Computer Based FE Analysis of Reinforced Concrete Walls by the Stringer Method

APPENDIX

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Appendix **A1**

Stringer Approach Illustrated by An Example

The calculation of structure 2 in the main report chapter 3 is shown with traditional hand calculations by means of the stringer method. For a more detailed description of the approach the reader should look into the main report.

The dimensions and loads for structure 2 is stated in table A1.1. Because of the thickness the example is representative for concrete walls.

Thickness	300 mm
Hight	1000 mm
Width	1400 mm
Q_1	75 kN
Q_2	135 kN

Table A1.1: *Dimensions and loads.*

For traditional hand calculation the shear and stringer forces are determined by using the following steps:

1. Select the stringer mesh, stringer lines must be placed at the center of the reactions and loads.
2. Chose direction of operational sign.
3. Determine the reactions if the system is static determinate, if not see step 5.
4. Calculate the shear stress in the fields.
    a) Determine the number of static indeterminate and which fields should be chosen freely regarding shear stress.
    b) Determine the shear stress in the remaining fields by vertical or horizontal projection.
    c) When the shear stress in the last field is calculated this value can be checked by equilibrium in the direction perpendicular.
5. Calculate the stringer forces by free body diagram for each stringer, the reactions can be found as well.
6. Determine the reinforcement and make sure that the compression can be carried in the stringers and shear fields.

A1.1 Stringer Mesh

The stringer mesh is placed such that the stringer lines follow the edge both external and around the hole, and in the center of the loads and reactions as well, cf. Figure A1.1. The direction of operational sign is marked with x-y axis.

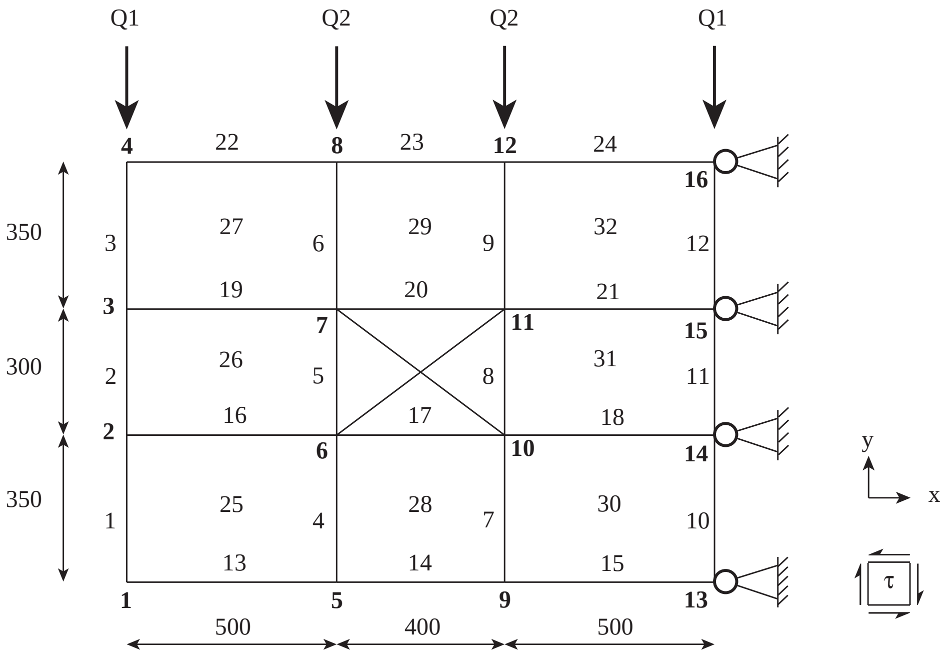


Figure A1.1: Structure 2converted to stringer system, nodes are numbered from 1 to 16 with bold, stringers from 1 to 24 and shear areas from 25 to 32.

A1.2 Stress in Shear Fields

The number of static indeterminate, N, is determined by equation (A1.1), and for the hand calculation N is interpreted as the number of static indeterminate shear areas.

$$\begin{aligned} \text{Variables}: &\quad 2 \cdot n_{stringers} + n_{shear\_areas} + n_{reactions} \\ \text{Equilibrium equations}: &\quad 2 \cdot n_{nodes} + n_{stringers} \\ \text{Statically indeterminate, } N: &\quad n_{variables} - n_{Equilibrium\_Equations} \end{aligned} \tag{A1.1}$$

For structure 2 N is determined as equation (A1.2).

$$N = (2 \cdot 24 + 1 \cdot 8 + 1 \cdot 8) - (2 \cdot 16 + 1 \cdot 24) = 8 \tag{A1.2}$$

Hereby it is determined that structure 2 is eight times statically indeterminate which lead to following options: The shear stresses in area 25 and 26 are chosen freely thus, the shear stresses in area 27 can be determined by vertical equilibrium in a cut through area 25, 26 and 27, cf. Figure A1.2.

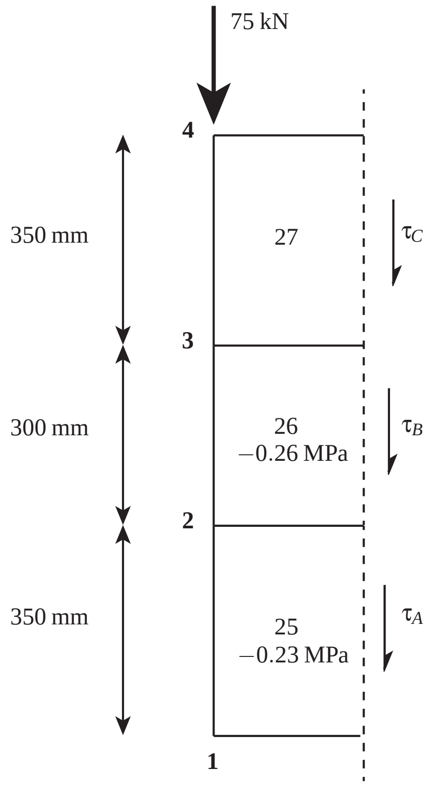


Figure A1.2: *Vertical cut through area 25, 26 and 27.*

The shear stresses in 25 and 26 are chosen relative equal so the stress in area 27 gets a value close to 25 and 26. $\tau_{25} = -0.23\,\text{MPa}$ and $\tau_{26} = -0.26\,\text{MPa}$ gives:

$$\begin{aligned}\tau_{27} &= \frac{-Q_1 - \tau_{25} \cdot h_{25} \cdot t - \tau_{26} \cdot h_{26} \cdot t}{h_{27} \cdot t} \\ &= \frac{-100\,\text{kN} - (-0.23\,\text{MPa}) \cdot 350\,\text{mm} \cdot 300\,\text{mm} - (-0.26\,\text{MPa}) \cdot 300\,\text{mm} \cdot 300\,\text{mm}}{350\,\text{mm} \cdot 300\,\text{mm}} \\ &= -0.261\,\text{MPa}\end{aligned}$$

By vertical cut through area 28 and 29, with $\tau_{28} = \tau_{29}$ as the third free choice, the last shear stresses in area 30, 31 and 32 are determined by horizontal equilibrium. The stresses are shown in Figure A1.3. The shear stresses are checked by a vertical cut through area 30, 31 and 32 just at the left of node 13,14,15 and 16.

-0.26	-1.00	-0.89
-0.26		-1.70
-0.23	-1.00	-0.93

Figure A1.3: *Shear stress in MPa.*

A1.3 Stringer Force and Reaction

The force in the stringer line through node 1, 2, 3 and 4 is calculated using vertical equilibrium and a free body diagram, cf. equation (A1.3) and Figure A1.4.

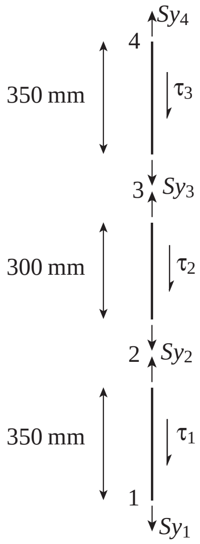


Figure A1.4: *Free body diagram for stringer line through node 1, 2, 3 and 4.*

$$
\begin{aligned}
Sy_1 &= 0 \\
Sy_2 &= -0.3\,\text{MPa} \cdot 350\,\text{mm} \cdot 300\,\text{mm} + 0 = -24.15\,\text{kN} \\
Sy_3 &= -0.35\,\text{MPa} \cdot 300\,\text{mm} \cdot 300\,\text{mm} + -31.5\,\text{kN} = -47\,\text{kN} \\
Sy_4 &= -0.3\,\text{MPa} \cdot 350\,\text{mm} \cdot 300\,\text{mm} + -63\,\text{kN} = -75\,\text{kN} \\
Sy_4 &= -Q_1\ OK
\end{aligned} \tag{A1.3}
$$

Same procedure follows for the remaining stringers. The calculations and appurtenant illustrations are placed at Appendix A7.1. The stringer forces are illustrated in Figure A1.5 and Figure A1.6.

