Aalborg University
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NON-LINEAR APPROXIMATION WITH BASES IN TRIEBEL-LIZORKIN AND BESOV SPACES

Non-linear approximation with bases in Triebel-Lizorkin and Besov spaces

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by

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Abstract: We study sufficient conditions for a decomposition system for $L^2(\mathbb{R}^d)$ such that it also forms a decomposition system for Triebel-Lizorkin and Besov spaces. Moreover we obtain a norm equivalence that allows us to distinguish the membership of a distribution in these spaces by the coefficients of its expansion. Particularly we show that a nice biorthogonal wavelet system forms a unconditional basis for Triebel-Lizorkin and Besov spaces. Afterwards we apply non-linear n-term approximation to these bases and fully characterize the approximation spaces in terms of interpolation spaces by Jackson and Bernstein inequalities. For decomposition systems we show that a Jackson inequality can still be obtained, yielding that the interpolation space is embedded in the approximation space. Finally we give a method for construction of an unconditional basis for Triebel-Lizorkin and Besov spaces by a finite linear combination of shifts and dilates of a single function with sufficient smoothness and decay and no vanishing moments. Applying n-term approximation to shifts and dilates of this function we again establish a Jackson inequality.

Summary

In recent years there has been great interest in non-linear wavelet approximation, among other things in numerical applications in statistics and signal and image processing. In this thesis we study non-linear n-term approximation in Triebel-Lizorkin and Besov spaces which include a wide range of function spaces for example L^p and the Sobolev spaces. We examine n-term approximation in three cases:

- Biorthogonal wavelet bases in $L^2(\mathbb{R}^d)$
- Decomposition systems in $L^2(\mathbb{R}^d)$
- Shifts and dilates of a single function.

In the case of nice biorthogonal wavelet bases we obtain Jackson and Bernstein inequalities which allows to characterize the approximation spaces completely by interpolation spaces. Here the requirements of nice becomes sufficient smoothness, decay and vanishing moments. In the second case we generalize the setting to nice decomposition systems, which includes frames. This generalization comes at the cost of the Bernstein inequality. The Jackson inequality still allows us to obtain that the interpolation spaces are embedded in the approximation spaces. In the third case we take a single function with sufficient smoothness and decay and no vanishing moments. We show that a Jackson inequality can also be obtained for n-term approximation by shifts and dilates of this single function. Although as in the second case we still lack the Bernstein inequality.
The prerequisites for reading this thesis is knowledge in the fields of distribution theory and function spaces, see for example [15] and [12].

1 Triebel-Lizorkin and Besov spaces **1**

We begin by introducing the Triebel-Lizorkin and Besov spaces (denoted $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$) with the aid of Littlewood-Paley operators and Bump functions ϕ. Next we study the convergence of Calderon's reproducing formula in S'/P_k and show that ϕ is a decomposition system for distributions f in Triebel-Lizorkin and Besov spaces with convergence in S'/P_k. From this we prove the norm equivalence between the $\dot{F}^s_{p,q}$-norm of f and the corresponding $\dot{f}^s_{p,q}$-norm of the coefficients of the expansion of f by ϕ. We conclude this section with showing that $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ are complete quasi-normed spaces by using the norm equivalence. This section is based on [8], [4] and [6].

2 Bounded operators and decomposition systems **15**

In this section we show the boundedness of the matrix associated with a nice system Θ and ϕ on $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$. Combinded with the norm equivalence of ϕ from the previous section this gives us a norm equivalence for Θ. Applied to a nice decomposition system Θ for $L^2(\mathbb{R}^d)$ we show that this gives an unconditional decomposition system for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$. We end this section with showing that the uniqueness of the expansion for a nice biorthogonal wavelet basis Ψ in $L^2(\mathbb{R}^d)$

carries on to $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ for coefficients in $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$. This section is based on [9] and [8].

3 Interpolation and approximation spaces 26

We start with introducing interpolation spaces and show a few properties of the interpolation space derived from the K-functional, especially the discretization of the norm. Next we prove a relation between the K-functional and the Jackson and Bernstein inequalities. This leads us to defining approximation spaces and under the assumptions of the Jackson and Bernstein inequalties we characterize the approximation spaces by interpolations spaces. Finally we apply this to the setting of non-linear n-term approximation from the wavelet bases for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ from the previous section. This section is based on [3], [1] and [7].

4 New bases by almost diagonal matrices 44

In this section we study a new nice system Θ sufficiently close to a wavelet bases Ψ from Section 2. By the use of almost diagonal matrices we prove that the matrix associated with Θ and Ψ has a bounded inverse. With this in hand we repeat the procedure of the last half of Section 2 to show that Θ is an unconditional wavelet basis for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ with the corresponding norm equivalence. Following this we show that a finite linear combination of shifts and dilates of a single nice function φ constitutes a nice system Θ, thereby forming a basis for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$. We end this section with proving Jackson inequalties for n-term approximation by shifts and dilates of φ. This section is based on [9] and [5].

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1 Triebel-Lizorkin and Besov spaces

In this section we introduce the bump function ϕ by which we define the Triebel-Lizorkin and Besov spaces and we show a norm equivalence between these spaces and a corresponding sequence space. We begin with some notation.
Denote by $S = S(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^d$ and by $S' = S'(\mathbb{R}^d)$ its dual, the space of tempered distributions. We also denote by S_k, $0 \leq k \leq \infty$ the subspace of S consisting of the Schwartz functions with k vanishing moments and by S'/P_k the space of equivalence classes of distributions in S' modulo polynomials of degree less then and equal to k. For the sake of notation we denote $S'/P_\infty = S'/P$ and $S_{-1} = S$.
We write D for the family of all dyadic cubes in $\mathbb{R}^d$ and D_m, $m \in \mathbb{Z}$ for the collection of all cubes $I \in D$ with sidelength $\ell(I) = 2^{-m}$. For any dyadic cube $I \in D$ we use x_I for its lower-left corner and $|I| = \ell(I)^d$ for its volume. We denote $\langle f, \eta \rangle$ the inner product $\int f\bar{\eta}$ of two functions when this makes sense and else the same notation is used for a distribution f taken on a Schwartz function $\bar{\eta}$. Also we denote the Fourier transform of a integrable function f by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \mathrm{d}x.$$

The Fourier transform is extended to S' by duality. Finally we write $Y \hookrightarrow X$ if Y is continuously embedded in X. If we furthermore have $X \hookrightarrow Y$ we write $\|\cdot\|_X \asymp \|\cdot\|_Y$ and $X = Y$.
We now introduce decomposition systems and wavelets for $L^2(\mathbb{R}^d)$. Let E be a finite set and $B = \{\theta_I^e, \widetilde{\theta}_I^e : e \in E, I \in D\}$ a family of functions in $L^2(\mathbb{R}^d)$. We say that B forms a decomposition system for $L^2(\mathbb{R}^d)$ if for $f \in L^2(\mathbb{R}^d)$,

$$f = \sum_{e\in E} \sum_{I\in D} \langle f, \widetilde{\theta}_I^e \rangle \theta_I^e,$$

in $L^2(\mathbb{R}^d)$. In the special case where $\left\langle \theta_I^e, \widetilde{\theta}_J^{e'} \right\rangle = \delta_{(I,e),(J,e')}$, where $\delta_{(I,e),(J,e')}$ is the Kronecker delta function and that

$$\theta_I^e(\cdot) = |I|^{-1/2} \theta^e \left(\frac{\cdot - x_I}{\ell(I)} \right),$$

then $\Theta = \{\theta^e, \widetilde{\theta}^e : e \in E\}$ forms a biorthogonal wavelet basis for $L^2(\mathbb{R}^d)$. In the case $\theta^e = \widetilde{\theta}^e$ we say that $\Theta = \{\theta^e\}_{e\in E}$ forms a orthonormal wavelet basis for $L^2(\mathbb{R}^d)$. Later on we shall show that B and Θ form unconditional decomposition systems and wavelet bases for the Triebel-Lizorkin and Besov spaces, and to this end we shall require some decay and vanishing moments for $B \subset C^r(\mathbb{R}^d)$

$$|(\theta_I^e)^{(\alpha)}(x)| \le C|I|^{-1/2-|\alpha|/d}\Big(1+\frac{|x-x_I|}{\ell(I)}\Big)^{-M}, \quad |\alpha| \le r_2,$$
$$|(\widetilde{\theta}_I^e)^{(\alpha)}(x)| \le C|I|^{-1/2-|\alpha|/d}\Big(1+\frac{|x-x_I|}{\ell(I)}\Big)^{-M}, \quad |\alpha| \le r_1,$$
$$\int_{\mathbb{R}^d} x^\alpha \theta_I^e(x)\mathrm{d}x = 0, \quad |\alpha| \le r_1 - 1,$$
$$\int_{\mathbb{R}^d} x^\alpha \widetilde{\theta}_I^e(x)\mathrm{d}x = 0, \quad |\alpha| \le r_2 - 1. \tag{1.1}$$

A function $\widetilde{\theta}_I^e$ is said to have $r_2 - 1$ vanishing moments if it satisfies (1.1). An example of a orthonormal wavelet in $\mathbb{R}$ which satisfies these conditions for any choice of r_1, r_2, M is the Meyer wavelet ψ [2, p.137] obtained from a multiresolution analysis. Its Fourier transform $\widehat{\psi} \in C^\infty(\mathbb{R})$ is supported on $[-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$, so it follows that $\psi \in S_\infty(\mathbb{R})$. The associated scaling function ϕ has a Fourier transform $\widehat{\phi} \in C^\infty(\mathbb{R})$ which is supported on $[-4\pi/3, 4\pi/3]$. We entend the basis to $\mathbb{R}^d$ using tensorproducts. Denote $\phi = \psi^0, \psi = \psi^1$ and

$$\psi^e(x) = \psi^{e_1}(x_1) \cdots \psi^{e_d}(x_d) \tag{1.2}$$

where $e = (e_1, \dots, e_d) \in E$ and E is the set of nonzero vertices of the unit cube in $\mathbb{R}^d$. By using [10, Proposition 5.2] we get that $\{\psi^e\}_{e\in E}$ forms a orthonormal wavelet basis for $L^2(\mathbb{R}^d)$ and it follows that $\psi^e \in S_\infty(\mathbb{R}^d)$.
In the following the Tribel-Lizorkin and Besov spaces are introduced, which will be the spaces of our main interest. Let $\phi \in S$ be such that for $\nu \in \mathbb{Z}$, $\phi_\nu(\cdot) = 2^{\nu d}\phi(2^\nu \cdot)$ satisfies the following conditions

$$\operatorname{supp} \widehat{\phi}_\nu(\xi) \subset \{\xi : 2^{\nu-1} \le |\xi| \le 2^{\nu+1}\}, \tag{1.3}$$
$$|\widehat{\phi}_\nu^{(\beta)}(\xi)| \le C2^{-\nu|\beta|}, \qquad \text{for } \beta \in \mathbb{N}^d, \tag{1.4}$$
$$\textstyle\sum_{\nu\in\mathbb{Z}} |\widehat{\phi}_\nu(\xi)|^2 = 1, \qquad \text{for } \xi \in \mathbb{R}^d\backslash\{0\}. \tag{1.5}$$

A function ϕ satisfying these conditions will be denoted a bump function and the existence follows by taking $g \in S$, supp $\widehat{g} \subseteq [1/2, 2]$ and defining $\widehat{\phi}(\cdot) = \widehat{g}(\cdot)/(\sum_{\nu\in\mathbb{Z}} |\widehat{g}_\nu(\cdot)|^2)^{1/2}$. Associated with a bump function we define the Littlewood-Paley operators $\Delta_\nu^\phi = \Delta_\nu$ as convolution with the functions $\phi_\nu(\cdot)$. Notice that $(\Delta_\nu f)^\frown = \widehat{\phi}_\nu \widehat{f}$, so $(\Delta_\nu f)^\frown$ has compact support which by Proposition A.1 gives that $\Delta_\nu f$ is a function in C^∞. This allow us to define the Triebel-Lizorkin and Besov spaces.

