Action Investment Games

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Synopsis:

We define the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets.

We have given a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We study the decision problem which is the foundation for reasoning about trade-offs and given a comprehensive analysis of complexity bounds for relevant problems within the formalism.

In the worst cases increases the complexity by two levels in the polynomial hierarchy compared to the complexity of the corresponding energy game.

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It is the job of any computer scientist and software engineer to develop software, but software does not always behave correctly or as intended. There can be a lot of reasons for flawed software but by applying formal techniques such as modelling, model-checking and verification it is possible to find bugs, correct errors and prove correctness, even before development process starts. [2]

Embedded software and embedded systems surround us all in our daily lives, making modelling and verification of these systems relevant, not only for our well-being but also for the companies producing these systems as correcting a bug or a flaw in thousands of deployed systems can be extremely costly if not impossible [4].

It is typical for an embedded system to have limited or constrained resources and to interact with uncontrollable or even hostile environments. When adding this to the ever increasing demand on functionality and reliability for the lowest possible price, several interesting problems emerge.

An example of such a system is a simple mobile phone where the constraining resource is the size of the battery. The required functionality could be that the phone must be able to make some number of calls before the battery is empty. It is possible to install different antennas in the phone, a good and a bad, where the bad antenna is less expensive than the good antenna, but it consumes more energy when used. Intuitively the battery needs to be larger if the budget limits the phone to have the bad antenna, than if the budget was high then the phone could have the good antenna.

The formalism for studying this kind of trade-off between a constrained resource of energy and a budget is the main topic of this thesis.

We define and study Action Investment Games (AIG), which are energy games, extended with budgets and prices on actions. An energy game [5] is played by two players on a finite graph, where Player 1 wins if he has a strategy such that the accumulated energy is constrained within an given interval, and Player 2 wins if Player 1 loses.

The extension captures the intuition from the mobile phone example, as Player 1 can make an investment not costing more than his budget and remove behaviour from the resulting energy game. It also adds the possibility of Player 2 to making an investment not costing more than his budget and enable possible behaviour.
We give a comprehensive analysis on the complexity for the decision problem; *is it possible for Player 1 to make an investment costing less than his budget, such that for any Player 2 investment costing less than his budget, Player 1 is the winner of the resulting energy game within a given interval?*

**Related work**

Energy games that are constrained by either a lower bound or an interval have attracted much attention in recent years. Energy games were introduced with and without time in [5] and generalised with mean-payoff games in [6]. Energy games with multiple weights were introduced in [7]. However, none of these introduce the concept of investments or the trade-offs involved with these.

In [3] a dual-price schema for modal transition systems is studied, this introduces a long-run average cost and a hardware investment cost, it gives a trade-off scenario similar to the one studied in this thesis. However, modal transition systems lack the concept of two players, meaning that there are no uncontrollable states to model an environment and it lack the concept of constraining as it gives a trade-off over a long-run average.

To the best of our knowledge this Theses provides the first study of energy games extended with budgets and prices on actions.
Chapter 2

Preliminaries

We start by defining boolean formulas and the polynomial hierarchy.

2.1 Boolean Formula

Boolean variables $X = \{x_1, x_2, \ldots\}$ are assigned true or false, by a valuation $\nu : X \rightarrow \{true, false\}$. A boolean formula $\varphi$ over $X$ consists variables combined with operators $\lor$ (logic OR), $\land$ (logic AND) and $\neg$ (logic NOT). A formula $\varphi$ is satisfiable if there exists an assignment $\nu$ such that $\varphi$ evaluates to true is under $\nu$.

2.2 The Polynomial Hierarchy

Definitions and results in this section is based on [1, Chap. 5] and [8, Chap. 17].

Alternating Turing Machines (ATM) are an extension of Turing Machines with existential ($\exists$) and universal ($\forall$) states. Existential ($\exists$) states capture non-determinism and universal ($\forall$) states capture co-non-determinism. The ATM accepts from a existential state ($\exists$) if one successor accepts and from a universal ($\forall$) states if all successors accept. ATMs give a characterisation of the polynomial-time hierarchy (PH).

**Definition 1.** Let $\Sigma_k^P$ be the class of languages $L$ accepted by an ATM that begins in a existential ($\exists$) state, alternates between existential and universal ($\forall$) states at most $k-1$ times.

**Definition 2.** Let $\Pi_k^P$ be the class of languages $L$ accepted by an ATM that begins in a universal ($\forall$) state, alternates between universal and existential ($\exists$) states at most $k-1$ times.

Note that for the first level is $\Sigma_1^P = \text{NP}$ and $\Pi_1^P = \text{coNP}$. It is also clear that $\Sigma_k^P \subseteq \Pi_{k+1}^P$ and $\Pi_k^P \subseteq \Sigma_{k+1}^P$, an illustration of this is shown in Figure 2.1.

**Definition 3.** The polynomial hierarchy is defined as $PH = \bigcup_k \Sigma_k^P = \bigcup_k \Pi_k^P$

**Theorem 4.** [8, Prop. 17.1] $PH \subseteq \text{PSPACE}$
We define quantified boolean formulas which have a finite number of alternations, formally $\Sigma_i^P$-SAT
\[ \exists \vec{x}_1 \forall \vec{x}_2 \exists \ldots \mathcal{Q} \vec{x}_i \varphi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_i) \]
and $\Pi_i^P$-SAT
\[ \forall \vec{x}_1 \exists \vec{x}_2 \forall \ldots \mathcal{Q} \vec{x}_i \varphi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_i) \]
where $\varphi$ is a boolean formula, each $\vec{x}_i$ is a vector of boolean variables, and $\mathcal{Q}$ is $\exists$ or $\forall$ depending on whether $i$ is odd or even.

QSAT problems is the decision problems whether a quantified boolean formulas is true for false. There are a QSAT problem represented on each level in the polynomial hierarchy.

**Theorem 5.** [8, Thm. 17.10] The $\Sigma_i^P$-SAT problem is $\Sigma_i^P$-complete and the $\Pi_i^P$-SAT problems is $\Pi_i^P$-complete for all $i \geq 0$
Chapter 3

Definitions

We defining Weighted Energy Games, runs, plays and strategies for this type of game, these are slight modification of the definitions in [5] and [6]. We then introduce Action Investment Game, which builds on top of Weighted Energy Games by giving each player a set of actions, a budget and introducing a cost for each action.

3.1 Energy Game

Definition 6 (Weighted Energy Game). A Weighted Energy Game (WEG) is a tuple \( \mathcal{G} = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0) \) where

- \( Q \) is a finite state set,
- \( Q_1, Q_2 \) are a disjoint splitting of \( Q \), \( Q = Q_1 \uplus Q_2 \),
- \( \Sigma \) is a finite set of actions,
- \( \rightarrow \subseteq Q \times \Sigma \times \mathbb{Z} \times Q \) is a successor relation where \( (q, \sigma, z, q') \in \rightarrow \) is written as \( q \xrightarrow{\sigma, z} q' \), and
- \( q_0 \in Q \) is the initial state.

A Weighted Energy Game is represented by a graph where each node represents a state, the node is circled if it is in \( Q_1 \) and squared if it is in \( Q_2 \). Edges represent a transition between states and the edge is labeled with an action name and a weight. Figure 3.1 is an example of a WEG.