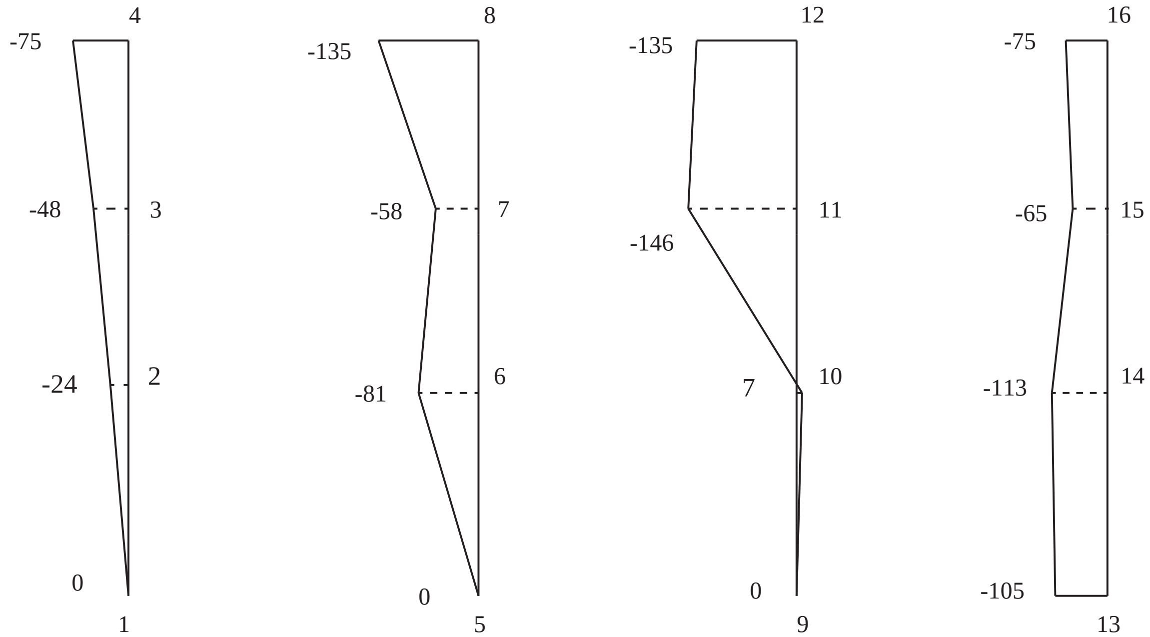


Figure A1.5: *Stringer forces in* kN.

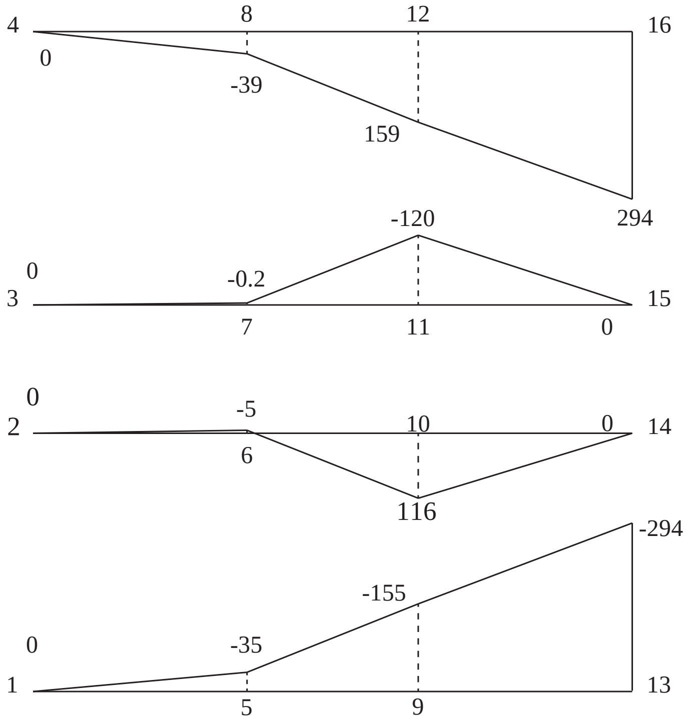


Figure A1.6: *Stringer forces in* kN.

A1.4 Determination of Reinforcement and Compression Capacity

The parameters in Table A1.2 are used for the reinforcement and checking the compression capacity.

Safety factor for normal control class	γ_c	1.45
Safety factor for reinforcing steel	γ_s	1.20
Compressive strength of concrete	f_c	25 MPa
Yield strength of reinforcement	f_y	350 MPa

Table A1.2: *Safety factors and strength values.*

The compression capacity is maintained in all stringers. Tensile stringers and shear fields has the following reinforcement, cf. Table A1.3 and the calculations is found in Appendix A7.1. The necessary reinforcement results in a total reinforcement amount of 40 kg.

Stringer line	Necessary reinforcement area $\left[mm^2\right]$	Reinforcement	Efficiency ratio Shear transmission	Efficiency ratio compression
1	-	-	-	15.6 %
2	-	-	-	35.1 %
3	16	4 Ø4	16 %	38 %
4	-	-	-	21.8 %
5	-	-	-	88 %
6	253	4 Ø10	17 %	2 %
7	-	-	-	42 %
8	642	6 Ø14	7 %	12 %
- x- and y-dir.	1113 $^{mm^2}/_{m}$	Ø18/ 160 mm		

Table A1.3: *Reinforcement in tensile stringers and shear fields.*

Appendix **A2**

Material parameters for example

The material parameters for the main report chapter 3.2 are calculated according to (Jensen, 2008).

Øvre grænse for trykstringer, indput for C_d

Gælder kun for eks.12.2.,*modificeret.* stringer 16,18,19, 21, Da den mindste højden på det tilstødende forskydningselement er 300.

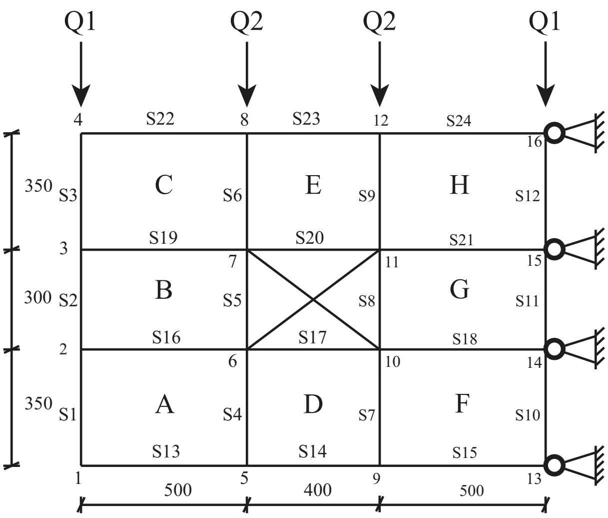
Normal kontrolklasse
Tabel 2.3,s38 — $\gamma_c := 1.45$

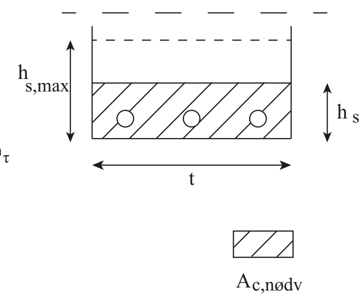
Beton styrke C25 — $f_c := 25\text{MPa}$

$$f_{cd} := \frac{f_c}{\gamma_c} = 17.241 \cdot \text{MPa}$$

Elementtykkelse — $t := 300\text{mm}$

Højde af forskydningsfelt vinkelret på stringeren — $h_\tau := 300\text{mm}$





Effektivitetsfaktoren
Formel 6.15, s.179

$$v_m := \max\left(0.98 - \frac{f_c}{500MPa}, 0.6\right) = 0.93$$

Anbefalede øvre grænse for stringerhøjden

$$h_{smax} := 0.2 \cdot h_\tau = 60 \cdot mm \quad \textbf{(1)}$$

Nødvendig betonareal i stingeren
vha. plastisk trykstyrke

$$A_{cnødv} := \frac{N_c}{v_m \cdot f_{cd}} \quad \textbf{(2)}$$

Stringerhøjde

$$h_s := \frac{A_{cnødv}}{t} \quad \textbf{(3)}$$

(1) indsættes i (3). A_c,nødv isoleres i (3) og indsættes i (2).
N_c isoleres i (2) hvormed N_cMAX haves da h_smax indgår

