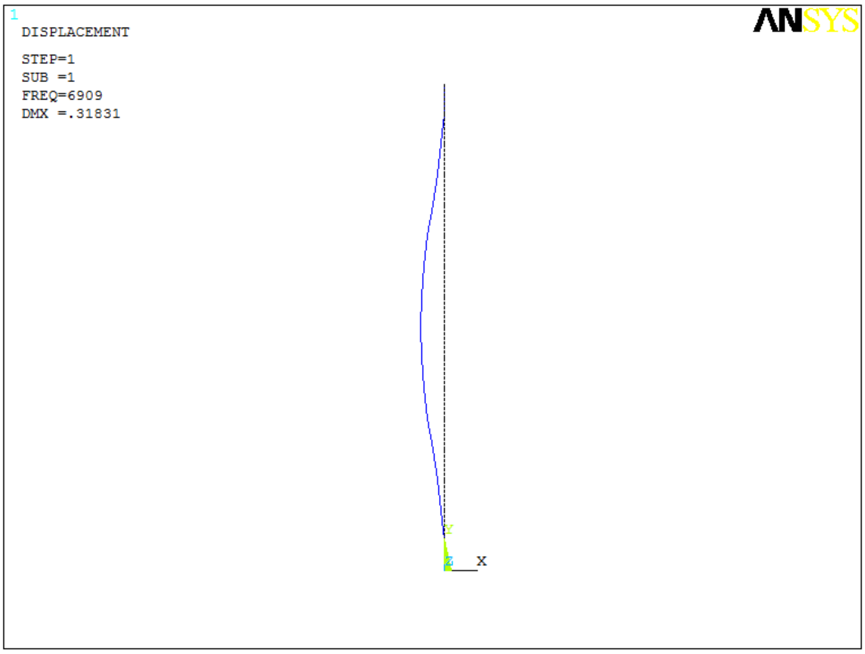


Figure 4-3: Buckling modeshape of a column with fixed support at the ends.

The results from the buckling analysis show that the column with fixed supports has the largest deviation from the analytical critical load. This is because the modeshape of the fixed column is more difficult to describe with few element since it has two deflection tangents.

This sensitivity analysis shows that a member division into 4 elements would be suitable for avoiding excessive calculation time.

4.1.2 Results When Using Shear-Flexible Beam Elements

In the classical beam theory (Bernoulli-Euler) only transverse deformations due to bending moments are taken into account, and it is assumed that the contribution from shear forces are negligible. This assumption is usual ok, but for stub beams the shear force contribution can be considerable. Stability problems occur in slender structures and the analytical critical load is found neglecting the shear force contribution.

Most commercial FEM programs is using shear-flexible beam elements (based on the Timoshenko beam theory) for the three dimensional structures. To illustrate the deviation from the analytical results when performing a linear buckling analysis using shear flexible elements, a simply supported column with length $1\ m$ and varying side length of a quadratic cross section is modeled. The analysis results are plotted in Figure 4-4.

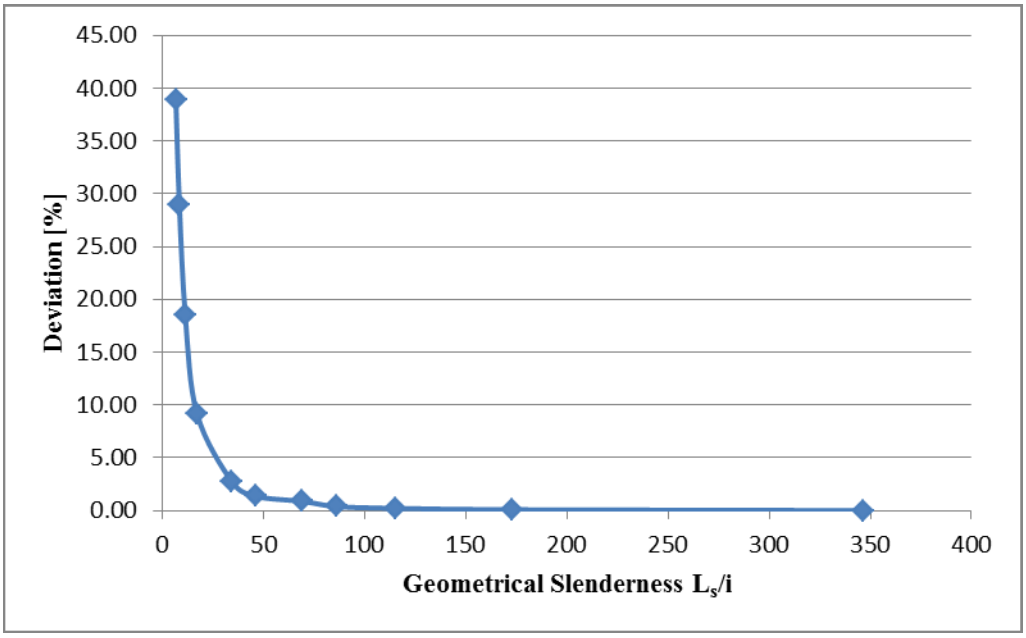


Figure 4-4 Deviation of results using shear-flexible beam elements compared to classic analytical theory

The FEM programs used in connection with this project uses shear-flexible beam element.

4.2 System Buckling Approach

A system buckling approach involves performing a linear buckling analysis. The individual K-factors associated with this type of analysis is found by multiplying the axial force in the members with the load multiplication factor for the entire system:

$$N_{cr,i} = N_i \cdot \lambda_{cr} \tag{36}$$

The K-factor can then be derived as shown in (25) in Section 2.3:

$$K_i = \sqrt{\frac{\pi^2 \cdot EI}{L^2 \cdot N_{cr,i}}} = \sqrt{\frac{\pi^2 \cdot EI}{L^2 \cdot N_i \cdot \lambda_{cr}}} \tag{37}$$

The K-factor is called a system K-factor as it is found by an analysis of the entire system. The multiplication factor is constant for the given load case, which means that the K-factor varies with the axial force in the members. This approach has the advantage that the entire system behavior is taken into account, and that the weakest member, i.e. most prone to buckling, will yield correct results. The disadvantage is the paradox that members with a small axial force will obtain a small critical buckling resistance and therefor a large K-factor.

These unexpected large K-factors are directly connected to the axial force, as the K-factor will approach infinity as the axial force reaches zero. However this doesn't necessarily mean that the member's strength is used up. In all cases the situation can be divided into to three categories as outline in (ASCE, 1996) [1]:

1) The axial loading effects in the member are negligible, and the member is for all practical purposes acting as a beam. As a beam, the member may or may not be contributing substantially to the buckling resistance of the system.

2) The member is a column, and the member's axial force is non-negligible, however, the member is not contributing significantly to the system buckling resistance. Nevertheless,

if the size of the member is reduced by a large enough amount, the buckling strength of the system might be controlled by a different buckling mode that does depend significantly on this member.

3) The member is a column that is contributing significantly to the computed system buckling resistance. In this case, if the size of the member is changed by even a small amount, the buckling strength of the system will change significantly.

In the first case, the beam shouldn't be penalized as a column in the design formulas, and therefore the reduction factor χ should be close to one. If the member is significantly influencing the buckling strength of the system by restraining other members that are subjected to significant compression, this restraint has already been accounted for within the buckling analysis. Since the buckling model is based on the bending moment that has been specified for this member, the buckling strength must be reevaluated if the member is restraining other members and its cross section is changed.

In the second case the members doesn't contribute significantly in the buckling of the system, but are still acting as column. This is the case for the most part of members with a small axial compression force compared to the critical bucking load with a K-factor of 1. In this case the K-factors should be computed in a more appropriate way.

In the third case the computed K-factors based on system buckling approach should be used, as the column, even though it only subjected to a small axial compression force, is at the limit of its buckling strength.

Whether a column belongs to category 2 or 3 can be hard to judge without sensitivity studies. Most often this comes down to the engineer's best judgment and experience.

4.3 The Paradox Concerning Large K-factors

In this section the system buckling approach (SBA) is applied to a system consisting of two members (the L-frame). The interaction between the members and the influence of tension/compression in the adjacent member is illustrated. As the axial compressive load becomes larger in the adjacent member, the K-factor becomes larger for the considered member as the axial load becomes relatively small.

Furthermore the K-factor paradox is verified by an example, where an apparently over utilized member is shown to get a reduction in the utilization ratio by approximately $10\,\%$ when fictitious extra load is applied.

4.3.1 L-frame – K-factor Dependence to Load Case

A model with two orthogonal members, known as the L-frame, is considered (see Figure 4-5). Both members have a geometrical length of $1\ m$ and a square cross section with a side length of $0.1\ m$. The members are simply supported in one end and rigidly jointed together in the other end. The critical length of the vertical column is investigated by different load cases letting the horizontal load vary.

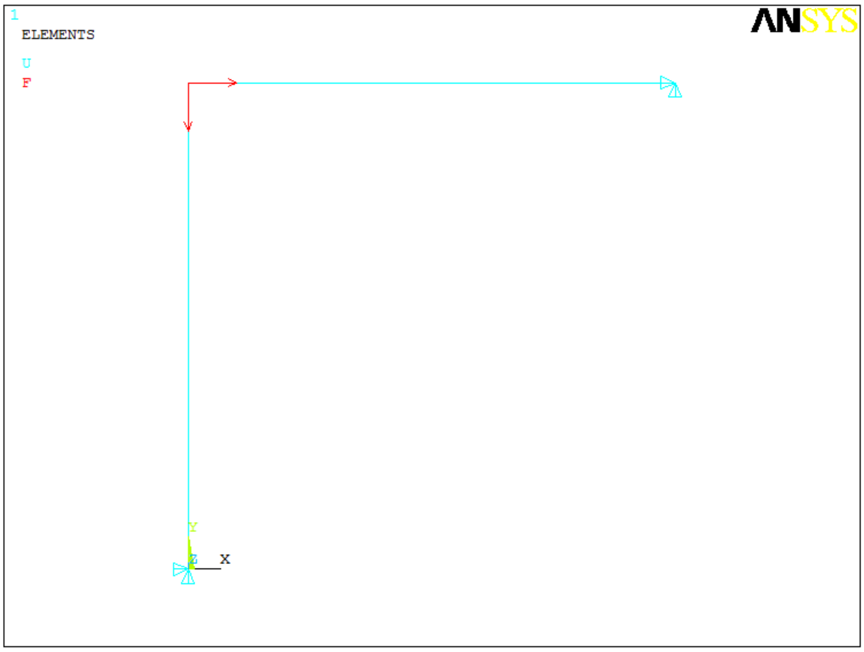


Figure 4-5 Model used for illustrating the influence of tension/compression in a member adjacent to a column

The critical axial load is found by linear buckling analysis, and the corresponding K-factor for varying horizontal load is shown in Figure 4-6. The reduction factor χ based on buckling curve c and a yield stress of 355 MPa is shown in Figure 4-7.

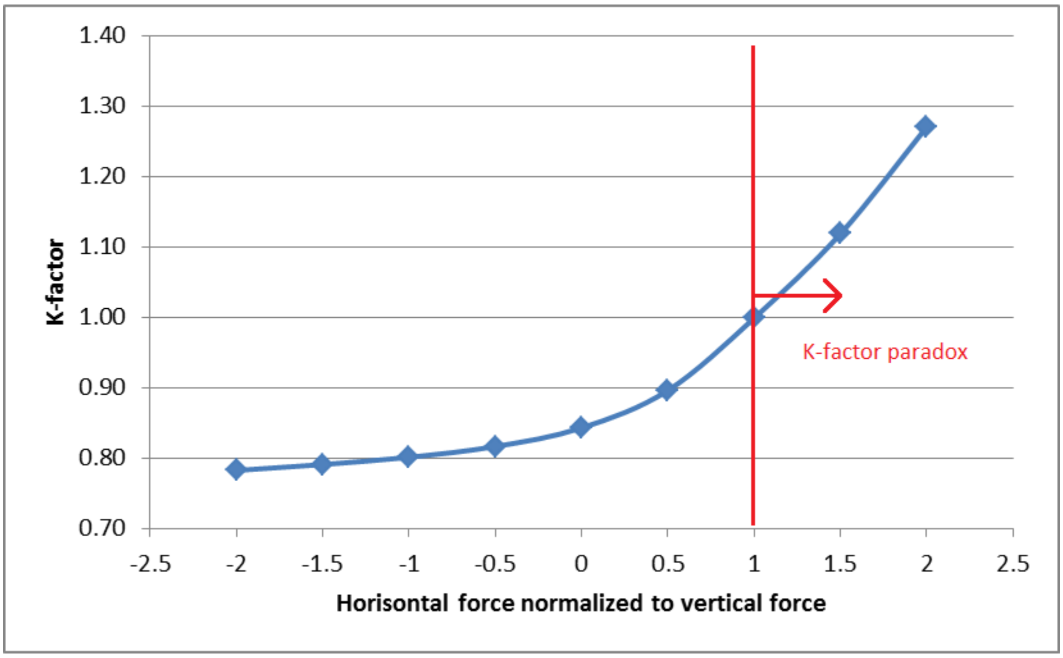


Figure 4-6 K-factor at various axial force in adjacent horizontal member

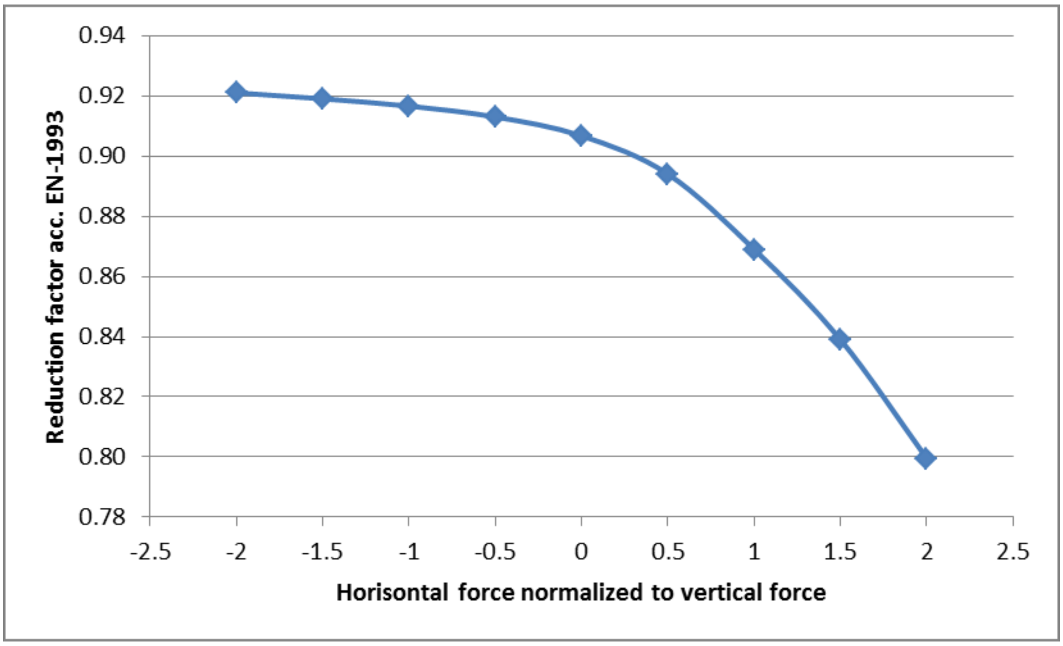


Figure 4-7 Reduction factor according to EN-1993 at various axial force in adjacent horizontal member

The main conclusions of these results are as follows:

1. When there are no axial force in the horizontal member, the K-factor is expected to be in the interval $0.7 < K < 1$, which is confirmed by the results where $K = 0.84$.

2. Applying tension in the horizontal member, the stiffness of the member is increased, and therefore the rotational stiffness of the joint is increased leading to a lower K-factor. The lower bound $K = 0.7$ is approached for very high tensile force in the horizontal member relative to the compressive force in the vertical member, where a normalized horizontal force of -1000 will give a K-factor of 0.7044. The mode shape is in this case similar to the one of a single column fixed in one end and pinned in the other (see Figure 4-8).

3. When the horizontal compressive force is equal to the vertical compressive force, the horizontal member does not contribute to any rotational stiffness since the members buckle with equal rotations at the ends (see Figure 4-9). This provides a K-factor of one for both members.

4. When the compressive horizontal force becomes large compared to the compressive vertical force, the K-factor is increased. This means that members in a system where the axial compressive force goes towards zero, the critical length will go towards infinity and thus the reduction factor goes towards zero. This is the paradox of the system buckling approach.

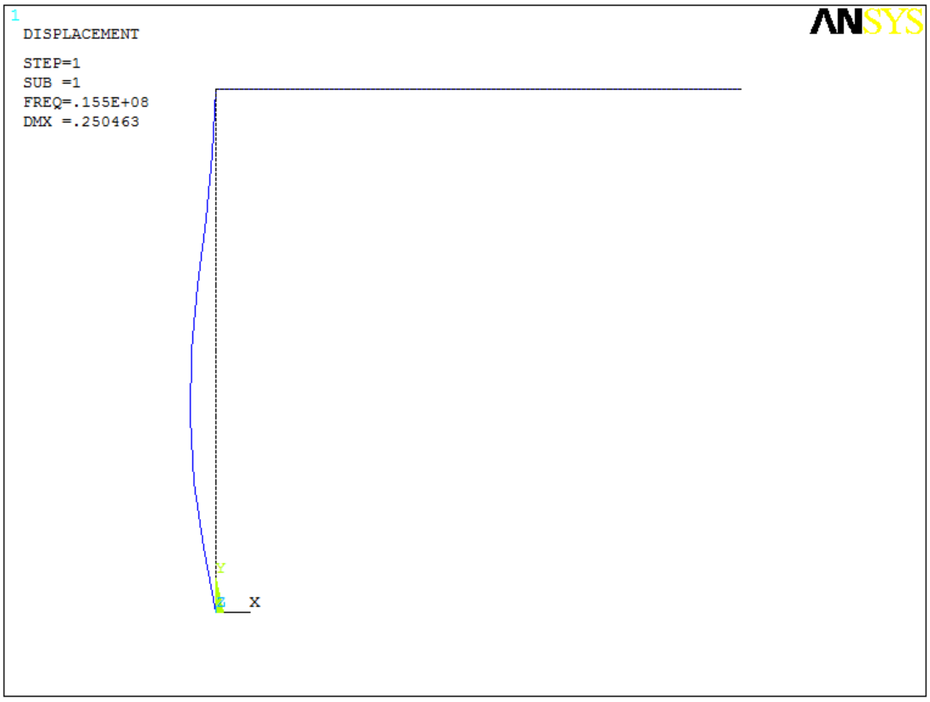


Figure 4-8 Mode shape when tension in horizontal member is very large

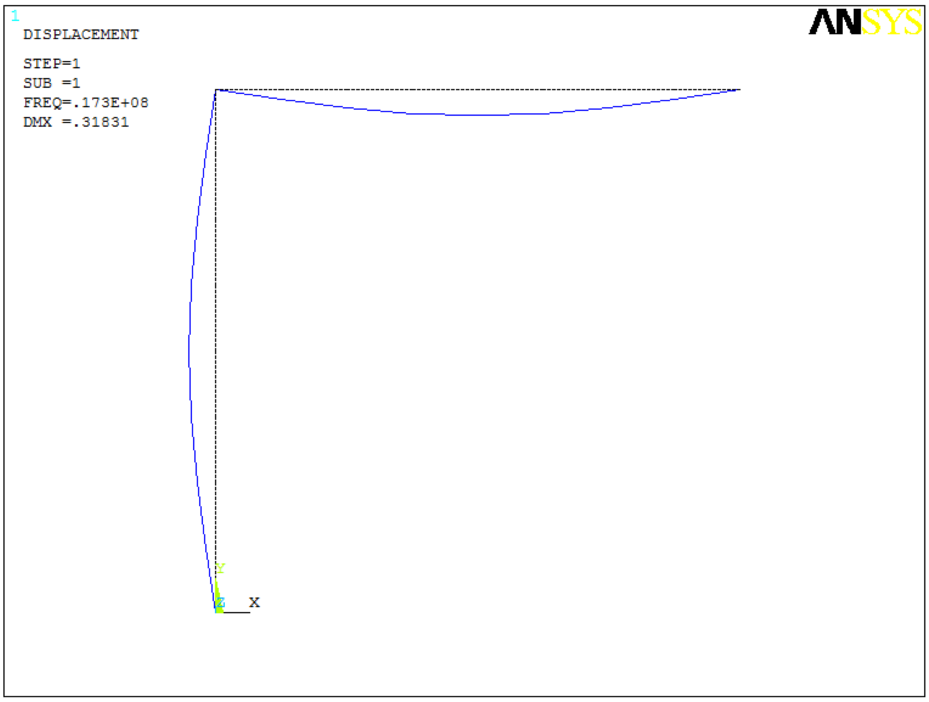


Figure 4-9 Mode shape when compressive axial force is equal in both members

The influence of the load case is one of the reasons why determining K-factors for individual members is challenging. Traditionally, engineers often think of the K-factor as dependent of the system stiffness, but forget that the stiffness is load dependent.

The SBA provides a conservative way of determining the K-factors, which can be implemented in FE-codes, however if more reasonable K-factors are needed the SBA is not recommendable.

4.3.2 Verification of the K-factor Paradox

Using the same model as in previous section, it is shown how the large critical length for a member with a small axial compressive force can be reduced without violating the structural integrity.

It is legal to increase the axial compressive force in a member as long as it does not change the system behavior significantly. If the additional force is increased to a level where the load multiplier only is increased slightly, the critical length of the member with small compressive axial force is reduced with only a slight increase in the critical length of members with large compressive force.

In the model the horizontal load is kept constant and linear buckling analyses are performed for various vertical loads. The critical load for each member is calculated using the SBA and the corresponding K-factor is seen in Figure 4-10. The reduction factor χ based on buckling curve c and a yield stress of 355 MPa is shown in Figure 4-11.