![Figure 3.1: Example of a simple energy game](image)

Informally a WEG is a board where an energy game is played on. The energy game is played by moving a token around on the board. The token starts in the initial state \( q_0 \). If the token is in a circle state, then Player 1 moves the token to a successor...
state. Likewise if the token is in a square state then Player 2 moves the token to a successor state. The sequence the token moves is called a run, this is formally defined as follows.

**Definition 7** (Run). A run in $G = (Q_1, Q_2, \Sigma, \rightarrow, q_0)$ starting from $q_0$ is a finite or infinite sequence $r = q_0 \xrightarrow{\sigma_0, z_0} q_1 \xrightarrow{\sigma_1, z_1} q_2 \xrightarrow{\sigma_2, z_2} \ldots$ where $q_i \in Q$ and $q_i \xrightarrow{\sigma_i, z_i} q_{i+1} \in \rightarrow$ for all $i$.

It is possible during run to determine the energy level $\gamma$ in each step by summing the weight on each transition up to that step in the run. Two runs on the WEG is shown in Figure 3.1 and their energy level through each step is seen in Figure 3.2.

Given a finite run, $r = q_0 \xrightarrow{\sigma_0, z_0} \ldots q_{n-1} \xrightarrow{\sigma_{n-1}, z_{n-1}} q_n$ let the last state of the run be denoted $\text{Last}(r) = q_n$. A finite run $r$ is maximal if $\text{Last}(r)$ has no successors.

**Definition 8** (Valid Run). A run $r$ is valid in the interval $[a, b]$, $a \in \mathbb{Z}$, $a \leq 0$, and $b \in \mathbb{Z} \cup \{\infty\}$, $b \geq a$ if $r = q_0 \xrightarrow{\sigma_0, z_0} \ldots q_n$ is a maximal finite run and

$$a \leq \sum_{i=0}^{n-1} z_i \leq b,$$

or if $r = q_0 \xrightarrow{\sigma_0, z_0} q_1 \xrightarrow{\sigma_1, z_1} \ldots$ is a infinite run and

$$a \leq \sum_{i=0}^{n} z_i \leq b \text{ for all } n \geq 0.$$

![Figure 3.2: Run on the WEG in Figure 3.1](image)

A run $r$ is said to be winning for Player 1 and loosing for Player 2 if it is valid. A Play on a WEG can produce several different runs depending on what strategy each player uses.

A strategy $\delta$ for Player $i$, where $i = \{1, 2\}$, maps a finite non-maximal run $r$ where $\text{Last}(r) = q_n$ and $q_n \in Q_i$ to a successor $q_n \xrightarrow{\sigma_n, z_n} q_{n+1}$. We denote a strategy for Player $i$ as $\delta_i$. 
A play on a WEG using strategy $\delta_1$ produces either a infinite run $q_0 \xrightarrow{\sigma_0,z_0} q_{n+1} \ldots$ where for any $n$, if $q_n \in Q_i$ then $q_n \xrightarrow{\sigma_n,z_n} q_{n+1} = \delta_i(q_0 \xrightarrow{\sigma_0,z_0} \ldots q_n)$. Or it produces a maximal finite run $q_0 \xrightarrow{\sigma_0,z_0} \ldots \xrightarrow{\sigma_{n-1},z_{n-1}} q_n$ where for $0 \leq m < n$ if $q_m \in Q_i$ then $q_m \xrightarrow{\sigma_m,z_m} q_{m+1} = \delta_i(q_0 \xrightarrow{\sigma_0,z_0} \ldots q_m)$.

We now define the interval bound problem, which is the question whether Player 1 wins or looses a given WEG.

**Interval bound problem:**
Given a WEG $G$ and an interval $[a, b]$, does there exist a strategy for Player 1, $\delta_1$, such that any play on $G$ using the strategy $\delta_1$ produces a run valid in the interval $[a, b]$?

Player 1 wins and Player 2 looses a WEG $G$ in the interval $[a, b]$ if the answer to the interval bound problem for $G$ in the interval $[a, b]$ is positive.

**Example 9.** Let us now turn to the example WEG in Figure 3.1 and consider this game with the interval $[-2, 4]$. We want to find the winner of the game, Player 1 wins if we can find a strategy which is a solution to the interval bound problem for $[-2, 4]$.

A strategy is a function which take a non-maximal run and gives the next move. But in this game we do not need the full run to describe the next move for Player 1, we only need the current energy level $\gamma$ and the current location.

The strategy covers behaviour in the state $A$ as this is the only state in $Q_1$. The strategy is as follows:

- If $\gamma \geq 3$ do $A \xrightarrow{a \cdot \gamma} B$
- If $\gamma = 2$ do $A \xrightarrow{a \cdot 2} A$
- If $\gamma < 2$ do $A \xrightarrow{a \cdot 3} A$

The solid run on Figure 3.2 is a run that can occur if Player 1 plays according to this strategy. It is possible to see that any play where Player 1 plays according to this strategy produces a valid run.

We now argue that this is true for the lower bound in the interval, let $\gamma = 3$ and the play has reached state $A$, then Player 1 takes the move to state $B$ changing $\gamma$ to 2, then if Player 2 takes the worst possible transition $B \xrightarrow{c \cdot -1} A$ do $\gamma = -2$ and the game will continue from $A$. The strategy tells us that Player 1 does only take transitions from $A$ to $B$ if the $\gamma \geq 3$, therefore we know that the lower bound has not been breached.

This strategy yields a positive solution to the interval bound problem, and therefore Player 1 is the winner of the WEG in Figure 3.1 in the interval $[-2, 4]$.

There are a less strict version of the interval bound problem called the lower-bound
problem which was studied in [5]. In this, a lower-bound problem is an interval bound problem for \([a, \infty]\).

Energy games in [5] have an initial energy level on \(c\) and the interval bound problem is solved for the interval \([0, b]\). In our definition the initial energy level is always 0, and the interval bound problem is solved for \([a, b]\) where \(a \in \mathbb{Z}, a \leq 0\). The two definitions are similar as our definition can model an initial energy level on \(c\) by changing the interval to \([a - c, b - c]\), and their definition can model an interval \([a, b]\) by setting the initial energy to \(|a|\) and the interval to \([0, b + |a|]\).

As these two definitions are similar can we use the complexity results from [5].

### 3.2 Action Investment Game

**Definition 10 (Action Investment Game).** An action investment game (AIG) is a tuple, \(\mathcal{A}_G = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0, \Sigma_1, \Sigma_2, \text{actCost}, B_1, B_2)\) where,

- \((Q, Q_1, Q_2, \Sigma, \rightarrow, q_0)\) is a WEG,
- \(\Sigma_1, \Sigma_2 \subseteq \Sigma\) are disjoint action sets \(\Sigma_1 \cap \Sigma_2 = \emptyset\)
- \(\text{actCost}\) is a function \(\text{actCost} : \Sigma_1 \cup \Sigma_2 \rightarrow \mathbb{N}_0\) and
- \(B_1, B_2 \in \mathbb{N}\) are two budgets.

An investment is a subset of actions \(I \subseteq \Sigma\). An investment for Player \(i\) is denoted \(I_i\), and \(I_i \subseteq \Sigma_i\). The cost of an investment is the sum of the cost of the actions in the investment \(\text{invCost}(I) = \sum_{\sigma \in I} \text{actCost}(\sigma)\).