Definition 1.1
For $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$ we define the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ as the set of all $f \in S'/P$ such that

$$\|f\|_{\dot{F}^s_{p,q}} = \left\{ \begin{array}{ll} \Big\| \Big(\sum_{\nu\in\mathbb{Z}} (2^{\nu s}|\Delta_\nu f|)^q \Big)^{1/q} \Big\|_{L^p}, & \text{if } q < \infty \\ \Big\| \sup_{\nu\in\mathbb{Z}} 2^{\nu s}|\Delta_\nu f| \Big\|_{L^p}, & \text{if } q = \infty \end{array} \right\} < \infty. \tag{1.6}$$

We also define the sequence space $\dot{f}^s_{p,q}$ consisting of all sequences $c = (c_I)_{I \in D}$ such that

$$\|c\|_{\dot{f}^s_{p,q}} = \Big\| \Big(\sum_{I \in D} (|I|^{-s/d-1/2} |c_I| \chi_I)^q \Big)^{1/q} \Big\|_{L^p} < \infty, \tag{1.7}$$

with an modification as in (1.6) for the case of $q = \infty$. For $s \in \mathbb{R}$, $0 < p, q \leq \infty$ we define the homogeneous Besov space as the set of all $f \in S'/P$ such that

$$\|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} \|\Delta_\nu f\|_{L^p})^q \Big)^{1/q} < \infty. \tag{1.8}$$

We also define $\dot{b}^s_{p,q}$ consisting of all sequences $c = (c_I)_{I \in D}$ such that

$$\|c\|_{\dot{b}^s_{p,q}} = \Big(\sum_{m \in \mathbb{Z}} \Big(\sum_{I \in D_m} (|I|^{-s/d+1/p-1/2} |c_I|)^p \Big)^{q/p} \Big)^{1/q} < \infty. \tag{1.9}$$

The notation $\|c\|_X = \|c_I\|_X$ is used when no confusion arises. ◇

The Triebel-Lizorkin and Besov spaces are linear and quasi-normed which follows by the properties of the ℓ^q- and L^p-norms. Later we shall prove a norm equivalence between $\dot{F}^s_{p,q}$ and $\dot{f}^s_{p,q}$ (and $\dot{B}^s_{p,q}$ and $\dot{b}^s_{p,q}$), Proposition 1.4, which yields the completeness of the Triebel-Lizorkin and Besov spaces, Proposition 1.5. It is also worth noting that the definition of the spaces $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ is independent of the specific choice of bump function. Two different functions ϕ and κ satisfying the conditions (1.3) - (1.5) will yield equivalent Triebel-Lizorkin and Besov quasi-norms [6, p.484 and p.482]. Certain well known function spaces are in fact particular Triebel-Lizorkin spaces. By the Littlewood-Paley characterization of the Sololev spaces [6, Section 6.2, p.424-433] one has that $W^p_s \asymp \dot{F}^s_{p,2}$ for $1 < p < \infty$ if one identifies the equivalence class $f + P$ with the distribution where the polynomial is 0. Especially the L^p-spaces for $1 < p < \infty$ can be viewed as Triebel-Lizorkin spaces, namely $\dot{F}^0_{p,2}$.

Norm equivalence by ϕ

Using the inverse Fourier transform one finds that (1.5) yields for $f \in L^2(\mathbb{R}^d)$ that

$$\sum_{\nu \in \mathbb{Z}} \widetilde{\phi}_\nu * \phi_\nu * f = f \tag{1.10}$$

with $\widetilde{f}(x) = \overline{f(-x)}$, where the convergence considered is in L^2. This equation is known as Calderon's reproducing formula. To study the convergence in S'/P_k for $f \in \dot{F}^s_{p,q}, \dot{B}^s_{p,q}$ we need the following lemma.

Lemma 1.2
Let $f, g \in S'$ and $k \geq -1$. Then the following three statements are equivalent:

1) $\sum_{\nu \in \mathbb{Z}} \widetilde{\phi}_\nu * \phi_\nu * f = g$ with convergence in $(S_k)'$.

2) There exist polynomials $p'_N, p \in P_k$ such that

$$\lim_{N\to\infty} \sum_{\nu=-N}^{\infty} \widetilde{\phi}_\nu * \phi_\nu * f + p'_N = g + p, \tag{1.11}$$

in S'.

3) For any $\alpha \in \mathbb{N}^d$ with $|\alpha| > k$ the series $\sum_{\nu=-\infty}^{0} (\widetilde{\phi}_\nu * \phi_\nu * f)^{(\alpha)}$ converges in S'.

Proof:

We shall assume that $k > -1$ since the case $k = -1$ is easily seen to be true. It will be shown that the first statement is equivalent to the other two statements.

1) ⇒ 2) Take $\kappa \in S$ with the properties that $\widehat{\kappa}(\xi) = 1$ for $|\xi| \leq 2$ and $\widehat{\kappa}(\xi) = 0$ when $|\xi| > 4$. Let us construct polynomials such that (1.11) holds. For $\nu > 0$ define $p_\nu = 0$ while for $\nu \leq 0$ define

$$p_\nu = \sum_{|\alpha|\leq k} c_{\alpha,\nu} x^\alpha, \quad \text{where } c_{\alpha,\nu} = -\frac{(-i)^{|\alpha|}}{\alpha!} \left\langle |\widehat{\phi}_\nu(\xi)|^2 \widehat{f}(\xi), \xi^\alpha \widehat{\kappa}(\xi) \right\rangle. \tag{1.12}$$

For $\eta \in S$, we have

$$\begin{aligned} &\lim_{N\to\infty} \sum_{\nu=-N}^{\infty} \left\langle \widetilde{\phi}_\nu * \phi_\nu * f + p_\nu, \eta \right\rangle \\ &= \sum_{\nu=1}^{\infty} \left\langle \widetilde{\phi}_\nu * \phi_\nu * f, \eta \right\rangle + \lim_{N\to\infty} \sum_{\nu=-N}^{0} \left\langle \widetilde{\phi}_\nu * \phi_\nu * f + p_\nu, \eta \right\rangle. \end{aligned} \tag{1.13}$$

For the first sum in (1.13) we use [6, Proposition 2.3.4 (b), p.110] which states that for $f \in S'$ there exists $r, s \in \mathbb{N}$ such that for $\phi \in S$ we have

$$|\langle f, \phi \rangle| \leq C \sum_{|\alpha|\leq r} \sum_{|\beta|\leq s} \sup_{x\in\mathbb{R}^d} |x^\alpha \phi^{(\beta)}(x)|.$$

This yields

$$\begin{aligned} \sum_{\nu=1}^{\infty} |\left\langle \widetilde{\phi}_\nu * \phi_\nu * f, \eta \right\rangle| &= \sum_{\nu=1}^{\infty} |\langle \widehat{f}, |\widehat{\phi}_\nu|^2 \widehat{\eta} \rangle| \\ &\leq C \sum_{\nu=1}^{\infty} \sum_{|\alpha|\leq r} \sum_{|\beta|\leq s} \sup_{\xi\in\mathbb{R}^d} |\xi^\alpha (|\widehat{\phi}_\nu(\xi)|^2 \widehat{\eta}(\xi))^{(\beta)}| \\ &\leq C \sum_{\nu=1}^{\infty} \sum_{|\alpha|\leq r} \sum_{|\beta|\leq s} \sup_{\xi\in\mathbb{R}^d} \sum_{\gamma\leq\beta} |(|\widehat{\phi}_\nu(\xi)|^2)^{(\gamma)}| |\xi^\alpha \widehat{\eta}^{(\beta-\gamma)}|. \end{aligned} \tag{1.14}$$

We shall consider (1.14) in two cases. First when $\gamma = 0$ we have by (1.5) and (1.3) that

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sup_{\xi\in R^d} |\widehat{\phi}_\nu(\xi)|^2 |\xi^\alpha \widehat{\eta}^{(\beta)}(\xi)| \\
\le &\sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sup_{\xi\in R^d} |\xi^\alpha \widehat{\eta}^{(\beta)}(\xi)| \sum_{\nu\in\mathbb{Z}} |\widehat{\phi}_\nu(\xi)|^2 \\
= &\sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sup_{\xi\in \mathbb{R}^d} |\xi^\alpha \widehat{\eta}^{(\beta)}(\xi)| \le C,
\end{aligned}$$

since $\eta \in S$. For the case when $\gamma > 0$ we have that (1.14) can be estimated using (1.4) as follows

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sup_{\xi\in \mathbb{R}^d} \sum_{0<\gamma\le\beta} |(|\widehat{\phi}_\nu(\xi)|^2)^{(\gamma)}||\xi^\alpha \widehat{\eta}^{(\beta-\gamma)}(\xi)| \\
\le &C \sum_{\nu=1}^{\infty} \sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sum_{0<\gamma\le\beta} 2^{-\nu|\gamma|} \sup_{\xi\in \mathbb{R}^d} |\xi^\alpha \widehat{\eta}^{(\beta-\gamma)}(\xi)| \\
\le &C \sum_{\nu=1}^{\infty} 2^{-\nu} \sum_{|\alpha|\le r} \sum_{|\beta|\le s} \sum_{0<\gamma\le\beta} \sup_{\xi\in \mathbb{R}^d} |\xi^\alpha \widehat{\eta}^{(\beta-\gamma)}(\xi)| \\
\le &C \sum_{\nu=1}^{\infty} 2^{-\nu} \le C,
\end{aligned}$$

which shows that the first sum in (1.13) converges absolutely in S'. For the second sum notice that by the properties of the Fourier transform we have that

$$\langle p_\nu, \eta\rangle = \sum_{|\alpha|\le k} c_{\alpha,\nu} \langle x^\alpha, \eta\rangle = \sum_{|\alpha|\le k} c_{\alpha,\nu} (-i)^{-|\alpha|} \overline{\widehat{\eta}^{(\alpha)}(0)}.$$