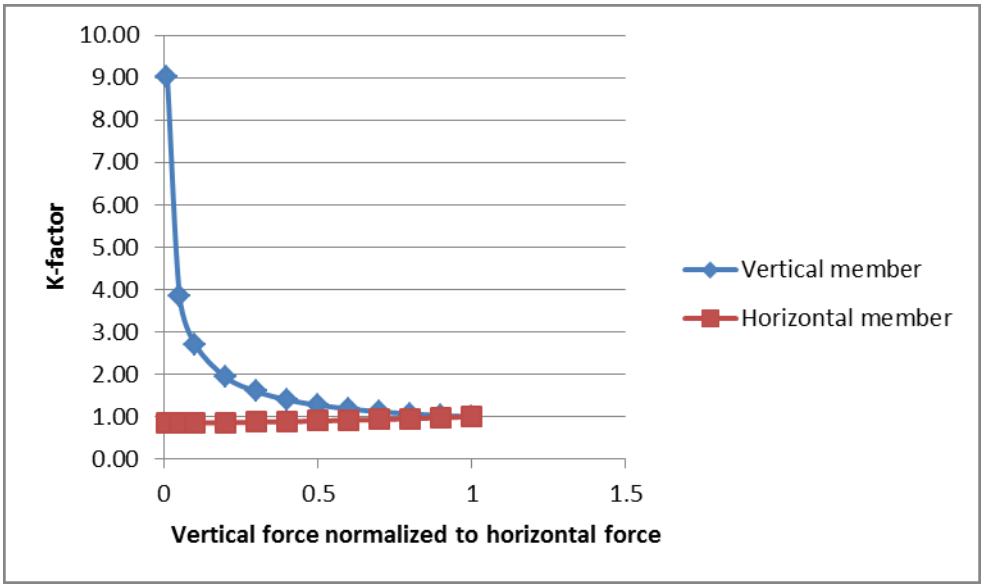


Figure 4-10 Calculated K-factors

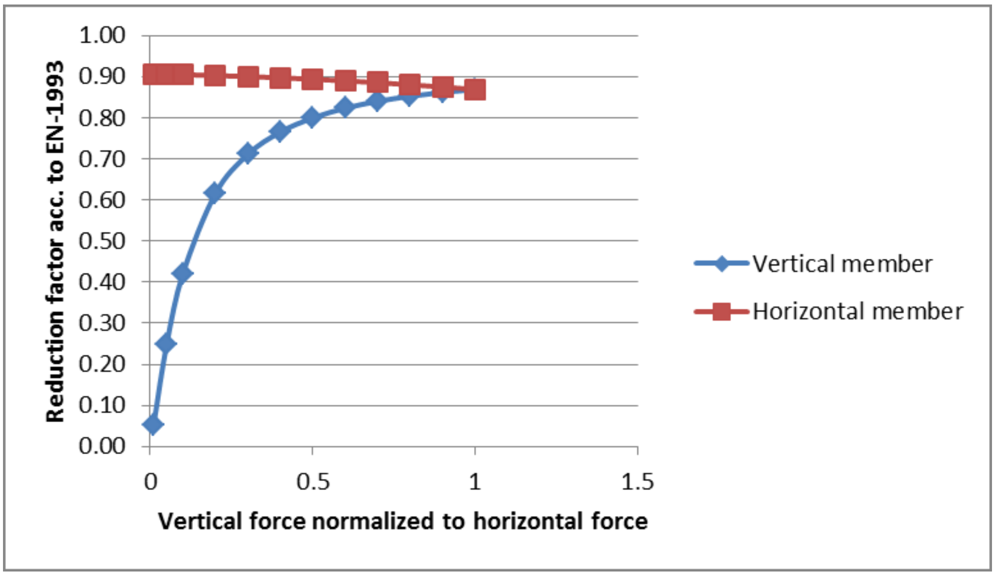


Figure 4-11 Calculated reduction factor according to EN-1993

From Figure 4-10 and Figure 4-11 it is seen that a small axial force in the vertical member leads to an unrealistic high K-factor and low reduction factor in this member. However, by increasing the axial force in the vertical member the K-factor decreases significantly, while only a small increase for the horizontal member is found.

In practice, this can be used to decrease the utilization ratio for the axial force in a member, leaving more capacity for actions caused by moments. To demonstrate this, we consider an extended version of the previous model. The horizontal member now has a geometrical length of $2\ m$, and the load case is a horizontal load of $250\ kN$ acting on the middle of the vertical column and a horizontal load of $500\ kN$ acting on the rigid joint (see Figure 4-12).

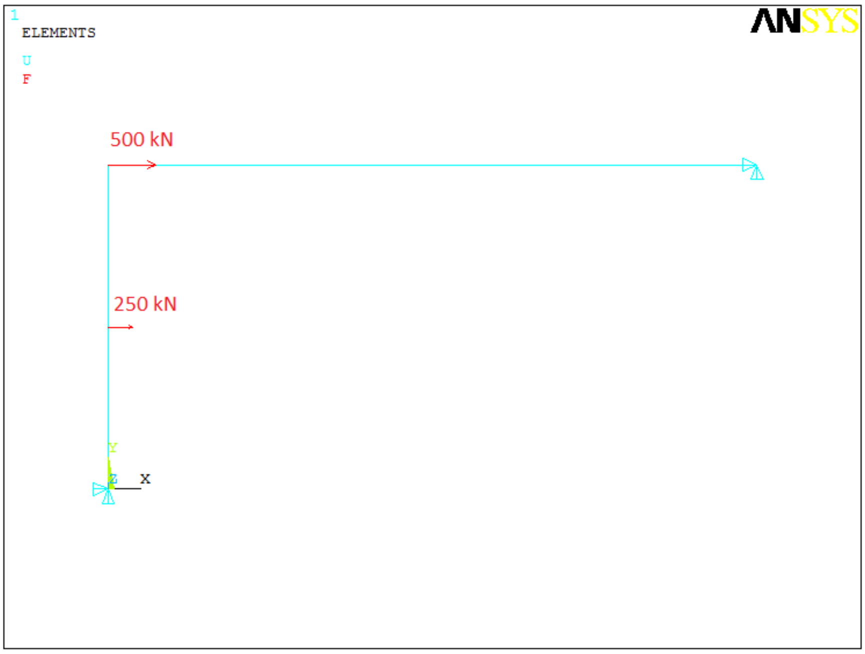


Figure 4-12 Model consisting of two members with small and large axial force respectively

A linear buckling analysis is performed and the mode shape and load multiplier is shown in Figure 4-13.

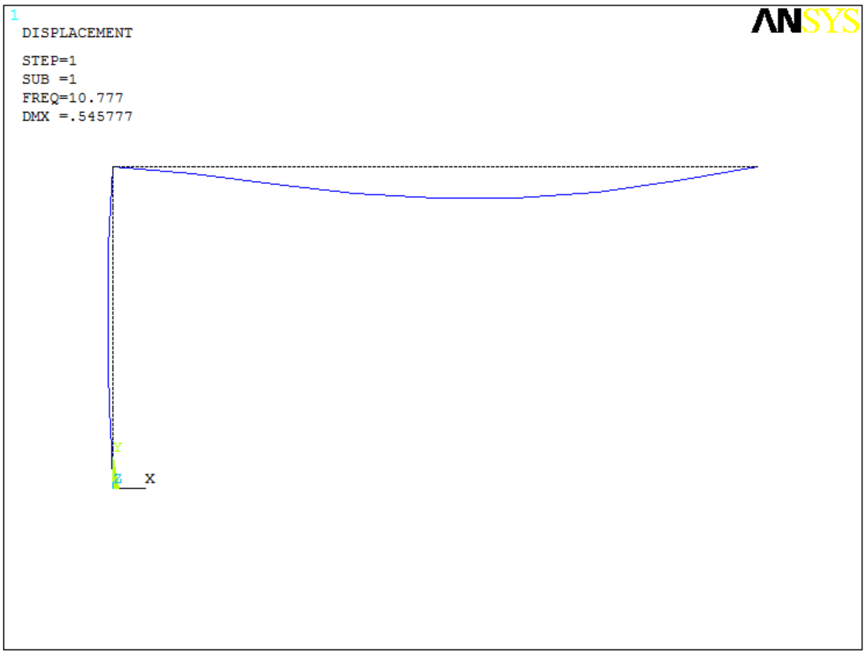


Figure 4-13 First mode

As the axial force in the vertical member is small, the critical load and reduction factor becomes small. In Table 4-3 results of sectional forces, reduction factors and utilization ratios (UR) are listed. The yield stress is taken as 355 MPa and an elastic distribution is used. The reduction factor χ is based on buckling curve c. The formulas used for calculating the utilization ratio are given by:

$$UR_{Axial} = \frac{N}{\chi \cdot A \cdot f_y} \tag{38}$$

and

$$UR_{Moment} = \frac{M}{W \cdot f_y} \tag{39}$$

where N and M are the sectional forces, W is the section modulus, A is the cross sectional area and f_y is the yield stress.

Member	Load multiplier	Axial force $[kN]$	Reduction factor χ	UR (Axial)	Moment $[kNm]$	UR (moment)	UR Total
Horizontal	10.78	640	0.71	0.25	14.6	0.25	0.50
Vertical		7.3	0.02	**0.10**	55.2	0.93	**1.04**

Table 4-3 Sectional forces, reduction factors and utilization ratios

It is found that the vertical member has a very low reduction factor and a utilization ratio larger than 1. To increase the reduction factor, a fictive additional load is applied to increase the axial force (and thereby the critical load) in the member. The additional load is taken as 10 times the real axial force in the vertical member and applied at the top of the column (see Figure 4-14). Results are listed in Table 4-4.

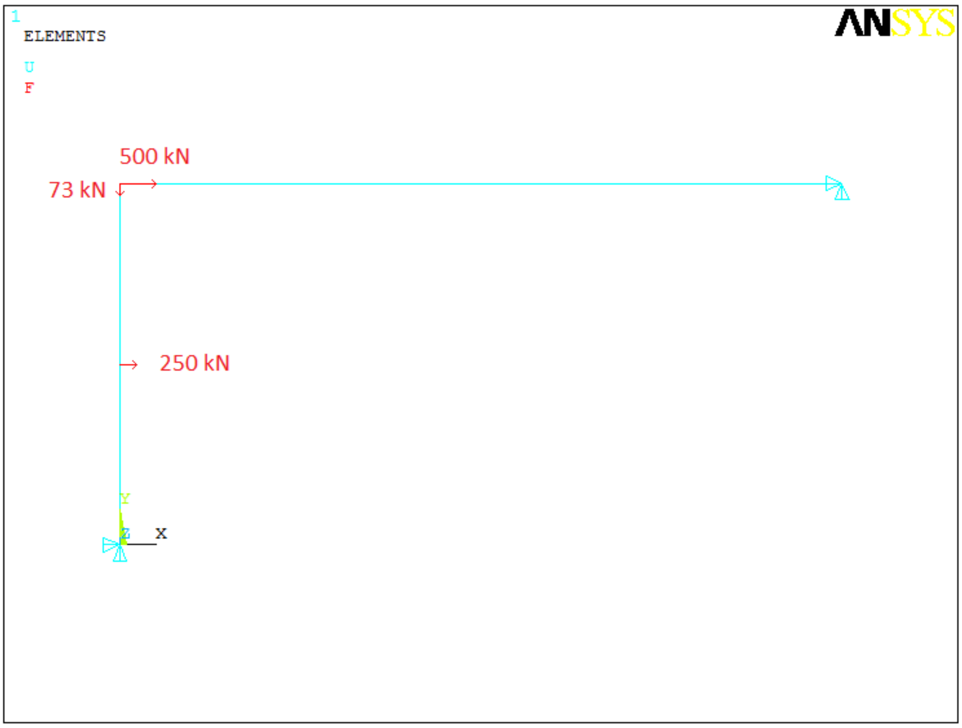


Figure 4-14 Additional load applied at the top of the vertical column

Member	Load multiplier	Axial force $[kN]$	Reduction factor χ	UR (Axial)	Moment $[kNm]$	UR (moment)	UR Total
Horizontal	10.72	640	0.71	0.25	14.6	0.25	0.50
Vertical		7.3 (+73)	0.19	**0.01**	55.2	0.93	**0.94**

Table 4-4 Sectional forces, reduction factors and utilization ratios after increasing the critical load in the vertical member

The results show the paradox, that it is possible to decrease the utilization for one member by applying extra load. However it has to be validated, that the added fictive load does not have significant influence on the system behavior e.g. there are no noticeable change in the sectional forces in the entire system besides the applied axial force.

The idea of using fictitious forces to decrease the K-factors for certain members with the disadvantage of a minor increase for others has been presented in earlier work by Choi and Yoo [7].

4.4 Iterative System Buckling Approach

The reason for excessively large K-factors when using the SBA (equation (37)) arises from the assumption that all members reach their buckling limit when the system buckles. Since not all members are close to their buckling limit when the system buckles equation (37) is only valid for the most critical member. As shown in the previous section an increase in axial in the weaker members only has a small effect on the system load multiplier. Choi and Yoo (2008) uses this in an iterative procedure where the axial force in the compression members is increased until an convergence criteria where the change in K-factor is sufficiently small is fulfilled. This eliminates the paradox that small axial forces give excessively large K-factors [7].

The most and least influential columns are to be determined by:

$$L\sqrt{\frac{P}{EI}} \tag{40}$$

Where P is the axial force in the member, E is the modulus of elasticity and I is the moment of inertia.

The K-factor is expressed as:

$$K_i = \sqrt{\frac{\pi^2 E_i I_i}{L_i^2 (P_i + \Delta P)\lambda}} \tag{41}$$

where ΔP is the final increase in axial force calculated as the sum of incremental changes:

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{K_{mi} L_{mi}}{\overline{K_{li}} L_{li}}\right)^2 P_{mi} - P_{li} \tag{42}$$

where the subscript li and mi refers to the least and most influential columns respectively. For an easy and stable iteration process, the terms K_{mi} and $\overline{K_{li}}$ are assumed to be equal to 1.0 in the first iteration step. The constant value of δP from this assumption is used in subsequent iterations for simplicity.

The convergence criterion is defined as:

$$\frac{K_i^j - K_i^{j-1}}{K_i^j} < \varepsilon \tag{43}$$

where j is the iteration number and ε is convergence criteria (ε =0.001)

4.5 Summary

In this chapter the significance of the load case is made clear. Furthermore the system buckling approach is exemplified with the implication of excessively large K-factors. The system buckling approach has the benefit that it is possible to calculate the K-factor for each individual member by an algorithm, however overly conservative values of K-factor for members with small axial force necessitate an improvement of the method.

For a simple system consisting of two members, it is shown that it is possible to reduce the K-factor by adding more axial force in the member with a small axial force. This does only have an insignificant influence on the load multiplier for the system. This is idea is used by Choi and Yoo (2008) as they have proposed an iterative system buckling approach. This method overcomes the K-factor paradox but lacks the possibility to be implemented in general software codes as the results of the iterative procedure is sensitive to both the incremental fictive force and the convergence criterion. This is also discussed in Appendix 3.

5. Proposed Method – The Energy Ratio Method

In this chapter the principals behind the Energy Ratio Method (ERM) is explained. The ERM is a numerical method with close connection to the System Buckling Approach (SBA). The difference between the ERM and the SBA is that the ERM circumvent the paradox of members with small axial force having extremely large K-factors.

The ERM could be considered as an extension of the SBA, where the idea behind the ERM is to account for the fact that all members in a system usually does not reach their individual buckling limit simultaneously. In the SBA the load multiplication factor is applied to the axial force of all compressive members (see Section 4.2), whereas the ERM uses a higher load multiplication factor for the members in which buckling is less pronounced. To assess how prone a member is to buckling, a ratio between stabilizing and destabilizing energy in the member is used. The energy calculations are based on the mode shape and load multiplier found from a linear buckling analysis. In Figure 5-1 the method is illustrated.

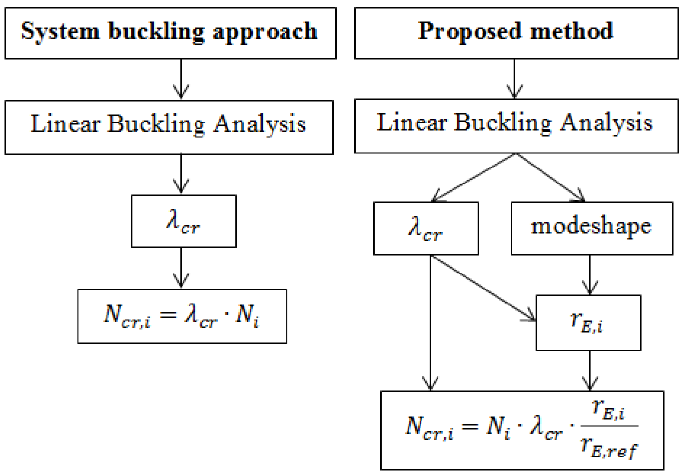


Figure 5-1 The difference between the system buckling approach and the proposed method

In the first section, the classical energy based stability considerations is presented. Known analytical formulas are used to validate a numerical calculation of energy. In the following sections, the philosophy behind the proposed method and formulas used for calculation of K-factors is outlined. Furthermore examples are used to verify the proposed method and illustrate the connection to the SBA.

5.1 Energy Based Stability Considerations of a Beam Member

The critical load of a beam member can be found by the considering the energy balance between inner and outer work. Due to bending of the beam inner work is done in the form of bending strain energy, which stabilizes the beam. The outer work is done by the applied load multiplied the displacement, which is destabilizing for the beam. Analytical and numerical methods for calculating the inner and outer energy are further explained by [3] and [10].

5.1.1 Analytical calculation of energy

In the calculation we consider a simply supported beam with length L, axial direction x and displacement field $u(x)$. The beam is subjected to an axial load P (positive in tension).

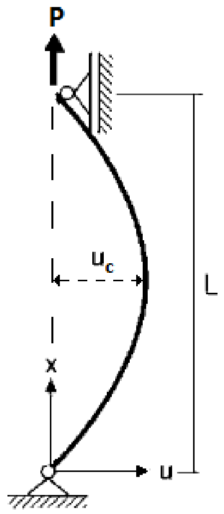


Figure 5-2 Column considered for energy calculation

At the critical load level, the internal and external energy is of equal size, where the internal energy corresponds to the strain energy in bending given in terms of the curvature $u_{,xx}$ by:

$$E_{internal} = \frac{1}{2}\int_0^L EI \cdot u_{,xx}^2 \; dx \tag{44}$$

and the external energy is the work done by the force P with the axial displacement given by:

$$E_{external} = -\int_0^L P \cdot \varepsilon_m \; dx \tag{45}$$

where the ε_m is the membrane strain. The beam only undergoes small displacements and the membrane strain can be approximated by:

$$\varepsilon_m \approx \frac{1}{2} u_{,x}^2 \tag{46}$$

and the external energy becomes:

$$E_{external} = -\frac{1}{2}\int_0^L P \cdot u_{,x}^2 \; dx \tag{47}$$

Using the criteria:

$$E_{internal} = E_{external} \tag{48}$$

the critical load can be found explicitly, and the solution is exact if the displacement field is exact.

5.1.2 Numerical calculation of energy

When performing the linear buckling analysis, a subdivision of each member into 4 elements is used. The internal and external energy in each member can be found as a summation of the energies in each element belonging to the member in question. The internal energy in an element can be found by:

$$E_{internal} = \frac{1}{2} \cdot \boldsymbol{s}^T \boldsymbol{k}\, \boldsymbol{s} \tag{49}$$

And the external energy can be found by:

$$E_{external} = \frac{1}{2} \cdot \boldsymbol{s}^T (\lambda_{cr} \boldsymbol{k_g}) \boldsymbol{s} \tag{50}$$

where $\boldsymbol{s}$ is a vector containing the elements 6 nodal displacements from the mode shape (translation in x and y direction and rotation about z axis) and λ_{cr} is the load multiplier. $\boldsymbol{k}$ and $\boldsymbol{k_g}$ are the elements local stiffness matrix and geometrical stiffness matrix respectively.

The internal energy is the bending strain energy, which is equal to the inner work. The external energy produces the bending stiffness reduction and the energy is coherent with the outer work.

The ERM is suitable for implementation in finite element codes, since the prerequisite linear buckling analysis is performed as a finite element method. In this project a 2 node beam element based on Bernoulli-Euler theory is used. A MATLAB script with a simple finite element code has been written to make the examples that demonstrates the ERM. The script can be found in Annex 2. Only 2D models are used to demonstrate the method to minimize the scope of the script; however the method can be extended to 3D.

5.1.3 Verification of Energy Calculation

In this section the analytical calculation of the energies from formulas (44) and (47) are compared to the ones calculated by the MATLAB script using formulas (49) and (50). The comparison is made of a beam with square section. The beam is simply supported and has the side length 0.1 m and length 1 m.

Analytical Calculated Energy

The mode shape of a simply supported beam during buckling has the form of a half sine. If the center deflection of the beam is noted u_c, the displacement field is given by:

$$u(x) = u_c \cdot \sin\left(\pi \frac{x}{L}\right) \tag{51}$$

and the first derivative is given by:

$$u_{,x} = u_c \cdot \frac{\pi}{L} \cdot \cos\left(\pi \frac{x}{L}\right) \tag{52}$$

and the second derivative is given by:

$$u_{,xx} = -u_c \cdot \left(\frac{\pi}{L}\right)^2 \cdot \sin\left(\pi \frac{x}{L}\right) \tag{53}$$

The integrals can be solved by using the following formulas:

$$\int_0^L \left(\sin\left(\pi \frac{x}{L}\right)\right)^2 dx = \left[\frac{x}{2} - \frac{\sin\left(2\frac{\pi}{L}x\right)}{4\frac{\pi}{L}}\right]_0^L \tag{54}$$

And

$$\int_0^L \left(\cos\left(\pi \frac{x}{L}\right)\right)^2 dx = \left[\frac{x}{2} - \frac{\sin\left(2\frac{\pi}{L}x\right)}{4\frac{\pi}{L}}\right]_0^L \tag{55}$$

The center deflection of the mode shape is $u_c = 0.3183\ m$ (found from the script in the following), and by using either formulas (44), (53) and (54) or formulas (47), (52) and (55), the analytical solution gives a total energy of:

$$E = 4317684\ Nm \tag{56}$$

Numerical Calculated Energy

A model of the considered beam is applied an axial load of $1\ N$, hence the load multiplier is the critical load in newton. The model is seen in Figure 5-3 and the external and internal energy is seen in Figure 5-4.

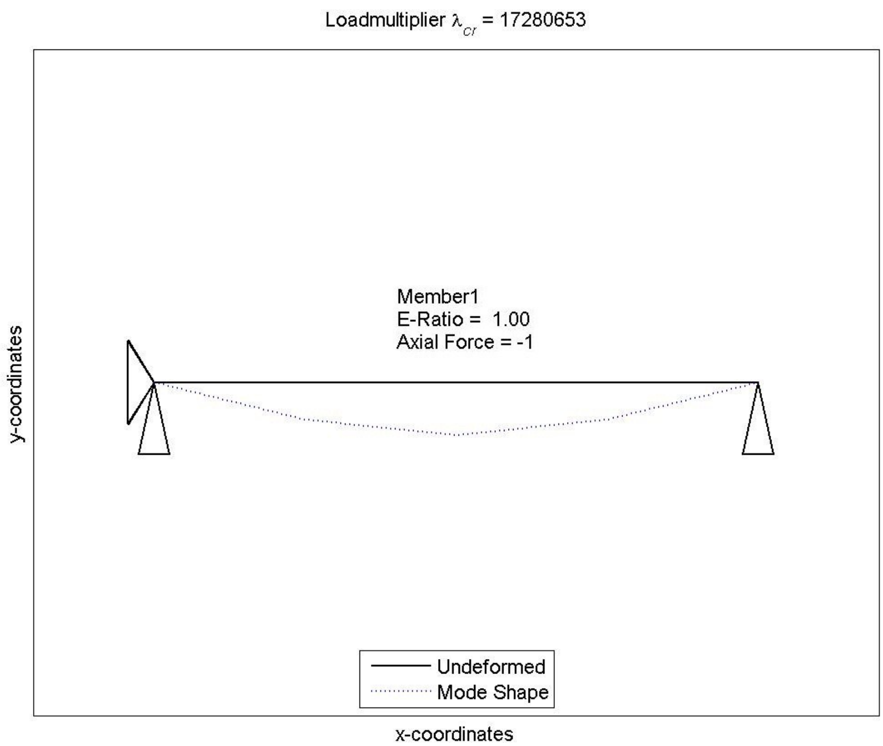


Figure 5-3 Model used for energy demonstration

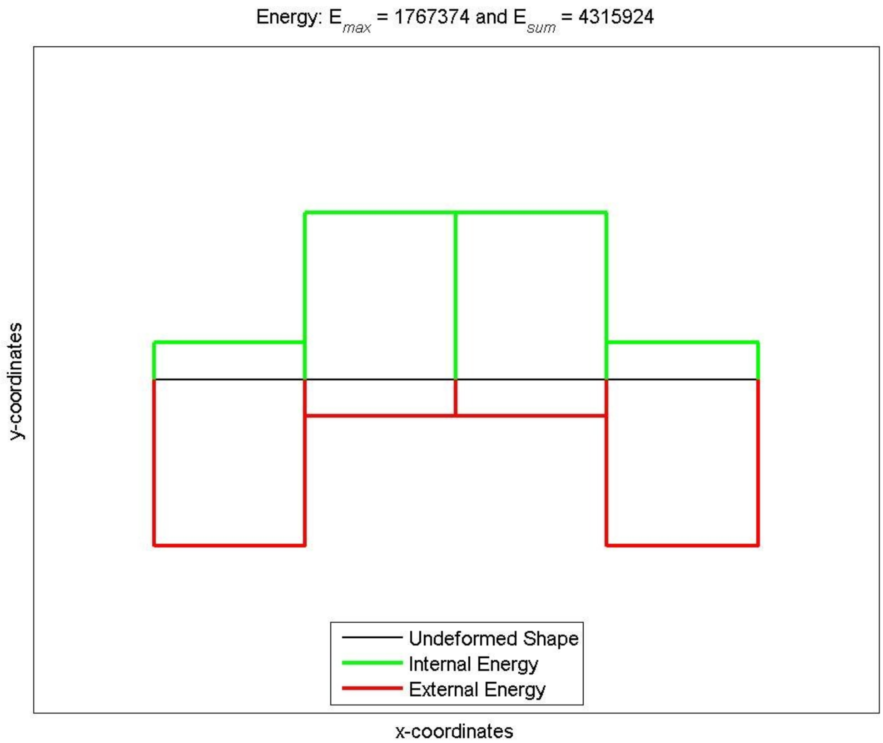


Figure 5-4 Energy calculated numerical by MATLAB script

The analytical calculated value is approximately the same as found in Figure 5-4. The numerical calculated energy approaches the analytical solution as the number of subdivisions increases. This is implied in Figure 5-5, where a subdivision of eight has been used.