**The AIG problem for an interval:**

Given AIG \(\mathcal{A}_G\), and an interval \([a, b]\) does there exist an initial investment \(I_1 \subseteq \Sigma_1\) where \(\text{invCost}(I_1) \leq B_1\), such that for all possible initial investment \(I_2 \subseteq \Sigma_2\) where \(\text{invCost}(I_2) \leq B_2\), Player 1 wins the WEG \(G' = (Q, Q_1, Q_2, \Sigma', \rightarrow', q_0)\) in the interval \([a, b]\), where \(\Sigma' = (\Sigma \setminus I_1) \cap ((\Sigma \setminus \Sigma_2) \cup I_2)\) and \(\rightarrow' = \rightarrow \cap (Q \times \Sigma' \times \mathbb{Z} \times Q)\)?

Player 1 is wins and Player 2 loses the AIG \(\mathcal{A}_G\) for the interval \([a, b]\) if the answer to the AIG problem for \(\mathcal{A}_G\) and the interval \([a, b]\) is positive. Meaning that Player 1 wins the AIG in the interval \([a, b]\) if he can find an investment \(I_1\) such that for any Player 2 investment \(I_2\), Player 1 has a strategy to win the resulting WEG in the interval \([a, b]\).

An action in \(I_1 \subseteq \Sigma_1\) is disabled from the resulting WEG, meaning that it is not possible for either player to take any transitions labeled with that action in the resulting WEG. Informally Player 1 can see his investment as a guarantee which ensures that some action can not happen.

An action in \(I_2 \subseteq \Sigma_2\) is enabled in the resulting WEG, meaning that it possible for either player to take any transitions labeled with that action in the resulting WEG.
Informally Player 2 enables actions and thereby possible behaviour in the resulting WEG by his investment.

It is clear that for an AIG where \( \Sigma_1 = \emptyset \) and \( \Sigma_2 = \emptyset \), the AIG problem for an interval is an interval bound problem, as neither player can make an investment.

It is also clear that for a AIG \( \mathcal{A}_G = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0, \Sigma_1, \Sigma_2, \text{actCost}, B_1, B_2) \) where \( B_1 = 0 \) and \( B_2 = 0 \) the AIG problem for an interval can be solved by solving a single interval bound problem for \( \mathcal{G}' = (Q, Q_1, Q_2, \Sigma', \rightarrow', q_0) \) where \( \Sigma' = \Sigma \setminus \Sigma_2 \) and \( \rightarrow' = \cap (Q \times \Sigma' \times \mathbb{Z} \times Q) \), as the only investment either player can make is the empty investment.
This is a motivational example of a network relay station. The network relay station contains a battery, a solar panel to charge the battery, and an antenna from which it can receive signals. It is clear that the amount of energy charged to the battery depends on the weather. There are two types of weather conditions in this study case, mild and extreme. If the weather is mild, and the station needs to charge there is added 2 or 3 energy units to the battery, if the weather is extreme then there is added 1 or 6 energy units to the battery.

When the network relay receives a signal, the signal can be good, normal, or bad. It requires more energy to decode the signal if it is bad than if it is normal or good, and less energy if it is good than normal. Furthermore the weather also influences how much energy is needed to receive and decode the signal. It is possible to make a test of the system, in a test no energy is added to or consumed from the battery.

On the station is it possible to install a better antenna and thereby remove the bad or normal signal, a better antenna naturally comes at a cost.

Figure 4.1 shows the AIG for the network relay station, where $\Sigma_1 = \{\text{bad, normal, good}\}$, $\Sigma_2 = \{\text{test, mild, extreme}\}$ and the actCost function is given in Table 4.1.

<table>
<thead>
<tr>
<th>$\sigma \in \Sigma_1$</th>
<th>bad</th>
<th>normal</th>
<th>good</th>
</tr>
</thead>
<tbody>
<tr>
<td>actCost($\sigma$)</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma \in \Sigma_2$</th>
<th>test</th>
<th>mild</th>
<th>extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td>actCost($\sigma$)</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 4.1: The actCost function for the AIG in Figure 4.1

The intuition is that if Player 1 has a high budget $B_1$ then can he buy a better antenna and ensure a better signal, this corresponds to Player 1 including the action bad in his investment, this implies that the station works with a smaller battery. If Player 2 has a high budget $B_2$ he can require that the station must work in extreme weather conditions, and the station needs a larger battery to work.

We now investigate two instances of the relay station to show this intuition.

1. For the budgets $B_1 = 6$, $B_2 = 3$ and the interval $[0, 4]$ Player 1 wins, but if the budget for Player 1 is lowered to $B_1 = 5$ then Player 1 loses. The reason is that for $B_1 = 6$ Player 1 can make the investment $I_1 = \{\text{bad, normal}\}$, and
ensure that the station only receives good signals, this is not possible if $B_1 = 5$ as $\text{actCost} \{\text{bad, normal}\} = 6$.

2. For the budgets $B_1 = 4$, $B_2 = 4$ and the interval $[0, 6]$ Player 1 wins, but if the budget for Player 2 $B_2$ is increased to $B_2 = 5$ then Player 1 loses. It is possible for Player 1 to win $B_1 = 4$ and $B_2 = 5$ but the interval needs to be $[0, 11]$, the reason is that the extreme actions could be enabled in the resulting WEG.
In this chapter we find the complexity of solving the AIG problem for an interval. We give results under different budget restrictions and for intervals with and without no upper bound. The chapter begins by introducing three gadgets, and defines how these can be linked, this is used when finding complexity bounds in the end of the chapter.

5.1 Gadgets for complexity bounds

Gadgets are basically constructors for AIGs. They are used later when satisfiability (SAT) problems are reduced to AIG problems for an interval.

In a gadget boolean variables are represented by actions. The boolean $x$ is represented by two actions $x$ and $x'$. The idea is that, a player has to choose his investment such that either the action $x$ or $x'$ is present in the resulting WEG, but not both. When the action $x$ is present the boolean variable $x$ assignment is true, $\nu(x) = true$, and when $x'$ is present $\nu(x) = false$. This creates a map from investments to the assignment of booleans.

**Definition 11 (Valid investment).** An investment $I_i \subseteq \Sigma_i$ is valid for Player $i$, where $i = \{1, 2\}$, if for all $x \in \Sigma_i$ either $x$ or $x'$ is in $I_i$, but not both $(x \in I_i \lor x' \in I_i) \land \neg(x \in I_i \land x' \in I_i)$.

It is clear that each player needs a sufficient budget to make a valid investment. This is defined as follows.

**Definition 12 (Sufficient Budget).** The budget $B_i$ is sufficient for Player $i$, where $i = \{1, 2\}$, if $\text{actcost}(I_i) \leq B_i$ for any valid investments $I_i \subseteq \Sigma_i$.

The two gadgets which are presented in the two following sections, have the property that a player needs to choose a valid investment or he risks loosing.