From this and the properties of κ we have for the second sum in (1.13) that

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{\nu=-N}^{0} \left\langle \widetilde{\phi}_\nu * \phi_\nu * f + p_\nu, \eta \right\rangle \\
= &\lim_{N\to\infty} \sum_{\nu=-N}^{0} \left\langle |\widehat{\phi}_\nu|^2 \widehat{f}, \widehat{\eta} \right\rangle + \langle p_\nu, \eta\rangle \\
= &\lim_{N\to\infty} \sum_{\nu=-N}^{0} \left\langle |\widehat{\phi}_\nu|^2 \widehat{f}, \widehat{\kappa\eta} \right\rangle + \sum_{|\alpha|\le k} c_{\alpha,\nu} (-i)^{-|\alpha|} \overline{\widehat{\eta}^{(\alpha)}(0)}.
\end{aligned}$$

From the definition of $c_{\alpha,\nu}$, see (1.12), we have

$$= \lim_{N\to\infty} \sum_{\nu=-N}^{0} \Big\langle |\widehat{\phi}_\nu(\xi)|^2 \widehat{f}(\xi), \widehat{\theta}(\xi)\big(\widehat{\eta}(\xi) - \sum_{|\alpha|\le k} \frac{\xi^\alpha}{\alpha!}\widehat{\eta}^{(\alpha)}(0)\big)\Big\rangle$$

$$= \lim_{N\to\infty} \sum_{\nu=-N}^{0} \big\langle |\widehat{\phi}_\nu(\xi)|^2 \widehat{f}(\xi), \widehat{\omega}(\xi)\big\rangle$$

$$= \sum_{\nu=-\infty}^{0} \big\langle \widetilde{\phi}_\nu * \phi_\nu * f, \omega\big\rangle, \tag{1.15}$$

where we have set $\widehat{\omega}(\xi) = \widehat{\theta}(\xi)\Big(\widehat{\eta}(\xi) - \sum_{|\alpha|\le k} \frac{\xi^\alpha}{\alpha!}\widehat{\eta}^{(\alpha)}(0)\Big)$. As we have that $S_k = \{\theta(x) \in S : \widehat{\theta}^{(\alpha)}(0) = 0, |\alpha| \le k\}$ one finds by using Leibniz' rule and differentiating the second factor that $\omega \in S_k$ and therefore that (1.15) is finite. As a consequence of the Banach-Steinhaus theorem [16, Theorem 2.8, p.46] we can define a distribution h in S' as

$$h = \lim_{N\to\infty} \Big(\sum_{\nu=-N}^{\infty} \widetilde{\phi}_\nu * \phi_\nu * f + p'_N\Big),$$

where $p'_N = \sum_{\nu=-N}^{0} p_\nu$. One has that $\operatorname{supp}(\widehat{h} - \widehat{g}) = \{0\}$, by using that if $\eta \in S$ and $0 \notin \operatorname{supp}\eta$ then $\widehat{\eta} \in S_\infty$. This implies that $h - g$ is a polynomial [6, Corollary 2.4.2., p.123], furthermore we have that it vanishes on S_k. Note that for $|\alpha| > k$ and $\eta \in S$ one has by partial integration that

$$\int_{\mathbb{R}^d} x^\beta \eta^{(\alpha)}(x)\mathrm{d}x = (-1)^{|\alpha|} C \int_{\mathbb{R}^d} \eta(x)(x^\beta)^{(\alpha)}\mathrm{d}x = 0, \quad \text{for } |\beta| \le k, \tag{1.16}$$

which shows that $\eta^{(\alpha)} \in S_k$. Thereby we have

$$\big\langle (h-g)^{(\alpha)}, \eta\big\rangle = (-1)^{|\alpha|}\big\langle h - g, \eta^{(\alpha)}\big\rangle = 0,$$

showing that $h - g = p \in P_k$.

2) ⇒ 1) A trivial consequence of the space $S_k \subset S$.

1) ⇒ 3) Assume that $\eta \in S$ and $|\alpha| > k$. As noted in (1.16) this implies that $\eta^{(\alpha)} \in S_k$. From 1) one finds

$$\sum_{\nu\in\mathbb{Z}} \big\langle (\widetilde{\phi}_\nu * \phi_\nu * f)^{(\alpha)}, \eta\big\rangle = (-1)^{|\alpha|} \sum_{\nu\in\mathbb{Z}} \big\langle \widetilde{\phi}_\nu * \phi_\nu * f, \eta^{(\alpha)}\big\rangle$$
$$= (-1)^{|\alpha|}\big\langle g, \eta^{(\alpha)}\big\rangle.$$

3) ⇒ 1) From the first part of the proof of 1) ⇒ 2) we already have that $\sum_{\nu>0} \widetilde{\phi}_\nu * \phi_\nu * f$ converges in S' therefore in S'_k. We now show that $\sum_{\nu\le 0} \widetilde{\phi}_\nu * \phi_\nu * f$ also converges in S'_k by using Taylor expansion. Let $\eta \in S_k$. A slight

alteration to the proof of Lagrange's remainder theorem [18, Theorem 7.52, p.212] using that

$$\sum_{|\beta|=k} x^\beta \eta^{(\beta)}(tx) = \eta_x^{(k)}(t)$$

where $\eta_x : t \to \eta(xt)$ shows that the remainder term can be written as

$$R_k^{\widehat{\eta},0}(x) = \sum_{|\beta|=k} \frac{\xi^\beta}{(k-1)!} \int_0^1 \widehat{\eta}^{(\beta)}(t\xi) t^{k-1} \mathrm{d}t.$$

Now by the vanishing moments of η and Taylor expansion of order k at 0 we have that

$$\widehat{\eta}(\xi) = \sum_{|\beta|=k+1} \frac{\xi^\beta}{k!} \int_0^1 \widehat{\eta}^{(\beta)}(t\xi) t^k \mathrm{d}t = \sum_{|\beta|=k+1} \xi^\beta \widehat{g}_\beta(\xi), \tag{1.17}$$

where we set $\widehat{g}_\beta(\xi) = \frac{1}{k!}\int_0^1 \widehat{\eta}^{(\beta)}(t\xi)t^k \mathrm{d}t$. Notice that $\widehat{g}_\beta(\xi)$ is a bounded function, since $\widehat{\eta}^{(\beta)}$ is bounded. Next we multiply (1.17) with $\kappa(x)$ from the beginning of the proof and use the inverse Fourier transform to get

$$\kappa * \eta = \sum_{|\beta|=k+1} \eta_\beta^{(\beta)},$$

where we define $\widehat{\eta}_\beta = (-i)^{|\beta|}\widehat{\kappa}\widehat{g}_\beta \in S$. From this we get

$$\begin{aligned}\sum_{\nu=-\infty}^{0} \langle \widetilde{\phi}_\nu * \phi_\nu * f, \eta \rangle &= \sum_{\nu=-\infty}^{0} \langle \widetilde{\phi}_\nu * \phi_\nu * f, \kappa * \eta \rangle \\ &= \sum_{|\beta|=k+1} \sum_{\nu=-\infty}^{0} \langle \widetilde{\phi}_\nu * \phi_\nu * f, \eta_\beta^{(\beta)} \rangle \\ &= \sum_{|\beta|=k+1} \sum_{\nu=-\infty}^{0} (-1)^{|\beta|} \langle (\widetilde{\phi}_\nu * \phi_\nu * f)^{(\beta)}, \eta_\beta \rangle,\end{aligned}$$

where the convergence of the last sums follows from property 3). ■

Note that in the scope of the lemma we have that $(S_k)' = S'/P_k$.
Now we examine the convergence of (1.10) when $f \in \dot{F}^s_{p,q}$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Since supp $(2^{\nu d}(\phi_\nu * f)\widehat{\ }(2^\nu x)) \subset \{x : |x| < 2\}$ we have from [17, Theorem, p.22] that

$$\|\phi_\nu * f(2^{-\nu}x)\|_{L^\infty} \leq C\|\phi_\nu * f(2^{-\nu}x)\|_{L^p}. \tag{1.18}$$

From (1.18) one has

$$\begin{aligned}\|(\widetilde{\phi}_\nu * \phi_\nu * f)^{(\alpha)}\|_{L^\infty} &\leq \|\widetilde{\phi}_\nu^{(\alpha)}\|_{L^1}\|\phi_\nu * f\|_{L^\infty} \\ &= C2^{\nu|\alpha|}\|\phi_\nu * f(2^{-\nu}x)\|_{L^\infty} \\ &\leq C2^{\nu(|\alpha|+d/p)}\|\phi_\nu * f\|_{L^p}.\end{aligned} \tag{1.19}$$

By the embedding of ℓ^q we have that $\dot{F}^s_{p,q_1} \subseteq \dot{F}^s_{p,q_2}$ if $q_1 \leq q_2$. From this embedding property and the estimate in (1.19) we have for $|\alpha| > s - d/p$ that

$$\begin{aligned}\sum_{\nu<0} \|(\widetilde{\phi}_\nu * \phi_\nu * f)^{(\alpha)}\|_{L^\infty} &\leq C \sup_{\nu<0} 2^{\nu s} \|\phi_\nu * f\|_{L^p} \\ &\leq C \|\sup_{\nu<0} 2^{\nu s} |\phi_\nu * f|\|_{L^p} \\ &= C\|f\|_{F^s_{p,\infty}} \leq C\|f\|_{\dot{F}^s_{p,q}} ,\end{aligned} \tag{1.20}$$

implying that $\sum_{\nu<0}(\widetilde{\phi}_\nu * \phi_\nu * f)^{(\alpha)}$ converges in S'. For the Besov space notice that the term on the right side of (1.20) is less then or equal to $C\|f\|_{\dot{B}^s_{p,q}}$.
Choosing $k = \max\{\lfloor s - d/p \rfloor, -1\}$ we can use Lemma 1.2 to find polynomials $p'_N \in P_k$ such that

$$\lim_{N\to\infty} \Big(\sum_{\nu=-N}^{\infty} \widetilde{\phi}_\nu * \phi_\nu * f + p'_N \Big) = g$$

converges in S'. We also have that $\operatorname{supp}(\widehat{g} - \widehat{f}) = \{0\}$, therefore there exists a polynomial $p \in P$ such that $g = f + p$ in S'. Which implies

$$\sum_{\nu\in\mathbb{Z}} \widetilde{\phi}_\nu * \phi_\nu * f = f, \qquad \text{in } S'/P.$$

The sum on the left-hand side converges to g modulo P_k, so by identifying the equivalence class $f + p$ with g, we write

$$\sum_{\nu\in\mathbb{Z}} \widetilde{\phi}_\nu * \phi_\nu * f = f, \qquad \text{in } S'/P_k. \tag{1.21}$$

Taking the class $f + p$ and identifying it with g modulo P_k, the Triebel-Lizorkin and Besov-spaces can be considered as subsets of the space S'/P_k.
Take $I \in D$ with $\ell(I) = 2^{-\nu}, x_I = 2^{-\nu}l$ and define

$$\phi_I(\cdot) = |I|^{-1/2} \phi\Big(\frac{\cdot - x_I}{\ell(I)}\Big) = 2^{-\nu d/2} \phi_\nu(\cdot - 2^{-\nu}l).$$

By using (1.21) we get the following lemma which shows that $\{\phi_I\}_{I\in D}$ is a decomposition system in S'/P_k. This will be crucial in proving the norm equivalence between $\dot{F}^s_{p,q}$ and $\dot{f}^s_{p,q}$ (and $\dot{B}^s_{p,q}$ and $\dot{b}^s_{p,q}$).