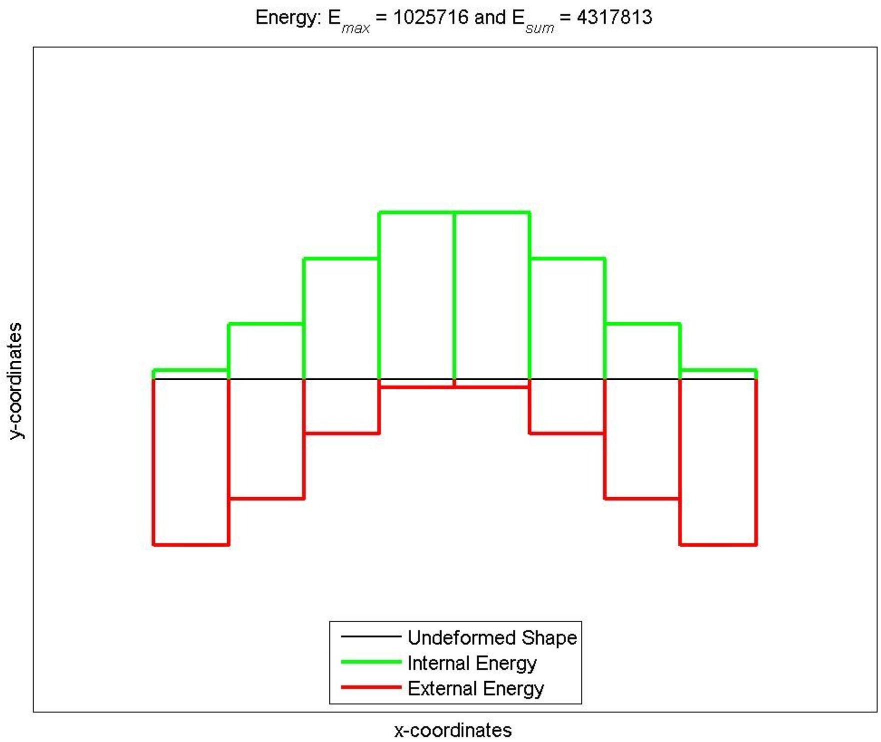


Figure 5-5 Energy calculated with twice the number of elements

In Table 5-1 comparison between the analytical calculated energy and the numerical calculated energy is made.

Number of Elements	Analytical value	ERM value	Deviation
4	$4317684\ Nm$	$4315924\ Nm$	0.04 %
8	$4317684\ Nm$	$4317813\ Nm$	0.003 %

Table 5-1 Comparison between analytically and ERM based energy calculations

It is assessed that a subdivision into four elements also is sufficient for the energy calculations.

5.2 The Proposed Method

The K-factor of a column can be found by considering the stabilizing internal energy (which is equivalent to the potential strain energy) and the destabilizing external energy (resulting from the outer work done by the axial forces). The K-factor is derived from the critical load, which is found from the criteria:

$$E_{internal} = E_{external} \tag{57}$$

The calculation of the energy requires the buckling shape, which is known for the basic cases, and when considering a system of members, the buckling shape is easily found by a linear buckling analysis as the mode shape (see Section 4.1).

When performing the linear buckling analysis on a system of members, some members are deforming as a result of the outer work being done by the axial force, and these members are in a state of buckling. Other members are more in a state of bending due to rigid connections to the members that are in a state of buckling. To assess "how much" a member is in the state of buckling, the following energy ratio is being used:

$$r_E = \frac{E_{internal}}{E_{external}} \tag{58}$$

The ratio r_E is calculated for each member, and the member with the lowest ratio is considered as the weakest member and therefore being 100% in the state of buckling. For this specific member, the critical load is defined in the same way as in the SBA, which is:

$$N_{cr} = N \cdot \lambda_{cr} \tag{59}$$

where N is the axial load in the member, and λ_{cr} is the load multiplication factor found by the linear buckling analysis.

The main point in the philosophy behind the proposed method is that for all other members, the load multiplication factor is increased relative to the respective members energy ratio compared to the lowest energy ratio. E.g. if a member has an energy ratio twice the size of the lowest energy ratio, the load multiplication factor should be increased by a factor of two. The lowest energy ratio is thereby a reference ratio, which is notated as $r_{E,ref}$ and the critical load for each member i in the system can be determined as

$$N_{cr,i} = N_i \cdot \lambda_{cr} \cdot \frac{r_{E,i}}{r_{E,ref}} \tag{60}$$

Now the critical load for each element can be calculated using formulas (58), (60), (49) and (50), and the K-factor for each element can be found using back calculations from the Euler-formula given by:

$$N_{cr} = \frac{\pi^2 \cdot EI}{(K \cdot L)^2} \tag{61}$$

Rewriting and inserting formula (60) yields:

$$K_i = \sqrt{\frac{\pi^2 \cdot EI}{L^2 \cdot N_{cr,i}}} = \sqrt{\frac{\pi^2 \cdot EI}{L^2 \cdot N_i \cdot \lambda_{cr} \cdot \frac{r_{E,i}}{r_{E,ref}}}} \tag{62}$$

In the special case where all members buckle at the same time, every member will have a ratio of one, and the ERM gives the same result as the SBA.

This method eliminates the paradox from the SBA, and gives more reasonable K-factors for all members in the system. However one must keep in mind that as for the SBA, the ERM produces load dependent K-factors, and therefore the method should be used for each load case when designing a structure.

5.3 Demonstration of the Energy Ratio Method

In this section the ERM is demonstrated. Comparison to the SBA is included to show the main advantage of the ERM, namely that the paradox that relatively small forces provide large K-factors in the SBA is avoided.

5.3.1 Simply Supported L-frame

In this example a model of an L-frame is used to demonstrate that:

- When the members buckle at the same time, the ratio r_E is equal to one for both members.
- The ERM provides the same results as the SBA for the member which is 100 % in the buckling state, i.e. the one that has the lowest energy ratio r_E.
- The K-factors is in the expected interval for all load cases, thus excluding the paradox from the SBA.

The model consists of two members with the same geometrical properties. The length of the members is $1\ m$ and the cross section is a square with side length $0.1\ m$. The supports allow rotations but no displacement. The model is the same L-frame as used in Section 4.3 to illustrate the paradox in the SBA.

The load cases considered is produced by letting the axial force in the vertical member (Member1) be fixed at $-1\ N$ (compression), and then varying the axial force in the horizontal member (Member2) as both compressive and tension. The model can be seen in Figure 5-6 where the axial force in Member2 is $-10\ N$ (compression).

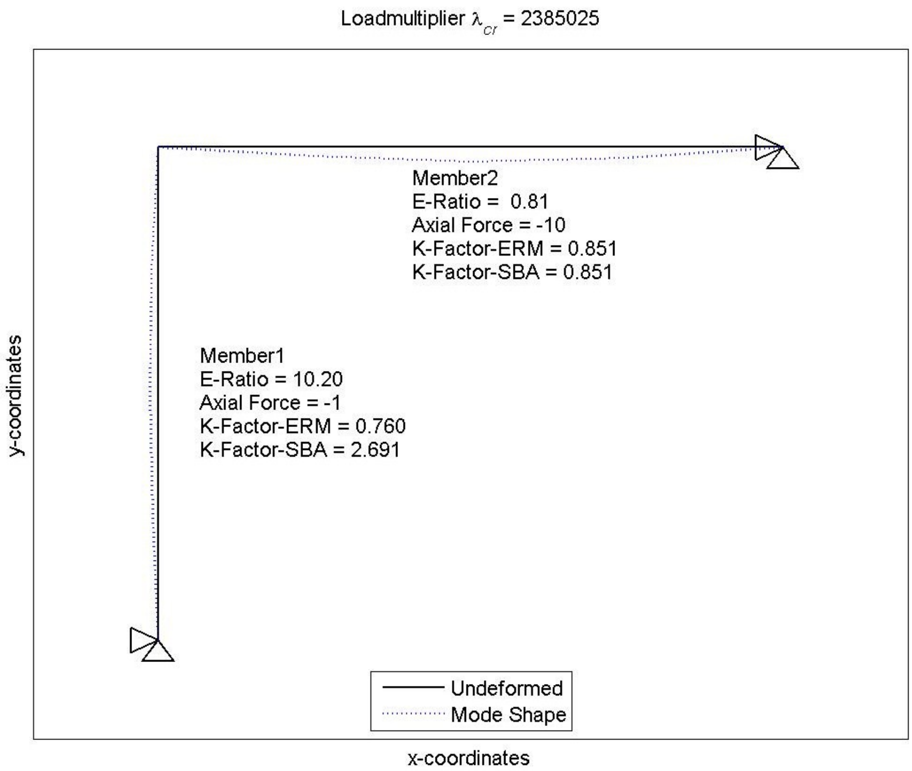


Figure 5-6 Model produced by MATLAB script providing K-factors by both ERM and SBA

The result from various load cases is seen in Figure 5-7 and Figure 5-8, where the y-axis reads the K-factor. The x-axis is the axial force in Member2 normalized to the axial force in member1, which means that negative values are equivalent to tension in Member2.

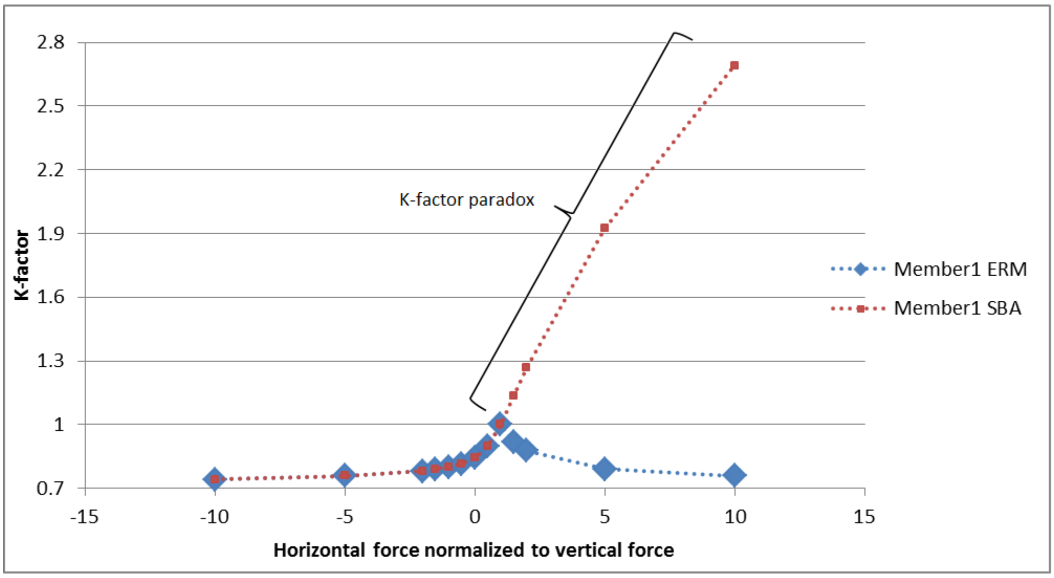


Figure 5-7 K-factors for Member1 found by ERM and SBA

Figure 5-7 shows how the K-factor for Member1 becomes large in the SBA when the axial force in Member 1 is small compared to the axial force in Member2, which is the paradox of the SBA. The ERM provides better results where the K-factor is in the expected interval $[0.7 < K < 1.0]$. The lower bound of 0.7 is found when tension in Member2 goes towards infinity, which prevent rotation at the rigid joint due to increased stiffness of Member2. This is in accordance with the basic case of a beam fixed at one end and roller supported at the other end (see Figure 2-2). In Figure 5-8 a zoom of Figure 5-7 is shown.

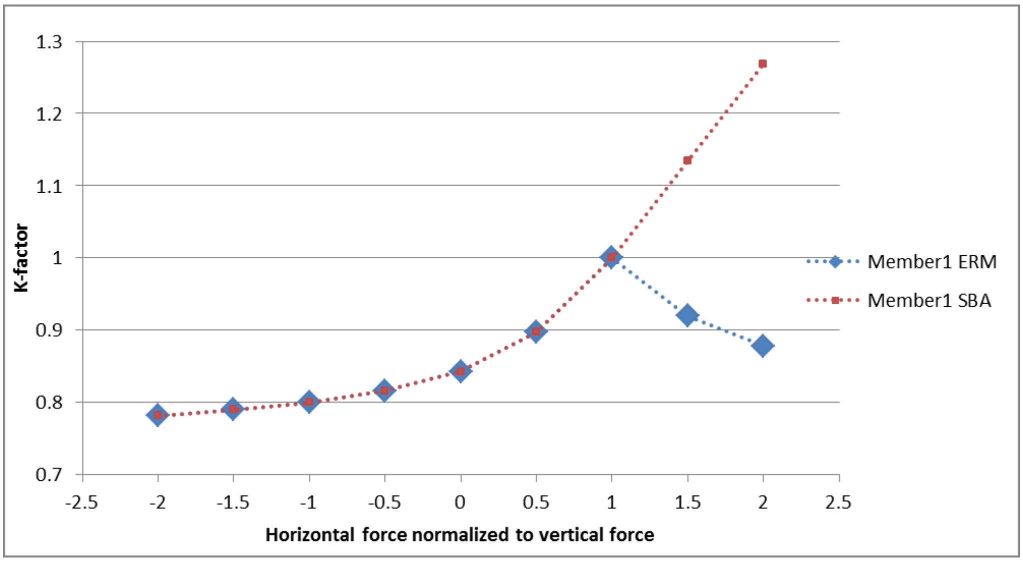


Figure 5-8 Plot of K-factors for Member1 in reduced interval of load cases

From Figure 5-8 it is seen that the ERM provides the same results of the K-factor for Member1 as the SBA when the axial compressive force in Member1 is the larger than in Member2. This is because the energy ratio r_E is smallest in Member1, i.e. Member1 buckles first, since the two members have the same properties and the force in Member1 is largest. When the axial compressive force in Member2 becomes the largest, the K-factor for Member 1 decreases according to ERM. This is because Member1 in these load cases function more as a stabilizing member by its bending stiffness, rather than a compressive member.

Furthermore it is seen, when the axial compressive force in both members is of same size, the members buckle at the same time, which provides a K-factor of one for both members. This is because each member focus all its internal energy on preventing buckling, leaving no bending stiffness to prevent rotation at the rigid joint. Therefore the K-factor is 1.0 according to the basic case of a beam which is simply supported at one end and roller supported at the other end. In Figure 5-9 the model with equal axial force in the members is shown, where it is seen that the energy ratio is one for both members.

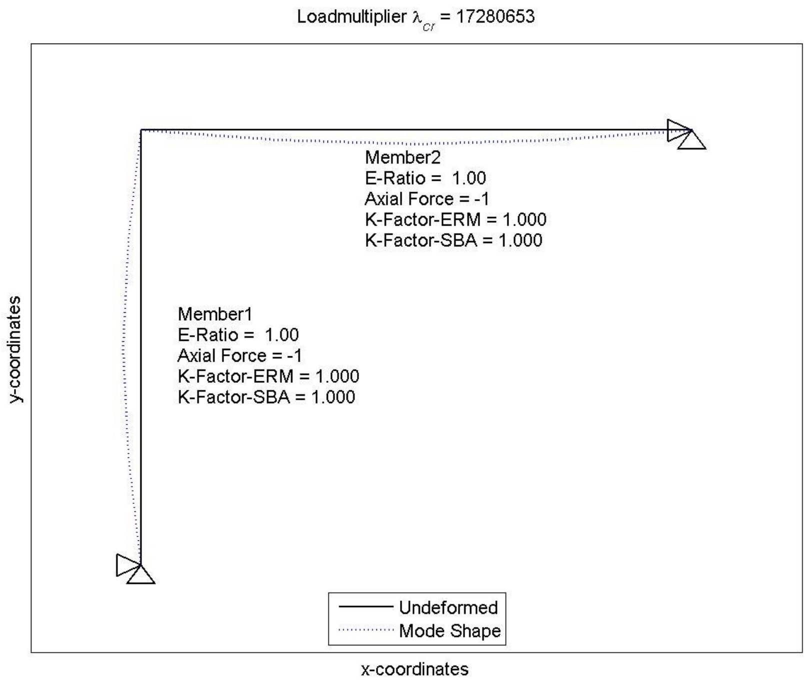


Figure 5-9 Model showing buckling at the same time for both members

In Figure 5-10 the K-factor for Member2 is shown.

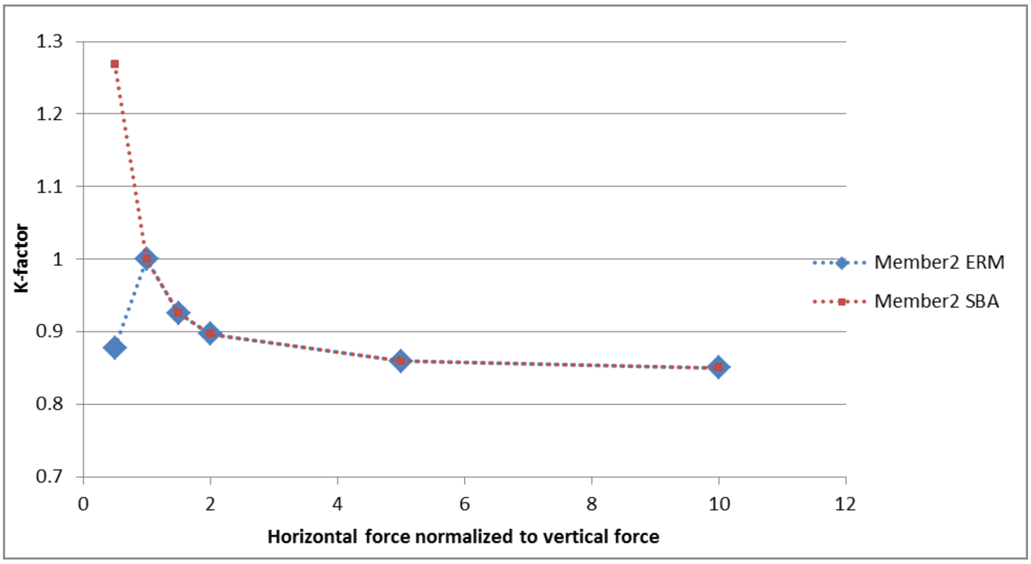


Figure 5-10 K-factors for Member2 found by ERM and SBA

From Figure 5-10 same conclusions for Member2 as for Member1 can be made. It shows that the ERM provides the same results for the K-factor of Member2 as the SBA, when Member2 is the member that buckles first. Furthermore when the axial force in Member1 is largest the ERM decreases the K-factor for Member2.

5.4 Other Application of the Energy Ratio Method

As illustrated in previous section, the energy ratios for each member in a system gives a picture of which members that are mainly in a buckling state, and which members that acts more as stabilizing member by their bending capacity.

Previous attempts to overcome the paradox in the SBA has been made by others, where fictitious axial force are applied iteratively to initiate buckling at the same time for all compressive members [7][8]. This method is a good alternative to the traditional SBA, where the K-factor decreases for the members that act as stabilizing bending members. However the method provides a larger K-factor for members which are most prone to buckling, because the fictitious forces applied yields a lower load multiplication factor.

The ERM can be used as an iterative method to calibrate the axial force in the compressive members so that the energy ratio r_E becomes one for all members, i.e. buckling occurs at the same time for all compressive members. As an example on this procedure the following example is provided.

The model is an L-frame where Member1 has the length $2\ m$ and a fixed support at the base, and Member2 has the length $1\ m$ and a simple support at the end. Both members have a square cross section with side length of $0.1\ m$. The load case considered is an axial force of $-10\ N$ in Member1 and $-1\ N$ in Member2. The model is seen in Figure 5-11.

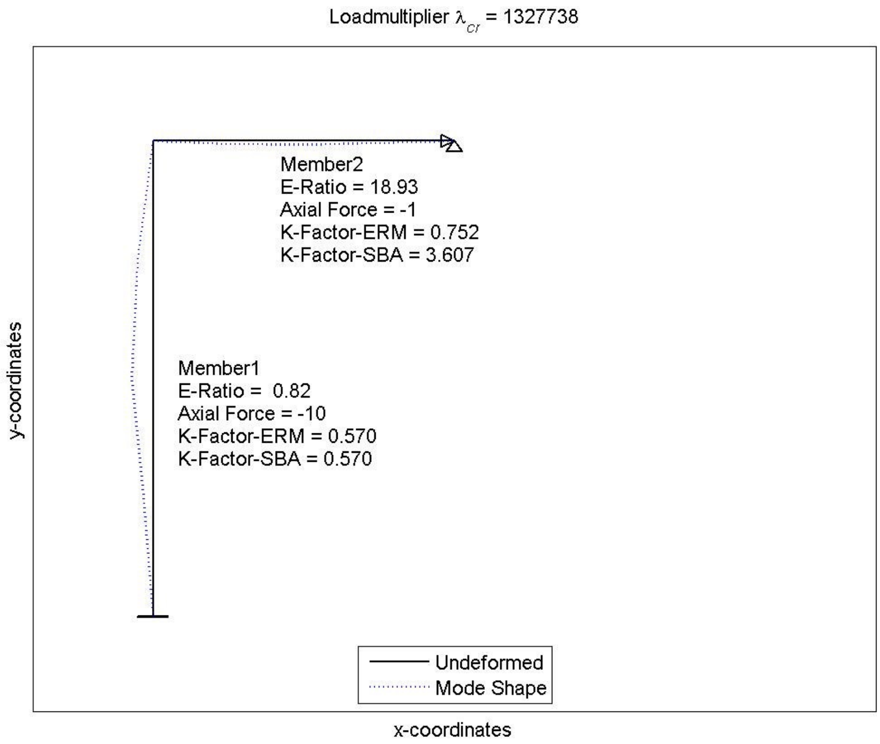


Figure 5-11 Model used to illustrate iterative procedure to find same time buckling

By multiplying the energy ratio r_E by the axial force in each member iteratively the energy ratio in both members becomes one in two iterations. The model of the two iteration steps is seen in Figure 5-12 and Figure 5-13. Note that in Figure 5-13 same time buckling occurs and the energy ratios r_E is equal to one.