5.1.1 Gadget $G_v(\vec{x})$

The purpose of the gadget $G_v(\vec{x})$ is to construct an AIG, $A_v$, from a vector of booleans $\vec{x} = (x_1 \ldots x_n)$. The property is that Player 2 needs to choose his invest-
ment as a valid investment, or Player 1 has a strategy to win any play. In addition if Player 2 choose a valid investment, any play starting from \( q_{in} \) reaches \( q_{out} \) or it is loosing for Player 1. Let \( \Sigma_2 = \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\} \) such that it is possible for Player 2 to make a valid investment. The construction of \( A_{\forall} \) and representation of the gadget is shown on Figure 5.1.

![Diagram of the gadget](image)

(a) Construction

(b) Representation

Figure 5.1: Gadget 2

**Lemma 13.**

(a) If \( I_2 \) is a valid investment for Player 2, then any play starting from \( q_{in} \) is either loosing for Player 1 or reaches \( q_{out} \).

(b) If \( I_2 \) is a valid investment for Player 2, then Player 1 has a strategy such that any play from \( q_{in} \) reaches \( q_{out} \).

(c) If \( I_2 \) is not a valid investment for Player 2, then Player 1 has a strategy to win any play starting from \( q_{in} \).

**Proof.** (a) Let \( I_2 \) be a valid investment for Player 2. We evaluate how a play can evolve in the first step from \( q_{in} \). \( I_2 \) is a valid investment therefore either \( x_1 \in I_2 \) or \( x'_1 \in I_2 \). We evaluate the outcome for each case. If \( x_1 \in I_2 \), Player 1 has two choices, a \( x_1 \) transition up and a \( x_1 \) transition right. If Player 1 takes the transition up he loses, as he will be in an infinite negative loop as \( x'_1 \notin I_2 \). If Player 1 takes the transition right the play is one step closer to \( q_{end} \). If \( x'_1 \in I_2 \) Player 1 can only
5.1. Gadgets for complexity bounds

take a transition right and the play is one step closer to \( q_{end} \).

Now we know how any play can evolve from \( q_{in} \) to be either loosing for Player 1 or one step closer to \( q_{end} \).

There are \( n \) steps from \( q_{in} \) to \( q_{out} \) and if the argument is repeated to cover all \( n \) steps, then it gives that any play from \( q_{in} \) either is loosing for Player 1 or reaches \( q_{out} \).

(b) Let \( I_2 \) be a valid investment for Player 2. Now we need to find a strategy for Player 1 which ensures that any play from \( q_{in} \) reaches \( q_{out} \). The Player 1 strategy is to take either the transition \( x_j \) or \( x'_j \) right for \( 0 < j \leq n \), this is possible as \( I_2 \) is valid.

Any play from \( q_{in} \) where Player 1 use this strategy reaches \( q_{out} \).

(c) Let \( I_2 \) be not a valid investment for Player 2. This means that there is a smallest \( i \) such that \( \{x_i, x'_i\} \in I_2 \) or \( \{x_i, x'_i\} \not\in I_2 \). Now we need to find a winning strategy for Player 1. The Player 1 strategy from \( q_{in} \) to \( q_i \) is to take a transition right, this is possible as \( x_j \in I_2 \) or \( x'_j \in I_2 \) for \( 1 \leq j < i \). From \( q_i \) there are two cases, if \( \{x_i, x'_i\} \in I_2 \) or if \( \{x_i, x'_i\} \not\in I_2 \). If \( \{x_i, x'_i\} \in I_2 \) then the strategy is to take two transition up, \( q_i \xrightarrow{x_i,0} r_i \) followed by \( r_i \xrightarrow{x'_i,0} s_i \) to \( s_i \), this is winning for Player 1 as \( s_i \) does not have any successors. If \( \{x_i, x'_i\} \not\in I_2 \) then Player 1 wins as \( q_i \) does not have any successors.

Any play from \( q_{in} \) where Player 1 use this strategy is Player 1 winning, as the resulting run always is a valid maximal run to \( q_i \) or \( s_i \).

\[ \square \]

5.1.2 Gadget \( G_3(\vec{x}) \)

The purpose of the gadget \( G_3(\vec{x}) \) is to construct an AIG, \( A_3 \), from a vector of booleans \( \vec{x} = (x_1 \ldots x_n) \). The property is that Player 1 needs to choose his investments as a valid investment, or Player 2 has a strategy to win any play. Let \( \Sigma_1 = \{x_1, \ldots x_n, x'_1 \ldots x'_n\} \) such that it is possible for Player 1 to make a valid investment. The constructed and representation of the gadget is shown in Figure 5.2.

Lemma 14.

(a) If \( I_1 \) is a valid investment for Player 1, then Player 1 has a strategy such that any play from \( q_{in} \) either reaches \( q_{out} \) or is loosing for Player 2.

(b) If \( I_1 \) is a valid investment for Player 1, then Player 2 has a strategy such that any play from \( q_{in} \) either reaches \( q_{out} \) or is loosing for Player 1.

(c) If \( I_1 \) is not a valid investment for Player 1, then Player 2 has a strategy to win any play starting from \( q_{in} \).
Proof. (a) Let $I_1$ be a valid investment for Player 1. We need to find a Player 1 strategy $\delta_1$ which ensures that any play from $q_{in}$ either reaches $q_{out}$ or is loosing for Player 2. The Player 1 strategy $\delta_1$ is; when the play reach a state in $Q_1 \setminus \{q_{end}\}$ then take the action in $\Sigma_2 \setminus I_2$ available from that state.

We now evaluate how any play can evolve starting from $q_{in}$ when Player 1 play according to $\delta_1$. As $q_{in} \in Q_1$ the play continues according to $\delta_1$ sending the play right, one step closer to $q_{out}$, and the play continues from a state in $Q_2$.

For this state there are two cases, if $x_1' \in I_1$ or if $x_1 \in I_1$. If $x_1' \in I_1$ then Player 2 has a choice, either to take a $x_1$ transition up or to take a $\tau$ transition right. If Player 2 takes the $x_1$ transition up he loses, as the run generated would be maximal and valid. If Player 2 takes the $\tau$ transition right the play continues from a state in $Q_1$, and the play will be one step closer to $q_{out}$.

If $x_1 \in I_1$ then the state only has one successor and Player 2 must take the $\tau$ transition right, the play will continue from a state in $Q_1$, and it will be one step closer to $q_{out}$.

Now we know how any play from $q_{in}$ where Player 1 plays according to $\delta_1$ either takes two transitions right, two steps closer to $q_{out}$, or is loosing for Player 2.

There are $2n$ steps from $q_{in}$ to $q_{out}$ and if the argument is repeated to cover all
2n steps, then it gives that any play from $q_{in}$ where Player 1 plays according to $\delta_1$ either reaches $q_{out}$ or is loosing for Player 2.

(b) Let $I_1$ be a valid investment for Player 1. We need to find a Player 2 strategy $\delta_2$ which ensures that any play from $q_{in}$ either reaches $q_{out}$ or is loosing for Player 1. The Player 2 strategy $\delta_2$ is; take the $\tau, 0$ transition right when possible.

We now evaluate how a play can evolve starting from $q_{in}$ when Player 2 plays according to $\delta_2$.

From $q_{in}$ Player 1 has a choice, either to take the $\tau, -1$ self-loop transition or move right along either a $x_1$ or $x'_1$ transition. Player 1 does eventually lose if enough $\tau, -1$ self-loop transition is taken, by breaking the lower bound of the interval.