Lemma 1.3
Suppose that $f \in \dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ then

$$f = \sum_{I\in D} \langle f, \phi_I \rangle \phi_I, \tag{1.22}$$

where the convergence is in S'/P_k with $k = \max\{\lfloor s - d/p \rfloor, -1\}$.

Proof:

By (1.21) we have

$$f = \sum_{\nu \in \mathbb{Z}} \phi_\nu * \widetilde{\phi}_\nu * f,$$

in S'/P_k. Hence (1.22) will follow if we can show that

$$\phi_\nu * \widetilde{\phi}_\nu * f = 2^{-\nu d} \sum_{l \in \mathbb{Z}^d} \Big\langle f, \phi_\nu(\cdot - 2^{-\nu} l) \Big\rangle \phi_\nu(\cdot - 2^{-\nu} l) = \sum_{I \in D_\nu} \langle f, \phi_I \rangle \, \phi_I,$$

in S'. As $\widehat{\phi}$ is compactly supported we have that $(\widetilde{\phi}_\nu * f)\widehat{\ }$ is also compactly supported so from Proposition A.1 we get that $(\widetilde{\phi}_\nu * f)(x)$ is slowly increasing and in C^∞. Take $\theta \in S$ with $\widehat{\theta}(0) = 1$, let $\theta_\varepsilon(x) = \varepsilon^d \theta(\varepsilon x)$ and δ_x the delta function in x, we have

$$\begin{aligned} \big\langle \delta_{2^{-\nu}l}, \overline{\theta_\varepsilon(-\cdot) * \widetilde{\phi}_\nu * f} \big\rangle &= \int_{\mathbb{R}^d} \theta_\varepsilon(x - 2^{-\nu} l) \widetilde{\phi}_\nu * f(x) \mathrm{d}x && (1.23) \\ &= \big\langle \widetilde{\phi}_\nu * f, \overline{\theta_\varepsilon(\cdot - 2^{-\nu} l)} \big\rangle \\ &= \big\langle f, \phi_\nu * \theta_\varepsilon(\cdot - 2^{-\nu} l) \big\rangle \\ &= \big\langle f, \phi_\nu(\cdot - 2^{-\nu} l) * \theta_\varepsilon \big\rangle. && (1.24) \end{aligned}$$

The left-hand side of (1.23) converges to $(\widetilde{\phi}_\nu * f)(2^{-\nu} l)$ as $\varepsilon \to \infty$ by [19, Lemma 1, p.157] and the term in (1.24) converges to $\big\langle f, \phi_\nu(\cdot - 2^{-\nu} l) \big\rangle$ by [14, Proposition, p.326], so we need to show that

$$\phi_\nu * \widetilde{\phi}_\nu * f = 2^{-\nu d} \sum_{l \in \mathbb{Z}^d} (\widetilde{\phi}_\nu * f)(2^{-\nu} l) \phi_\nu(\cdot - 2^{-\nu} l) \tag{1.25}$$

in S'. Take

$$f_{\nu,\delta}(x) = (\widetilde{\phi}_\nu * f)(x) \prod_{i=1}^d \left(\frac{\sin(\delta x_i)}{\delta x_i} \right)^j,$$

with j large enough so that $f_{\nu,\delta} \in L_2(\mathbb{R}^d)$. We have $\mathrm{supp} \widehat{f}_{\nu,\delta} \subset \{\xi : |\xi| < 2^\nu \pi\}$ if δ is sufficiently small, which follows from

$$\widehat{g}(x) = \Big(\prod_{i=1}^d \tfrac{1}{2\delta} \chi_{[-\delta,\delta]}(\xi_i) \Big)\widehat{\ }(x) = \prod_{i=1}^d \frac{\sin(\delta x_i)}{\delta x_i},$$

$$\begin{aligned} \big\langle (\widetilde{\phi}_\nu * f)\widehat{\ }, \eta * \underbrace{g * \cdots * g}_{j \text{ times}} \big\rangle &= \left\langle \widetilde{\phi}_\nu * f, \widehat{\eta} \prod_{i=1}^d \left(\frac{\sin(\delta x_i)}{\delta x_i} \right)^j \right\rangle \\ &= \int_{\mathbb{R}^d} \widehat{f}_{\nu,\delta}(x) \eta(x) \mathrm{d}x \end{aligned}$$

and supp $\widehat{\phi}_\nu \subset \{\xi : |\xi| < 2^{\nu+1}\} \subset \{\xi : |\xi| < 2^\nu \pi\}$. This we can use together with

$$(\phi_\nu * f_{\nu,\delta})(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}_{\nu,\delta}(\xi) \hat{\phi}_\nu(\xi) e^{ix\xi} \mathrm{d}\xi, \tag{1.26}$$

by extending $\widehat{\phi}_\nu e^{ix\xi}$ periodically with period $2^{\nu+1}\pi$ in each variable and represent it as its Fourier series

$$\begin{aligned}\widehat{\phi}_\nu(\xi)e^{ix\xi} &= \sum_{l\in\mathbb{Z}^d}(2^{\nu+1}\pi)^{-d}\int_{\mathbb{R}^d}\widehat{\phi}_\nu(y)e^{ixy}e^{-i2^{-\nu}ly}\mathrm{d}y\, e^{i2^{-\nu}l\xi}\\ &= 2^{-\nu d}\sum_{l\in\mathbb{Z}^d}\phi_\nu(x-2^{-\nu}l)e^{i2^{-\nu}l\xi} \qquad (1.27)\end{aligned}$$

for $|\xi| < 2^\nu\pi$ almost everywhere, [6, Proposition 3.1.14 p.165]. By inserting (1.27) in (1.26) we get

$$\begin{aligned}(\phi_\nu * f_{\nu,\delta})(x) &= (2^{\nu+1}\pi)^{-d}\int_{\mathbb{R}^d}\widehat{f}_{\nu,\delta}(\xi)\sum_{l\in\mathbb{Z}^d}\phi_\nu(x-2^{-\nu}l)e^{i2^{-\nu}l\xi}\mathrm{d}\xi\\ &= 2^{-\nu d}\sum_{l\in\mathbb{Z}^d}f_{\nu,\delta}(2^{-\nu}l)\phi_\nu(x-2^{-\nu}l),\end{aligned}$$

using Fubini's theorem. Now we let $\delta\to 0$ which gives us

$$(\phi_\nu * \widetilde{\phi}_\nu * f)(x) = 2^{-\nu d}\sum_{l\in\mathbb{Z}^d}(\widetilde{\phi}_\nu * f)(2^{-\nu}l)\phi_\nu(x-2^{-\nu}l) \qquad (1.28)$$

almost everywhere, where the convergence follows from $\widetilde{\phi}_\nu * f$ being slowly increasing and the dominated convergence theorem with the counting measure applied on the right-hand side. We finally get (1.25) by justifying that (1.28) also converges in S'. This follows again from the dominated convergence theorem where we use $(|\phi_\nu| * |\widetilde{\phi}_\nu * f|)(x)|\eta(x)|$ to dominate the right-hand side. ■

The following theorem shows the norm equivalence between an element $f\in\dot{F}^s_{p,q}$ and the sequence $(\langle f,\phi_I\rangle)_{I\in D}\in\dot{f}^s_{p,q}$ from (1.22) and thereby the connection between the space of distributions $\dot{F}^s_{p,q}$ and the sequence space $\dot{f}^s_{p,q}$ by ϕ. Similary for $\dot{B}^s_{p,q}$ and $\dot{b}^s_{p,q}$.

Proposition 1.4
Let $s\in\mathbb{R}$ and for $\dot{F}^s_{p,q}$ take $0<p<\infty$, $0<q\leq\infty$ and for $\dot{B}^s_{p,q}$ take $0<p,q\leq\infty$. Given $f=\sum_{J\in D}\langle f,\phi_J\rangle\phi_J$ in S'/P then $f\in\dot{F}^s_{p,q}$ if and only if $\big(\langle f,\phi_J\rangle\big)_{J\in D}\in\dot{f}^s_{p,q}$ and if affirmative, one has

$$\|\langle f,\phi_J\rangle\|_{\dot{f}^s_{p,q}} \asymp \|f\|_{\dot{F}^s_{p,q}}$$

Proof:
Assume that $\langle f,\phi_J\rangle_{J\in D}\in\dot{f}^s_{p,q}$ and $f=\sum_{J\in D}\langle f,\phi_J\rangle\phi_J$ in S'/P. Let Δ^κ_j be the Littlewood-Paley operator associated with the bump function κ (1.3)-(1.5). Since $\phi\in S$ then for every $J\in D_m$ we have that

$$|\phi_J^{(\gamma)}(y)| \leq C2^{-md/2}\frac{2^{|\gamma|m+md}}{(1+2^m|y-x_I|)^M},$$

for every $\gamma \in \mathbb{N}^d$ and $M \in \mathbb{N}$. We also have that the bump function κ satisfies the estimate

$$|\kappa_j^{(\gamma)}(y-x)| \le C \frac{2^{j|\gamma|+jd}}{(1+2^j|y-x|)^M} ,$$

for every $\gamma \in \mathbb{N}^d$ and $M \in \mathbb{N}$. Since both functions have Fourier transforms that are compactly supported away from the origin, we have by using Lemma A.2 twice

$$|\Delta_j^\kappa(\phi_J)(x)| \le C 2^{-md/2} \frac{2^{\min\{j,m\}d-|m-j|L}}{(1+2^{\min\{j,m\}}|x-x_J|)^M} , \tag{1.29}$$

regardless of whether $j \le m$ or $m < j$, and where L can be chosen as large as necessary for M large enough. Let $t = \min\{1, p, q\}$ and choose $L > \max\{d/t - d - s, s\}$. By a similar proof as the one for Lemma A.4 one has for $M > d/t$, $n \in \mathbb{Z}$ and $x \in I \in D_n$ that

$$\begin{aligned} &\sum_{J\in D_m} |\langle f, \phi_J\rangle| \Big(1 + \frac{|x-x_J|}{2^{-\min\{j,m\}}}\Big)^{-M} \\ &\le C 2^{\max\{m-j,0\}d/t} M_t\Big(\sum_{J\in D_m} |\langle f, \phi_J\rangle| \chi_J\Big)(x), \end{aligned}$$

where

$$M_t(f)(x) = \Big(\sup_{\{Q: x\in Q\}} |Q|^{-1}\int_Q |f(y)|^t \mathrm{d}y\Big)^{1/t},$$

the supremum being taken over all cubes Q with sidelength parallel to the axes. Using this estimate and (1.29) one has

$$2^{js}\sum_{m\in\mathbb{Z}}\sum_{J\in D_m} |\langle f, \phi_J\rangle| |\Delta_j^\kappa(\phi_J)(x)| \tag{1.30}$$

$$\begin{aligned} \le C \sum_{m\in\mathbb{Z}} & 2^{\min\{j,m\}d} 2^{-|j-m|L} 2^{-md} 2^{(j-m)s} 2^{\max\{m-j,0\}d/t} \\ & \cdot M_t\Big(\sum_{J\in D_m} |\langle f, \phi_J\rangle| |J|^{-s/d-1/2}\chi_J\Big)(x). \end{aligned} \tag{1.31}$$