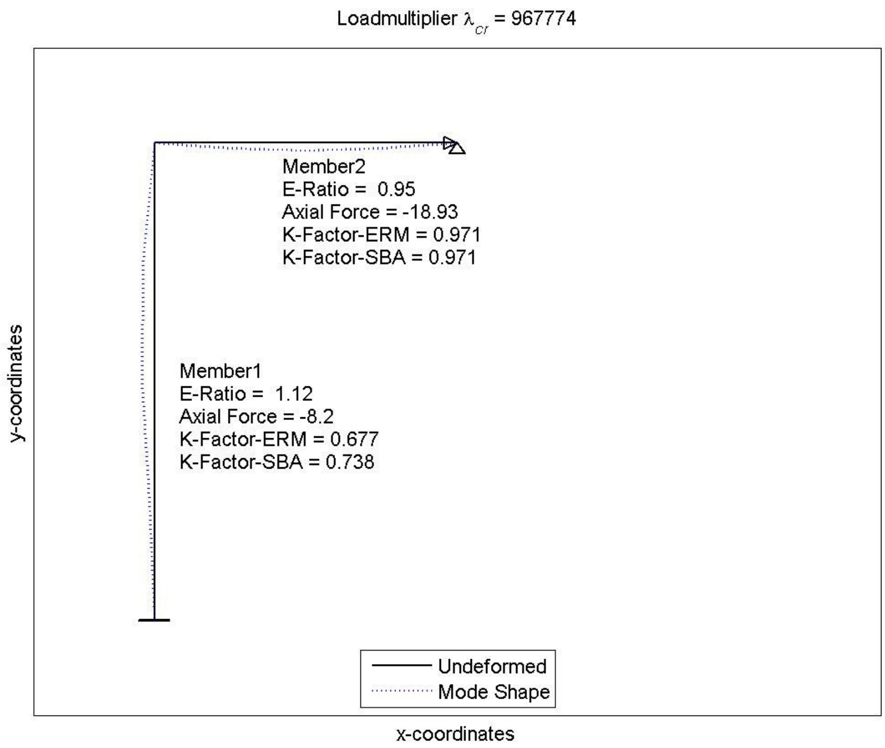


Figure 5-12 First iterative step: Energy ratios approaches one

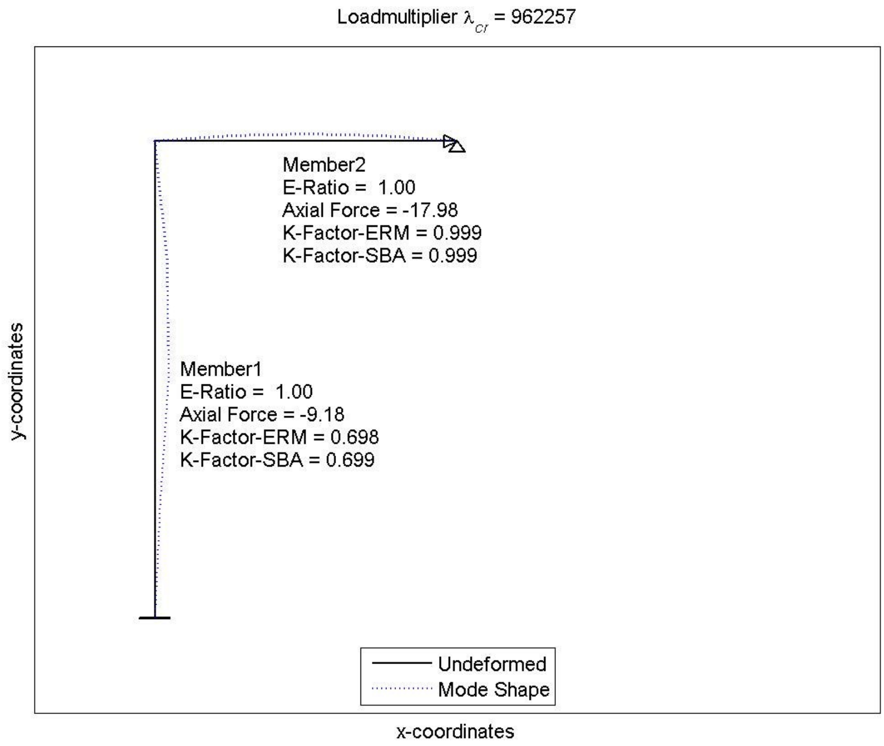


Figure 5-13 Second iterative step: Energy ratios have become one for both members

As an addition to this example, the difference between the traditional SBA, the iterative SBA according to [7], and the ERM can be seen by comparing the K-factors found in Figure 5-11 and Figure 5-13. The K-factors are listed in Table 5-2 and it is seen that the ERM in this example provides more realistic K-factors.

K-factors	Traditional SBA	SBA (same time buckling)	ERM
Member1	0.57	0.70	0.57
Member2	3.61	1.00	0.75

Table 5-2 K-factors found by traditional SBA, improved SBA and ERM

The results from this example illustrates, that same time buckling and the SBA, in cases were the system consists of all compressive members, may provide overly conservative K-factor for the members that are in the buckling state. In cases were the system consists of both tension and compression members the improved might give better results than the ERM from a design point of view.

5.5 Summary

In this chapter, a method to avoid the paradox of the System Buckling Approach has been proposed. The paradox, of members with relatively small axial force having excessively large K-factors, is circumvented by considering members as acting partly as a compression member and partly as a bending member.

Based on the mode shape found by the traditional linear buckling analysis, the internal and external energy can be found in each member, and the energy ratio indicates whether a member is stability sensitive to the load case. A low ratio means the axial forces is performing a relatively large destabilizing outer work.

The one member which has the lowest ratio is considered as being 100 % critical in the given load case, thus the load multiplier defines the critical load for member directly by multiplication to the axial force in the member. This member is chosen as the reference member, and the load multiplier is increased for the other members in the system by a factor corresponding to the ratio between the energy ratio of the considered member and the reference energy ratio of the member being 100 % critical. Thereby the critical load can be found as:

$$N_{cr,i} = N_i \cdot \lambda_{cr} \cdot \frac{r_{E,i}}{r_{E,ref}} \tag{63}$$

The proposed method is easily implemented in traditional FE-codes, and the method is not sensitive to extremely small axial forces.

6. Comparison of Proposed Method to Other Known Methods

In this example, the Energy Ratio Method is applied to a model of a three story building, which has already been investigated by Choi and Yoo (2008) [7]. The proposed method of Choi and Yoo is the aforementioned iterative System Buckling Approach (iSBA) (see Section 4.4), and in their example the method is compared to traditional SBA and story methods.

The purpose of this example is to compare the Energy Ratio Method to the other widely used and newly proposed methods.

6.1 Model Description

The model of the three story building is a 2D frame consisting of vertical members (HEB360) and horizontal members (IPE400). Both the HEB360 and the IPE400 has bending about the strong axis, and sectional properties can be found in Table 6-1. The modulus of elasticity is set to $210.000\ MPa$.

Sectional properties	**HEB360**	**IPE400**
Moment of inertia	$4.319 \cdot 10^{-4}\ m^4$	$2.313 \cdot 10^{-4}\ m^4$
Sectional Area	$1.810 \cdot 10^{-2}\ m^2$	$0.845 \cdot 10^{-2}\ m^2$

Table 6-1 Sectional properties

The story height is $10\ m$, which gives a total height of $30\ m$, and the bay width is $20\ m$. Concentrated loads are imposed on the joints of each story, where the load $P = 100\ kN$. The horizontal loads shown with dotted arrows are only used in the story stiffness method. The model is seen in Figure 6-1.

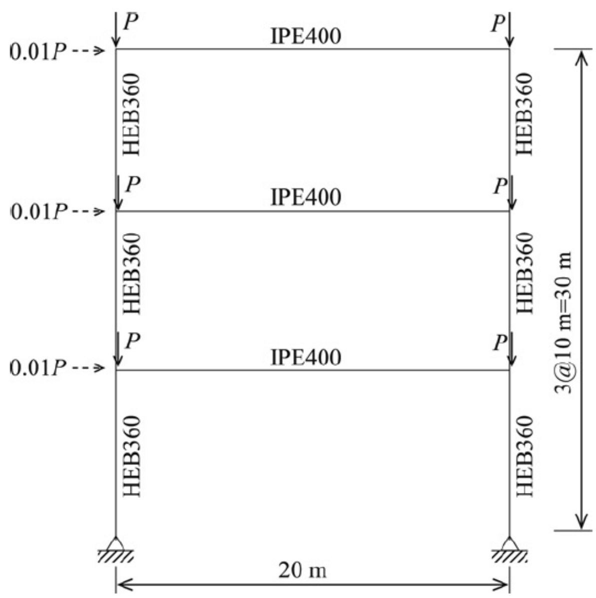


Figure 6-1 Model of the three story building

6.2 Results Compared to the Energy Ratio Method

In the paper presented by Choi and Yoo (2008) the story buckling method, the story stiffness method, the method suggested by Gantes and Mageirou (2005) [12] and the system buckling approach is compared to their proposed method: The iterative system buckling approach. A vague method description has led to questions about the method; however it has been possible to recreate the results from the paper (see Appendix 3).

The Energy Ratio Method (ERM) has been applied to the model and the results are included in Table 6-2 with results from the paper of Choi and Yoo. Furthermore the results are plotted in Figure 6-2, where the Energy Ratio Method is highlighted in red color.

K-factors	Story 1	Story 2	Story 3
Story Buckling Method	2.831	2.643	2.267
Story Stiffness Method	2.989	2.713	2.664
Gantes & Mageirou	2.969	2.775	2.250
System Buckling Approach	2.971	3.639	5.146
Choi & Yoo (iSBA)	3.087	2.345	2.016
Energy Ratio Method	2.971	2.591	2.695

Table 6-2 Comparison of K-factors

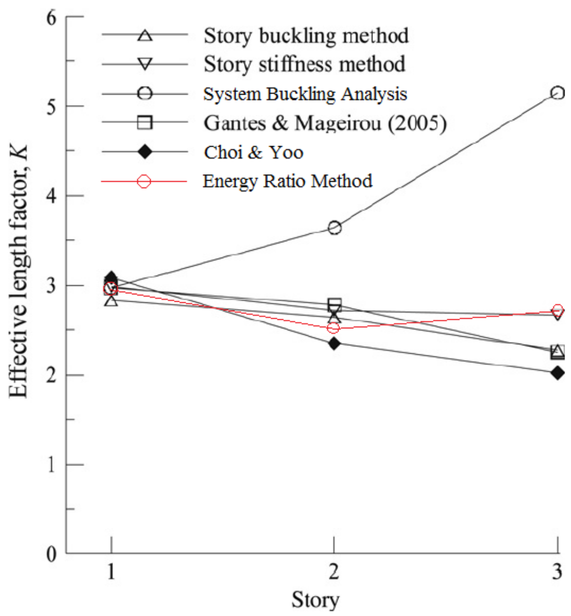


Figure 6-2 Results graphically displayed

The ERM is calculated using a MATLAB script, where beam elements have rectangular cross sections and is based on Bernoulli-Euler beam theory. In Appendix 2 it is verified that results are independent on cross section type and only depends on sectional properties.

6.2.1 Similar Results Found by the ERM Compared to Other Known Methods

The results show the SBA paradox of the top stories having excessively large K-factors. It is clear, that the other methods provide much more realistic results. The method proposed by Choi and Yoo gives the most favorable results for story 2 and story 3; however it is on the cost of some of the strength of story 1. This is related to the slight decrease in the load multiplier. When only considering story frames with no axial compressive force in the horizontal members, the proposed method by Choi and Yoo provides favorable results, however if there is a relatively small axial compressive force in the horizontal members the applied fictitious forces will tend to decrease load multiplier significantly. This would lead to conservatively large K-factors for the vertical members in story 1. One of the strength of the ERM is that it is provides intuitive realistic K-factors for all members in the structure. E.g. if an axial compressive force of $0.01P$ where applied to the horizontal members the vertical members would get a non-noticeable increase in the K-factor and the horizontal members would be assigned a K-factor of approximately 0.2.

6.2.2 Inconsistent Results Found by the ERM Compared to Other Known Methods

The ERM stands out in the result by the fact that it provides a larger K-factor for Story 3 than for story 2. It is assessed, that this is caused by the fact that the internal strain energy becomes relatively smaller in story 3 than story 2 because of the relative transverse displacement of each story seen in the mode shape (see Figure 6-3). This effect is also indicated when comparing the Story Buckling Method to the Story Stiffness Method (see Table 6-2), where the decrease of the K-factor is small by the Story Stiffness Method, because it considers the transverse displacements.

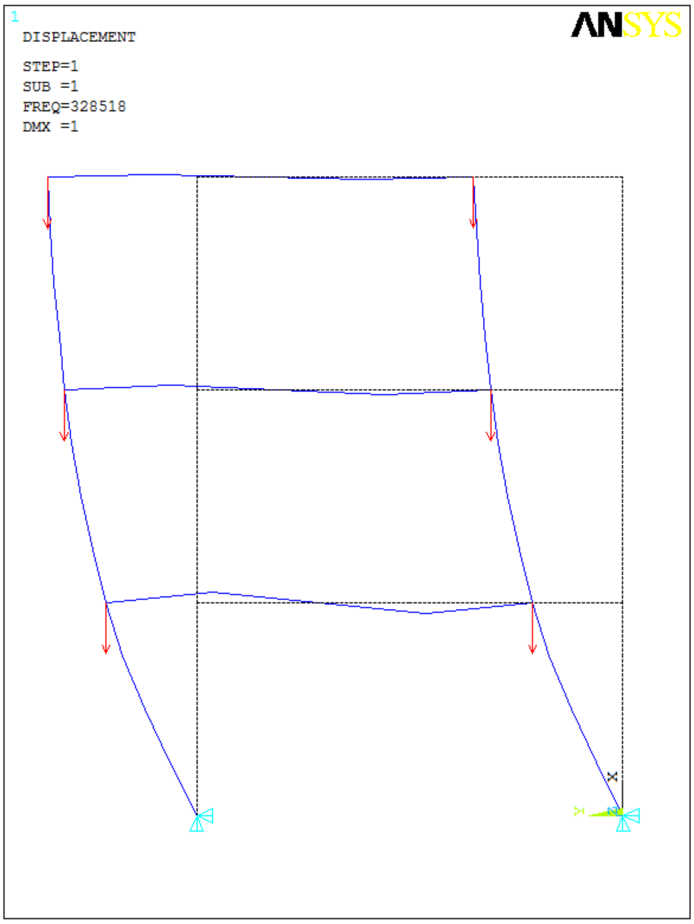
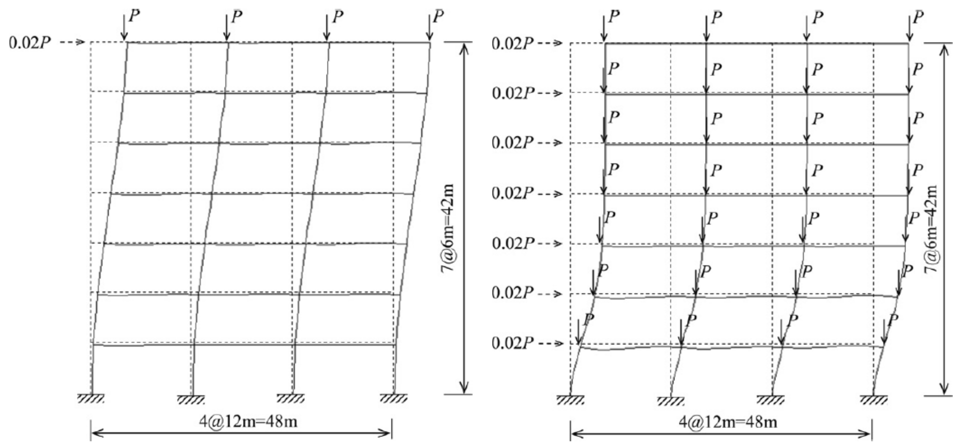


Figure 6-3 Modeshape of three story building where axial load increases down the columns

The main difference in sway when the axial load increases down the columns compared to a constant axial load is emphasized in another example by Choi and Yoo, where a seven story three bay frame is considered (see Figure 6-4). Note the increase (marked with red circle) in the K-factor from story 6 to story 7 in load case 2 is found by the story stiffness method.



(a) Load case 1. (b) Load case 2.

Fig. 1. A seven-story three-bay frame and its buckled modes under different loading configurations. (Note: Lateral loads are used only in the story-stiffness method.)

Table 1
Comparison of the effective length factors of the outer columns in the seven-story three-bay frame.

Method	Alignment chart (Eq. (2))	Story buckling method (Eq. (4))	Story stiffness method (Eq. (5))		Elastic buckling analysis (Eq. (9))	
Story	LC 1 and LC 2	LC 1 and LC 2	LC 1	LC 2	LC 1	LC 2
1	1.564	1.467	1.617	1.557	2.045	1.704
2	2.466	2.105	2.150	2.118	2.045	1.840
3	2.466	2.105	2.251	2.236	2.045	2.016
4	2.466	2.105	2.272	2.268	2.045	2.254
5	2.466	2.105	2.268	2.280	2.045	2.602
6	2.466	2.105	2.243	2.302	2.045	3.187
7	2.130	1.853	2.063	2.452	2.045	4.507

Figure 6-4 Example from Choi and Yoo (2008)

6.3 Summary

This example illustrates how the ERM provides realistic K-factors in accordance to known analytical methods. Furthermore it has been noted, that the method is not sensitive to members with relatively small axial force, wherefore the method is suitable for implementation in FE-codes.

7. Conclusion

The critical length is a key parameter when designing compressive members according to the Eurocode. Most engineers have a reasonable idea of, in which interval the K-factor is expected to be, however as the complexity of the structures increases the risk of incorrect stability assessment becomes imminent.

New structures are often designed with some degree of spare capacity, therefore exact K-factors are not that crucial, however the risk of overlooking a general behavior, that yields larger K-factor than expected, could lead to a disastrous design. This can be avoided by performing a linear buckling analysis, which will show the general behavior of the system.

When existing structures are reassessed because of change in the anticipated load exposure or design safety, some members might become over utilized according to the design formulas. By evaluating more accurate K-factors than previously conservative values, one might avoid the need to strengthen the structure.

Analytical correct K-factors are seldom obtainable, wherefore methodical approaches are widely used to give estimates of K-factors. These methods can provide realistic K-factors in typical structures such as story-frames, but due to various assumptions in the methods the usability is primarily limited to regular frames.

In complex structures where methodical approaches cannot be used, numerical approaches like the system buckling approach are useful. The method provides an exact solution of the member most prone to buckling, however for members with a low axial force that acts mainly as stabilizing bending members the method yields excessively large K-factor. This is a known paradox and the reason why, the method is not implemented as an automatically determination of K-factors in code check software.

The main objective of this thesis was to propose a method that overcomes the paradoxical weakness of the system buckling approach. The proposed method is based on energy considerations, where the load multiplier from a linear buckling analysis is weighed to the individual members. It was found that method provides realistic K-factors that are not subject to the paradox of the traditional system buckling approach.

The proposed method is deduced from physical arguments, where the energy balance between inner and outer work is used as an indicator for how much the individual members are sensitive to additional load in the critical state. Only 2D verification of the method was performed, however further thorough investigation and application of the method is believed to have the potential of the development of a method, which can be implemented with into code check in existing code check software.

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APPENDIX 1

Examples of Methodical and Experience Based Approaches

1. Example 1: Offshore Bridge between Gorm C and Gorm E

An often used type of offshore bridge consists of three chords and a system of braces. A cross section of this type of bridge forms a triangle, where some bridges are orientated with the apex chord upwards as the bridge between Gorm C and Gorm E (see *Figure 1-1*), and some are turned upside down.

Figure 1-1 Typical offshore bridge between Gorm C and Gorm E in the North Sea.

The bridge between Gorm C and Gorm E has a span of 102.5 m. The chord in the apex has a larger cross sectional area than each of the two chords carrying the deck. The braces have the largest cross sectional area at the ends of the bridge and smallest at the middle, which is logically chosen when thinking of the shear distribution of a uniform loaded beam.

In the present example the choice of critical length for the chords and braces in the bridge is discussed. Appropriate default values, which have been used for the actual design of the bridge, are outlined and a linear buckling analysis of a beam model is used for comparison. To see the difference between this type of bridge having the apex chord in the top or bottom, the same linear buckling analysis is performed were the beam model is turned upside down.

The purpose of this example is to:

- Validate the use of empirical K-factors on the members that are most prone to buckling.
- Illustrate that in some cases the general behavior with a global sway mode becomes determinative for the buckling capacity, so that the use of empirical K-factors for individual members no longer applies.

1.1 Empirical Values of K-factor for Calculation of the Critical Length

For the design of the bridge empirical default values of the K-factor have been used. All chords are assigned a K-factor of 1 and all braces are assigned a K-factor of 0.7. These values can be found in the FEM package ROSAP, which was used for the design of the bridge.

1.2 Description of Beam Model

The geometry is modeled according to the structural drawings found in Annex 1.

The support conditions of the bridge is pinned in one end (no displacement but rotations allowed in all directions) and roller supported at the other end of the two deck chords (same as pinned but displacements allowed in the bridge direction).

Point loads is applied the keypoints connecting braces to the two deck chords. This represents a uniform load distribution, which is typical for an offshore bridge.

The model of the bridge is shown in *Figure 1-2* and *Figure 1-3* orientated normally and upside down respectively.

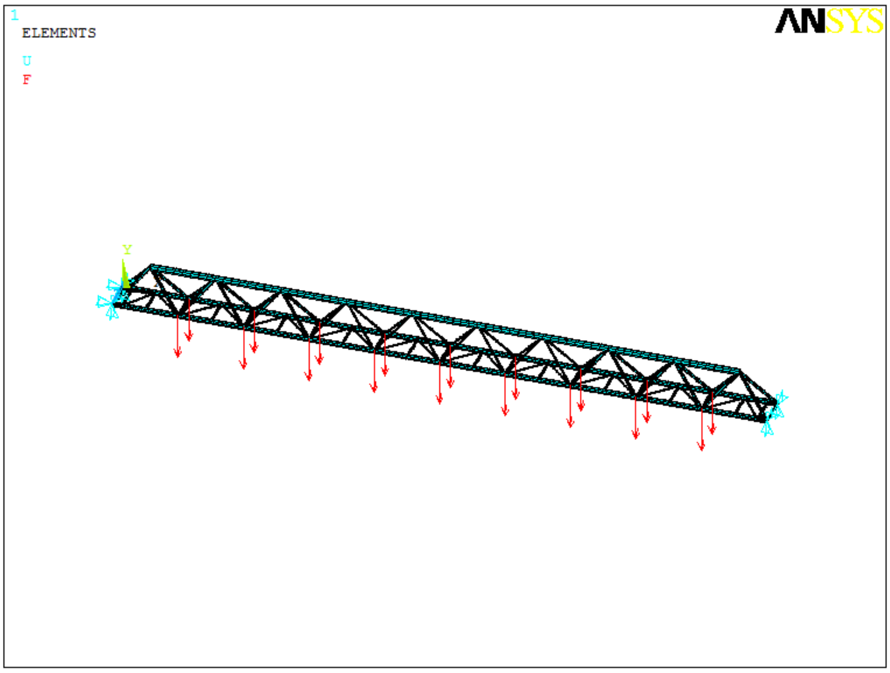

Figure 1-2 Beam model of the bridge between Gorm C and Gorm E.