If Player 1 takes a transition right the play continues from a state in $Q_2$, one step closer to $q_{out}$. From there Player 2 plays according to $\delta_2$ and takes $\tau, 0$ transition right to a state in $Q_1$, again one step closer to $q_{out}$.

Now we know how any play from $q_{in}$ where Player 2 plays according to $\delta_2$, Player 1 loses or the play evolves right, two steps closer to $q_{out}$.

The argument can be repeated to cover all $2n$ steps through the game, and gives that any play from $q_{in}$ where Player 2 plays according to $\delta_2$ either reaches $q_{out}$ or is loosing for Player 1.

(c) Let $I_1$ not be a valid investment for Player 1. This means that there is a smallest $i$ such that $\{x_i, x'_i\} \in I_2$ or $\{x_i, x'_i\} \notin I_2$ and for $1 \leq j < i$ is either $x_j \in I_1$ or $x'_j \in I_1$, but not both. We need to find a Player 2 strategy which ensures that Player 2 wins any play starting from $q_{in}$.

The Player 2 strategy is; take $\tau, 0$ transition right when possible until the play reach $q_i$. By the similar argument as in (b) we know that any play from $q_{in}$ reaches $q_i$ or is winning for Player 2.

If the play reaches $q_i$ is there two possibility, either $\{x_i, x'_i\} \in I_1$ or $\{x_i, x'_i\} \notin I_1$. We will now find a Player 2 strategy for each possibility. If $\{x_i, x'_i\} \notin I_2$ wins Player 2 as Player 1 is forced to do a infinite amount of $q_i \xrightarrow{\tau,-1} q_i$, transitions

If $\{x_i, x'_i\} \notin I_2$ wins Player 2 by taking two transitions up to $t_i$ as any play from $t_i$ gives a infinite invalid run which is winning for Player 2. \qed
5.1.3 Gadget $\mathcal{G}_\varphi(\varphi)$

The purpose of the gadget $\mathcal{G}_\varphi(\varphi)$ is to construct an AIG, $\mathcal{A}_\varphi$, from a boolean formula $\varphi(x_1 \ldots x_n)$. We assume without loss of generality that all negations are pushed to the variables in $\varphi$.

The gadget is inductively construct using the four figures, in Figure 5.3.

![Building blocks for $\mathcal{G}_\varphi$](image)

The representation of the gadget is shown in Figure 5.4.

![Representation of $\mathcal{G}_\varphi$](image)
Recall that $\Sigma_1 \cap \Sigma_2 = \emptyset$.

**Lemma 15.** Let $I_1, I_2$ be a valid investments and let

$$v(x) = \begin{cases} 
  \text{true} & \text{if } x' \in I_1 \text{ or } x \in I_2 \\
  \text{false} & \text{if } x \in I_1 \text{ or } x' \in I_2
\end{cases}$$

(a) If $\varphi$ is true under $v$ then Player 1 has a strategy to win any play starting from $q_{in}$ in $A_\varphi$.

(b) If $\varphi$ is false under $v$ then Player 2 is the winner of any play starting from $q_{in}$ in $A_\varphi$.

**Proof.** (a) Let $\varphi$ be true under $v$ and we prove by induction in the structure of $A_\varphi$ that Player 1 has a strategy to win any play starting from $q_{in}$ on $A_\varphi$.

**Basis:** We have two base cases, $\varphi = x$ and $\varphi = \neg x$.

If $\varphi = x$, then $A_\varphi$ is as in Figure 5.3a and the Player 1 strategy needs to cover the first state where a $x, 0$ and a $\tau, -1$ transition is possible. The Player 1 strategy is to take the $x, 0$ transition. The $x, 0$ action is possible as $\varphi$ is true under $v$. Any play with this strategy is winning for Player 1 as the resulting run is valid and maximal.

If $\varphi = \neg x$ then $A_\varphi$ is as in Figure 5.3b and the Player 1 strategy needs to cover the first state where a $x, 0$ and a $\tau, -1$ transition is possible. The Player 1 strategy is to take the $x', 0$ transition. The $x', 0$ action is possible as $\varphi$ is true under $v$. Any play with this strategy is winning for Player 1 as the resulting run is valid and maximal.

**Induction Step:** Assume by induction hypothesis (IH) that for $\varphi = F$ Player 1 has a strategy to win any play on $A_\varphi$.

We now prove that Player 1 has a strategy to win any play where $\varphi = F_1 \land F_2$ and where $\varphi = F_1 \lor F_2$.

If $\varphi = F_1 \land F_2$, we know that $\varphi$ is true under $v$, this implies that $F_1$ and $F_2$ is true, and we know by the IH that Player 1 has a strategy to win both $F_1$ and $F_2$, therefor he also has a strategy to win when $\varphi = F_1 \land F_2$.

If $F_1 \lor F_2$ we know that $\varphi$ is true under $v$, this implies that either $F_1$ or $F_2$ is true. We know by the IH that if $F_1$ is true Player 1 has a winning strategy for $F_1$ or if $F_2$ is true Player 1 has a winning strategy for $F_2$. The Player 1 strategy is to take the transition Left $\tau, 0$ if $F_1$ is true and take Right $\tau, 0$ if $F_2$ is true. Player 1 now has a strategy to win if either $F_1$ or $F_2$ is true.

We now know by induction that Player 1 has a winning strategy to win any play on $A_\varphi$ if $\varphi$ is true under $v$.

(b) Let $\varphi$ be false under $v$ and we prove by induction in the structure of $A_\varphi$ that any play starting from $q_{in}$ on $A_\varphi$ is Player 2 winning.

**Basis:** We have two base cases, $\varphi = x$ and $\varphi = \neg x$. 

If $\varphi = x$, then $A_{\varphi}$ is as in Figure 5.3a and

If $\varphi = x$, then $A_{\varphi}$ as Figure 5.3a We know $\varphi$ is false under $v$ meaning that either $x \in I_1$ or $x' \in I_2$ implying that no $x$ transition is present in the resulting WEG. Any play is therefore Player 2 winning it produces infinite and invalid run.

If $\varphi = \neg x$, then $A_{\varphi}$ as Figure 5.3b We know $\varphi$ is false under $v$ meaning that either $x' \in I_1$ or $x \in I_2$ implying that no $x'$ transition is present in the resulting WEG. Any play is therefore Player 2 winning it produces infinite and invalid run.

**Induction Step:** Assume by induction hypothesis (IH) that for $\varphi = F$ is any play on $A_{\varphi}$ Player 2 winning.

We now prove that any play on $A_{\varphi}$ where $\varphi = F_1 \land F_2$ and where $\varphi = F_1 \lor F_2$ is Player 2 winning.

If $F_1 \land F_2$, we know that $\varphi$ is false if either $F_1$ or $F_2$ is false under $v$. By the IH do and the fact that $F_1$ and $F_2$ are in sequence wins Player 2 any play as any produces run is infinite and invalid.

If $F_1 \lor F_2$, we know that $\varphi$ is false if both $F_1$ and $F_2$ is false under $v$. By the IH do and the fact that $F_1$ and $F_2$ are in parallel wins Player 2 any play as any produces run is infinite and invalid. \qed
5.1. Gadgets for complexity bounds

5.1.4 Linking gadgets

Gadgets can be linked together in sequence and thereby create a combined AIG. This is done by adding a $\tau,0$ transition from $q_{out}$ in one gadget to the state $q_{in}$ in another gadget. If it is necessary then rename states and updating successor relation, union action sets, union state sets, union successor relation, sum budgets.