In (1.30) we now take the l^q-norm over j and the L^p-norm over x (with the usual modification for $q = \infty$). This gives

$$\begin{aligned} &\|f\|_{\dot{F}^s_{p,q}} \\ &\le \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} \delta(j-m) M_t\Big(\sum_{J\in D_m} |\langle f, \phi_J\rangle| |J|^{-s/d-1/2}\chi_J\Big)(x)\Big)^q\Big)^{1/q}\Big\|_{L^p}, \end{aligned} \tag{1.32}$$

where we define

$$\delta(j-m) = C 2^{\min\{j-m,0\}(d-d/t)} 2^{-|j-m|L} 2^{(j-m)s}.$$

Notice that $\delta(j-m)$ does not depend on a specific values of j or m but on the difference between them. For $0 < q \leq 1$ we can assess the term inside the L^p-norm from (1.32) as

$$\Big(\sum_{j\in\mathbb{Z}}\delta(j)^q\Big)^{1/q}\Big(\sum_{m\in\mathbb{Z}}M_t\Big(\sum_{J\in D_m}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)^q\Big)^{1/q}.$$

In the case where $1 < q$ we use Minkovski's inequality to find

$$\begin{aligned}
&\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\delta(j-m)M_t\Big(\sum_{J\in D_m}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)\Big)^q\Big)^{1/q}\\
=&\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\delta(-m)M_t\Big(\sum_{J\in D_{m+j}}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)\Big)^q\Big)^{1/q}\\
\leq&\sum_{m\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}\Big(\delta(-m)M_t\Big(\sum_{J\in D_{m+j}}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)\Big)^q\Big)^{1/q}\\
=&\Big(\sum_{m\in\mathbb{Z}}\delta(m)\Big)\Big(\sum_{j\in\mathbb{Z}}M_t\Big(\sum_{J\in D_j}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)^q\Big)^{1/q}.
\end{aligned}$$

In both cases we have that the first factor can be estimated as a constant since $L > \max\{d/t-d-s,s\}$. Together with Fefferman Stein's maximal inequality (A.4) this implies that

$$\begin{aligned}
\|f\|_{\dot{F}^s_{p,q}}\leq&C\Big\|\Big(\sum_{m\in\mathbb{Z}}M_t\Big(\sum_{J\in D_m}|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)(x)^q\Big)^{1/q}\Big\|_{L^p}\\
\leq&C\Big\|\Big(\sum_{m\in\mathbb{Z}}\sum_{J\in D_m}\Big(|\langle f,\phi_J\rangle||J|^{-s/d-1/2}\chi_J\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
=&C\|\langle f,\phi_J\rangle\|_{\dot{f}^s_{p,q}}.
\end{aligned}$$

Conversly we have by (1.23) and (1.24) that

$$\langle f,\phi_J\rangle = 2^{-md/2}\widetilde{\varphi}_m * f(2^{-m}k),$$

where $x_J = k2^{-m}$. This we use together with the fact that for any $x\in\mathbb{R}^d$ there is only one $J\in D_m$ that contains x

$$\begin{aligned}
&\sum_{J\in D_m}\Big(|J|^{-s/d-1/2}|\langle f,\phi_J\rangle|\chi_J\Big)^q\\
\leq&\sum_{J\in D_m}\Big(2^{ms}\sup_{y\in J}|(\widetilde{\varphi}_m * f)(y)|\chi_J\Big)^q\\
\leq&C\sup_{|y|\leq 2^{-m}\sqrt{d}}\big(2^{ms}(1+2^m|y|)^{-b}|\widetilde{\varphi}_m * f(x-y)|\big)^q(1+2^m|y|)^{bq}\\
\leq&C\big(2^{ms}M^{**}_{b,m}(f;\widetilde{\varphi})(x)\big)^q,
\end{aligned}$$

where

$$M^{**}_{b,m}(f;\phi)(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\widetilde{\phi}_m * f)(x-y)|}{(1+2^m|y|)^{-b}}.$$

If we now sum over all $m \in \mathbb{Z}$, then raising to the power $1/q$ and taking the L^p-norm we find by using [6, Theorem 6.5.6., p.483] with $b > d/\min\{p,q\}$ that

$$\begin{aligned}
\|\langle f, \overline{\phi_J}\rangle\|_{\dot{f}^s_{p,q}} &\leq C\Big\| \Big(\sum_{m\in\mathbb{Z}} |2^{ms} M^{**}_{b,m}(f;\widetilde{\phi})(x)|^q\Big)^{1/q}\Big\|_{L^p} \\
&\leq C\Big\| \Big(\sum_{m\in\mathbb{Z}} (2^{ms}|\Delta^{\phi}_m(f)|)^q\Big)^{1/q}\Big\|_{L^p} = C\|f\|_{\dot{F}^s_{p,q}}.
\end{aligned}$$

■

The norm equivalence between $\dot{F}^s_{p,q}$ and $\dot{f}^s_{p,q}$ (and $\dot{B}^s_{p,q}$ and $\dot{b}^s_{p,q}$), can be used to show that $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ are complete as $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$ are complete, whereby $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ are complete quasi-normed spaces. The proof follows by using Lemma 2.3 which is placed later for continuity. This approach seems new compared to earlier proofs.

Proposition 1.5
Let $s \in \mathbb{R}$ and $1 \leq p,q < \infty$. Then $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ are complete quasi-normed spaces.

Proof:
We only give the proof for $\dot{F}^s_{p,q}$ since the proof for $\dot{B}^s_{p,q}$ follows similarly. As pointed out that earlier $\dot{F}^s_{p,q}$ is quasi-normed follows from the properties of the l^q, L^p-norms. So completeness remains. Take a Cauchy sequence $f_n \in \dot{F}^s_{p,q}$ and let $\varepsilon > 0$ be given. Fix J and then by the norm equivalence of the previous theorem we have

$$\begin{aligned}
\|f_n - f_m\|_{\dot{F}^s_{p,q}} &\geq C\|\langle f_n, \phi_J\rangle - \langle f_m, \phi_J\rangle\|_{\dot{f}^s_{p,q}} \\
&\geq C\||J|^{-s/d-1/2}(\langle f_n, \phi_J\rangle - \langle f_m, \phi_J\rangle)\chi_J\|_{L^p} \\
&= C|J|^{-s/d+1/p-1/2}|\langle f_n, \phi_J\rangle - \langle f_m, \phi_J\rangle|.
\end{aligned}$$

As f_n is Cauchy in $\dot{F}^s_{p,q}$ we can for every $J \in D$ find N such that for $n, m > N$ we have that

$$\varepsilon > |J|^{s/d-1/p+1/2}\|f_n - f_m\|_{\dot{F}^s_{p,q}} \geq |\langle f_n, \phi_J\rangle - \langle f_m, \phi_J\rangle|.$$

This shows that $\langle f_n, \phi_J\rangle$ is Cauchy in $\mathbb{C}$ and therefore convergent. Its limit we shall denote by $\langle f, \phi_J\rangle$. From Lemma 2.3 we have that $\sum_{J\in D}\langle f, \phi_J\rangle\phi_J =$

f in S'/P. Then by Fatou's lemma with the counting measure we find

$$\begin{aligned}&\Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q}\\ &\leq \liminf_n \Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f_n,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q} \end{aligned}\tag{1.33}$$

Using Fatou's lemma with the Lebesgue measure yields

$$\begin{aligned}&\|\liminf_n \Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f_n,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q}\|_{L^p}\\ &\leq \liminf_n \|\Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f_n,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q}\|_{L^p}. \end{aligned}\tag{1.34}$$

Combining Proposition 1.4 and (1.33), (1.34) we have that

$$\begin{aligned}\|f\|_{\dot{F}^s_{p,q}} &\leq C\|\Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q}\|_{L^p}\\ &\leq \liminf_n C\|\Big(\sum_{I\in D}\big(|I|^{-s/d-1/2}(\langle f_n,\phi_J\rangle)\chi_J\big)^q\Big)^{1/q}\|_{L^p}\\ &\leq \liminf_n C\|f_n\|_{\dot{F}^s_{p,q}} < \infty,\end{aligned}$$

which shows that $f \in \dot{F}^s_{p,q}$. By repeating the calculations with $f_m - f$ instead of f and $\langle f_n, \phi_J\rangle - \langle f_m, \phi_J\rangle$ instead of $\langle f_n, \phi_J\rangle$ we find

$$\|f - f_m\|_{\dot{F}^s_{p,q}} \leq \liminf_n C\|f_m - f_n\|_{\dot{F}^s_{p,q}}.$$

Since f_n is Cauchy in $\dot{F}^s_{p,q}$ we can find N such that for $m, n > N$ we have $\|f_m - f_n\|_{\dot{F}^s_{p,q}} < \varepsilon C^{-1}$ which shows that $\dot{F}^s_{p,q}$ is complete. ■

2 Bounded operators and decomposition systems

In this section we shall prove one of our main results: namely that a nice decomposition system for $L^2(\mathbb{R}^d)$ is also a decomposition system for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$. The proof of this will in no small part depend on the boundedness of operators on the spaces $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$ and the fact that we have a norm equivalence with $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ by ϕ.

Let $B = \{\theta^e_I, \widetilde{\theta}^e_I : I \in D, e \in E\}$ be a decomposition system for $L^2(\mathbb{R}^d)$, we then have

$$\phi_I = \sum_{e\in E}\sum_{J\in D} \langle \phi_I, \widetilde{\theta}^e_J\rangle \theta^e_J \quad , \text{for } I \in D.$$

At the same time we have by Lemma 1.3

$$\theta^e_I = \sum_{J\in D} \langle \theta^e_I, \phi_J \rangle \phi_J \quad , \text{for } e \in E, I \in D$$

in S'/P_k. We shall first prove that the matrices of coefficients of one decomposition system with respect to the other

$$\widetilde{\boldsymbol{A}}_e = \left(\langle \phi_J, \widetilde{\theta}^e_I\rangle\right)_{I,J\in D} \qquad \boldsymbol{A}_e = \left(\langle \theta^e_J, \phi_I\rangle\right)_{I,J\in D}. \tag{2.1}$$

are bounded operators on $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$ for a certain range of the indices s, p, q. To prove this the following lemma will be needed.