$$N_{cMAX} := v_m \cdot f_{cd} \cdot t \cdot h_{smax} = 288.621 \cdot kN$$

Øvre grænse for forskydning i betonfelt

Normal kontrolklasse
Tabel 2.3,s38

$$\gamma_c := 1.45$$

Beton styrke C25

$$f_c := 25MPa$$

$$f_{cd} := \frac{f_c}{\gamma_c} = 17.241 MPa$$

Effektivitetskoefficienten
Formel 5.22 s.140

$$v := \max\left(0.7 - \frac{f_c}{200MPa}, 0.45\right) = 0.575$$

Øvre grænse for betonspænding
Plastisk betontryk
Formel 12.9, s.326

$$øg_\sigma := v \cdot f_{cd} = 9.914 MPa$$

$$\tau_{max} := \frac{øg_\sigma}{2} = 4.957 MPa$$

Appendix **A3**

Linear Programming

`Opti_String` is written in MATLAB and uses the built-in function linprog which is a linear programming algorithm. This chapter contains all the input matrices and vectors for structure 1 which is a simple and clear example. Input for both lower and upper bound is shown.

A3.1 Input for Lower Bound Method

This section provides an example of the input for the lower bound method. Structure 1 is used for exemplification of the input and is shown in Figure A3.1.

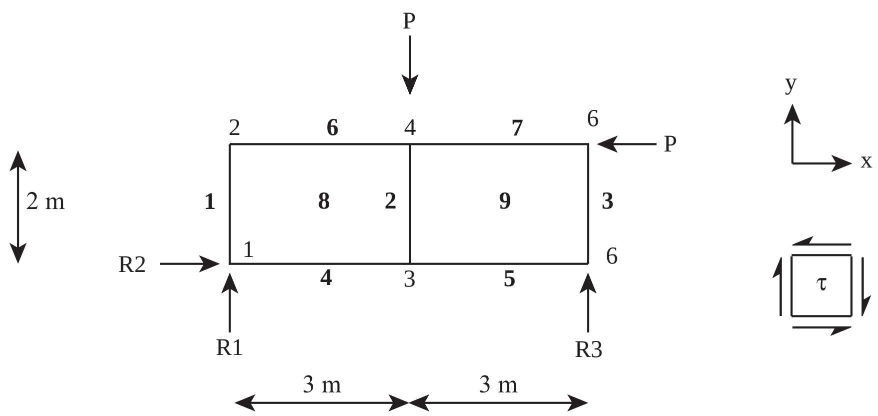


Figure A3.1: *Stringer system for exemplification of input into the linprog function in MATLAB. After (Jensen, 2008, example 12.1).*

The linear programming problem is specified by equation (A3.1) for the MATLAB function.

$$\min\ f^T x \text{ such that } \begin{cases} A\,x \leq b \\ Aeq\,x = beq \\ lb \leq x \leq ub \end{cases} \tag{A3.1}$$

where

f	Linear objective function
A	Matrix for linear inequalities constraints
b	Vector for linear inequalities constraints
Aeq	Matrix for linear equality constraints
beq	Vector for linear equality constraints
lb	Vector for lower bounds
ub	Vector for upper bounds
$x0$	Initial point for algorithm

The syntax for the linprog function in MATLAB is shown in equation (A3.2).

$$\text{x = linprog(f, A, b, Aeq, beq, lb, ub, x0, options)} \tag{A3.2}$$

The limits, *lb* and *ub*, are put in as empty matrices as no limits are set but the function needs the input in order to carry out the calculation. The initial point for the algorithm, $x0$, is set to zero. The value is not used because linprog uses a built-in starting point but it needs the input value in order to carry out the calculation. Using options the algorithm for the calculation is selected. This can either be Simplex or Large-Scale, cf. section A3.3.

Load optimisation

The objective of load optimisation using the lower bound method is to maximise the load to find the ultimate load bearing capacity of the structure which is mathematical described in equation (A3.3) (Damkilde, 1995, equation (15)).

$$\begin{aligned} \text{maximise:} \quad & \lambda \\ \text{restrictions:} \quad & -\mathbf{H}\,\boldsymbol{\beta} + \lambda\,\mathbf{R} = -\mathbf{R}_0 \\ & \mathbf{C}\,\boldsymbol{\beta} \leq \mathbf{C}_d \end{aligned} \tag{A3.3}$$

where

$\boldsymbol{\beta}$	Vector with stress parameters
$\mathbf{H}$	Flexibility matrix
$\mathbf{R}$	Load vector
$\mathbf{R}_0$	Constant load vector
$\mathbf{C}$	Constraint matrix
$\mathbf{C}_d$	Material constraint matrix

The input into MATLAB of matrices and vectors for the example are shown in the following. Table A3.1 indicates the symbolic connection between the syntax for MATLAB and the load optimisation problem.

MATLAB	Load optimisation
f	$\mathbf{c}$
x	x
A	$\mathbf{C}$
b	$\mathbf{C}_d$
Aeq	$\mathbf{H}$
beq	$\mathbf{R}_0$
$x0$	0

Table A3.1: *Symbols for MATLAB syntax and load optimisation.*

The object function, **c**, is made as a zero vector with the last values set to one and the length of the vector corresponds to the number of columns in the flexibility matrix, where **x** contains the load parameter λ which is the variable to maximise, cf. equation (A3.4).

$$\mathbf{c}^T\mathbf{x} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \right\}^T \left\{ \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \\ \lambda \end{array} \right\} = \left\{ \begin{array}{cccc} 0 & \dots & 0 & 1 \end{array} \right\} \left\{ \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \\ \lambda \end{array} \right\} \tag{A3.4}$$

where

c	Object function regarding load optimisation
x	Vector with variables to be determined containing $\boldsymbol{\beta}$ and λ

The first restriction in equation (A3.3), which expresses the equilibrium equation, can be rewritten as equation (A3.5). The matrix corresponds to the left hand side in Figure A3.2 and $\mathbf{R}_0$ is the right hand side in Figure A3.2.

$$\left[\begin{array}{cc} -\mathbf{H} & \mathbf{R} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\beta} \\ \lambda \end{array} \right\} = -\mathbf{R}_0 \tag{A3.5}$$

The rows in flexibility matrix, **H**, cf. Figure A3.2, consists of equilibrium equations corresponding to the number of rows. Following is interpreted:

$$\begin{aligned} n_{rows} &= 2 \cdot n_{nodes} + n_{stringers} \\ n_{columns} &= 2 \cdot n_{stringers} + n_{shear\_areas} \end{aligned}$$

			Stringers 1		Stringers 2		Stringers 3		Stringers 4		Stringers 5		Stringers 6		Stringers 7		Shear fields A	Shear fields B	Load		
			1	2	1	2	1	2	1	2	1	2	1	2	1	2	τ_A	τ_B	R		R_0
Nodes	N1x	1							-1										R3		0
	N1y	2	-1																R1		0
	N2x	3											-1								0
	N2y	4		1																	0
	N3x	5								1	-1										0
	N3y	6			-1																0
	N4x	7												1	-1						0
	N4y	8				-1													-P		0
	N5x	9										1									0
	N5y	10					-1												R2		0
	N6x	11														1			-P	=	0
	N6y	12						1													0
Vertical	S1	13	-1	1													-2				0
	S2	14			-1	1											2	-2			0
	S3	15					-1	1										2			0
Horizontal	S4	16							-1	1							-3				0
	S5	17									-1	1						-3			0
	S6	18											-1	1			3				0
	S7	19													-1	1		3			0

Figure A3.2: *Matrix **A** containing **H** and **R** for load optimisation.*

Rows containing a support are removed in order to solve the system. Each stringer constitutes two columns, one for each end while shear areas constitute one for each area. This is due to the fact that stringer force varies linear from one end to the other while the shear stress is constant, cf. the main report chapter 3.1. The load vector, **R**, is included in the last column.

The constraint matrix, **C**, cf. Figure A3.3 contains rows and columns as following:

$$n_{rows} = 4 \cdot n_{stringers} + 2 \cdot n_{shear\_areas}$$
$$n_{columns} = 2 \cdot n_{stringers} + n_{shear\_areas}$$

The material constraint vector, $\mathbf{C}_d$, contains material constrains equalling compression and tension for stringers and shear strength for areas.