The short hand notation of linking gadgets is an arrow. The representation of three linked gadgets $G_{\exists}(\vec{x}) \rightarrow G_{\forall}(\vec{y}) \rightarrow G_{\varphi}(\varphi)$ is shown in Figure 5.5.

The construction of linking gadgets is useful for reducing (Q)SAT problems to AIG problems. A construction of a reduction from a given QSAT problem to a AIG problem is shown in Example 16.

**Example 16.** Reduction from the QSAT problem $\exists x_1, x_2 \forall y_1, y_2 (x_1 \lor (\neg y_2 \land y_1) \land (\neg y_1 \lor y_2 \lor \neg x_2)) = 1$ to an AIG problem. The construction is three gadgets linked together, $G_{\exists}(\vec{x}) \rightarrow G_{\forall}(\vec{y}) \rightarrow G_{\varphi}(\varphi)$, the representation of this shown in Figure 5.5.

The full drawn out AIG is shown in Figure 5.6.

The intuitions is that if there exists a $x_1, x_2$ for which the QSAT problem is true, then Player 1 is the winner of the AIG. Similar if there does not exists a $x_1, x_2$ for which the QSAT problem is true then Player 2 is the winner of the AIG.
We now argue that this transformation can be done in polynomial time. For a problem with $n$ literals is at most $5n$ states introduced in the game by the first two gadgets, and for the last gadget with the worst possible $\varphi$, all $n$ literals disjoint introduce $4n + 2$ states. Therefore the AIG constructed is at most $20n + 2$ larger than the QSAT problem, which is definitely polynomial in the size of the QSAT problem.
5.2 Complexity results

This section covers all complexity results.

<table>
<thead>
<tr>
<th>Budget restrictions</th>
<th>Interval</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Existential $Q_2 = \emptyset$</td>
<td>Game</td>
</tr>
<tr>
<td>$B_1 = 0, B_2 = 0$</td>
<td>$[a, \infty]$</td>
<td>$\in P; [5]$</td>
</tr>
<tr>
<td></td>
<td>$[a, b]$</td>
<td>$\in UP \cap \text{coUP}; [5]$</td>
</tr>
<tr>
<td>$B_2 = 0$</td>
<td>$[a, \infty]$</td>
<td>NP-Complete; Lem. 18</td>
</tr>
<tr>
<td></td>
<td>$[a, b]$</td>
<td>NP-Hard, $\in P$; Lem. 19</td>
</tr>
<tr>
<td></td>
<td>$\in PSPACE$, Lem. 24</td>
<td>EXPTIME-complete; Lem. 27</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>$[a, \infty]$</td>
<td>$\Pi_1^P$-complete; Lem. 20</td>
</tr>
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<td></td>
<td>$[a, b]$</td>
<td>$\Pi_2^P$-hard; Claim 21, $\in PSPACE$, Lem. 22</td>
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<td></td>
<td>$\in \Sigma_2^P$-complete; Lem. 26</td>
<td></td>
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<tr>
<td>$-$</td>
<td>$[a, \infty]$</td>
<td>$\Pi_1^P$-complete; Lem. 23</td>
</tr>
<tr>
<td></td>
<td>$[a, b]$</td>
<td>$\Pi_2^P$-hard; Claim 21, $\in PSPACE$, Lem. 22</td>
</tr>
<tr>
<td></td>
<td>$\in \Sigma_2^P$-complete; Lem. 27</td>
<td></td>
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</tbody>
</table>

Table 5.1: Complexity for AIG problems for an interval

**Proposition 17.** Let $\mathcal{A}_G$ be an AIG where $Q_2 = \emptyset$. The complexity of solving the AIG problem for $\mathcal{A}_G$ in an interval with no restrictions on $B_1$ and is equivalent to the complexity of solving the AIG for problem for an $\mathcal{A}_G$ interval where $B_1 = 0$.

**Proof.** Let $\mathcal{A}_G$ be an AIG where $Q_1 = \emptyset$ and with no restrictions on $B_1$. Now we solve the AIG problem for an $\mathcal{A}_G$ interval. Player 1 choses an investment $I_1 \subseteq \Sigma_1$ where $actCost(I_1) \leq B_1$. But all states are in $Q_1$, meaning that Player 1 controls any play on the resulting WEG, hence Player 1 does not benefit from making an investment different from $I_1 = \emptyset$ as this could remove options for him in the resulting WEG.

Therefore the budget $B_1$ can be ignored and restricted to 0, as Player 1 in any AIG where $Q_2 = \emptyset$ for any budget makes the investment $I_1 = \emptyset$. This makes the two problems equivalent, and the budget $B_1$ can be set to 0 when $Q_2 = \emptyset$.

**Lemma 18.** The AIG problem where $Q_2 = \emptyset$ and $B_2 = 0$, for the interval $[a, \infty]$ is in $P$.

**Proof.** This follows from Proposition 17 and the fact that the AIG problem where $Q_2 = \emptyset$, $B_2 = 0$ and $B_1 = 0$ for the interval $[a, \infty]$ is in $P$.

**Lemma 19.** The AIG problem where $Q_2 = \emptyset$, and $B_2 = 0$, for the interval $[a, b]$ is NP-Hard and in $PSPACE$. 


Chapter 5. Complexity

Proof. This follows from Proposition 17 and the fact that the AIG problem where \( Q_2 = \emptyset, B_2 = 0 \) and \( B_1 = 0 \) for the interval \([a, \infty)\) is NP-Hard and in PSPACE. \(\square\)

Lemma 20. The AIG problem where \( Q_2 = \emptyset \) and \( B_1 = 0 \), for the interval \([a, \infty)\) is \(\Pi^P_1\)-complete.

Proof. Lower bound: The AIG problem where \( Q_2 = \emptyset \) and \( B_1 = 0 \) for the interval \([a, \infty)\) is \(\Pi^P_1\)-hard by reduction from \(\Pi^P_1\)-SAT.

Let \( \forall \bar{x} \varphi(\bar{x}) = 1 \) be a \(\Pi^P_1\)-SAT problem. We construct the AIG \(A_G\) given by \(G_{\forall}(\bar{x}) \rightarrow G_{\varphi}(\varphi)\) and define \(B_2\) to be any sufficient budget.

We want to show:

(a) If \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is true, Player 1 wins \(A_G\).

(b) If \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is false, Player 2 wins \(A_G\).

(a) Suppose that \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is true under the assignment of \(v\). We now want to show that Player 1 wins the AIG \(A_G\).

Player 2 can pick either an invalid or valid investment \(I_2\). If \(I_2\) is an invalid investment we know from Lemma 13 (c) that Player 1 has a strategy to win any play starting from \(q_{in}\) in the AIG \(A_{\forall}\) constructed by \(G_{\forall}(\bar{x})\). If \(I_2\) is a valid investment we know from Lemma 13 (b) that Player 1 has a strategy such that any play from \(q_{in}\) reaches \(q_{out}\) in \(A_{\forall}\).

If the play reaches \(q_{out}\) in \(A_{\varphi}\) then by the construction of linking the play continues in the AIG \(A_{\varphi}\) constructed from \(G_{\varphi}(\varphi)\).