Lemma 2.1

Let $I, J \in D$, with $|J| \le |I|$, and let η_J be a measurable function on $\mathbb{R}^d$ and $\theta_J \in C^r(\mathbb{R}^d)$ with the properties that for some $r \ge 0$ and $M > d + r$,

$$\int_{\mathbb{R}^d} x^\alpha \eta_J(x)\mathrm{d}x = 0, \quad |\alpha| \le r-1, \text{ which is void if } r = 0, \tag{2.2}$$

$$|\eta_J(x)| \le C|J|^{-1/2}\left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{-M}, \tag{2.3}$$

$$|\theta^{(\alpha)}_I(x)| \le C|I|^{-1/2-|\alpha|/d}\left(1 + \frac{|x - x_I|}{\ell(I)}\right)^{-M}, \quad |\alpha| \le r. \tag{2.4}$$

Then

$$|\langle \theta_I, \eta_J\rangle| = C\left(\frac{\ell(J)}{\ell(I)}\right)^{r+d/2}\left(1 + \frac{|x_I - x_J|}{\ell(I)}\right)^{-M}. \tag{2.5}$$

Proof:

In the first part of the proof we shall assume that $r \ge 1$. At the end of the proof we shall comment on the case when η has no vanishing moments. By the vanishing moments of η, (2.2) we can add a polynomial in the integral free of charge

$$
\begin{aligned}
|\langle\theta_I,\eta_J\rangle| &= \left|\int_{\mathbb{R}^d}\theta_I(x)\eta_J(x)\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}^d}\left|\left(\theta_I(x)-\sum_{|\alpha|<r}\frac{(x-x_J)^\alpha}{\alpha!}(\theta_I)^{(\alpha)}(x_J)\right)\right|\left|\eta_J(x)\right|\mathrm{d}x.
\end{aligned}
$$

By a change of variable $x=\ell(I)x+x_I$ we find that

$$
\begin{aligned}
= \int_{\mathbb{R}^d}&\left|\theta_I(\ell(I)x+x_I)-\sum_{|\alpha|<r}\frac{(\ell(I)x+x_I-x_J)^\alpha}{\alpha!}(\theta_I)^{(\alpha)}(x_J)\right| \\
&\cdot\left|\eta_J(\ell(I)x+x_I)\right|\ell(I)^d\mathrm{d}x.
\end{aligned}
\tag{2.6}
$$

At this point we shall split the integral into two, by integrating over the area $A=\{x:|x-x_{I,J}|\ge 1\}$ and its complement, where $x_{I,J}=\frac{x_J-x_I}{\ell(I)}$. First we deal with the integral over the area A, where we employ (2.3) and (2.4) to obtain

$$
\begin{aligned}
&C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{-d/2}\int_A\left((1+|x|)^{-M}+\frac{\left|x-x_{I,J}\right|^{r-1}}{(1+|x_{I,J}|)^M}\right)\left(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\right)^{-M}\mathrm{d}x \\
=&B_1+B_2,
\end{aligned}
$$

where

$$
B_1=C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{-d/2}\int_A(1+|x|)^{-M}\left(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\right)^{-M}\mathrm{d}x
$$

and

$$
B_2=C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{-d/2}\int_A\frac{\left|x-x_{I,J}\right|^{r-1}}{(1+|x_{I,J}|)^M}\left(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\right)^{-M}\mathrm{d}x.
$$

In the case of B_1 we once again consider two cases. In the first case we consider $|x|\le\frac{1}{2}|x_{I,J}|$. Then we have that $\left|x_{I,J}-x\right|\ge\frac{1}{2}|x_{I,J}|$, and by the definition of the area A

$$
|x-x_{I,J}|\ge\tfrac{1}{2}(1+|x-x_{I,J}|)\ge\tfrac{1}{4}(1+|x_{I,J}|).
$$

Using this we find

$$
\begin{aligned}
&C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{-d/2}\int_{A\cap\{x:|x|\le\frac{1}{2}|x_{I,J}|\}}(1+|x|)^{-M}\left(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\right)^{-M}\mathrm{d}x \\
\le&C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{M-d/2}(1+|x_{I,J}|)^{-M}\int_{\mathbb{R}^d}(1+|x|)^{-M}\mathrm{d}x \\
\le&C\left(\tfrac{\ell(J)}{\ell(I)}\right)^{M-d/2}(1+|x_{I,J}|)^{-M}.
\end{aligned}
$$

On the other hand if $|x| > \frac{1}{2}|x_{I,J}|$ then $(1+|x|)^{-M} \leq 2^M(1+|x_{I,J}|)^{-M}$. For the complementary area we also note that if $|x| \geq c > 0$ and $h \in \mathbb{R}_+$ then

$$\begin{aligned} 1+\frac{|x|}{h} &= \Big(\frac{|x|}{|x|+1}\Big)\Big(1+\frac{|x|}{h}+\frac{1}{|x|}+\frac{1}{h}\Big) \\ &\geq \Big(\frac{c}{c+1}\Big)\Big(\frac{1+|x|}{h}\Big), \end{aligned} \tag{2.7}$$

which, together with the first observation, yields

$$\begin{aligned} &C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{-d/2}\int_{A\cap\{x:|x|>\frac{1}{2}|x_{I,J}|\}}(1+|x|)^{-M}\Big(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\Big)^{-M}\mathrm{d}x \\ \leq{}& C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{-d/2}(1+|x_{I,J}|)^{-M}\int_{\mathbb{R}^d}\Big(\frac{1+\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\Big)^{-M}\mathrm{d}x \\ \leq{}& C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{M-d/2}(1+|x_{I,J}|)^{-M}. \end{aligned}$$

Turning our attention to B_2 we utilize that $\left|x-x_{I,J}\right| \geq 1$ and $M > d+r$ to obtain

$$\begin{aligned} B_2 &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r-1-d/2}(1+|x_{I,J}|)^{-M}\int_A\Big(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\Big)^{r-1-M}\mathrm{d}x \\ &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{M-d/2}(1+|x_{I,J}|)^{-M}. \end{aligned}$$

Combining these three estimates and the fact that we assumed $M - d/2 > r + d/2$ we find that (2.6), over the area A, can be bounded by $C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r+d/2}(1+|x_{I,J}|)^{-M}$. For the integral over A^c we use Taylor's formula

$$\begin{aligned} &\int_{A^c}\Big|\theta_I(\ell(I)x+x_I)-\sum_{|\alpha|<r}\frac{(\ell(I)x+x_I-x_J)^\alpha}{\alpha!}(\theta_I)^{(\alpha)}(x_J)\Big| \\ &\qquad\qquad\qquad\cdot\left|\eta_J(\ell(I)x+x_I)\right|\ell(I)^d\mathrm{d}x \\ ={}&\int_{A^c}\Big|\sum_{|\alpha|=r}\frac{(\ell(I)x+x_I-x_J)^\alpha}{\alpha!}(\theta_I)^{(\alpha)}(x_0)\Big|\left|\eta_J(\ell(I)x+x_I)\right|\ell(I)^d\mathrm{d}x, \end{aligned}$$

where x_0 is a point on the linesegment between x_J and $\ell(I)x+x_I$. By the assumptions (2.3) and (2.4) one finds

$$\begin{aligned} &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{-d/2}\int_{A^c}|x-x_{I,J}|^r \\ &\qquad\cdot\sup_{z\in l(x_J,\ell(I)x+x_I)}\Big(1+\frac{|z-x_I|}{\ell(I)}\Big)^{-M}\Big(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\Big)^{-M}\mathrm{d}x \\ &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r-d/2}\int_{A^c}\sup_{z\in l(x_{I,J},x)}(1+|z|)^{-M}\Big(1+\frac{\left|x-x_{I,J}\right|}{\ell(J)/\ell(I)}\Big)^{-M+r}\mathrm{d}x. \end{aligned} \tag{2.8}$$

The properties of the set $A^c = \{x : |x - x_{I,J}| < 1\}$ imply that

$$1 + |x_{I,J}| \leq 1 + |z - x_{I,J}| + |z| \leq 2(1 + |z|).$$

This in turn yields the following equation

$$\sup_{z \in l(x_{I,J}, x)} (1 + |z|)^{-M} \leq 2^M (1 + |x_{I,J}|)^{-M}. \tag{2.9}$$

Substituting (2.9) into (2.8) we find

$$\begin{aligned} &C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r-d/2} (1 + |x_{I,J}|)^{-M} \int_{A^c} \left(1 + \frac{|x - x_{I,J}|}{\ell(J)/\ell(I)}\right)^{-M+r} \mathrm{d}x \\ &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r+d/2} (1 + |x_{I,J}|)^{-M} \int_{\mathbb{R}^d} (1 + |x|)^{-M+r} \mathrm{d}x \\ &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{r+d/2} (1 + |x_{I,J}|)^{-M}, \end{aligned}$$

where we have used that $M > d + r$ in the last inequality to assess the integral, thereby yielding the desired conclusion for $r \geq 1$.
In the case of no vanishing moments of η, B_1 from the first part estimates the integral over A. For $\int_{A^c} |\theta_I(x)||\eta_J(x)|\mathrm{d}x$ we use that $1 + |x_{I,J}| \leq 2(1 + |x|)$, which together with (2.3) and (2.4) shows that

$$\begin{aligned} |\langle \theta_I, \eta_J \rangle| &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{-d/2} \int_{A^c} (1 + |x|)^{-M} \left(1 + \frac{|x - x_{I,J}|}{\ell(J)/\ell(I)}\right)^{-M} \mathrm{d}x \\ &\leq C\Big(\tfrac{\ell(J)}{\ell(I)}\Big)^{d/2} (1 + |x_{I,J}|)^{-M}. \end{aligned}$$

■

For the remainder of this section we will assume that for $r_1, r_2 \in \mathbb{N}$, $M > d + \max\{r_1, r_2\}$ the functions $\eta_I \in C^{r_1}(\mathbb{R}^d)$ and $\theta_I \in C^{r_2}(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} x^\alpha \theta_I(x) \mathrm{d}x = 0, \qquad |\alpha| \leq r_1 - 1, \text{ which is void if } r_1 = 0, \tag{2.10}$$

$$|\theta_I^{(\alpha)}(x)| \leq C|I|^{-1/2-|\alpha|/d} \left(1 + \frac{|x - x_I|}{\ell(I)}\right)^{-M}, \qquad |\alpha| \leq r_2, \tag{2.11}$$

$$\int_{\mathbb{R}^d} x^\alpha \eta_I(x) \mathrm{d}x = 0, \qquad |\alpha| \leq r_2 - 1, \text{ which is void if } r_2 = 0, \tag{2.12}$$

$$|\eta_I^{(\alpha)}(x)| \leq C|I|^{-1/2-|\alpha|/d} \left(1 + \frac{|x - x_I|}{\ell(I)}\right)^{-M}, \qquad |\alpha| \leq r_1, \tag{2.13}$$

thereby guaranteeing a suitable decay of $|\langle \theta_J, \eta_I \rangle|$ regardless of the relative size of $|I|$ and $|J|$ by the previous lemma. As the next propositions will show, choosing r_1, r_2 and M large enough will insure that the infinite matrix

$$\boldsymbol{A} = \big(\langle \theta_J, \eta_I \rangle\big)_{I,J \in D} \tag{2.14}$$

is a bounded operator on the $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$.

Proposition 2.2
Let $\{\theta_I\}_{I\in D}$, $\{\eta_I\}_{I\in D}$ be families of functions satisfying (2.10)-(2.13) for $r_1, r_2 \in \mathbb{N}$, $M > d + \max\{r_1, r_2\}$. Moreover let $s \in \mathbb{R}$, $0 < q \leq \infty$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $L = d/\min\{1, p, q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p \leq \infty$, $L = d/\min\{1, p\}$. If $r_2 > s$, $r_1 > L - d - s$ and $M > L$ then the infinite matrix defined in (2.14) is a bounded operator on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$.