			Stringers														Shear fields			C_d
			1		2		3		4		5		6		7		A	B		
			start	end	start	end	start	end	start	end	start	end	start	end	start	end	τ_A	τ_B		
Stringers	1	*l*	-1																	2886,21
		u	1																	350,00
		l		-1																2886,21
		u		1																350,00
	2	*l*			-1															2886,21
		u			1															350,00
		l				-1														2886,21
		u				1														350,00
	3	*l*					-1													2886,21
		u					1													350,00
		l						-1												2886,21
		u						1												350,00
	4	*l*							-1											2886,21
		u							1										≤	350,00
		l								-1										2886,21
		u								1										350,00
	5	*l*									-1									2886,21
		u									1									350,00
		l										-1								2886,21
		u										1								350,00
	6	*l*											-1							2886,21
		u											1							350,00
		l												-1						2886,21
		u												1						350,00
	7	*l*													-1					2886,21
		u													1					350,00
		l														-1				2886,21
		u														1				350,00
Shear fields	8	*l*															-1			4,96
		u															1			4,96
	9	*l*																-1		4,96
		u																1		4,96

Figure A3.3: *Constraint matrix,* **C** *for load optimisation, l = lower value for material strength, u = upper value for material strength.* $\boldsymbol{C}_d$ *values for stringers are in kN and MPa for shear areas.*

Material optimisation

The linear programming problem can be used for minimising the material, e.g. by weight or cost, of the structure and is described as equation (A3.6) (Damkilde, 1995, equation (24)).

$$\begin{aligned} \text{minimise}: \quad & \mathbf{w}^T\,\mathbf{d} \\ \text{restrictions}: \quad & \mathbf{H}\,\boldsymbol{\beta} = \mathbf{R} \\ & \mathbf{C}\,\boldsymbol{\beta} - \mathbf{C}_d\,\mathbf{d} \leq \mathbf{C}_0 \\ & \mathbf{d} \geq 0 \end{aligned} \tag{A3.6}$$

where

w	Object function regarding material optimisation containing weighting parameters **w**
d	Vector with variables to be determined containing $\boldsymbol{\beta}$ and design variables **d**

$d_{stringer}$ = kN
d_{shear_area} = MPa

Equation (A3.6) is similar to equation (A3.1) except the limits for the variable, which must be positive. The input into MATLAB of matrices and vectors for the example is shown in the following. Table A3.2 indicates the symbolic connection between the syntax for MATLAB and the material optimisation problem.

MATLAB	Material optimisation
f	$\mathbf{c}$
x	x
A	$\mathbf{C}$ & $\mathbf{C}_d$
b	$\mathbf{C}_0$
Aeq	$\mathbf{H}$
beq	$\mathbf{R}$
$x0$	0

Table A3.2: *Symbols for MATLAB syntax and material optimisation.*

Compared to load optimisation the object function, **c**, contains material weights. When minimising $\mathbf{w}^T\mathbf{d}$ the optimal design parameters, **d**, are determined. The vector **c** contains first zeros corresponding to the number of columns in the matrix **C** and then weighting parameters corresponding to the number of columns in $\mathbf{C}_d$, cf. Figure A3.4. **c** and **x** are in general written as equation (A3.7).

$$\mathbf{c}^T\mathbf{x} = \begin{Bmatrix} 0_1 \\ \vdots \\ 0_{ns} \\ w_1 \\ \vdots \\ w_{var} \end{Bmatrix}^T \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ d_1 \\ \vdots \\ d_{ns} \end{Bmatrix} = \left\{ \begin{matrix} 0_1 & \dots & 0_{ns} & w_1 & \dots & w_{ns} \end{matrix} \right\} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ d_1 \\ \vdots \\ d_{ns} \end{Bmatrix} \tag{A3.7}$$

where

$$ns = 2 \cdot n_{stringers} + n_{shear\_areas}$$

The number of variants can be reduced if elements should have same design parameters, e.g. one common compression strength and tension strength for all stringers and one for all shear areas. In this case three weighting parameters which gives three design variables; d_1 for compression strength of stringers, $F_{c,max}$, d_2 for tension strength of stringers, $F_{t,max}$, and d_3 for shear areas, τ_{max}, cf. equation (A3.8). **c** contains zeros as the first 16 values and the weighting parameters for the last three values.

$$\mathbf{c}^T\mathbf{x} = \begin{Bmatrix} 0 \\ \vdots \\ 0_{16} \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix}^T \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{16} \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \{\ 0 \quad \dots \quad 0_{16} \quad w_1 \quad w_2 \quad w_3\ \} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{16} \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{A3.8}$$

The second restriction in equation (A3.6) is rewritten as equation (A3.9), where the matrix is the left hand side in Figure A3.4 and $\mathbf{C}_0$ is the right hand side. $\mathbf{C}$ and $\mathbf{C}_d$ for structure 1 with a simplified $\mathbf{C}_d$ is illustrated in Figure A3.5.

$$\begin{bmatrix} \mathbf{C} & -\mathbf{C}_d \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta} \\ \mathbf{d} \end{Bmatrix} \leq \mathbf{C}_0 \tag{A3.9}$$

		Stringers														Shear field		Cd																	C_0
		1		2		3		4		5		6		7		A	B																	
		start	end	start	end	start	end	start	end	start	end	start	end	start	end	τ_A	τ_B	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16		
Stringers 1	l	-1															-1																0
	u	1																-1															0
	l		-1														-1																0
	u		1															-1															0
2	l			-1															-1														0
	u			1																-1													0
	l				-1														-1														0
	u				1															-1													0
3	l					-1															-1												0
	u					1																-1											0
	l						-1														-1												0
	u						1															-1											0
4	l							-1															-1										0
	u							1																-1								≤	0
	l								-1														-1										0
	u								1															-1									0
5	l									-1															-1								0
	u									1																-1							0
	l										-1														-1								0
	u										1															-1							0
6	l											-1															-1						0
	u											1																-1					0
	l												-1														-1						0
	u												1															-1					0
7	l													-1															-1				0
	u													1																-1			0
	l														-1														-1				0
	u														1															-1			0
Shear fields 8	l															-1															-1		0
	u															1															-1		0
9	l																-1															-1	0
	u																1															-1	0

Figure A3.4: *Constraint matrix **C** for material optimisation.*

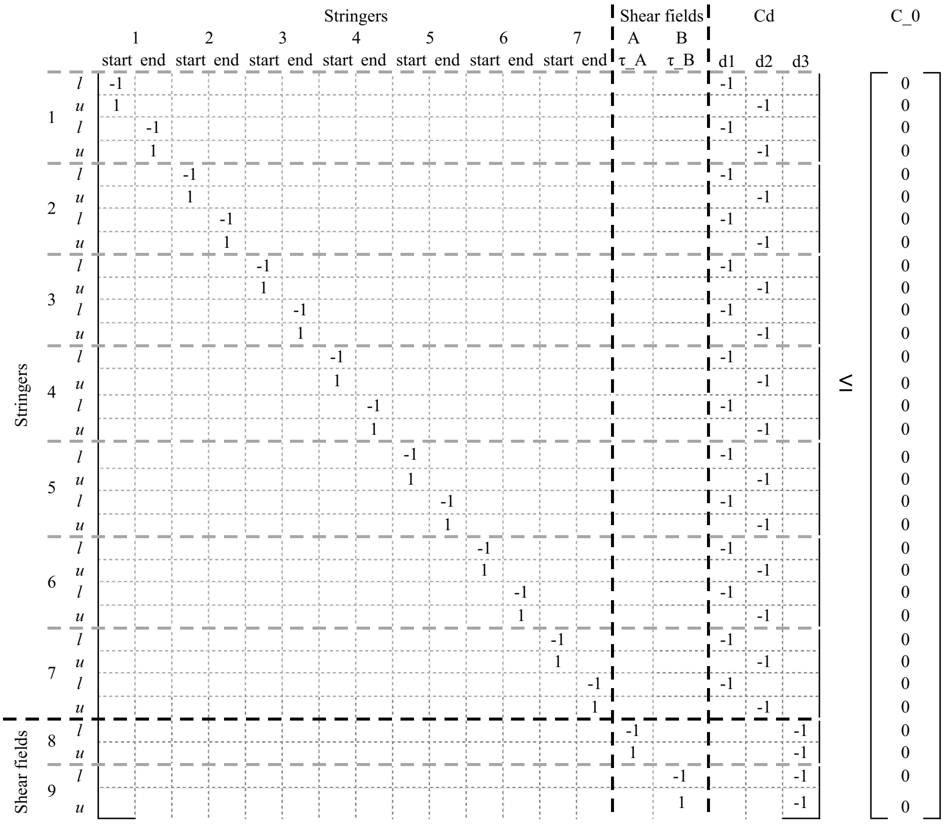


Figure A3.5: *Simplified constraint matrix, **C**, for material optimisation.*

The load, R, is removed from the flexibility matrix, **H**, and instead put in as the right-hand side of the equality in equation (A3.6). Three columns containing zeros are added to the flexibility matrix as the number of columns must match the number of columns in the constraint matrix.