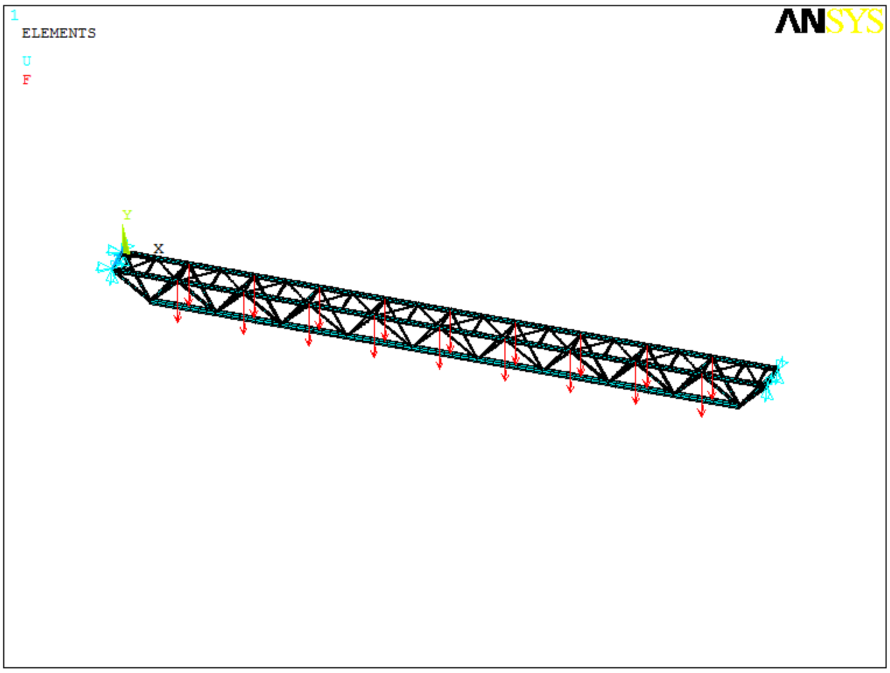

Figure 1-3 Beam model of the bridge turned upside down.

1.3 Results from buckling analysis

In the buckling analysis it is found that the first and second mode in principle has the same deformation pattern. The difference in the two modes is the direction in which the deformation happens. The load multiplier for the first and second mode is $0.390 \cdot 10^7$ and $0.401 \cdot 10^7$ respectively. In *Figure 1-4* and *Figure 1-5* the two modes are seen.

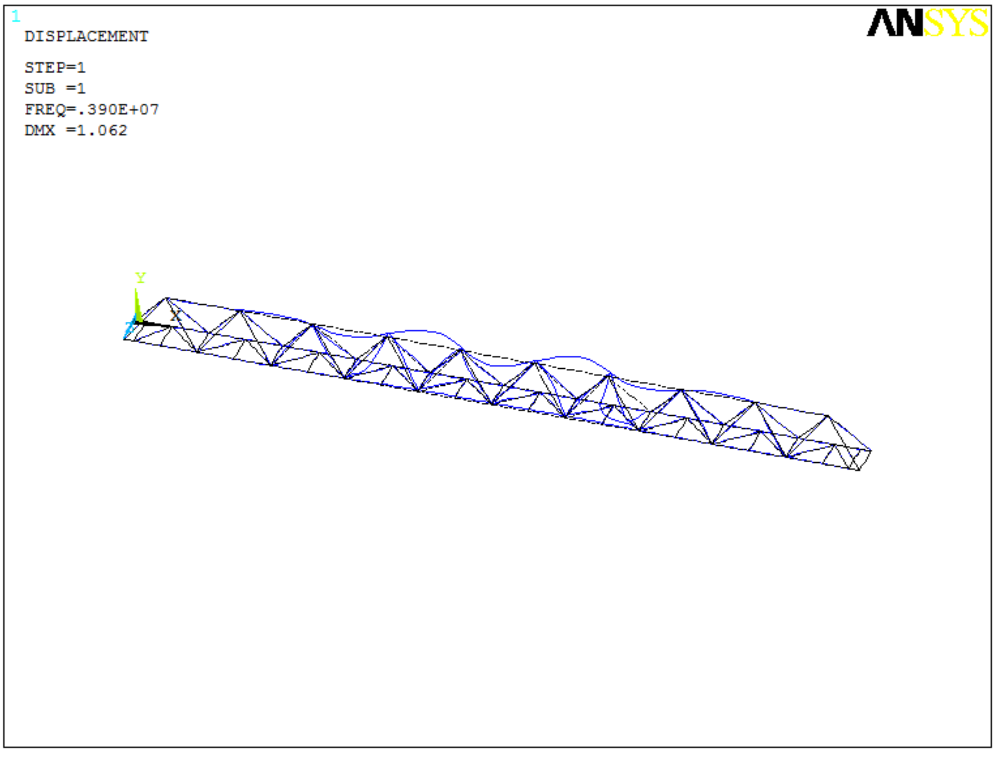

Figure 1-4 First mode. Apex chord deforming in the horizontal plane.

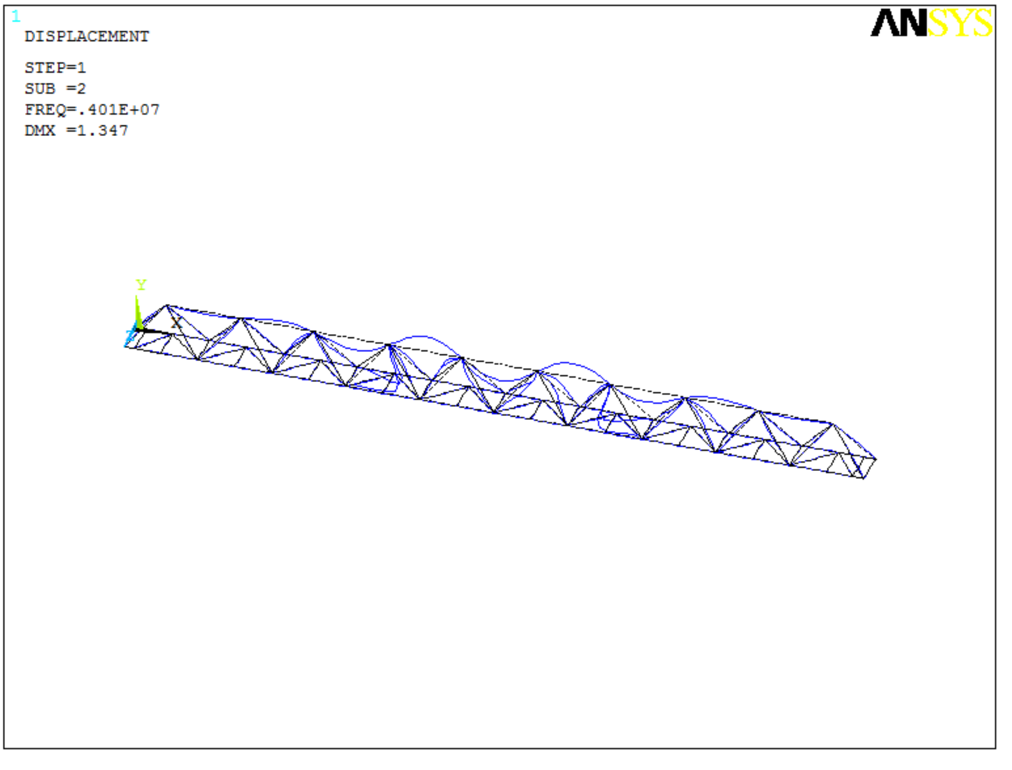

Figure 1-5 Second mode. Apex chord deforming in the vertical plane.

To see the distinct deformations the second mode is chosen to be seen from the side (see *Figure 1-6*).

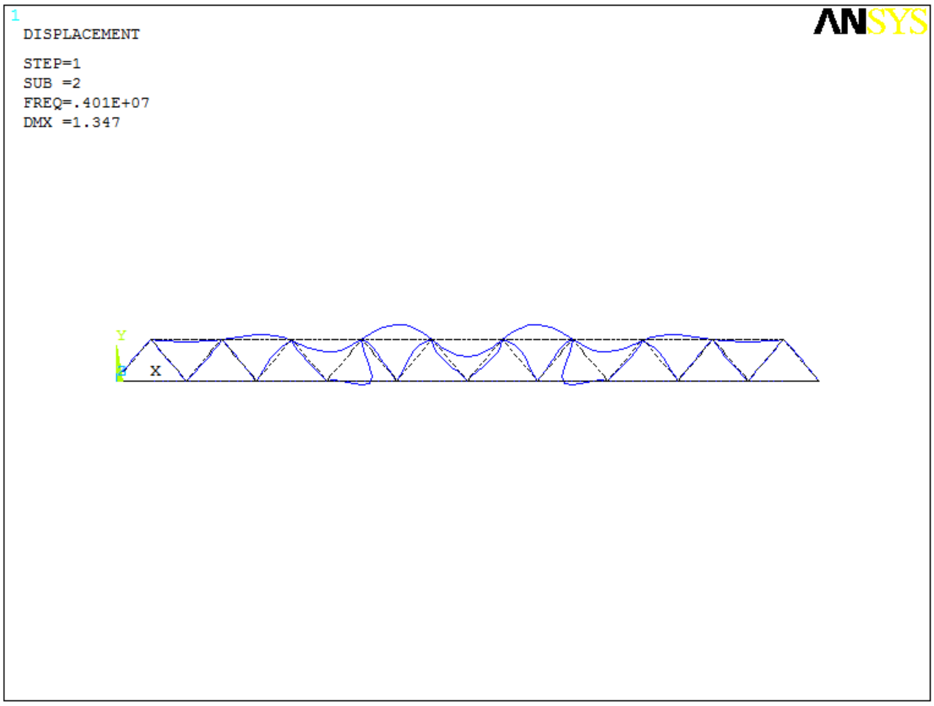

Figure 1-6 Side view of the second mode shows distinct deformations of the chord and braces.

It is noted that the most slender braces furthest from the middle receives the largest deformations. This indicates that critical stress level is reached first in these elements when the load is gradually increased. Also deformations of the apex chord are significant, which indicates that the stress level is close to the critical at the time when the brace buckles. In the following, axial forces in the aforementioned braces and chord at the critical load level are compared to the critical axial forces calculated from default critical lengths. The load multiplier of $0.390 \cdot 10^7$ is used and results are given in *Table 1-1*.

Critical Axial Force	Axial Force in element at critical load level	Critical axial force calculated from default critical length	Deviation
Brace (largest deformation)	$7.92 \cdot 10^6$	$4.41 \cdot 10^6$	44 %
Chord (middle of the bridge)	$1.65 \cdot 10^8$	$1.61 \cdot 10^8$	2 %

Table 1-1 Comparison of critical load level to critical axial forces determined by default K-factors.

It is assessed that the large deviation at the brace is due to a conservatively taken default value of the K-factor. If the K-factor is calculated backwards from the axial force in the braces at the critical load level, a value of 0.52 is found. Compared to the default value of 0.7 it seems so that the element boundary conditions is closer to a column fixed in both ends. This is in agreement with the fact that the brace buckles before the chord, wherefore the rotational stiffness is high at the brace ends. The default value of 0.7 insures the situation where the load in the brace and the chord reaches a critical level at the same time, which allows rotation at one of the brace ends.

When using empirical values it is important to assess if it is reasonable values. In this case the position of the keypoints connecting the braces to the chords is fairly fixed in the buckling modeshape, since the chord in compression (the apex chord) is maintained by braces in two transverse directions. This means that the general behavior is not determinative for the buckling capacity, and therefore K-factors between 0.5 and 1 is expected according to the basic cases (see Figure 2-2 section 2.1 of the main report). In the following section a hypothetical situation where the bridge is turned upside down is analyzed to emphasize when care should be taken.

1.4 Upside Down Bridge

When turning the bridge upside down the deck chords are in compression. The keypoints at the deck chords are not fixed if the bridge has a global mode shape corresponding to a horizontal transverse deformation of the deck. Therefore the bridge may fail by buckling at a lower load level than expected when using default K-factors. In *Figure 1-7* the first modeshape of the upside down bridge is seen.

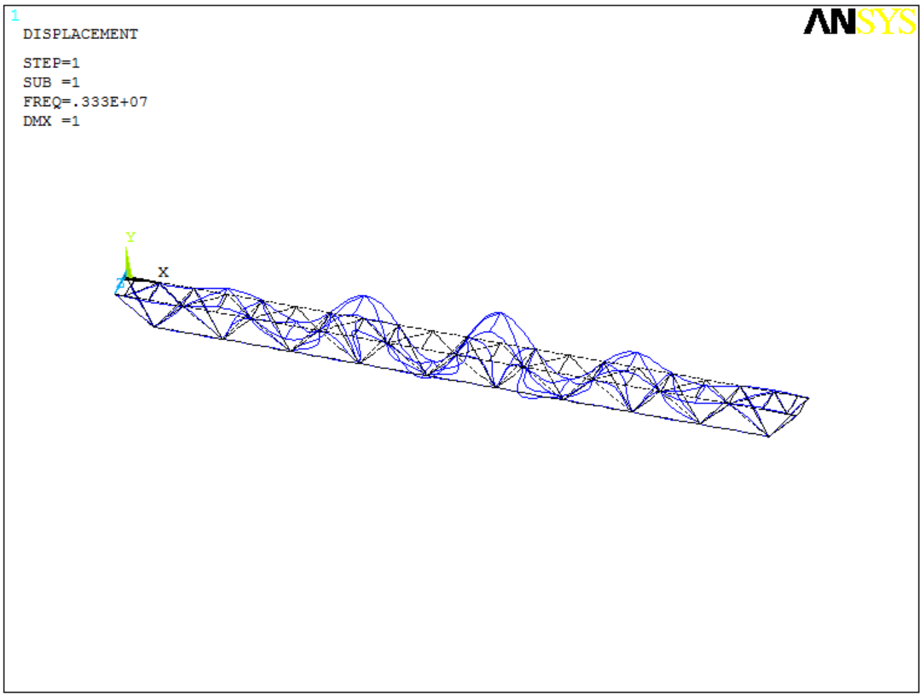

Figure 1-7 First mode of the upside down bridge.

The same comparison of the K-factor as in section 1.3 is made in *Table 1-2* for the first modeshape.

Critical Axial Force	Axial Force in element at critical load level	Critical axial force calculated from default critical length	Deviation
Brace (largest deformation)	$6.70 \cdot 10^6$	$4.41 \cdot 10^6$	34 %
Chord (middle of the bridge)	$6.81 \cdot 10^7$	$6.33 \cdot 10^7$	7 %

Table 1-2 Comparison of critical load level to critical axial forces determined by default K-factors.

The same conclusions as in section 1.3 concerning the K-factor can be made. However the deviation for the brace is smaller, because the ratio r_{total} is closer to 1 (see *Table 1-3*). The ratio is given by:

$$r_{total} = r_{cr}/r_A$$

where r_A is the ratio of the axial force in the chord and brace, and r_{cr} is the ratio of the critical axial force for the chord and brace. If $r_{total} = 1$ the elements buckle at the same time with element boundary conditions corresponding to the K-factors used for calculating the critical axial forces. Comparison of the ratios can be seen in *Table 1-3*.

Ratio table	Axial force ratio chord/brace r_A	Critical force ratio chord/brace r_{cr}	Total ratio r_{total}
Bridge normally orientated	20.8	36.5	1.75
Bridge turned upside down	10.2	14.4	1.41

Table 1-3 Comparison of ratios for assessment of sametime buckling of chord and brace.

The first modeshape did not show the global sway modeshape with a low critical level, but the modeshape appears as the second mode (see *Figure 1-8*).

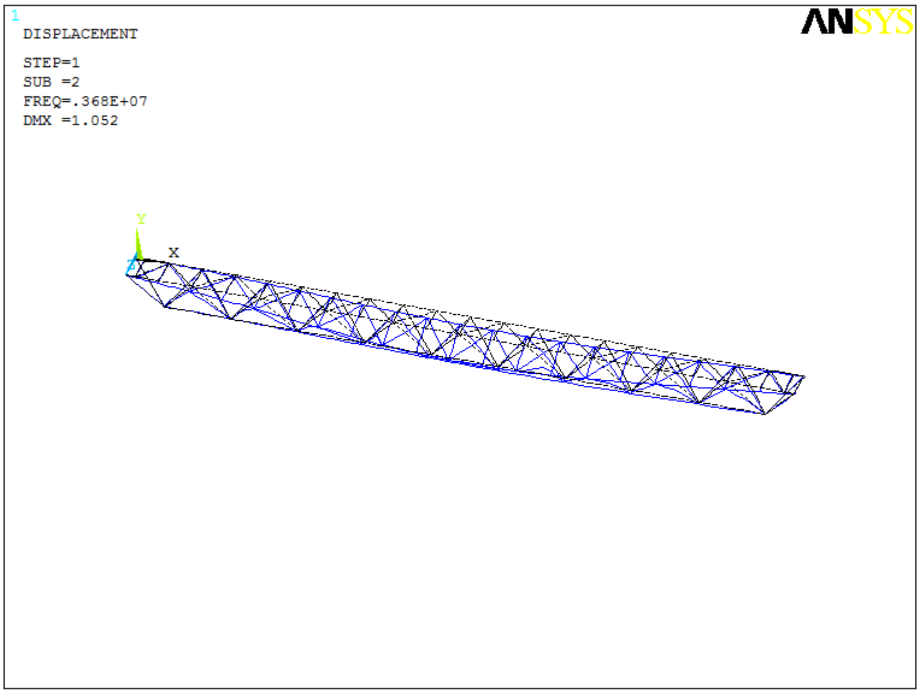

Figure 1-8 Second mode of upside down bridge. Global modeshape with keypoints deformed out of initial position.

The second modeshape shows a global modeshape where the keypoints is deformed out of initial position. However the deck chords and deck braces provides enough stiffness so the global sway modeshape is not the first critical mode. If the bridge deck is narrowed to half the width, a load multiplication of $0.172 \cdot 10^7$ is found (see *Figure 1-9*), which gives a considerable lower critical load level than the first mode in *Figure 1-7*.

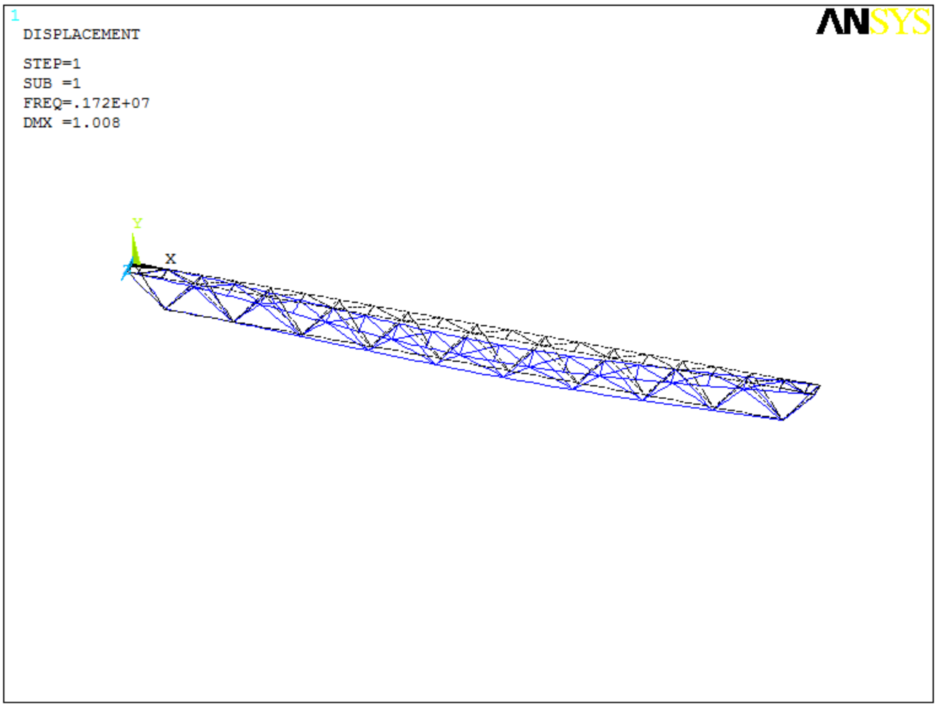

Figure 1-9 Narrow bridge with a global sway modeshape at a low critical level.

1.5 Summary of Example 1

Using the linear buckling analysis on a beam model of the bridge between Gorm C and Gorm E the default values of K-factors for chords and braces is explained. The risk of global sway modes is clarified with an upside down narrow version of the bridge that gives a low critical load level. A general behavior with a global sway modeshape might be overlooked by the designing engineer, however the linear buckling analysis provides an effective check of this, so that unstable designs can be avoided.

2. Example 2: Support Structure for Galleries

The support structure for this gallery is an example of an uncommon structure where the general behavior of the system is hard to predict. The support structure consists of I-profile beams while the bracing is made with tubes. The structure is characterized by being slender and unbraced in one plane and thick and semi brace in the other.

In the following the subassembly method is used to evaluate the stability of the structure and the results are compared to a linear buckling analysis. Furthermore common engineering practice is used by simplified assumptions.

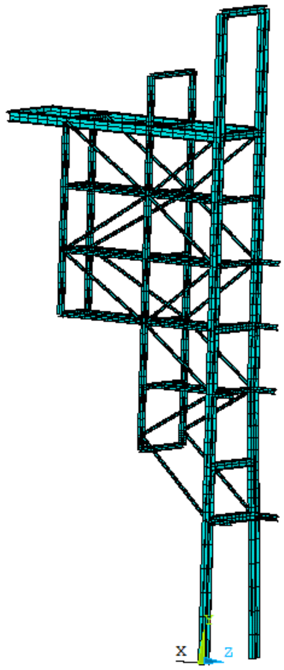

Figure 2-1 Support structure for galleries

2.1 Isolated Subassembly Approach

Based on the pre-EN1993 standard described in section 3.2 of the main report the critical column length for the straight bottom column is calculated using the isolated subassembly method.

The column is a HEB340 as sketched on *Figure 2-2* with the weak axis out of plane.

Based on the structure, it is assumed that the column is well braced about its weak axis, which means a critical column length about the weak axis with a value between 0.5 and 1 times the actual length. The value of 1.0 is chosen. This simplification was made in the original design.