Proof:
We prove the result for $\dot{F}^s_{p,q}$ where $q < \infty$ as the proof for $\dot{B}^s_{p,q}$ follows similarly and the case $q = \infty$ follows in exactly the same way as $q < \infty$, with ℓ^∞ instead of ℓ^q. For every $h \in \dot{f}^s_{p,q}$ we have $(\boldsymbol{A}h)_I = \sum_{J\in D} \langle \theta_J, \eta_I \rangle h_J$. We take the $\dot{f}^s_{p,q}$-norm and split the sequence in two

$$\begin{aligned}\|\boldsymbol{A}h\|_{\dot{f}^s_{p,q}} \leq & \Big\| \Big(\sum_{I\in D} \Big(|I|^{-s/d-1/2} \sum_{J\in D} |\langle \theta_J, \eta_I \rangle| |h_J| \chi_I \Big)^q \Big)^{1/q} \Big\|_{L^p} \\ \leq & C(\sigma_1 + \sigma_2),\end{aligned}$$

where

$$\begin{aligned}\sigma_1 = & \Big\| \Big(\sum_{I\in D} \Big(|I|^{-s/d-1/2} \sum_{|J|\leq|I|} |\langle \theta_J, \eta_I \rangle| |h_J| \chi_I \Big)^q \Big)^{1/q} \Big\|_{L^p} \\ \sigma_2 = & \Big\| \Big(\sum_{I\in D} \Big(|I|^{-s/d-1/2} \sum_{|J|>|I|} |\langle \theta_J, \eta_I \rangle| |h_J| \chi_I \Big)^q \Big)^{1/q} \Big\|_{L^p}.\end{aligned}$$

Choose $0 < t < \min\{1, p, q\}$ such that $M > d/t$ and $r_1 > d/t - d - s$ and denote $|I|^{-s/d-1/2}\chi_I = \lambda_I$. For σ_1 the inequality $|J| \leq |I|$ holds and thus Lemma 2.1 can be applied in this case. Together with Lemma A.4 we infer

$$\begin{aligned}\sigma_1 \leq C & \Big\| \Big(\sum_{I\in D} \Big(\sum_{|J|\leq|I|} \Big(\frac{\ell(J)}{\ell(I)} \Big)^{r_1+d/2} \Big(1 + \frac{|x_I - x_J|}{\ell(I)} \Big)^{-M} |h_J| \lambda_I \Big)^q \Big)^{1/q} \Big\|_{L^p} \\ = C & \Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{I\in D_n} \Big(\sum_{m\geq n} 2^{(n-m)(r_1+d/2)} \\ & \qquad \cdot \sum_{J\in D_m} \Big(1 + \frac{|x_I - x_J|}{\ell(I)} \Big)^{-M} |h_J| \lambda_I \Big)^q \Big)^{1/q} \Big\|_{L^p} \\ \leq C & \Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{I\in D_n} \Big(\sum_{m\geq n} 2^{(n-m)(r_1+d/2-d/t)} \\ & \qquad \cdot M_t \Big(\sum_{J\in D_m} |h_J| \chi_J \Big)(x) \lambda_I \Big)^q \Big)^{1/q} \Big\|_{L^p} \\ = C & \Big\| \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{m\geq n} 2^{(n-m)(r_1+d-d/t+s)} M_t \Big(\sum_{J\in D_m} |h_J| \lambda_J \Big)(x) \Big)^q \Big)^{1/q} \Big\|_{L^p},\end{aligned}$$

where in the last equality we used $\sum_{I\in D_n}\chi_I = 1$. Next we change n to $-n$, m to $-m$ and use Lemma A.5 with $\mu = 1$, $\lambda = 0$, $a_m = M_t\Big(\sum_{J\in D_{-m}}|h_J|\lambda_J\Big)(x)$ and
$b_n = \sum_{m\le n} 2^{(m-n)(r_1+d-d/t+s)}M_t\Big(\sum_{I\in D_{-m}}|h_I|\lambda_I\Big)(x)$ which yields

$$\le C\Big\|\Big(\sum_{n\in\mathbb{Z}}\Big(M_t\Big(\sum_{I\in D_n}|h_I|\lambda_I\Big)(x)\Big)^q\Big)^{1/q}\Big\|_{L^p} \le C\|h\|_{\dot{f}^s_{p,q}}\ ,$$

where we in the last inequality have imployed Fefferman Stein's maximal inequality (A.4) and the fact that $|\sum_{I\in D_n} h_I\lambda_I|^q = \sum_{I\in D_n}|h_I\lambda_I|^q$, since the elements of D_n are disjoint. For the other half, were $|J|>|I|$, we interchange the roles of η_J and θ_I in Lemma 2.1 together with Lemma A.4 to find

$$\begin{aligned}
\sigma_2 \le& C\Big\|\Big(\sum_{I\in D}\Big(\sum_{|J|>|I|}\Big(\frac{\ell(I)}{\ell(J)}\Big)^{r_2+d/2}\Big(1+\frac{|x_I-x_J|}{\ell(J)}\Big)^{-M}|h_J|\lambda_I\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
=&C\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{I\in D_n}\Big(\sum_{m<n}2^{(m-n)(r_2+d/2)}\\
&\qquad\cdot\sum_{J\in D_m}\Big(1+\frac{|x_I-x_J|}{\ell(J)}\Big)^{-M}|h_J|\lambda_I\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
\le& C\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{I\in D_n}\Big(\sum_{m<n}2^{(m-n)(r_2+d/2)}M_t\Big(\sum_{J\in D_m}|h_J|\chi_J\Big)(x)\lambda_I(x)\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
=&C\Big\|\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{m<n}2^{(m-n)(r_2-s)}M_t\Big(\sum_{J\in D_m}|h_J|\lambda_J\Big)(x)\Big)^q\Big)^{1/q}\Big\|_{L^p}.
\end{aligned}$$

Since $r_2 > s$ we can use Lemma A.5 and Fefferman Stein's maximal inequality (A.4) as before to find that

$$\begin{aligned}
\le& C\Big\|\Big(\sum_{n\in\mathbb{Z}}M_t\Big(\sum_{I\in D_n}|h_I|\lambda_I\Big)(x)\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
\le& C\|h\|_{\dot{f}^s_{p,q}}\ .
\end{aligned}$$

■

Decomposition systems and wavelet bases

Assuming that $\{\theta_I\}_{I\in D}$ satisfies (2.10)-(2.11) and $\{\widetilde{\theta}_I\}_{I\in D}$ satisfies (2.12)-(2.13) we now have by Proposition 2.2 that $\boldsymbol{A}_e$ and $\widetilde{\boldsymbol{A}}_e$ in (2.1) are bounded operators on $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$. The last detail we need to examine before proving that B forms a decomposition system for $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ is the convergence of the series $\sum_{I\in D}\theta_I$ in S'/P_k.

Lemma 2.3
Let $s \in \mathbb{R}$, $0 < q \leq \infty$. For $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p \leq \infty$, $L = d/\min\{1,p\}$ and for both spaces $k = \max\{\lfloor s - d/p \rfloor, -1\}$. If $\{\theta_I\}_{I \in D}$ satisfies (2.10) and (2.11) for $r_1, r_2 \in \mathbb{N}$ with $r_1 > L - d - s$, $r_2 > s$ and $M > \max\{L, d + r_1, d + r_2\}$, then for $a \in \dot{f}^s_{p,q}$ or $a \in \dot{b}^s_{p,q}$ and $\eta \in S_k$

$$\sum_{I \in D} |a_I| \left| \langle \theta_I, \eta \rangle \right| < \infty$$

so the series $\sum_{I \in D} a_I \theta_I$ converges in S'/P_k.

Proof:
We only prove the result for $\dot{F}^s_{p,q}$ as the result for $\dot{B}^s_{p,q}$ follows similarly. By Lemma 2.1 we have for $|I| > 1$

$$|\langle \theta_I, \eta \rangle| \leq C \ell(I)^{-(k+1+d/2)} \left(1 + \frac{|x_I|}{\ell(I)} \right)^{-M},$$

because $r_2 \geq k + 1$, while for $|I| \leq 1$,

$$|\langle \theta_I, \eta \rangle| \leq C \ell(I)^{r_1 + d/2} (1 + |x_I|)^{-M}. \tag{2.15}$$

We also have

$$|I|^{-s/d - 1/2 + 1/p} |a_I| = \|a_I\|_{\dot{f}^s_{p,q}} \leq \|a\|_{\dot{f}^s_{p,q}},$$

which we will use to estimate the series for $|I| > 1$.

$$\begin{aligned} \sum_{|I|>1} |a_I| \, |\langle \theta_I, \eta \rangle| &\leq C \sum_{|I|>1} \ell(I)^{s+d/2-d/p-(k+1+d/2)} \left(\frac{1+|x_I|}{\ell(I)} \right)^{-M} \\ &= \sum_{n>0} 2^{n(s-k-1-d/p)} \sum_{j \in \mathbb{Z}^d} (1+|j|)^{-M} \\ &\leq \sum_{n>0} 2^{n(s-k-1-d/p)} < \infty, \end{aligned}$$

where we have used Lemma A.6 and $k + 1 > s - d/p$. To estimate the series for $|I| \leq 1$ we use Lemma A.4 with t chosen such that $0 < t < \min\{1, p, q\}$,

$M > d/t$ and $r_1 > d/t - d - s$,

$$\begin{aligned}
\sum_{|I|\leq 1} |a_I|\,|\langle\theta_I,\eta\rangle| &\leq \sum_{|I|\leq 1} \ell(I)^{r_1+d/2}|a_I|(1+|x_I|)^{-M}\\
&\leq C\sum_{n\geq 0} 2^{-n(r_1+d/2)} \sum_{I\in D_n} |a_I|(1+|x_I|)^{-M}\\
&\leq C\sum_{n\geq 0} 2^{-n(r_1+d/2-d/t)} M_t\Big(\sum_{I\in D_n}|a_I|\chi_I\Big)(x)\\
&= C\sum_{n\geq 0} 2^{-n(r_1+s+d-d/t)} M_t\Big(\sum_{I\in D_n}|I|^{-s/d-1/2}|a_I|\chi_I\Big)(x)\\
&\leq C\sup_{n\geq 0} M_t\Big(\sum_{I\in D_n}|I|^{-s/d-1/2}|a_I|\chi_I\Big)(x)\\
&\leq C\Big(\sum_{n\geq 0}\Big(M_t\Big(\sum_{I\in D_n}|I|^{-s/d-1/2}|a_I|\chi_I\Big)(x)\Big)^q\Big)^{1/q},
\end{aligned}$$

where x is in the unit cube I_0 and the usual change is made if $q = \infty$. Taking the $L^p(I_0)$ norm on both sides and using Fefferman Stein's maximal inequality (A.4) we get

$$\begin{aligned}
\sum_{|I|\leq 1} |a_I|\,|\langle\theta_I,\eta\rangle| &\leq C\Big\|\Big(\sum_{n\geq 0}\Big(M_t\Big(\sum_{I\in D_n}|I|^{-s/d-1/2}|a_I|\chi_I\Big)(x)\Big)^q\Big)^{1/q}\Big\|_{L^p(I_0)}\\
&\leq C\|a\|_{\dot{f}^s_{p,q}}.
\end{aligned}$$

■

Remark 2.4
Observe that the series $\sum_{I\in D} a_I\,\langle\theta_I,\eta\rangle$ in Lemma 2.3 converges not only for $\eta\in S_k$ but for any η with k vanishing moments and satisfying

$$|\eta^{(\alpha)}(x)| \leq C(1+|x|)^{-M}, \quad |\alpha|\leq r_1.$$

Therefore if $f = \sum_{I\in D} a_I\theta_I$ we may define

$$\langle f,\eta\rangle = \sum_{I\in D} d_I\,\langle\theta_I,\eta\rangle\,,$$

dispite the fact that $\eta\notin S_k$. ○

We are now ready to state and prove that a nice decomposition system for $L^2(\mathbb{R}^d)$ is also a decomposition system for the Triebel-Lizorkin and Besov space if θ_I^e, $\widetilde{\theta}_I^e$ have adequate decay and vanishing moments.