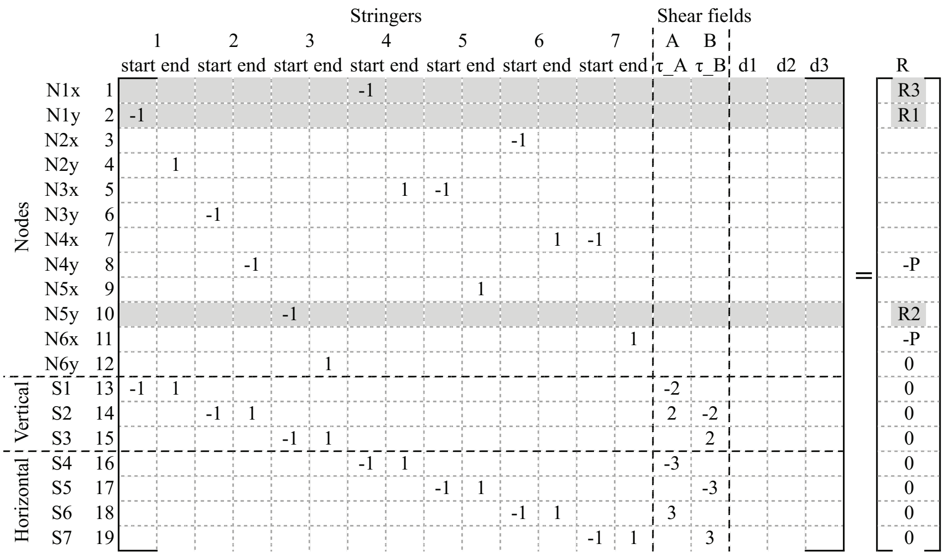


Figure A3.6: *Flexibility matrix, **H**, for material optimisation.*

The last restriction in equation (A3.6) are given to *linprog* as equation (A3.10) and equation (A3.11).

$$\left\{ \; lb \; \right\} = \left\{ \; [\,] \quad \ldots \quad [\,] \; \right\} \tag{A3.10}$$

$$\left\{ \; ub \; \right\} = \left\{ \; [\,] \quad \ldots \quad [\,] \; \right\} \tag{A3.11}$$

A3.2 Input for Upper Bound Method

This section provides the matrices and vectors used when `Opti_String` is performing optimisation regarding the upper bound method. Structure 1 is used for illustration and is shown in Figure A3.1.

Load optimisation

The LP problem for finding an upper value for the load optimisation is stated in equation (A3.12).

$$\begin{aligned} \text{minimise}: \quad & \mathbf{C}_d^T\ \mathbf{\Psi} - \mathbf{R}_0^T\ \mathbf{V} \\ \text{restrictions}: \quad & -\mathbf{H}^T\ \mathbf{V} + \mathbf{C}^T\ \mathbf{\Psi} = 0 \\ & \mathbf{R}^T\ \mathbf{V} = 1 \\ & \mathbf{\Psi} \geq 0 \end{aligned} \tag{A3.12}$$

where

$\mathbf{C}_d$	Vector with material constraints, strength values
$\mathbf{\Psi}$	Vector with plastic strain variables
$\mathbf{R}_0$	Constant load vector
$\mathbf{V}$	Vector with displacements for each node
$\mathbf{H}$	Flexibility matrix
$\mathbf{C}$	Constraint matrix
$\mathbf{R}$	Load vector

By rewriting the object function with the appurtenant variables in equation (A3.12) the object function for *linprog* is shown to the left which is multiplied with the variables to minimise, cf. equation (A3.13).

$$\left\{ \; \mathbf{C}_d^T \quad -\mathbf{R}_0^T \; \right\} \left\{ \begin{array}{c} \mathbf{\Psi} \\ \mathbf{V} \end{array} \right\} \tag{A3.13}$$

The object function, $\mathbf{c}$, is illustrated in Figure A3.7 where the first 32 rows corresponds to the $\mathbf{C}_d$ vector with four values per stringer and two per area. The values are similar to the $\mathbf{C}_d$ vector in Figure A3.3 where;

row_1 = $F_{c,max}$ for stringer 1 in start node
row_2 = $F_{t,max}$ for stringer 1 in start node
row_3 = $F_{c,max}$ for stringer 1 in end node
row_4 = $F_{t,max}$ for stringer 1 in end node
row_{29} = τ_{min}, lower values for shear stress in area 8
row_{29} = τ_{max}, upper values for shear stress in area 8

$\mathbf{c}$ =

1	2886,21	1	Stringer	C_d^T
2	350,00	1	Stringer	C_d^T
3	2886,21	1	Stringer	C_d^T
4	350,00	1	Stringer	C_d^T
⋮	⋮		Stringer	C_d^T
25	2886,21	7	Stringer	C_d^T
26	350,00	7	Stringer	C_d^T
27	2886,21	7	Stringer	C_d^T
28	350,00	7	Stringer	C_d^T
29	4,96	8	shear areas	C_d^T
30	4,96	8	shear areas	C_d^T
31	4,96	9	shear areas	C_d^T
32	4,96	9	shear areas	C_d^T
1	0,00			R_0^T
⋮	⋮			R_0^T
16	0,00			R_0^T

Figure A3.7: *Object function for load optimisation regarding upper bound method.*

The $\mathbf{R}_0$ vector contains zero since the self weight is neglected.

The first restriction represents the equality constrains and can be rewritten as equation (A3.14). The matrices is shown in Figure A3.2 and A3.3 respectively.

$$\begin{bmatrix} \mathbf{C}^T & -\mathbf{H}^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} = 0 \tag{A3.14}$$

The second restriction is linearised to $\mathbf{R}^T\mathbf{V} \leq 1$ and $-\mathbf{R}^T\mathbf{V} \leq -1$ and is written as equation (A3.15).

$$\begin{bmatrix} \mathbf{0} & \mathbf{R}^T \\ \mathbf{0} & -\mathbf{R}^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} \leq \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{A3.15}$$

Equation (A3.15) is illustrated in Figure A3.8.

Plastic strains Ψ													Displacements. Nodes V													Displacements. Stringers V		
1					7				8		9		1		2		3		4		5		6				
start		end			start		end		L	U	L	U	x	y	x	y	x	y	x	y	x	y	x	y	1		7
0	0	0	0	···	0	0	0	0	0	0	0	0	R1	R2	0	0	0	0	0	2	0	R3	2	0	0	···	0
0	0	0	0	···	0	0	0	0	0	0	0	0	R1	R2	0	0	0	0	0	-2	0	R3	-2	0	0	···	0

Figure A3.8: *Matrix **A** containing zeros for plastic strains and the displacements including loads. Gray columns contains supports and are removed.*

The last restriction in equation (A3.12) which state that all strains must be positive is given to *linprog* as equation (A3.16) and equation (A3.17).

$$\left\{ \; lb \; \right\} = \left\{ \; 0 \quad \ldots \quad [\,] \; \right\} \tag{A3.16}$$

$$\left\{ \; ub \; \right\} = \left\{ \; [\,] \quad \ldots \quad [\,] \; \right\} \tag{A3.17}$$

Material optimisation

The LP problem for finding an upper value for the material optimisation is stated in equation (A3.18)

$$\begin{aligned} \text{maximise :} \quad & \mathbf{R}^T \; \mathbf{V} - \mathbf{C}_0^T \; \mathbf{\Psi} \\ \text{restrictions :} \quad & \mathbf{H}^T \; \mathbf{V} - \mathbf{C}^T \; \mathbf{\Psi} = 0 \\ & \mathbf{C}_d^T \; \mathbf{\Psi} \leq \mathbf{w} \\ & \mathbf{\Psi} \geq 0 \end{aligned} \tag{A3.18}$$

All the matrices and vector are known from load optimisation.

A3.3 Algorithms

A comparison of the Simplex and Large-Scale algorithms is made. Based on the comparison one algorithm is chosen for the following calculations. The comparison is done using the simple system in Figure A3.1. For the first comparison the material limits for all the stringers and shear fields have been set to the same value. Using the two algorithms leads to the collapse mechanisms shown in Figure A3.9. Notice, only the mode of the collapse mechanisms are found, thus the size of the displacements are unknown as the calculations are done with perfect plastic material behaviour. Beside the collapse mechanisms the plastic strains are examined.