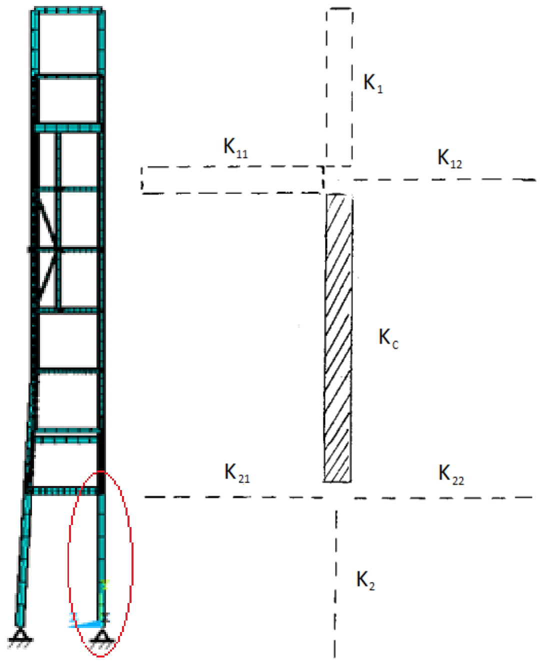

Figure 2-2 Column designations

The column is simple supported at the bottom which means K_2, K_{21} and K_{22} are equal to zero. This also applies to K_{12}. The value of effective beam stiffness K_{11} is calculated based on the most conservative approach as described in section 3.2 of the main report. The horizontal beam and the top column are also HEB340 profiles.

$$K_C = \frac{I}{l} = \frac{366.6 \cdot 10^6}{5780} = 63{,}426 \frac{N}{mm^5} \tag{64}$$

$$K_1 = \frac{I}{l} = \frac{366.6 \cdot 10^6}{2200} = 166{,}636 \frac{N}{mm^5} \tag{65}$$

$$K_{11} = \frac{0.5 \cdot I}{l} = \frac{366.6 \cdot 10^6}{3155} = 58{,}098 \frac{N}{mm^5} \tag{66}$$

$$\eta_1 = \frac{K_C + K_1}{K_C + K_1 + K_{11} + K_{12}} = \frac{63{,}426 + 166{,}636}{63{,}426 + 166{,}636 + 58{,}098} = 0.80 \tag{67}$$

$$\eta_2 = \frac{K_C + K_2}{K_C + K_2 + K_{21} + K_{22}} = \frac{63{,}426}{63{,}426} = 1.0 \tag{68}$$

Using the formula for sway mode the following K-factor is obtained.

$$\frac{l_{s,strong}}{l} = K_{strong} = \left[\frac{1 - 0.2\,(0.80 + 1.0) - 0.12\;0.80\;1.0}{1 - 0.8\,(0.80 + 1.0) + 0.6\;0.80\;1.0}\right]^{0{,}5} = 3.7 \tag{69}$$

To sum up the following K-factors are found using a conventional way of establishing critical lengths: $K_{strong} = 3.69$ and $K_{weak} = 1.0$

2.2 ANSYS Model

In order to perform a linear buckling analysis of the structure a model is build up in ANSYS APDL.

The structure is simply supported at the bottom and this is sketched on *Figure 2-3*.

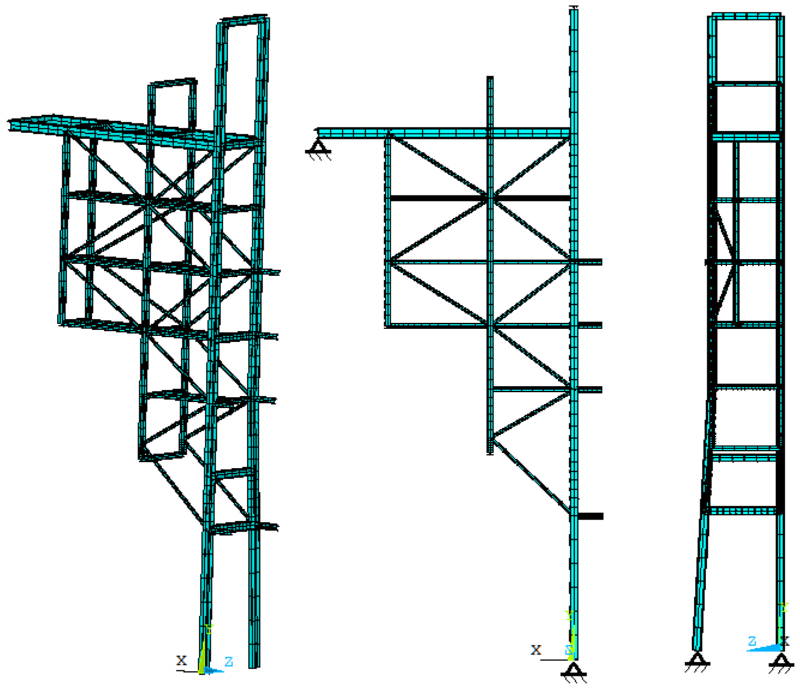

Figure 2-3 Support structure for galleries

Two point loads of 0.5 N each in the vertical direction are applied to the joints shown on the figure below.

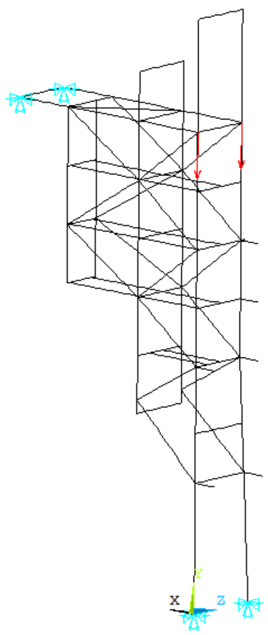

Figure 2-4 Application of loads

A buckling analysis is performed and the load multiplier and axial force of the column is outputted.

Load multiplier λ_1	**Axial force** N
$0.947 \cdot 10^7$	$0.514\ N$

Table 2-1 ANSYS Output for first buckling mode

The corresponding mode shape is shown in the following figures.

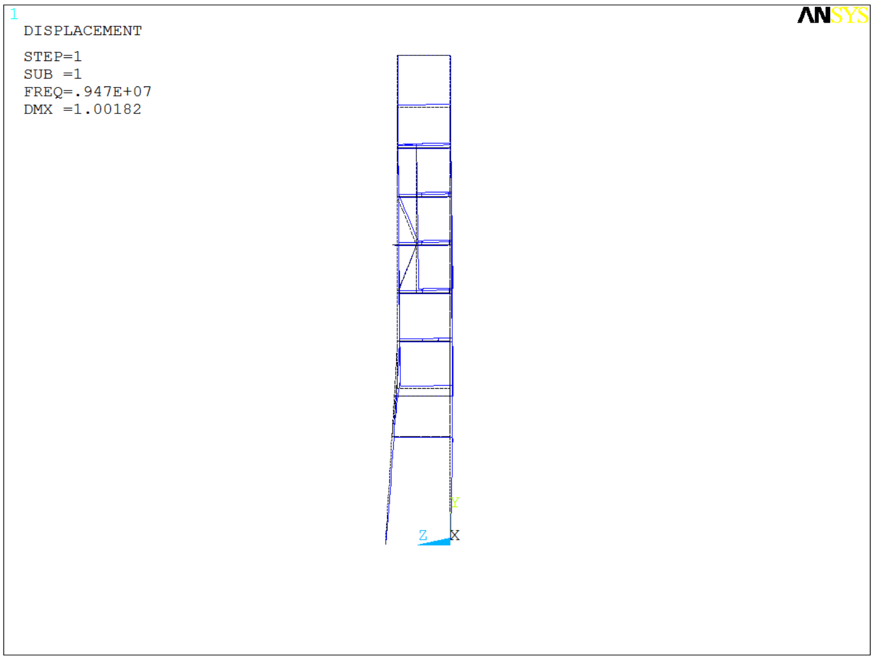

Figure 2-5 First buckling mode - front view

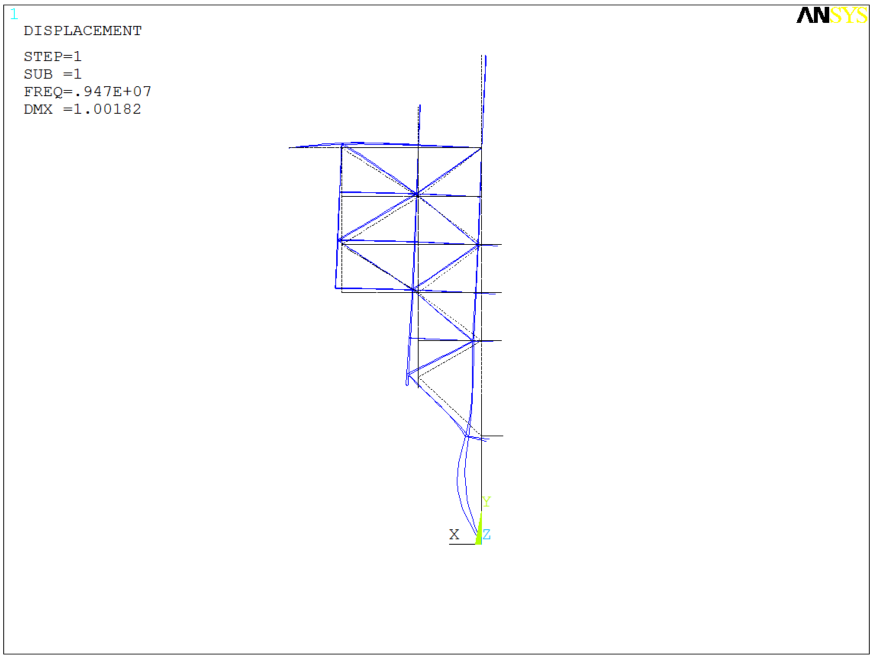

Figure 2-6 First buckling mode - side view

The output gives a critical load for the column, and based on this load, the critical column length for the found buckling mode is derived.

$$N_{cr} = \lambda_1 \cdot N = 4.868 \cdot 10^6 N \tag{70}$$

Critical length about the weak axis:

$$K_{weak} = \sqrt{\frac{\pi^2 \; E \; I}{N_{cr} \; l^2}} = \sqrt{\frac{\pi^2 \cdot 2.1 \cdot 10^5 \cdot 96.9 \cdot 10^6}{4.868 \cdot 10^6 \cdot 5780^2}} = 1.1 \tag{71}$$

In order to calculate the critical column length about the strong axis the buckling mode with the lowest multiplier corresponding to this deflection is found. This is the case with the third buckling mode as shown on the figures below.

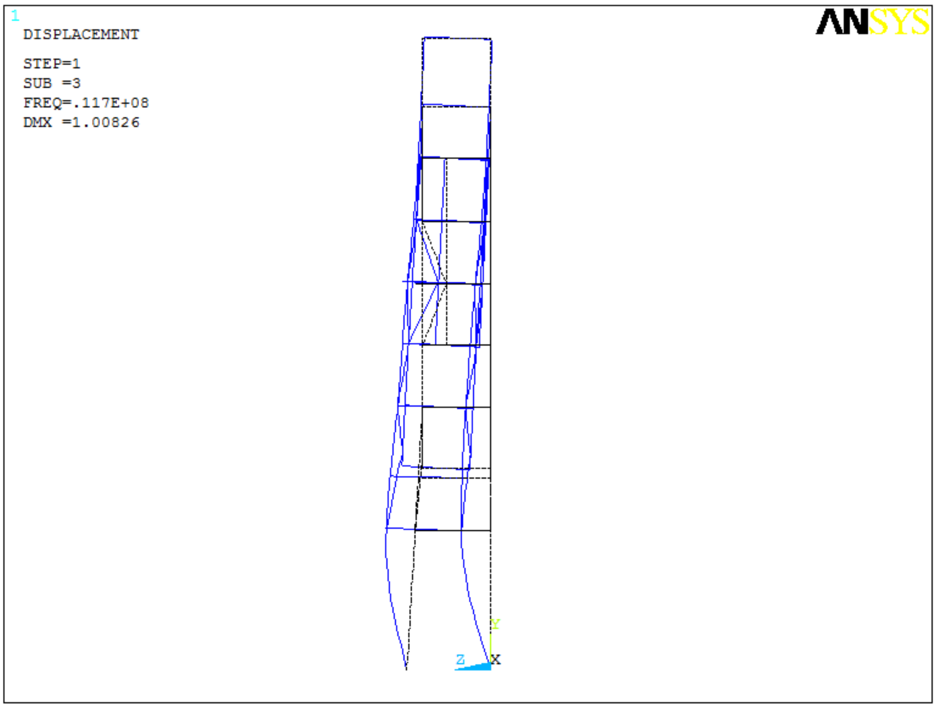

Figure 2-7 Third buckling mode - front view

e

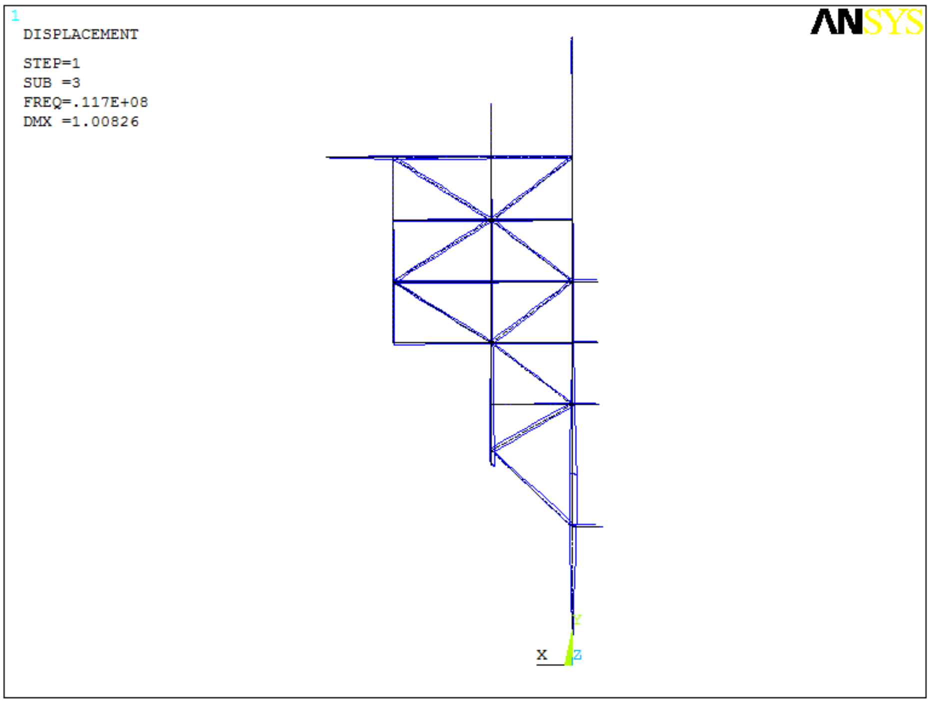

Figure 2-8 Third buckling mode - side view

Load multiplier λ_3	**Axial force** N
$1.17 \cdot 10^7$	$0.514\ N$

Table 2-2 ANSYS Output for third buckling mode

This gives a critical buckling load of:

$$N_{cr} = \lambda_3 * N = 6.0138 \cdot 10^6 N \tag{72}$$

The critical column length about the strong axis can then be calculated as:

$$K_{strong,ANSYS} = \sqrt{\frac{\pi^2\ E\ I}{N_{cr}\ l^2}} = \sqrt{\frac{\pi^2 \cdot 2{,}1 \cdot 10^5 \cdot 366{,}6 \cdot 10^6}{6{,}0138 \cdot 10^6 \cdot 5780^2}} = 1.9 \tag{73}$$

2.3 Comparison of Isolated Subassembly Method and ANSYS

In the following chart the results from the two methods are summarized.

	Isolated Subassembly	**ANSYS**	**Deviation**
Weak axis	1.0	1.1	9,7 %
Strong axis	3.7	1.9	90 %

Table 2-3 Comparison of K-factors

About the weak axis the assumption that the column was well braced, and therefor has a critical column length of one times the actual length of column turns out to be non-conservative. When used in further calculations to determine the load capacity, the ANSYS critical length gives a reduction factor $\chi = 0.51$ and the prEN1993 gives $\chi = 0.56$.

About the strong axis the difference is much larger. A K-factor of 3.7, as found by the subassembly approach, seems to be a very conservative value. This is partly due to the fact that the structure is semi-braced against sway at the very top of the structure. This means that the overall sway mode is a mix between non-sway and sway. This is not accounted for in the method. By looking at formula (67) and (68) it becomes apparent that the only way to reduce the critical length is to maximize the value of K_{11} and minimize the value of K_1. It can be argued that the value of K_1 should be reduced as the resistance against rotation at the top by default is assumed fixed. Reducing the value of K_1 and maximizing the value of K_{11} would at best give the following distribution factors.

$$\eta_1 = \frac{K_C + K_1}{K_C + K_1 + K_{11}} = \frac{63{,}426 + \frac{166{,}636}{2}}{63{,}426 + \frac{166{,}636}{2} + 58{,}098 \cdot 1.5} = 0.63 \tag{74}$$

The distribution factor gives a K-factor of 2.83 which is still larger than the obtained ANSYS value of 1.9. This supports the claim that the method does not scale to this type of problem where the structure is a mixture of sway and non-sway.

An alternative way of calculating the critical length of the bottom columns is to see the bottom frame as an isolated frame, thereby assuming that the structure above doesn't have any influence on the motion of the bottom frame. This means that the K_1 and K_{11} in formula (67) are equal to zero. By calculating formula (67) and (68) the following distribution factors are obtained:

$$\eta_1 = \frac{K_C + K_1}{K_C + K_1 + K_{11} + K_{12}} = \frac{63{,}426}{63{,}426 + 58{,}098} = 0.52 \tag{75}$$

$$\eta_2 = \frac{K_C + K_2}{K_C + K_1 + K_{21} + K_{22}} = \frac{63{,}426}{63{,}426} = 1.0 \tag{76}$$

Thereby the K-factor becomes:

$$\frac{l_{s,weak}}{l} = K_{weak} = \left[\frac{1 - 0.2\,(0.52 + 1.0) - 0.12\;0.52\;1.0}{1 - 0.8\,(0.52 + 1.0) + 0.6\;0.52\;1.0}\right]^{0.5} = 2.6 \tag{77}$$

So by using this alternative approach to calculating K-factor a value of 2.6 is obtained. This value is closer to value from the buckling analysis but still higher. This is again because of the semi braced nature against sway at the top of the structure.

In the table below the reduction factors, χ, about the strong and weak axis for both isolated subassembly method and ANSYS is calculated. The reduction factor for strong axis buckling for isolated subassembly is based on the lowest value of K found using the alternative approach.

Reduction factor χ	Isolated Subassembly	ANSYS
Strong axis	0.42	0.56
Weak axis	0.56	0.51

Table 2-4 Summary of reduction factors

It quickly becomes apparent that estimating the critical column lengths can lead to either too large or too small values. This is primarily due to the fact that the general behavior of the system becomes hard to predict with the increasing complexity. Furthermore it should be noted the results found in ANSYS using the linear buckling analysis only is valid for the current load case, whereas the isolated subassembly method provides a K-factor that does not depend on the load case.

2.4 Summary of Example 2

In this example the K-factor has a large variation based on the method used.

When using the isolated subassembly method from prEN1993 it is found that the largest K-factor is about the strong axis. A linear buckling analysis in ANSYS shows the same, but with different values.

In this case the isolated subassembly method from preEN1993 is overly conservative with a reduction factor 18% lower than what system buckling approach in ANSYS suggests.

APPENDIX 2

Verification of Cross-section input for 2D Elements in Strain Energy Calculations

1. Verification of Cross-section Type Independence

This appendix contains a comparison between a rectangular cross section using Bernoulli-Euler beam theory and an I-profile cross section using Timoshenko beam theory. The point of which is showing that the type of cross section, i.e. rectangular and I-profile, provides the same result for the energy calculations as long as the area and moment of inertia are identical.

1.1 General model

The frame is a three story frame. The columns are HEB360 and the horizontal girders are IPE400. The frame has a total height of 30m and a width of 20m. The frame is simply supported at the base and a load of 1 N is placed at each story.

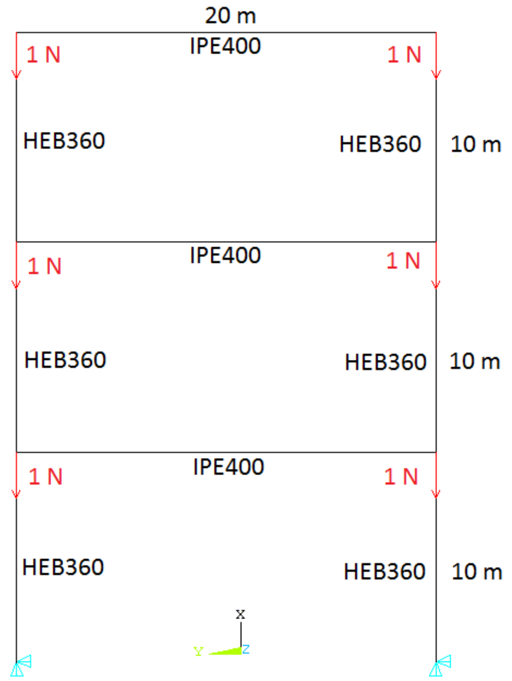

Figure 1-1 Three story frame

The sectional properties are listed in the table below:

	HEB360	IPE400
Area	$0.017438\ m^2$	$0.008068\ m^2$
Moment of inertia	$0.418 \cdot 10^{-3} m^4$	$0.219 \cdot 10^{-3} m^4$

Table 1-1 Sectional properties

1.2 2D Frame

The 2D model is based on beam elements with a rectangular cross section with the sectional properties of an I-profile about its strong axis.

This gives the following result:

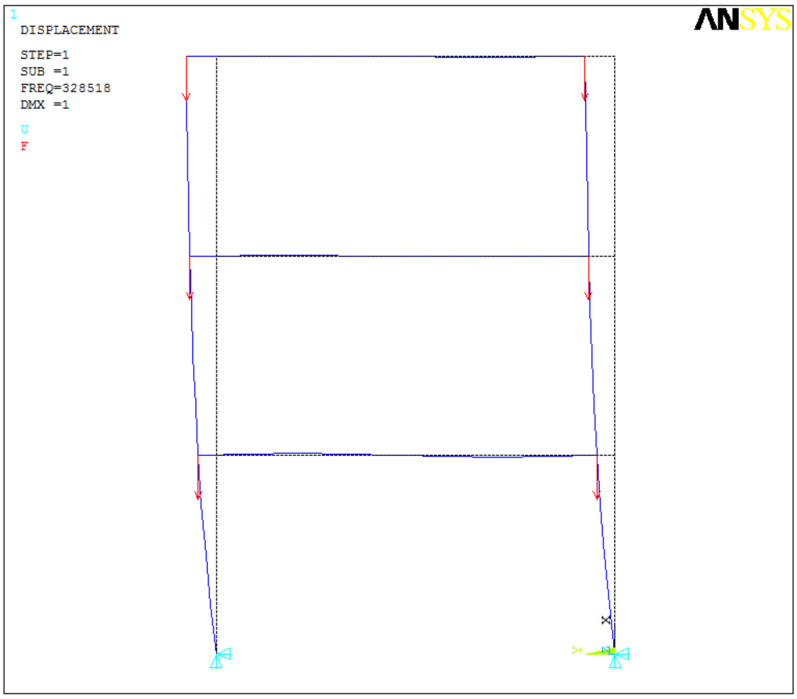

Figure 1-2 Buckling analysis of 2D frame

1.3 3D Frame

The 3D model is based on beam elements with the I-profile cross section. As this model has additional degrees of freedom further constraints are needed in order to obtain the same system behavior as the 2D frame. This means supporting against rotation about the X (torsional) and Y (out of plane) axis at the base of the frame.

The relevant modeshape is seen at modeshape number 5 as this model has different sectional properties based on the direction of the cross section.