We know that \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is true. Hence for every \(v\) this imply by Lemma 15 (a) that Player 1 win any play starting from \(q_{in}\) in \(A_{\varphi}\).

Therefore do Player 1 have a strategy to win no what investment \(I_2\) Player 2 does.

(b) Suppose that \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is false. There there is an assignment \(v\) such that \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is false. We now want to show that Player 2 wins the AIG \(A_G\).

Player 2 chooses the valid investment \(I_2 = \{x \mid v(x) = true\} \cup \{x' \mid v(x) = false\}\).

By Lemma 13 (a) we know that any play starting in \(q_{in}\) gets to \(q_{out}\) in the AIG \(A_{\forall}\) constructed by \(G_{\exists}(\bar{x})\), or Player 2 wins.

If the play reaches \(q_{out}\) in \(A_{\varphi}\) then by the construction of linking the play continues in the AIG \(A_{\varphi}\) constructed from \(G_{\varphi}(\varphi)\).

We know that \( \forall \bar{x} \varphi(\bar{x}) = 1 \) is false under the assignment of \(v\) and therefore by Lemma 15 (b) we know that Player 2 is winning any play starting from \(q_{in}\) in \(A_{\varphi}\).

We have now shown that Player 2 wins any on play on \(A_G\).
This construction shows that a solution to the AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$ for the interval $[a, \infty]$ can solve any $\Pi_1^P$-SAT problem, hence the AIG problem is $\Pi_1^P$-hard.

Upper bound:
The AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$ for the interval $[a, \infty]$ is in $\Pi_1^P$ by the following algorithm.

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve interval bound problem for the resulting WEG ($Q_2 = \emptyset$) in the interval $[a, \infty]$ if all results are positive return yes, else return no.

The algorithm uses one universal quantifier over a polynomial time problem, therefore in $\Pi_1^P$.

\begin{claim}
The AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is $\Pi_1^P$-hard.
\end{claim}

\begin{lemma}
The AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is in PSPACE.
\end{lemma}

\begin{proof}
We want to prove that the AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is in PSPACE, this is done by finding an algorithm in PSPACE which solves the problem. The algorithm is

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve interval bound problem for the resulting WEG ($Q_2 = \emptyset$) in the interval $[a, b]$ if all results are positive return yes, else return no.

Step 1 is a universal quantifier over step 2. Step 2 is a problem in PSPACE, hence the algorithm is in PSPACE.
\end{proof}

\begin{lemma}
The AIG problem where $Q_2 = \emptyset$, for the interval $[a, \infty]$ is $\Pi_1^P$-complete.
\end{lemma}

\begin{proof}
This follows from Proposition 17 and Lemma 20.
\end{proof}

\begin{lemma}
The AIG problem where $B_2 = 0$ for an interval $[a, \infty]$ is NP-complete.
\end{lemma}

\begin{proof}
Lower bound:
The AIG problem where $B_2 = 0$ for the interval $[a, \infty]$ is NP-hard by reduction from SAT.

Let $\exists \vec{x} \varphi(\vec{x}) = 1$ be a SAT problem. We construct the AIG $A_G$ given by $G_\exists(\vec{x}) \rightarrow G_\varphi(\varphi)$ and define $B_1$ to be any sufficient budget.

We want to show:

\begin{enumerate}
\item[(a)] If $\exists \vec{x} \varphi(\vec{x}) = 1$ is true, Player 1 wins $A_G$.
\end{enumerate}
(b) If $\exists \vec{x} \varphi(\vec{x}) = 1$ is false, Player 2 wins $A_G$.

(a) Suppose that $\exists \vec{x} \varphi(\vec{x}) = 1$ is true under some assignment $\nu$. We now want to show that Player 1 wins the AIG $A_G$. Player 1 chooses the valid investment $I_1 = \{ x' \mid \nu(x) = true \} \cup \{ x \mid \nu(x) = false \}$.

Player 1 has by Lemma 14 (a) a strategy such that any play starting in $q_{in}$ gets to $q_{out}$ in the AIG $A_3$ constructed by $G_3(\vec{x})$, or Player 1 wins.

If the play reaches $q_{out}$ in $A_3$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\varphi)$.

We know that $\exists \vec{x} \varphi(\vec{x}) = 1$ is true under the assignment of $\nu$ and therefore Player 1 in $A_\varphi$ by Lemma 15 (a) has a strategy such that any play starting in $q_{in}$ is winning for Player 1.

We have now shown that Player 1 has an investment $I_1$ and a strategy to win such that he wins $A_G$.

(b) Suppose that $\exists \vec{x} \varphi(\vec{x}) = 1$ is false under any assignment $\nu$. We now want to show that Player 2 wins the AIG $A_G$ no matter what Player 1 does.

Player 1 can pick either an invalid or valid investment $I_1$. If $I_1$ is an invalid investment we know from Lemma 14 (c) that Player 2 has a strategy to win any play starting from $q_{in}$ in the AIG $A_3$ constructed by $G_3(\vec{x})$.

If $I_1$ is a valid investment we know from Lemma 14 (b) that Player 2 has a strategy such that any play from $q_{in}$ either reaches $q_{out}$ in $A_3$ or is winning for Player 2.

If the play reaches $q_{out}$ in $A_3$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\varphi)$.

We know that $\exists \vec{x} \varphi(\vec{x}) = 1$ is false under the assignment of $\nu$ this implies by Lemma 15 (b) that Player 2 wins any play starting from $q_{in}$ in $A_\varphi$.

Therefore Player 2 has a strategy to win no matter what Player 1 does.

This construction shows that a solution to the AIG problem where $B_2 = 0$ for the interval $[a, \infty]$ can solve any SAT problem, hence the AIG problem is NP-hard.

Upper bound:
The AIG problem where $B_2 = 0$ for an interval $[a, \infty]$ is in NP by the following algorithm.

1. Guess $I_1 \subseteq \Sigma_1$ where $actCost(I_1) \leq B_1$, let $I_2 = \emptyset$ and construct the resulting WEG.

2. Solve the interval bound problem for the resulting WEG in the interval $[a, \infty]$, if the result is positive return yes, else return no.

Step 1 is a guess of polynomial size, this is done in polynomial time. Step 2 is problem in UP $\cup$ coUP, which is a subclass of NP. Therefore the algorithm is in
5.2. Complexity results

NP.

Lemma 25. The AIG problem where $B_1 = 0$ for an interval $[a, \infty]$ is $\Pi_P^2$-complete.

Proof. Lower bound: The AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ is $\Pi_P^2$-hard by reduction from $\Pi_P^2$-SAT.

Let $\forall \bar{x} \varphi(x) = 1$ be a $\Pi_P^2$-SAT problem. We construct the AIG $A_G$ given by $G_\varphi(x) \rightarrow G_{\varphi}(\varphi)$ and define $B_2$ to be any sufficient budget.

The proof for the two following properties of the construction is similar to the proof in Lemma 20.

(a) If $\forall \bar{x} \varphi(x) = 1$ is true, Player 1 wins $A_G$.
(b) If $\forall \bar{x} \varphi(x) = 1$ is false, Player 2 wins $A_G$.

This construction shows that a solution to the AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ can solve any $\Pi_P^2$-SAT problem, hence the AIG problem is $\Pi_P^2$-hard.