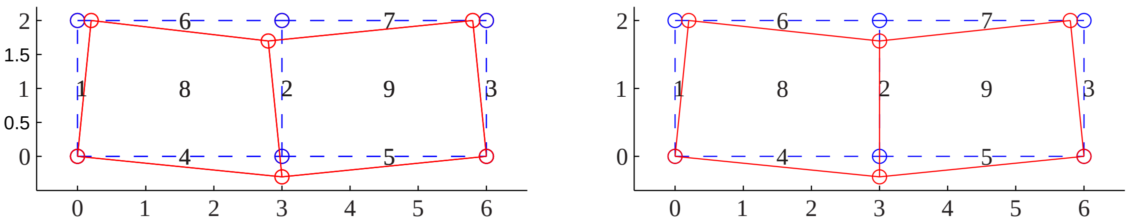


Figure A3.9: *Collapse mechanisms using Simplex (left) and Large-Scale (right) algorithms. Notice, only the mode of the collapse mechanisms are found, thus the size of the displacements are unknown.*

The collapse mechanism using Simplex provides a non-symmetric mechanism while Large-Scale provides a symmetric. Analysing which elements are exposed to plastic strains supports the shape of the two collapse mechanisms. Simplex only leads to plastic strains in stringer 6 while Large-Scale leads to plastic strains in both stringer 6 and 7.

An examination of the stress parameters, β, shows the two algorithms provides the exact same values.

Based on this simple comparison the Large-Scale algorithm is chosen for the following calculations as it entails a more balanced use of the materials.

Appendix **A4**

Practial design for material optimisation

This chapter covers the input for `Opti_String` *when carry out material optimisation with regard to material strengths. The chapter contains an enhancement of structure 1 from chapter A3 and the use in a context is shown in chapter 8.2.*

A4.1 Object Function

The object function **w** is shown in Figure A4.1. The number of variants is $2 \cdot n_{stringers} + n_{shear}$. The $\boldsymbol{\beta}$ vector contains two stress parameters for each stringer, including start and end node with the unit kN. After this follows the design parameters; two for each stringer and one for each area, kN for stringers and MPa for shear areas.

Object function = w^T * $\boldsymbol{\beta}$

	w^T	$\boldsymbol{\beta}$
1	0	1.1
2	0	1.2
3	0	2.1
4	0	2.2
5	0	3.1
6	0	3.2
7	0	4.1
8	0	4.2
9	0	5.1
10	0	5.2
11	0	6.1
12	0	6.2
13	0	7.1
14	0	7.2
15	0	8
16	0	9
w1	1	d_1.min
w2	1	d_1.max
w3	1	d_2.min
w4	1	d_2.max
w5	1	d_3.min
w6	1	d_3.max
w7	1	d_4.min
w8	1	d_4.max
w9	1	d_5.min
w10	1	d_5.max
w11	1	d_6.min
w12	1	d_6.max
w13	1	d_7.min
w14	1	d_7.max
w15	1	d_8.min/max
w16	1	d_9.min/max

Figure A4.1: *Object function to minimise.*

A4.2 Linking Design Variables

The number of row in the flexibility matrix, H, is increased to the number of links between elements, cf. equation (A4.1). The links are expressed in E.

$$\begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta} \\ \mathbf{d} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{e}_0 \end{Bmatrix} \tag{A4.1}$$

In this example **E** consist of five rows which ensure stringer four and five to be one stringer line with the same material strength for both compression and tension and the same for six and seven. Moreover, both shear areas have the same material parameters. The location of **E** is due to the length and entries in **w**. H is illustrated in Figure A4.2 for structure 1.

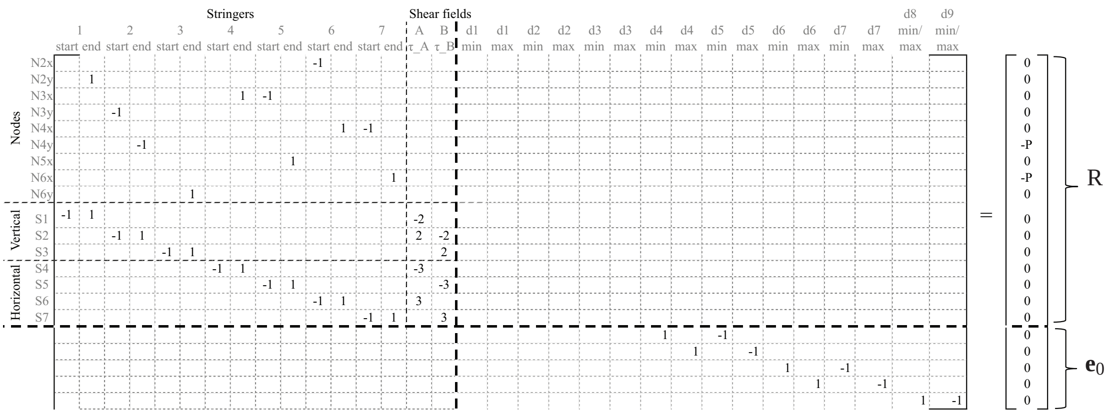


Figure A4.2: *Equality matrix for material optimisation regarding elements with equal strength parameters.*

A4.3 Implementation of Material Limits in Material Optimisation

The extra inequalities are added up to the inequalities from equation (A3.6) which afterwards is formulated by equation (A4.2).

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta} \\ \mathbf{d} \end{Bmatrix} - \begin{Bmatrix} \mathbf{C}_d \\ 0 \end{Bmatrix} \mathbf{d} \leq \begin{Bmatrix} \mathbf{C}_0 \\ -\mathbf{m}_0 \end{Bmatrix} \tag{A4.2}$$

The matrix for inequalities is illustrated in Figure A4.3. The width of $\mathbf{C}_d$ and **M** corresponds to the number of variants **d** in the $\boldsymbol{\beta}$ vector. It is noticed that **M** is multiplied with minus one because the inequality sign is turned.

	Elm.		Stringers 1 start	1 end	2 start	2 end	3 start	3 end	4 start	4 end	5 start	5 end	6 start	6 end	7 start	7 end	Shear fields A τ_A	B τ_B	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	≤
Stringers	1	l	-1																-1																0
		u	1																	-1															0
		l		-1															-1																0
		u		1																-1															0
	2	l			-1																-1														0
		u			1																	-1													0
		l				-1															-1														0
		u				1																-1													0
	3	l					-1																-1												0
		u					1																	-1											0
		l						-1															-1												0
		u						1																-1											0
	4	l							-1																-1										0
		u							1																	-1									0
		l								-1															-1										0
		u								1																-1									0
	5	l									-1																-1								0
		u									1																	-1							0
		l										-1															-1								0
		u										1																-1							0
	6	l											-1																-1						0
		u											1																	-1					0
		l												-1															-1						0
		u												1																-1					0
	7	l													-1																-1				0
		u													1																	-1			0
		l														-1															-1				0
		u														1																-1			0
Shear fields	8	l															-1																-1		0
		u															1																	-1	0
	9	l																-1															-1		0
		u																1																-1	0
Stringers	1	l																	-1																F_c
		u																		1															F_t
		l																	-1																F_c
		u																		1															F_t
	2	l																			-1														F_c
		u																				1													F_t
		l																			-1														F_c
		u																				1													F_t
	3	l																					-1												F_c
		u																						1											F_t
		l																					-1												F_c
		u																						1											F_t
	4	l																							-1										F_c
		u																								1									F_t
		l																							-1										F_c
		u																								1									F_t
	5	l																									-1								F_c
		u																										1							F_t
		l																									-1								F_c
		u																										1							F_t
	6	l																											-1						F_c
		u																												1					F_t
		l																											-1						F_c
		u																												1					F_t
	7	l																													-1				F_c
		u																														1			F_t
		l																													-1				F_c
		u																														1			F_t
Shear fields	8	l																															-1		τ_{min}
		u																																1	τ_{max}
	9	l																															-1		τ_{min}
		u																																1	τ_{max}

Figure A4.3: *Inequality matrix for material optimisation regarding material parameters.*

A4.4 Results for Linking Design Variables

By linking design variables in the main report chapter 8.1 the following results are obtained. Based on the calculated forces in the stringers and shear areas the amount of reinforcement is found, cf. Table A4.1. The reinforcement represents a weight of 38 kg.