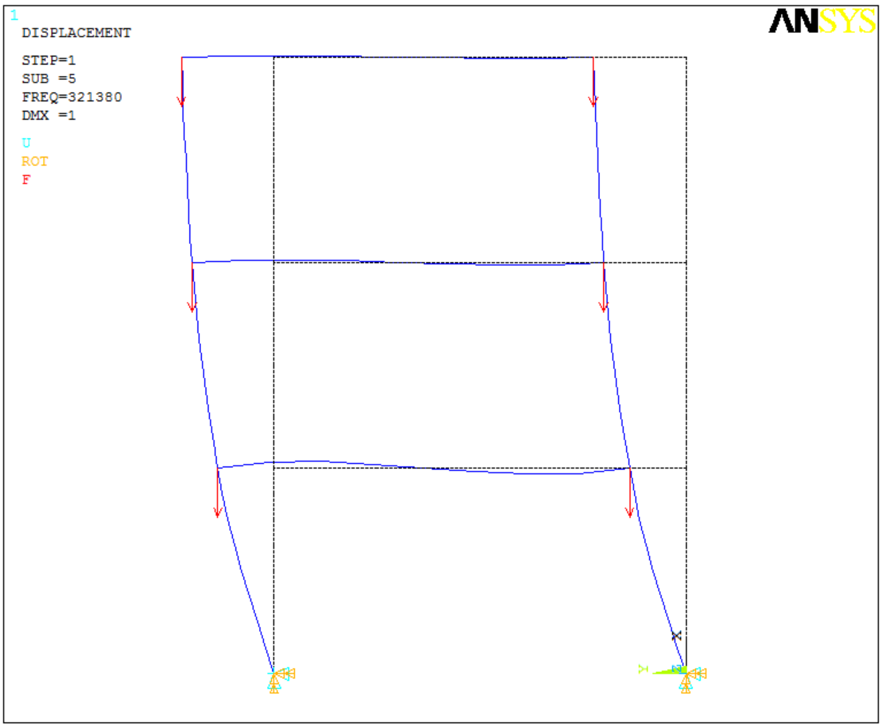

Figure 1-3 Buckling analysis of 3D frame

1.4 Comparison of 2D and 3D

The outputs considered in this report are the load multiplier and the stain energy (internal energy). Therefore these two outputs are compared. The strain energy for the lower column is summarized for the comparison.

	2D Frame	3D Frame	Deviation
Load multiplier	326518	321380	1.6 %
Strain energy	7654	7763	-1.4 %

Table 1-2 Comparison between 3D and 2D elements

The deviation is ascribed to the difference between Bernoulli-Euler and Timoshenko beam theory, as the 2D frame is based on Bernoulli-Euler elements and the 3D elements is based on Timoshenko elements. This is also described in section 4.1.2 is the main report.

APPENDIX 3

Critics of Iterative System Buckling Approach

1. Critics of the iterative system buckling approach

This appendix contains critics of the iterative system buckling approach proposed by Choi and Yoo [7]. The example used as reference example in the original article is used, as the results of the article are tried replicated. The method description is vague in a few essential key points, where assumptions concerning the explanation have been made.

1.1 Application of iterative system buckling approach

The model is a three story one-bay frame, with a width of 20m and a story height of 10m. The columns are made of HEB360 and the girders are made of IPE400. A point load, $P = 100kN$, is applied at the joints of story. The frame is pinned at the supports and a modulus of elasticity of $210{,}000\ MPa$ is used. The element properties are listed in the table below.

	HEB360	**IPE400**
Moment of inertia	$431.9 \cdot 10^6 mm^4$	$231.3 \cdot 10^6\ mm^4$
Area	$18.1 \cdot 10^3\ mm^2$	$8.45 \cdot 10^3\ mm^2$

Table 1-1 Member properties

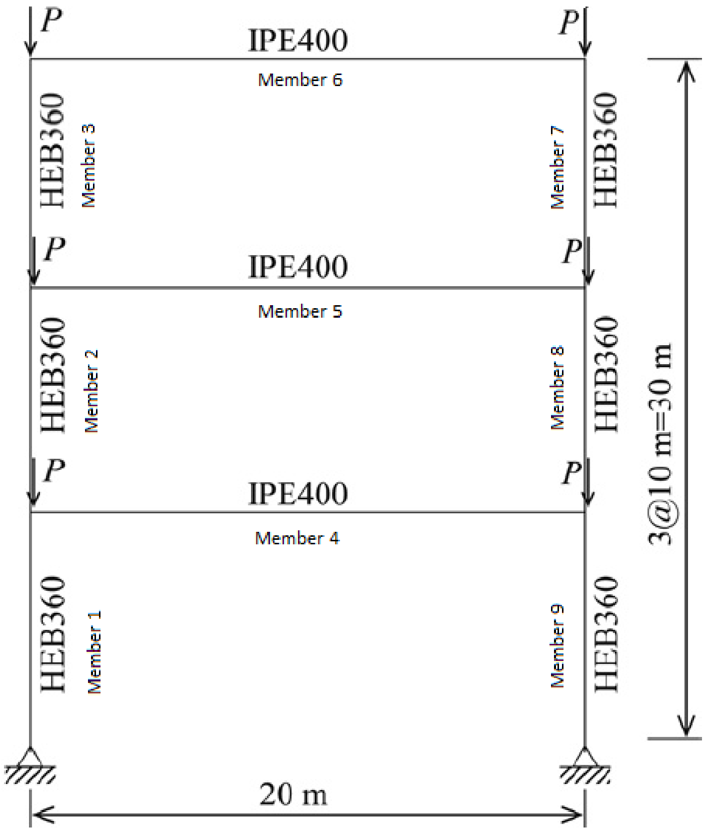

Figure 1-1 Reference example - 3 story one-bay frame

On Figure 1-1 the frame with member designation can be seen. A conventional system buckling approach is made using ANSYS. A load multiplier $\lambda = 3.3801$ is found and the results are summarized in the table and figure below.

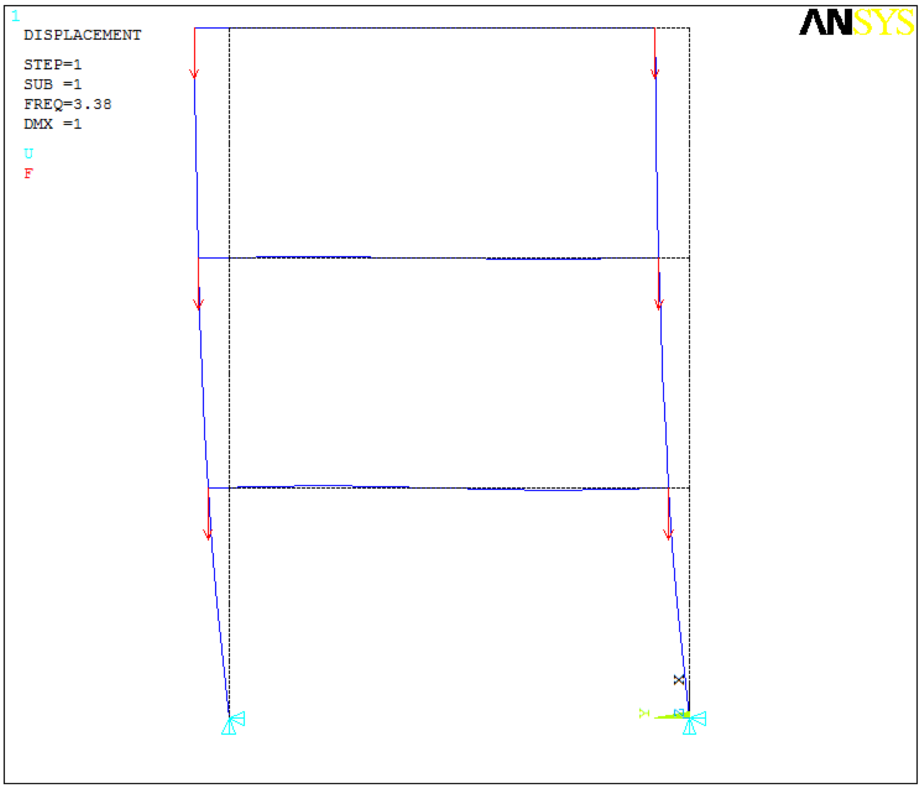

Figure 1-2 Result of linear buckling analysis

Member	**Axial force**	**Critical load**	**K-factor**
1 and 9 (story 1)	$300\ kN$	$1014\ kN$	2.971
2 and 8 (story 2)	$200\ kN$	$676\ kN$	3.639
3 and 7 (story 3)	$100\ kN$	$338\ kN$	5.146

Table 1-2 Results of system buckling approach

To determine the most and least influential member the stiffness parameter is calculated as:

$$L\sqrt{P/EI}$$

where L is the actual length of the member, P is the axial force in the member, E is the modulus of elasticity and I is the moment of inertia.

In Table 1-3 the calculated stiffness parameters are listed.

Member	**Stiffness parameter**
1 and 9 (story 1)	0.58
2 and 8 (story 2)	0.47
3 and 7 (story 3)	0.33

Table 1-3 Stiffness parameters for story 1-3

It is assumed that the determination of the most and least influential member should be taken amongst the compressive member. If the girder were included, the axial force of zero would lead to a K-factor of infinity. This would give implications in the following calculation of the increment of the fictitious axial force. Therefor the most influential member is Member 1 and the least influential is Member 3.

The method description is clear up to this point, but the explanation of the next step becomes unclear. The increment of the fictitious axial force is to be determined as:

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{K_{mi} L_{mi}}{\bar{K}_{li} L_{li}} \right)^2 P_{mi} - P_{li}$$

where subscript "mi" and "li" indicates the most and least influential member respectively. K is the K-factor found from the elastic buckling analysis and $\bar{K}$ is the modified K-factor found from the iterative elastic buckling analysis.

The unclearness in the article is that the iterative elastic buckling analysis have not yet been performed wherefore $\bar{K}$ does not exist. This is also seen in the flowchart from the article (see Figure 1-3).

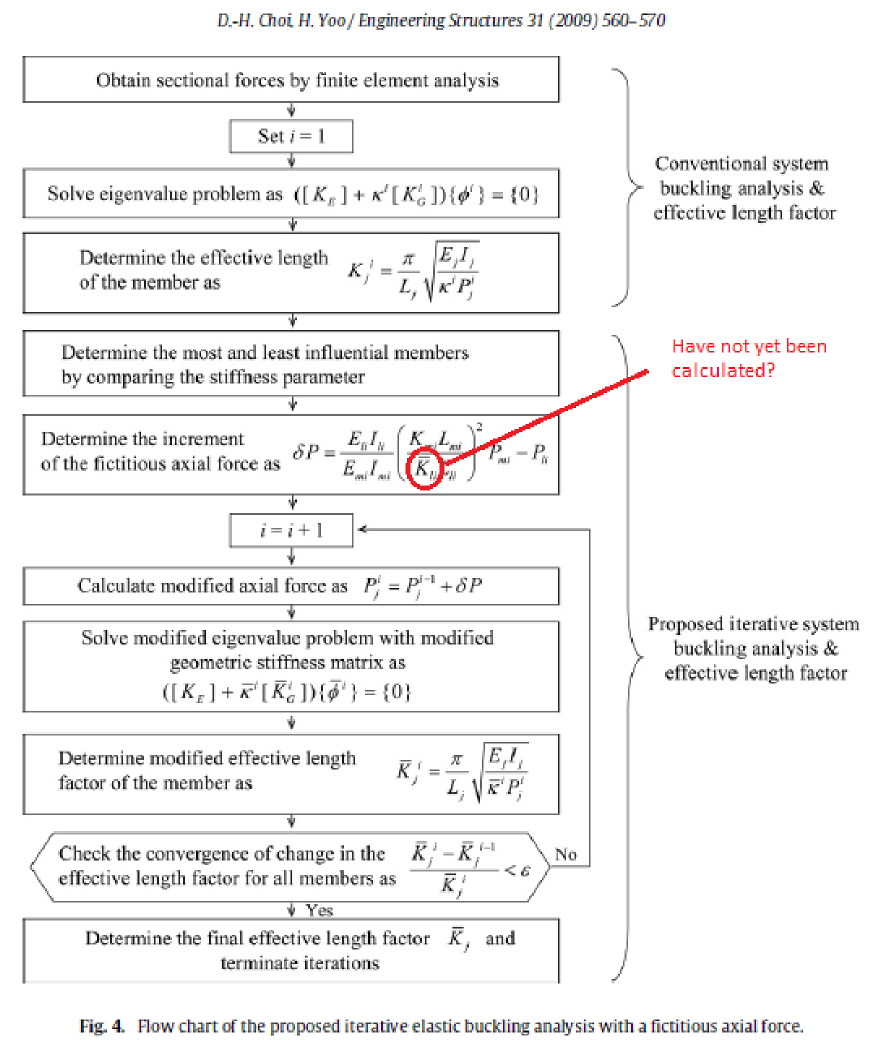

Figure 1-3 Flowchart of iterative system buckling approach [7]

In the iteration scheme i is the iteration number and j is the member number. It is seen that $\bar{K}$ is calculated for $i = 2, 3, \ldots$ and not for $i = 1$ because that is the conventional buckling analysis. However an extract from [7] reads:

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{K_{mi} L_{mi}}{\bar{K}_{li} L_{li}} \right)^2 P_{mi} - P_{li}. \tag{15}$$

For an easy and stable iteration process, the terms K_{mi} and $\bar{K}_{li}$ are assumed to be equal to 1.0 in the first iteration step. The constant value of δP from this assumption is used in subsequent iterations for simplicity. The most and least influential members are distinguished, by comparing the stiffness parameters ($L\sqrt{P/EI}$) among the members of the frame. While iteration is in progress, the axial force of each member is increased by the increment of the fictitious axial force, which causes the eigenvalue from Eq. (13) to change. The effective length factor of each member is subsequently

Figure 1-4 Extract from article

As no further description is made it is assessed that K_{mi} should be taken from the conventional buckling analysis ($i = 1$) and $\bar{K}_{li}$ is calculated from a buckling analysis where the loads have been increased by:

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{L_{mi}}{L_{li}} \right)^2 P_{mi} - P_{li}$$

Since the members have the same properties the first increase becomes:

$$\delta P = P_{mi} - P_{li} = 300\ kN - 100\ kN = 200\ kN$$

The modified K-factor for the least influential member is then calculated as $\bar{K}_{li} = 3.901$, and from the conventional buckling analysis we have $K_{mi} = 2.971$. The increment in axial force in the other members for the subsequent iterations is:

$$\delta P = \left(\frac{2.971}{3.901} \right)^2 300\ kN - 100\ kN = 74\ kN$$

For every iteration step the following criteria is checked for all members:

$$\frac{\bar{K}_j^i - \bar{K}_j^{i-1}}{\bar{K}_j^i} < \varepsilon$$

where $\varepsilon = 0.001$ is a convergence limit. For members where the iteration criterion is met, no more increments of fictitious axial force are added. It is not clear in the method description whether or not to store the K-factor for each member in the specific iteration step where convergence for the member is met, however this is assumed since the K-factor for story 1 (Member 1 and Member 9) becomes very large.

When performing the iterations convergence for all members is met at $i = 121$, and the results are seen in Table 1-4.

Member	**Recreated results of K-factors**	**Results of K-factors from [7]**	**Deviation [%]**
1 and 9 (story 1)	3.068	3.087	−0.6
2 and 8 (story 2)	2.362	2.345	0.7
3 and 7 (story 3)	2.039	2.016	1.1

Table 1-4 Results of the recreation of results from [7]

The description of the method is vague, however recreation of the results has been possible with some assumptions to the method description. The deviations in Table 1-4 might originate in rounding of properties and/or the programs choice of beam theory.

To sum up, the essential doubtful key points where:

- Which value is assigned for δP?
- Should the K-factors for each member be taken in the iteration step where the convergence criteria is met for the single member, or should the K-factor be taken in the final iteration step where all members has converges?

1.2 Comments

In Table 1-5 K-factors is seen where the iteration scheme has been performed with a higher and lower increments of the fictitious axial force $\delta P = 74\ kN$.

Increment	1 and 9 (story 1)	2 and 8 (story 2)	3 and 7 (story 3)
$\frac{1}{100}\delta P$	3.022	3.378	3.899
$\frac{1}{10}\delta P$	3.023	2.482	2.188
$\frac{1}{5}\delta P$	3.050	2.376	2.061
δP	3.068	2.362	2.039
$5\delta P$	3.094	2.338	2.001
$10\delta P$	3.101	2.328	2.332
$100\delta P$	3.109	2.304	2.305

Table 1-5 Results of K-factors with various iteration step size

The results from Table 1-5 shows that increasing the increment of fictitious axial force provides lower K-factors for story 2 and story 3, however this is at the expense of a higher K-factor for story 1. If the increment becomes too large, the results show non-consistency which depends on the convergence criteria. If the increment is too small, the convergence criteria might be met before an actual convergence is reached.

It should be noted that results also depends on the criteria $\varepsilon = 0.001$. In Table 1-6 K-factors has been listed where the criteria is reduced. The increment of $\delta P = 74\ kN$ has been used in these calculations.

Criteria	1 and 9 (story 1)	2 and 8 (story 2)	3 and 7 (story 3)
$\varepsilon = 0.001$	3.068	2.362	2.039
$\varepsilon = 0.0001$	3.098	2.325	1.985
$\varepsilon = 0.00001$	3.111	2.288	1.927

Table 1-6 Results of K-factors with different convergence criterion

The results from the method proposed as described in the article are very much dependent on the increment in fictitious axial force and convergence criteria. Even though results that match those of the article are obtained, the method is hard to implement and use on other examples, the results vary greatly depending on the combination of criterion and step size.

In later published articles [8][9] the method is slightly changed, which again would lead to different results. The changes will not be outlined since the example used for comparison only appears in [7].

APPENDIX 4

Journal Paper (DRAFT)

Energy modified system buckling approach for flexural stability design of general beam systems

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Abstract

When determining the effective length of a column in a general beam system, the system buckling approach is the most common method used. The system buckling approach does however yield excessively large K-factors for members with a relatively small axial force. This paper illustrates a new method that overcomes this paradox by modifying the system load multiplier for each member based on energy considerations. The method distinguishes between members being prone to buckling and those that are not sensitive to additional axial load. The method is verified by comparing results from the proposed method with results from previously known methods used on frames. The results show the proposed method gives reasonable effective lengths for all members in the frames. The proposed method's prerequisites are already calculated for the linear buckling analysis which makes the method eligible to be included in software algorithms.

Keywords: flexural buckling, effective length, system buckling approach, energy, excessively large K-factors

1 Introduction

Almost all structures consist of compression members which mean a risk of instability. Instability is a non-linear phenomenon, which often can be handled with linear considerations. In the European standards, a stability check often comprise of a check of the individual members, where the non-linear behavior is included by a reduction depending on the elastic critical load (CEN, 2009). The critical length of the compression member is needed, as it defines the critical load (Timoshenko, 1961). The critical length can be determined in a number of ways, either by methodical desk calculation or computer based methods. Most desk calculation methods are based on a number of assumptions in order to obtain an analytical solution. The isolated subassembly approach considers the rotational stiffness among adjacent members, but assumes all members have the same stiffness parameters and therefore buckles at the same time (ASCE, 1997). Gantes and Mageirou have developed a more comprehensive approach considering all rotation and translation at the beam ends. This approach yields good results for frames (Gantes and Mageirou, 2005). The story-methods allows for shear to be transferred between columns in a story, which means that the weaker columns are supported by the stronger ones. But this method suffers from the same assumptions depending on how the critical load is calculated (ASCE, 1997). It also means that these methods are made for specific cases, which for the most part mean regular frames. Therefor the usability of these methods is limited.

When evaluating structures that cannot be classified as a regular frame by a number of stories and/or bays, the critical length are either determined by experience and standard values or by computer based numerical calculations such as a system buckling analysis (ASCE, 1997). As personal computers are a basic tool for the engineer, analysis such as a system buckling analysis becomes more common. The system buckling analysis is not limited to specific cases and therefor offers a wide usability. The system buckling analysis assumes that all members reach their buckling limit when the system buckles, this leads to the paradox that compression members with a small axial force will give excessively long critical lengths. This means that the method is often used as a supplement to the analytical approaches. A new method has been proposed where a fictitious axial force is used in an iterative manner in order to assess this problem (Choi and Yoo, 2008, 2009, 2010). The results of this method do however rely on the increment in fictitious force and the convergence criterion of the iteration process.

In this paper, a new method based on the system buckling analysis is proposed, which is easily programmable into existing software. The method uses a modified load multiplier for each member based on their individual buckling resistance calculated using energy considerations. In order to verify the proposed method the K-factors are calculated for different cases and compared to results from numerical methods and analytical methods where relevant.

2 Current Methods

The K-factor is defined from the Euler load:

$$N_{cr} = \frac{\pi^2 \cdot EI}{(K \cdot l)^2} \qquad => \qquad K = \sqrt{\frac{\pi^2 \cdot EI}{N_{cr} \cdot l^2}} \tag{1}$$

The critical load can be determined analytically using the beam-column differential equation assuming all end restraints are well defined. This is easily done for the commonly known end restraints just as pinned and fixed. When a column is a part of a larger system of members the end restraints depends on the behavior of the entire system. This means that a column acting in a system needs to have defined four different restraints, as seen on Figure 1, in order to obtain an analytical solution.

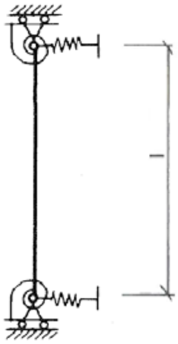

Figure 1 General column restraints

Determining these end restraints are often a cumbersome process and several methodical approaches are often used instead.

2.1 Isolated Subassembly Approach

The isolated subassembly approach is a commonly used approach for determining K-factors for a single column. It is also known as the alignment chart method, G-factor method or nomograph method. The method is characterized by the determination of stiffness factors, or G-factors, at each end of the column. The stiffness factor is a function of the adjacent beams stiffness coefficients. The stiffness factors at each end of the column are used in junction with either an alignment chart, nomograph or approximated formulas. These are based on a number of assumptions which must be fulfilled in order to obtain correct results (ASCE, 1997). Based on the origin of the method the stiffness factors can be calculated in a number of ways, where the European pr-EN1993 standard uses the following equations (CEN, 1992):

$$\eta_1 = \frac{K_C + K_1}{K_C + K_1 + K_{11} + K_{12}} \tag{2}$$

$$\eta_2 = \frac{K_C + K_2}{K_C + K_2 + K_{21} + K_{22}} \tag{3}$$

where K_C is the column stiffness coefficient, K_i are the stiffness coefficients for the adjacent columns and K_{ij} is the effective girder stiffness coefficient. Based on the state of sway the K-factor can be obtained as:

Non-sway mode:

$$\frac{l_s}{l} = K = \left[\frac{1 + 0{,}145\ (\eta_1 + \eta_2) - 0{,}265\ \eta_1\ \eta_2}{2 - 0{,}364\ (\eta_1 + \eta_2) - 0{,}247\ \eta_1\ \eta_2}\right] \tag{4}$$

Sway mode:

$$\frac{l_s}{l} = K = \left[\frac{1 - 0{,}2\ (\eta_1 + \eta_2) - 0{,}12\ \eta_1\ \eta_2}{1 - 0{,}8\ (\eta_1 + \eta_2) + 0{,}6\ \eta_1\ \eta_2}\right]^{0{,}5} \tag{5}$$

2.2 Story methods

The story buckling approach differs from the isolated subassembly approach in the fact that it can account for the transfer of shear forces between columns. This means that if a column in any given story is stronger than some of the other columns, then this column will provide the weaker column with some of its buckling resistance. All story based procedures can be formulated bases on the following equation:

$$\lambda_{story} \cdot \sum_{all} P_u = \sum_{non-leaner} P_{cr(story)} \tag{6}$$

where λ_{story} is the load multiplier that the axial loads P_u in the respective story must be scaled by to achieve story sidesway buckling, and $P_{cr(story)}$ represents the contribution from each column to the story sidesway buckling resistance.