Upper bound: The AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ is in $\Pi_P^1$ by the following algorithm.

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve the interval bound problem for the resulting WEG in the interval $[a, \infty]$, if all results are positive return yes, else return no.

Step 1 is a universal guess over Step 2. Step 2 is problem in UP $\cup$ coUP, which is a subclass of coNP = $\Pi_P^1$. Since the algorithm only uses universal quantifiers, it is in $\Pi_P^1$.

Lemma 26. The AIG problem for an interval $[a, \infty]$ is $\Sigma_P^2$-complete.

Proof. Lower bound: The AIG problem for the interval $[a, \infty]$ is $\Sigma_P^2$-hard by reduction from $\Sigma_P^2$-SAT.

Let $\exists \bar{x} \forall \bar{y} \varphi(x, y) = 1$ be a $\Sigma_P^2$-SAT problem. We construct the AIG $A_G$ given by $G_{\exists \bar{x} \forall \bar{y}}(\bar{x}, \bar{y}) \rightarrow G_{\varphi}(\varphi)$ and define $B_1$ and $B_2$ to be any sufficient budget.

We want to show:

(a) If $\exists \bar{x} \forall \bar{y} \varphi(x, y) = 1$ is true, Player 1 wins $A_G$.
(b) If $\exists \bar{x} \forall \bar{y} \varphi(x, y) = 1$ is false, Player 2 wins $A_G$.

(a) Suppose that $\exists \bar{x} \forall \bar{y} \varphi(x, y) = 1$, is true. Then $v$ is an assignment of the booleans in $\bar{x}$ such that for any assignments of $\bar{y}$, $\varphi$ is true. We now want to show that Player 1 wins the AIG $A_G$. 

Player 1 chooses the valid investment $I_1 = \{x' \mid v(x) = true\} \cup \{x \mid v(x) = false\}$.

Player 1 has by Lemma 14 (a) a strategy such that any play starting in $q_{in}$ gets to $q_{out}$ in the AIG $A_3$ constructed by $G_3(\vec{x})$, or Player 1 wins.

If the play reaches $q_{out}$ in $A_3$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\vec{y})$, where the play starts in $q_{in}$.

Player 2 can pick either an invalid or valid investment $I_2$. If $I_2$ is an invalid investment we know from Lemma 13 (c) that Player 1 has a strategy to win any play starting from $q_{in}$ in the AIG $A_\varphi$.

If $I_2$ is a valid investment we know from Lemma 13 (b) that Player 1 has a strategy such that any play from $q_{in}$ reaches $q_{out}$ in $A_\varphi$.

If the play reaches $q_{out}$ in $A_\varphi$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\varphi)$.

We know that $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ is $true$, under the assignment $v$ this implies by Lemma 15 (a) that Player 1 wins any play starting from $q_{in}$ in $A_\varphi$.

Therefore Player 1 has an investment $I_1$ and a strategy to win no matter what investment $I_2$ and strategy Player 2 uses.

(b) Suppose that $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ is $false$. Then $v$ is the assignment which, for any assignment of $\vec{x}$, assigns the booleans in $\vec{y}$ such that $\varphi$ is false. We now want to show that Player 2 wins the AIG $A_G$ no matter what Player 1 does.

Player 1 can pick either an invalid or valid investment $I_1$. If $I_1$ is an invalid investment we know from Lemma 14 (c) that Player 2 has a strategy to win any play starting from $q_{in}$ in the AIG $A_3$ constructed by $G_3(\vec{x})$.

If $I_1$ is a valid investment we know from Lemma 14 (b) that Player 2 has a strategy such that any play from $q_{in}$ either reaches $q_{out}$ in $A_3$ or is winning for Player 2.

If the play reaches $q_{out}$ in $A_3$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\vec{y})$.

Player choose the valid investment $I_2 = \{x \mid v(x) = true\} \cup \{x' \mid v(x) = false\}$.

By Lemma 13 (a) we know that any play starting in $q_{in}$ gets to $q_{out}$ in $A_\varphi$, or Player 2 wins.

If the play reaches $q_{out}$ in $A_\varphi$ then by the construction of linking the play continues in the AIG $A_\varphi$ constructed from $G_\varphi(\varphi)$.

We know that $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ is $false$ under any assignment $v$ and therefore by Lemma 15 (b) we know that Player 2 is winning any play starting from $q_{in}$ in $A_\varphi$.

We have now shown that Player 2 wins any on play on $A_G$. 
5.2. Complexity results

Upper bound:
The AIG problem where for the interval \([a, \infty]\) is in \(\Sigma^P_2\) by the following algorithm.

1. Guess \(I_1 \subseteq \Sigma_1\) where \(\text{actCost}(I_1) \leq B_1\).
2. For all \(I_2 \subseteq \Sigma_2\) where \(\text{actCost}(I_2) \leq B_2\), and construct the resulting WEG.
3. Solve the interval bound problem for the resulting WEG in the interval \([a, \infty]\) if all results are positive return yes, else return no.

Step 1 is a guess of polynomial size. Step 2 is a universal guess over Step 3. Step 3 is problem in \(\text{UP} \cup \text{coUP}\), which is a subclass of \(\text{coNP} = \Pi^P_1\).

The algorithm uses one existential followed by two universal quantifiers, and is therefore in \(\Sigma^P_2\).

Lemma 27. The AIG problem for an interval \([a, b]\) is EXPTIME-complete.

Proof. Lower bound
The EXPTIME-hard lower bound is given by the AIG problem for an interval \([a, \infty]\) where \(B_1 = 0\) and \(B_2 = 0\).

Upper bound:
The EXPTIME upper bound is given by the following argument. There can be exponential many different combinations of investments \(I_1, I_2\), this depends on the budget for each player, and for each combination we need to solve an EXPTIME problem, this gives an algorithm in EXPTIME.
We have introduced the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets. We have given a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We have given a comprehensive analysis of the complexity of the decision problem which is the foundation for reasoning about these trade-offs. The complexity problem is studied in different cases. In the existential case, where all states are controlled by Player 1, under the restriction that the budget for Player 2 is 0 then the complexity bounds follow those of energy games. In all other cases increases the complexity by at most two levels in the polynomial hierarchy.

It is expected that the complexity increases by two levels in the polynomial hierarchy as the problem introduces a existential choice and a universal quantifier before the energy game.

The future work is to write full proof of the claims, that the complexity for the existential action investment games problem for a closed interval is $\Pi_2^P \text{ – hard}$. It is also of high priority to find a case study where the investment of Player 2 is used to model a more controllable phenomena than the weather.
We define the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets.

The formalism is an extended of energy games with budgets and prices on actions. An energy game are played by two players on a finite graph, where Player 1 wins if he has a strategy such that the accumulated energy is constrained within an given interval, and Player 2 wins if Player 1 loses.

We use the formalism in a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We given a comprehensive analysis of the complexity of the decision problem which is the foundation for reasoning about these trade-offs. The complexity problem is studied in different cases. In the existential case, where all states are controlled by Player 1, under the restriction that the budget for Player 2 is 0 then the complexity bounds follow those of energy games.

In the worst cases increases the complexity by two levels in the polynomial hierarchy compared to the complexity of the corresponding energy game.
Bibliography