Nodes	Necessary reinforcement area $\left[mm^2\right]$	Reinforcement
1	-	-
2	-	-
3	-	-
4	199	6 Ø8
5	-	-
6	-	-
7	263	6 Ø8
8	594	6 Ø12
Shear fields, x- and y-dir.	982 $^{mm^2}/_{m}$	Ø12/ 100 mm

Table A4.1: *Reinforcement in tensile stringers and shear fields.*

A4.5 Load Combinations

By having two load cases, **I** and **II**, equation (A3.6) is rewritten to equation (A4.3) for material optimisation based on the lower bound method. Notice all vectors and matrices are grown to the double size.

$$
\begin{aligned}
\text{minimise}: \quad & \mathbf{w}^T \cdot \mathbf{d} \\
\text{restrictions}: \quad & \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \beta_I \\ \beta_{II} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_I \\ \mathbf{R}_{II} \end{Bmatrix} \\
& \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \beta_I \\ \beta_{II} \end{Bmatrix} - \begin{Bmatrix} \mathbf{C}_d \\ \mathbf{C}_d \end{Bmatrix} \mathbf{d} \leq \begin{Bmatrix} \mathbf{C}_0 \\ \mathbf{C}_0 \end{Bmatrix} \\
& \mathbf{d} \geq 0
\end{aligned}
\tag{A4.3}
$$

where

w	Object function with material weights **w**
d	Vector with variables to be determined containing $\boldsymbol{\beta}$ and **d**

The expression can be developed by implementing the practical design restrictions from section A4.2 and A4.3. For the two load cases equation (A4.3) can be developed to the expression in Figure A4.4 by including equation (A4.1) and A4.2.

minimise: $\mathbf{c}^T\ \mathbf{x}$

restrictions:
$$\left[\begin{array}{cc|cc} \mathbf{H} & \mathbf{0} & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{0} & \mathbf{E} & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{H} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{E} \end{array}\right] \begin{Bmatrix} \beta_I \\ \mathrm{d1} \\ \beta_{II} \\ \mathrm{d2} \end{Bmatrix} = \begin{Bmatrix} \mathrm{R1} \\ \mathrm{e0} \\ \mathrm{R2} \\ \mathrm{e0} \end{Bmatrix}$$

$$\left[\begin{array}{cc|cc} \mathbf{C} & \mathbf{0} & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{0} & \mathbf{-M} & & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{C} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{-M} \end{array}\right] \begin{Bmatrix} \beta_I \\ \mathrm{d1} \\ \beta_{II} \\ \mathrm{d2} \end{Bmatrix} - \begin{Bmatrix} \mathbf{C}_{dI} \\ 0 \\ \mathbf{C}_{dII} \\ 0 \end{Bmatrix} \quad \mathbf{d} \quad \leq \begin{Bmatrix} \mathbf{C}_{0I} \\ -\mathbf{m}_{0I} \\ \mathbf{C}_{0II} \\ -\mathbf{m}_{0II} \end{Bmatrix}$$

$\mathbf{d} \geq 0$

Figure A4.4: *Two load combinations for structure 1 with respect to matrix **M** and **E**.*

Output from `Opti_String`

This chapter contains the output from `Opti_String` when performing load optimisation based on the lower bound method. The input are shown in chapter A3. The primal value from material optimisation are also shown since they differ because of the design variables. The treatment of the values are described in the main report chapter 9.

A5.1 Primal Values

The Primal values are shown in Figure A5.1. The treatment of the values are described in the main report chapter 9.3.

Load Optimisation

	Primal
1	133,33
	0,00
2	0,00
	160,00
3	26,67
	0,00
4	160,00
	-40,00
5	-40,00
	0,00
6	0,00
	200,00
7	200,00
	160,00
8	0,22
9	-0,04
λ	16,00

Material Optimisation

		Primal
stringers	1	-8,333
		0,000
	2	0,000
		-10,000
	3	-1,667
		0,000
	4	-10,000
		2,500
	5	2,500
		0,000
	6	0,000
		-12,500
	7	-12,500
		-10,000
shear areas	8	-0,014
	9	0,003
	d_1	12,500
	d_2	2,500
	d_3	0,014

Figure A5.1: *Primal values from* `Opti_String` *using linprog.*

A5.2 Dual Values

The dual values also called the shadow prices are shown in Figure A5.2. The treatment of the values are described in the main report chapter 9.3. When using *linprog* the shadow prices consist of the dual inequalities and dual equalities. The dual inequalities are shown to the left in Figure A5.2 and the dual equalities are shown to the right.

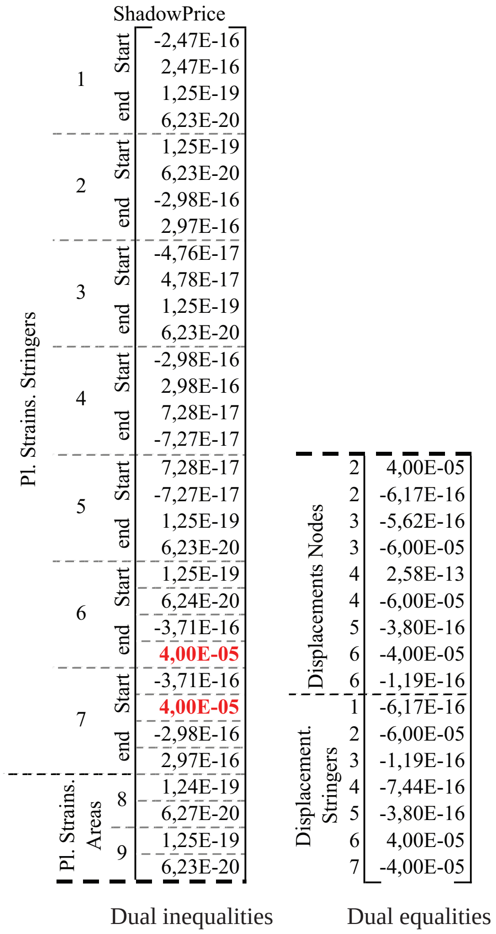


Figure A5.2: *The vector ShadowPrices from `Opti_String` for structure 1. The elements with plastic strains are marked with bold, read. The vector is split in two because of the length.*

A5.3 Material Parameters

`Opti_String` automatic generate the matrix `mat` regarding strength values stated in the Eurocode for stringer method, cf. Figure A5.3. `mat` is used for generating C_d in load optimisation cf. equation (A3.3) and Figure A3.3. Moreover are `mat` used for generating $\mathbf{m}_0$ when performing advanced material optimisation, cf. Figure A4.3.

mat

stringers	1	2886,21	350,00
	2	2886,21	350,00
	3	2886,21	350,00
	4	1924,14	350,00
	5	1924,14	350,00
	6	1924,14	350,00
	7	1924,14	350,00
shear areas	8	4,96	4,96
	9	4,96	4,96

Figure A5.3: *Matrix generated by* `Opti_String` *for the example in chapter A3. Stringer strength [* kN*]. Stress in areas[* MPa*]*

A5.4 Practical Design

This section contains the vectors which `Opti_String` handle from *linprog* when performing practical design for material optimisation. Two cases will be illustrated;

1. Material optimisation regarding matrix **M** and **E**

2. Load combinations with two load cases, *I* and *II* and regarding matrix **M** and **E**

Material Optimisation Regarding Practical Design Restrictions

Design variables		Design variables after linking
d_1.min		d_1.min
d_1.max		d_1.max
d_2.min		d_2.min
d_2.max		d_2.max
d_3.min		d_3.min
d_3.max		d_3.max
d_4.min		max(d_4min;d_5_min)
d_4.max		max(d_4max;d_5.max)
d_5.min		max(d_4min;d_5_min)
d_5.max	⇒	max(d_4max;d_5.max)
d_6.min		max(d_6min;d_7_min)
d_6.max		max(d_6max;d_7.max)
d_7.min		max(d_6min;d_7_min)
d_7.max		max(d_6max;d_7.max)
d_8.min/max		max(d_8;d_9)
d_9.min/max		max(d_8;d_9)

Figure A5.4: *Design variables are collected in a vector after the property matrix* **E** *has linked relevant elements*

By introducing material matrix **M** design variables are restricted in the optimisation. Elements in the stringer system must respect design requirements stated in Eurocodes which is ensured by the automatically generated matrix `mat` in `Opti_String`.