From this equation the K-factor for a member in the story can be derived:

$$K_{story} = \sqrt{\frac{1}{P_u} \cdot \frac{\pi^2 EI}{L^2} \cdot \frac{\sum_{all} P_u}{\sum_{non-leaner} P_{cr(story)}}} \tag{7}$$

The different story methods differs primarily in the way $P_{cr(story)}$ is calculated.

The Story Buckling Method, also known as just story buckling, assumes that the buckling capacity of the story is equal to the sum the column buckling loads computed using a K-factor based on isolated assembly method. This however has the same limitation and must fulfill the same assumptions.

Another method is known as the Story Stiffness Method or Practical Story Based Effective Length Factor. This method provides more realistic results when there is significant violation of the assumptions on which the isolated subassembly approach is based.

Here the K-factor is derived as:

$$K_{story} = \sqrt{\frac{1}{P_u} \cdot \frac{\pi^2 EI}{L^2} \cdot \frac{\Delta_{oh} \cdot \sum_{all} P_u}{(0.85 + 0.15 \cdot R_L) \cdot \sum_{non-lean}}} \tag{8}$$

Where H is the lateral displacement forces, L is the length of the column and Δ_{oh} is the 1. order lateral displacement of the story, and R_L is defined by:

$$R_L = \frac{\sum_{Leaner} P_u}{\sum_{all} P_u} \tag{9}$$

2.3 System Buckling Approach

The system buckling approach uses a linear buckling analysis to determine the critical load level for system. The governing equation of a linear buckling analysis can be expressed as:

$$(K + \lambda \cdot K_g)V = 0 \tag{10}$$

Where K and K_g are the elastic and geometrical stiffness matrices. λ is the eigenvalues or load multiplier, and V is the eigenmode vector. The stiffness matrices can be derived from the principles of virtual work (Bazant, 2010). The system buckling approach assumes that all members reach their buckling limits at the buckling of the overall structure. The buckling load of member *i* can then be expressed as:

$$P_{cr,i} = \lambda P_i = \frac{\pi^2 E_i I_i}{(K_i L_i)^2} \tag{11}$$

Where E_i, I_i and L_i are the modulus of elasticity, moment of inertia, and the length of member i. P_i and K_i are the axial force and the effective length factor of member i respectively.

$$K_i = \sqrt{\frac{\pi^2 E_i I_i}{L_i^2 P_i \lambda}} \tag{12}$$

3 The K-factor paradox

It has been well established that the system buckling approach, which is the only method that offers usability in general beam systems, also introduces a paradox when used on members with a small axial force. The reason for excessively large K-factors when using equation (12) arises from the assumption that all members reach their buckling limit when the system buckles. Since not all members are close to their buckling limit when the system buckles, means that equation (12) is only valid for the most critical member. This can be verified by applied extra axial force in a member with a small axial force. Here it can be seen that the load multiplier for the system is almost unchanged, which indicates that the added axial force does not affect the system behavior. This problem has been addressed by Choi and Yoo (Choi and Yoo, 2008, 2009, 2010)

3.1 Iterative System Buckling approach

Choi and Yoo[ref] uses fictitious axial forces in an iterative procedure for story frames where the axial force in the compression members is increased until the change in K-factor is sufficiently small. This eliminates the paradox that small axial forces give excessively large K-factors.

The most and least influential columns are to be determined by:

$$L\sqrt{\frac{P}{EI}} \tag{13}$$

Where P is the axial force in the member is, E is the modulus of elasticity and I is the moment of inertia.
The K-factor is expressed as

$$K_i = \sqrt{\frac{\pi^2 E_i I_i}{L_i^2 (P_i + \Delta P)\lambda}} \tag{14}$$

where ΔP is the final increase in axial force calculated as the sum of incremental changes:

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{K_{mi} L_{mi}}{\overline{K_{li}} L_{li}}\right)^2 P_{mi} - P_{li} \tag{15}$$

where the subscript li and mi refers to the least and most influential columns respectively. For an easy and stable iteration process, the terms K_{mi} and $\overline{K_{li}}$ are assumed to be equal to 1.0 in the first iteration step. The constant value of δP from this assumption is used in subsequent iterations for simplicity.
The convergence criterion is defined as:

$$\frac{K_i^j - K_i^{j-1}}{K_i^j} < \varepsilon \tag{16}$$

where j is the iteration number and ε is convergence criteria (=0.001)

3.2 Comparison of methods

In order to illustrate the inconsistency of the system buckling approach a three story one bay frame used by both Gantes and Mageirou(2005) and Choi and Yoo(2008) is presented.
The model of the three story building is a 2D frame consisting of columns (HEB360) and girders (IPE400). Both the HEB360 and the IPE400 has bending about the strong axis. The moment of inertia for columns and girders are $4.319 \cdot 10^{-4}\,m^4$ and $2.313 \cdot 10^{-4}\,m^4$ respectively. The modulus of elasticity is $210.000\,MPa$ and the frame is pinned at the base. The story height is $10\,m$, which gives a total height of $30\,m$, and the bay width is $20\,m$. Concentrated loads are imposed on the joints of each story, where the load $P = 100\,kN$. The horizontal loads shown with dotted arrows are only used in the story stiffness method.

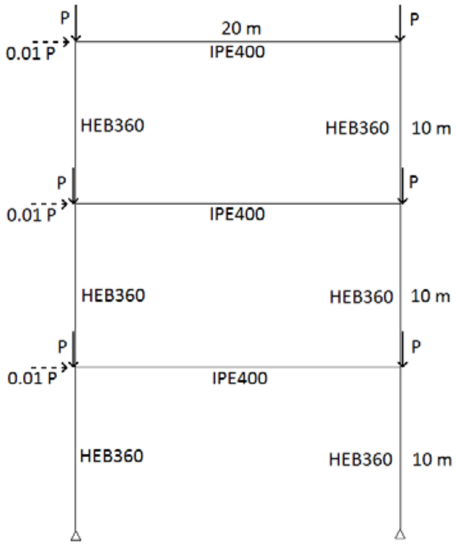

Figure 2 Three story one bay frame

In Table 1 and Figure 3 a comparison of the results can be seen.

K-factors	Story 1	Story 2	Story 3
Story Buckling Method	2.831	2.643	2.267
Story Stiffness Method	2.989	2.713	2.664
Gantes & Mageirou	2.969	2.775	2.250
System Buckling Approach	2.971	3.639	5.146
Choi & Yoo	3.087	2.345	2.016

Table 1 Effective length factors for three story one bay frame

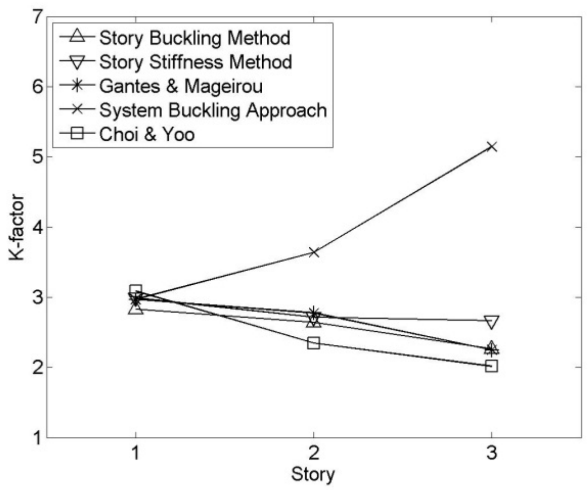

Figure 3 Comparison of effective length factors for three story one bay frame

In Figure 3 it is seen that the system buckling approach yield significantly larger K-factors for the top two stories. As the axial force increases down the columns of the building so does the critical load according to (11). This goes against intuition as the columns are identical in length and properties.

4 Proposed Method

The primary reason why the system buckling approach gives erroneous results is due to the assumption that all members reach their buckling limit when the system buckling. This becomes apparent when considering the frame example (Figure 2). In such a case the assumption is clearly wrong, as one would expect the top and bottom columns to have approximately the same critical load. Since not all members reach their buckling limits when the system buckles, a way of assessing how prone each member is to buckling is sought.

4.1 Energy based stability considerations

The energy in a member consists of the internal energy, which is the stabilizing energy, or potential strain energy, and the external energy resulting from the outer work done by the axial forces, which is the destabilizing energy. At the critical load level there is a balance between the total external energy and the total internal energy.

$$E_{internal} = E_{external} \tag{17}$$

The internal and external energy in the individual member are not equal to each other, except in the case where all members reach their buckling limit when the entire system buckles.

The calculation of the energy requires the buckling shape, which is known for the basic cases, and when considering a system of members, the buckling shape is found by a linear buckling analysis as the mode shape. The internal and external energy for each member can be found as a summation of the energies in each sub divided element belonging to the member in question. The internal energy in an element can be found by:

$$E_{internal} = \frac{1}{2} \cdot \boldsymbol{s}^T \boldsymbol{k}\, \boldsymbol{s} \tag{18}$$

And the external energy can be found by:

$$E_{external} = \frac{1}{2} \cdot \boldsymbol{s}^T \boldsymbol{k_g} \boldsymbol{s} \tag{19}$$

where $\boldsymbol{s}$ is a vector containing the elements 6 nodal displacements from the mode shape (translation in x and y direction and rotation about z axis). $\boldsymbol{k}$ and $\boldsymbol{k_g}$ are the elements local stiffness matrix and geometrical stiffness matrix respectively (Cook, 2002):

$$k_g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6P}{5L} & \frac{P}{10} & 0 & -\frac{6P}{5L} & \frac{P}{10} \\ 0 & \frac{P}{10} & \frac{2PL}{15} & 0 & -\frac{P}{10} & -\frac{P}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6P}{5L} & -\frac{P}{10} & 0 & \frac{6P}{5L} & -\frac{P}{10} \\ 0 & \frac{P}{10} & -\frac{P}{30} & 0 & -\frac{P}{10} & \frac{2PL}{15} \end{bmatrix}$$

$$
\boldsymbol{k}=\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0\\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2}\\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L}\\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0\\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2}\\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
$$

When performing the linear buckling analysis on a system of members, some members are deforming as a result of the outer work being done by the axial force, and these members are in a state of buckling. Other members are more in a state of bending due to rigid connections in the system and acts as a stabilizing member. To assess how prone a member is to buckle, the following energy ratio is being used:

$$r_E=\frac{E_{internal}}{E_{external}} \tag{20}$$

A higher ratio means a higher resistance to buckling of the member and therefore the member is less prone to buckling. When looking at a system of members the ratio for a single member does not mean anything by itself, but it solely expresses the buckling resistance compared to other members. This means that the energy ratio can be used to compare the buckling resistance of individual member in a system based on the current load level.

The ratio r_E is calculated for each member, and the member with the lowest ratio is considered as the weakest member and therefore being 100% in the state of buckling. The lowest energy ratio is taken as a reference ratio, which is notated as $r_{E,ref}$ and the critical load for each member i in the system can be determined as

$$N_{cr,i}=N_i\cdot\lambda_{cr}\cdot\frac{r_{E,i}}{r_{E,ref}} \tag{21}$$

In the special case where all members buckle at the same time, every member will have a ratio of one, and the energy modified system buckling approach gives the same result as the system buckling approach.
The K-factor for each member i can then be found as:

$$K_i=\sqrt{\frac{\pi^2\cdot E_iI_i}{{L_i}^2\cdot N_{cr,i}}}=\sqrt{\frac{\pi^2\cdot E_iI_i}{{L_i}^2\cdot N_i\cdot\lambda_{cr}\cdot\frac{r_{E,i}}{r_{E,ref}}}} \tag{22}$$

where E is the modulus of elasticity and I is the moment of inertia.

The difference between the system buckling approach and the proposed method is illustrated on Figure 4. It is noted that no extra analysis is needed, as post-processing is done using only information from the existing linear buckling analysis.

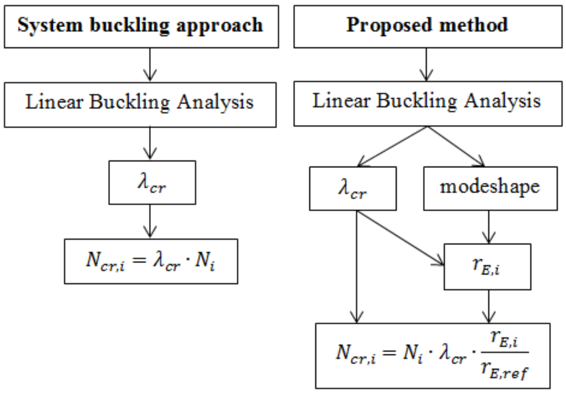

Figure 4 Flowchart illustrating difference between system buckling approach and proposed method

5 Applications

5.1 3 Story 1 bay frame

The first test model is identical to the model used for illustrating the flaws of the system buckling approach (Figure 2). The proposed method is applied to the model and the results are seen in Table 2 and in Figure 5.

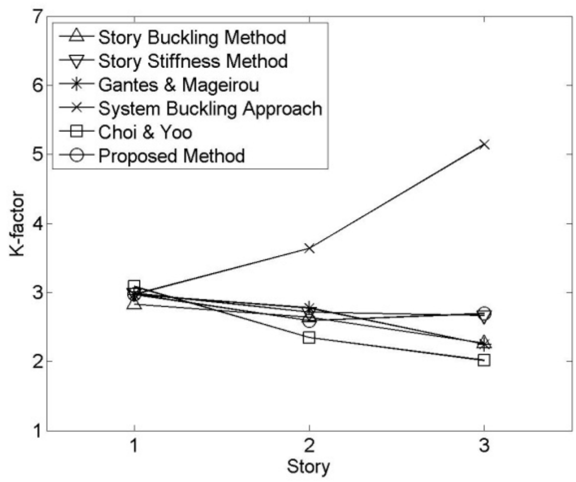

Figure 5 Comparison of effective length factors for three story one bay frame including results from proposed method

K-factors	Story 1	Story 2	Story 3
Story Buckling Method	2.831	2.643	2.267
Story Stiffness Method	2.989	2.713	2.664
Gantes & Mageirou	2.969	2.775	2.250
System Buckling Approach	2.971	3.639	5.146
Choi & Yoo	3.087	2.345	2.016
Proposed Method	2.971	2.591	2.695

Table 2 Comparison of effective length factors for three story one bay frame including results from proposed method

The method proposed by Choi and Yoo (2008) gives the most favorable results for story 2 and story 3; however it is on the cost of some of the buckling resistance of story 1. This is related to the slight decrease in the load multiplier. When only considering story frames with no axial compressive force in the horizontal members, the proposed method by Choi and Yoo(2008) provides favorable results, however if there is a relatively small axial compressive force in the horizontal members the applied fictitious forces will tend to decrease load multiplier significantly. This would lead to conservatively large K-factors for the vertical members. One of the strength of the proposed method is that it is provides intuitive realistic K-factors for all members in the structure. E.g. if an axial compressive force of $0.01P$ where applied to the horizontal members the vertical members would get a non-noticeable increase in the K-factor and the horizontal members would be assigned a K-factor of approximately 0.2.

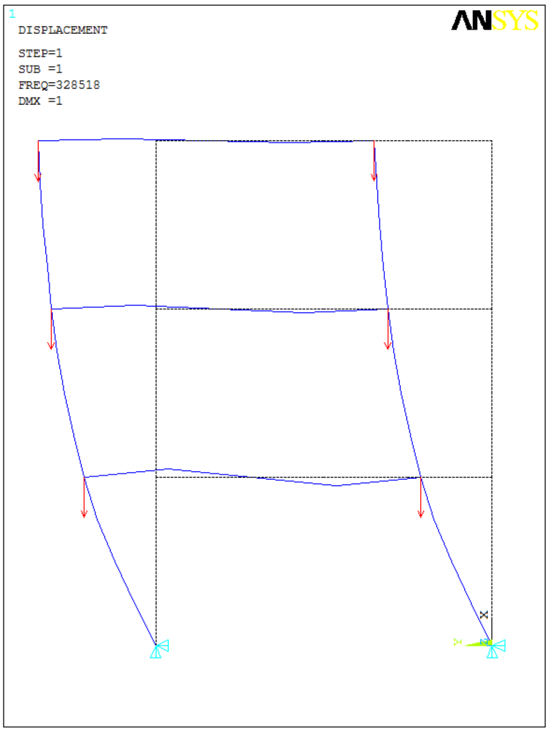

Figure 6 Buckling analysis of three story one bay frame showing the transvers displacement of girders

The proposed method stands out in the result by the fact that it provides a larger K-factor for Story 3 than for story 2. It is assessed, that this is caused by the fact that the internal strain energy becomes relatively smaller in story 3 than story 2 because of the relative transverse displacement of each story seen in the mode shape (see Figure 6). This effect is also indicated when comparing the Story Buckling Method to the Story Stiffness Method, where the decrease of the K-factor is small by the Story Stiffness Method, because it considers the transverse displacements.

5.2 Mixed story 2 bay frame

In the second example, a model of an asymmetrical frame with unequal column length is investigated. Both vertical and horizontal loads are imposed, and all members are subjected to compression. Amongst the known methods of determining the K-factor, the system buckling analysis would be chosen. In this example results using the system buckling analysis approach are compared to the proposed method.

Story 1 consists of two bays with span $20 + 20 = 40\ m$. In story 1 the left and middle column has a length of $7\ m$ and the right column has a length of $10\ m$. In Story 2 the right bay has been removed, and the column lengths are $10\ m$. The frame is pinned at the base. The model is seen in Figure 7.

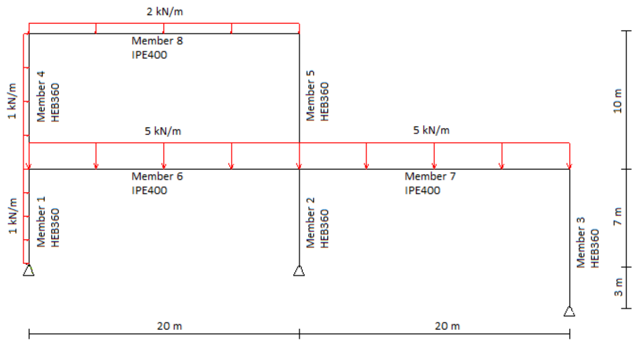

Figure 7 Mixed story two bay frame

All column profiles are HEB360 ($A = 1.810 \cdot 10^{-2}\ m^2$ and $I = 4.319 \cdot 10^{-4}\ m^4$) and all girder profiles are IPE400 ($A = 0.845 \cdot 10^{-2}\ m^2$ and $I = 2.313 \cdot 10^{-4}\ m^4$). The axial forces in the members where obtained by a static analysis, and the load multiplier and mode shape where obtained by a linear buckling analysis. The distorted model from the mode shape is seen in Figure 8.

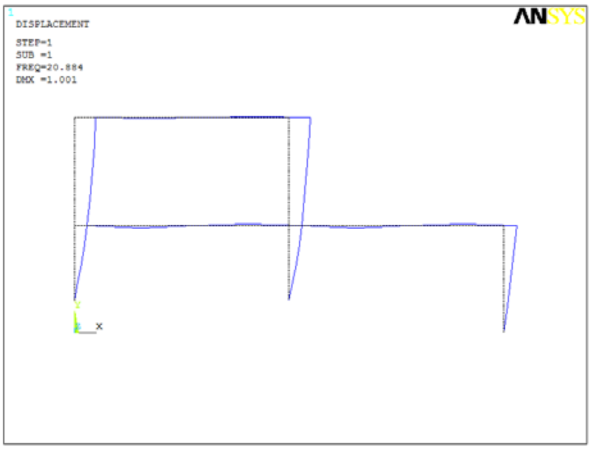

Figure 8 Modeshape for mixed story two bay frame

As the girders has a double curvature (see Figure 8) low K-factors is expected for these members. In story 2, the right column (Member 5) is subjected to the largest axial load and is therefore expected to be weakest and have a larger K-factor than the left column (Member 4). In story 1, it is hard to predict which column is the weakest. The right column (Member 3) is subjected to the lowest axial load but has largest geometrical length. The middle column (Member 2) is subjected to the highest axial load, but contrary to the other two columns, two girders provide rotational stiffness at the top joint. In Table 3 K-factors according to the system buckling approach and the proposed method are given, and the results are plotted in Figure 9.

K-factors	System Buckling Approach	Proposed Method
Member 1	*3.792*	*3.276*
Member 2	*2.607*	*2.455*
Member 3	*2.916*	*2.916*
Member 4	*5.008*	*2.157*
Member 5	*4.325*	*3.198*
Member 6	*2.967*	*0.181*
Member 7	*1.920*	*0.183*
Member 8	*2.056*	*0.182*

Table 3 Comparison of K-factors for mixed story two bay frame

The results shows, that the right column in story 1 was the member most prone to buckling since the proposed method and the system buckling approach gave the same result. The proposed method provides more favorable K-factors for all other members since the load multiplication factor is increased by a factor relative to the energy ratios. Especially for the girders much better results are provided by the proposed method, as the K-factor should be low in compression members that act mainly as bending members. For story 2, the proposed method provides K-factors that are consistent with the intuition that the weakest member (Member 5) in the analysis should be assigned the largest critical length. This is in contrast to the system buckling analysis, which assumes same time buckling and therefore buckling at a lower load for the member with the lowest axial force (Member 4).

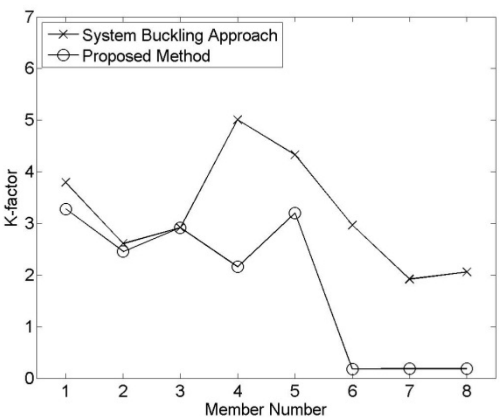

Figure 9 Comparison of K-factors for mixed story two bay frame

6 Conclusion

This paper proposes a new method to calculate the effective length factors for columns in a general beam system. Previously used methods such as the isolated subassembly approach, story approaches and system buckling approach are used to verify the results the proposed by comparison. This shows excessively large K-factors when using the system buckling approach, where complications arises from the assumption that all members reach their buckling limit at the same time of the system.

A recently proposed method uses a fictive axial force in each member in an iterative procedure in order to modify the results of the system buckling approach. This shows good results compared to other the methods, but results depend on the convergence criterion and the increment of fictitious axial force, wherefore the results require extra attention if implemented in software algorithms.

Instead a new method is proposed where the load multiplier from the system buckling approach is modified for each member based on energy considerations. The method distinguishes between members being prone to buckling and members that are not sensitive to additional axial load by looking at the inner and outer work being done in the members.

A three story one bay frame is analyzed using previously mentions methods. Here it is shown that while the system buckling approach gives excessively large K-factors for some members, the proposed method yields reasonable results in good agreement with other methods.

A mixed story two bay is analyzed using the system buckling approach and the proposed method. Here it is shown that the proposed method gives reasonable results for all members, while the system buckling approach yields excessively large K-factors for all but the weakest member in the structure.

It is concluded that the method provides a wide usability as it is not limited to frames and does not have the same problem as the system buckling approach when members have a relatively small axial force. Furthermore the method is eligible to be programmed into existing numerical software as all prerequisites for the energy considerations are calculated as a normal part of a linear system buckling analysis.

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