$$\begin{bmatrix} d\_1.min \\ d\_1.max \\ d\_2.min \\ d\_2.max \\ d\_3.min \\ d\_3.max \\ d\_4{,}5.min \\ d\_4{,}5.max \\ d\_6{,}7.min \\ d\_6{,}7.max \\ d\_9{,}9.min/max \end{bmatrix} \leq \begin{bmatrix} m\_1.min \\ m\_1.max \\ m\_2.min \\ m\_2.max \\ m\_3.min \\ m\_3.max \\ m\_4{,}5.min \\ m\_4{,}5.max \\ m\_6{,}7.min \\ m\_6{,}7.max \\ m\_9{,}9.min/max \end{bmatrix} = \begin{bmatrix} 2886{,}21 \\ 350{,}00 \\ 2886{,}21 \\ 350{,}00 \\ 2886{,}21 \\ 350{,}00 \\ 1924{,}14 \\ 350{,}00 \\ 1924{,}14 \\ 350{,}00 \\ 4{,}96 \end{bmatrix}$$

Designvariables $\qquad \mathbf{m}_0 \qquad$ `mat`

Figure A5.5: *Design variables subjected to material strengths by **m** equal matrix* `mat` *from* `Opti_String`*. mat is in kN.*

Material Optimisation Regarding Two Load Cases and Practical Design Restrictions

$$\begin{array}{c} \text{Loadc1} \\ \\ \\ \text{Loadc2} \end{array} \begin{bmatrix} \beta_{i,I} \\ \vdots \\ \beta_{j,I} \\ d_{i,I} \\ \vdots \\ d_{j,I} \\ \hdashline \beta_{i,II} \\ \vdots \\ \beta_{j,II} \\ d_{i,I} \\ \vdots \\ d_{j,II} \end{bmatrix} = \begin{bmatrix} \max(\beta_{i,I}, \beta_{i,II}) \\ \vdots \\ \max(\beta_{j,I}, \beta_{j,II}) \\ \hdashline \max(d_{i,I}, d_{i,II}) \\ \vdots \\ \max(d_{j,I}, d_{j,II}) \end{bmatrix}$$

primal variables $\qquad$ Applied variables

Figure A5.6: *Filtration of primal variables regarding two load cases.*

Application of `Opti_String` for Complex Structure

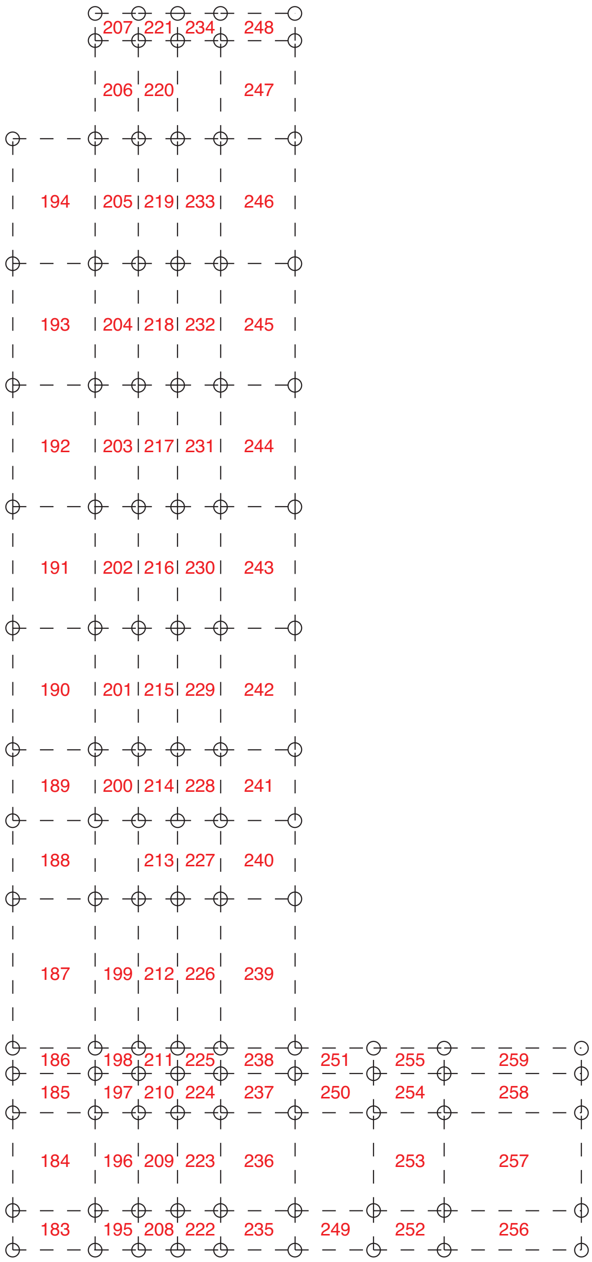


Figure A6.1: *Shear area numbers for structure 4.*

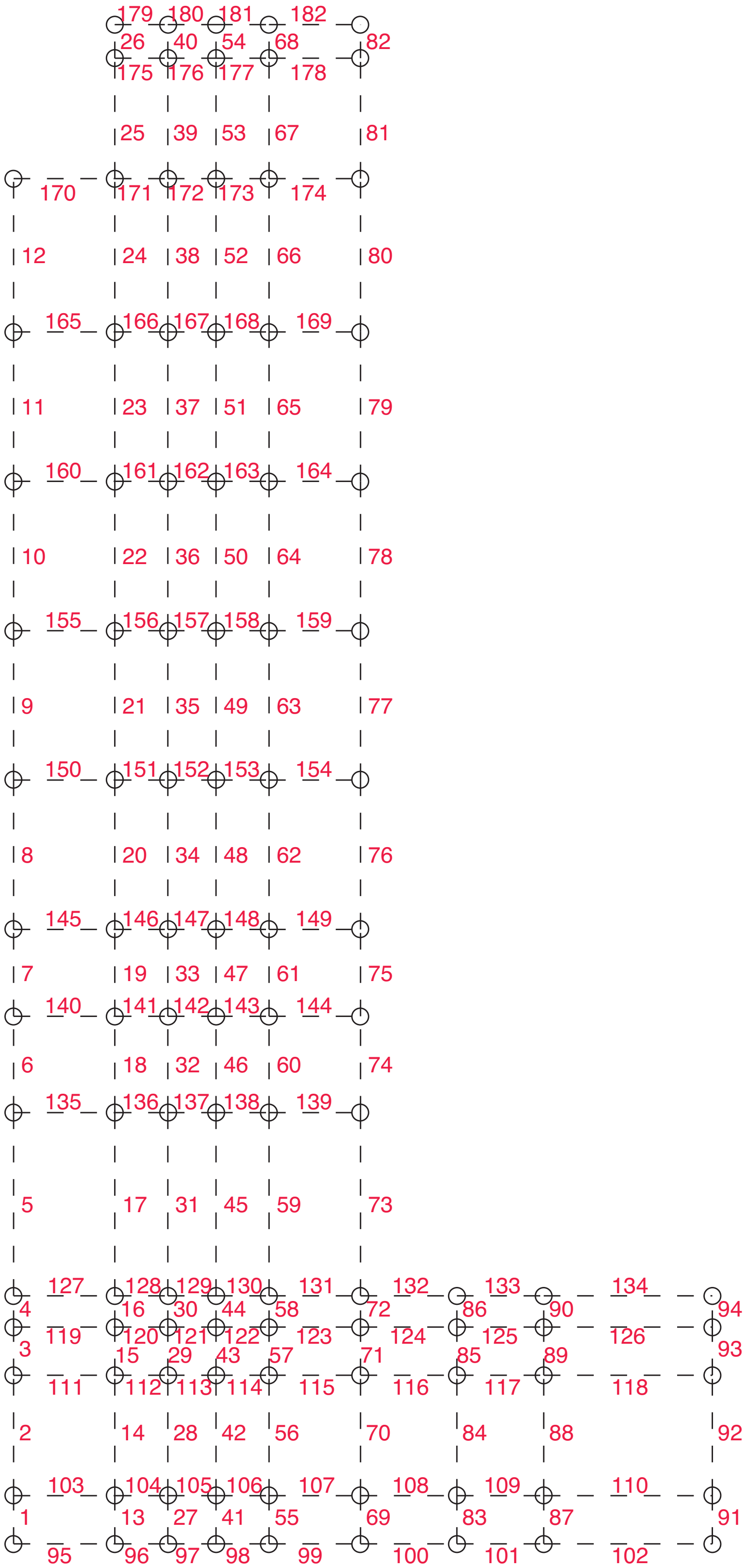


Figure A6.2: Stringer numbers for structure 4.

Appendix CD

A7.1 Stringer Method Hand Calculations

Shear Stress and Stringer Force

reinforcement and compression capacity

A7.2 `Opti_String` - Final Version

Functions and Data Files

A7.3 `Opti_String` - Lower Bound Method

Functions and Data Files

A7.4 `Opti_String` - Upper Bound Method

Functions and Data Files

Bibliography

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