Theorem 2.5
Let $s\in\mathbb{R}$, $0<q\leq\infty$. For $\dot{F}^s_{p,q}$ let $0<p<\infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0<p\leq\infty$, $L = d/\min\{1,p\}$ and for both spaces $k = \max\{\lfloor s-d/p\rfloor, -1\}$.

Assume that $\{\theta_I^e, \widetilde{\theta}_I^e : I \in D, e \in E\}$ forms a decomposition system for $L^2(\mathbb{R}^d)$ satisfying (2.10)-(2.13) for $r_1, r_2 \in \mathbb{N}$ with $r_1 > L - d - s$, $r_2 > s$ and $M > \max\{L, d + r_1, d + r_2\}$. Then for $f \in \dot{F}^s_{p,q}$

$$f = \sum_{e\in E}\sum_{J\in D} \langle f, \widetilde{\theta}_J^e\rangle \theta_J^e,$$

unconditionally in S'/P_k (see page 8), where $\langle f, \widetilde{\theta}_J^e\rangle = \sum_{I\in D} \langle f, \phi_I\rangle\langle \phi_I, \widetilde{\theta}_J^e\rangle$. We also have

$$\|f\|_{\dot{F}^s_{p,q}} \asymp \sum_{e\in E} \|\langle f, \widetilde{\theta}_I^e\rangle\|_{\dot{f}^s_{p,q}} \tag{2.16}$$

and for $q \neq \infty$ the series also converges unconditionally in $\dot{F}^s_{p,q}$, where a similar statement and norm equivalence applies for $\dot{B}^s_{p,q}$ and convergence in $\dot{B}^s_{p,q}$ is guaranteed if $p, q \neq \infty$.

Proof:

We only prove the result for $\dot{F}^s_{p,q}$ as the result for $\dot{B}^s_{p,q}$ follows in a similar fashion. We begin by showing the convergence in S'/P_k. Since θ_I^e, $\widetilde{\theta}_I^e$ is a decomposition system for $L^2(\mathbb{R}^d)$ we have

$$\phi_I = \sum_{e\in E}\sum_{J\in D} \langle \phi_I, \widetilde{\theta}_J^e\rangle \theta_J^e,$$

with convergence in S'. Together with Lemma 1.3 we formally get

$$\begin{aligned} f = \sum_{I\in D} \langle f, \phi_I\rangle\, \phi_I &= \sum_{I\in D}\sum_{e\in E}\sum_{J\in D} \langle f, \phi_I\rangle\, \langle \phi_I, \widetilde{\theta}_J^e\rangle \theta_J^e \\ &= \sum_{e\in E}\sum_{J\in D}\sum_{I\in D} \langle f, \phi_I\rangle \langle \phi_I, \widetilde{\theta}_J^e\rangle \theta_J^e \qquad (2.17) \\ &= \sum_{e\in E}\sum_{J\in D} \langle f, \widetilde{\theta}_J^e\rangle \theta_J^e, \qquad (2.18) \end{aligned}$$

with convergence in S'/P_k. To justify (2.17) we first note from Proposition 1.4 that $(\langle f, \phi_I\rangle)_{I\in D} \in \dot{f}^s_{p,q}$. From Proposition 2.2 we have that $\widetilde{\boldsymbol{A}}_e = \big(\langle \phi_J, \widetilde{\theta}_I^e\rangle\big)_{I,J\in D}$ is bounded on $\dot{f}^s_{p,q}$, so $\big(\langle f, \widetilde{\theta}_J^e\rangle\big)_{J\in D} = \widetilde{\boldsymbol{A}}_e(\langle f, \phi_I\rangle)_{I\in D} \in \dot{f}^s_{p,q}$. Now using Lemma 2.3 we have that

$$\sum_{J\in D} |\langle f, \widetilde{\theta}_J^e\rangle||\langle \theta_J^e, \eta\rangle| < \infty,$$

for $\eta \in S_k$ which allows us to interchange the order of summation.
Next we prove (2.16) and from (2.18) we get

$$\langle f, \phi_I\rangle = \sum_{e\in E}\sum_{J\in D} \langle f, \widetilde{\theta}_J^e\rangle\langle \theta_J^e, \phi_I\rangle,$$

so we have

$$(\langle f, \phi_I\rangle)_{I\in D} = \sum_{e\in E} \boldsymbol{A}_e\big(\langle f, \widetilde{\theta}_J^e\rangle\big)_{J\in D}.$$

From Proposition 2.2 it follows that $\boldsymbol{A}_e$ is bounded on $\dot{f}^s_{p,q}$ and together with the boundedness of $\widetilde{\boldsymbol{A}}_e$ this gives that

$$\begin{aligned}\|(\langle f,\phi_I\rangle)_{I\in D}\|_{\dot{f}^s_{p,q}} &= \Big\|\sum_{e\in E}\boldsymbol{A}_e^T(\langle f,\widetilde{\theta}^e_J\rangle)_{J\in D}\Big\|_{\dot{f}^s_{p,q}} \le C\sum_{e\in E}\|(\langle f,\widetilde{\theta}^e_J\rangle)_{J\in D}\|_{\dot{f}^s_{p,q}}\\ &= C\sum_{e\in E}\|\widetilde{\boldsymbol{A}}_e^T(\langle f,\phi_I\rangle)_{I\in D}\|_{\dot{f}^s_{p,q}} \le C\|(\langle f,\phi_I\rangle)_{I\in D}\|_{\dot{f}^s_{p,q}}.\end{aligned}$$

Using Proposition 1.4 we get (2.16).
To see the unconditional convergence in $\dot{F}^s_{p,q}$ for $q\neq\infty$ we take a sequence $(H)_{N\in\mathbb{N}}$ of finite subsets of D such that $H_N\subseteq H_M$ if $N\le M$ and $\cup_{N\in\mathbb{N}}H_N = D$ and use (2.16)

$$\begin{aligned}\Big\|f-\sum_{e\in E}\sum_{I\in H_N}\langle f,\widetilde{\theta}^e_I\rangle\theta^e_I\Big\|_{\dot{F}^s_{p,q}} &\le C\sum_{e\in E}\Big\|\sum_{I\in D\setminus H_N}\langle f,\widetilde{\theta}^e_I\rangle\theta^e_I\Big\|_{\dot{F}^s_{p,q}}\\ &\le C\sum_{e\in E}\|\langle f,\widetilde{\theta}^e_I\rangle\chi_{I\in D\setminus H_N}\|_{\dot{f}^s_{p,q}}.\end{aligned}$$

We recall from (1.7) that

$$\|\langle f,\widetilde{\theta}^e_I\rangle\chi_{I\in D\setminus H_N}\|_{\dot{f}^s_{p,q}} = \Big\|\Big(\sum_{I\in D}(|I|^{-s/d-1/2}|\langle f,\widetilde{\theta}^e_I\rangle|\chi_{I\in D\setminus H_N}\chi_I)^q\Big)^{1/q}\Big\|_{L^p}, \tag{2.19}$$

fix a point in I and use the dominated convergence theorem with the counting measure on the l^q-norm, next use the dominated convergence theorem on the L^p-norm to get that (2.19) goes to zero for $N\to\infty$. ■

In the case where our decomposition system is a biorthogonal wavelet basis, we can sharpen the result, such that the coefficients in $\dot{f}^s_{p,q}, \dot{b}^s_{p,q}$ are unique.

Proposition 2.6
Let $s\in\mathbb{R}$, $0<q\le\infty$. For $\dot{F}^s_{p,q}$ let $0<p<\infty$, $L=d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0<p\le\infty$ and $L=d/\min\{1,p\}$ and $k=\max\{\lfloor s-d/p\rfloor,-1\}$ for both spaces. Assume that Ψ forms a biorthogonal wavelet basis for $L^2(\mathbb{R}^d)$ satisfying

$$\begin{gathered}|(\psi^e)^{(\alpha)}(x)|\le C(1+|x|)^{-M},\quad |\alpha|\le r_2\\ |(\widetilde{\psi}^e)^{(\alpha)}(x)|\le C(1+|x|)^{-M},\quad |\alpha|\le r_1\\ \int x^\alpha\psi^e(x)\mathrm{d}x = 0,\quad |\alpha|\le r_1-1\\ \int x^\alpha\widetilde{\psi}^e(x)\mathrm{d}x = 0,\quad |\alpha|\le r_2-1,\end{gathered}$$

for $r_1,r_2\in\mathbb{N}$ with $r_1>L-d-s$, $r_2>s$ and $M>\max\{L,d+r_1,d+r_2\}$. Then for all $f\in\dot{F}^s_{p,q}$ there exists unique coefficients in $\dot{f}^s_{p,q}$ such that

$$f=\sum_{e\in E}\sum_{J\in D}\langle f,\widetilde{\psi}^e_J\rangle\psi^e_J, \tag{2.20}$$

unconditional in S'/P_k where $\langle f, \widetilde{\psi}_J^e \rangle = \sum_{I\in D} \langle f, \phi_I \rangle \langle \phi_I, \psi_J^e \rangle$. We also have

$$\|f\|_{\dot{F}^s_{p,q}} \asymp \sum_{e\in E} \|\langle f, \widetilde{\psi}_I^e \rangle\|_{\dot{f}^s_{p,q}} \tag{2.21}$$

and for $q \neq \infty$ the series (2.20) also converges unconditionally in $\dot{F}^s_{p,q}$, where a similar statement and norm equivalence applies for $\dot{B}^s_{p,q}$, and convergence in $\dot{B}^s_{p,q}$ is guaranteed if $p, q \neq \infty$.

Proof:

We prove the result for $\dot{F}^s_{p,q}$ as the proof for $\dot{B}^s_{p,q}$ follows in a similar way. Note that Ψ fulfills the requirements of Theorem 2.5 proving the proposition except for the uniqueness of the coefficients. Assume that there exists $(c_J^e)_{J\in D} \in \dot{f}^s_{p,q}$ such that

$$f = \sum_{e\in E} \sum_{J\in D} c_J^e \psi_J^e$$

with convergence in S'/P_k or $\dot{F}^s_{p,q}$. This implies

$$0 = \Big\| f - \sum_{e\in E} \sum_{J\in D} c_J^e \psi_J^e \Big\|_{\dot{F}^s_{p,q}} \asymp \Big\| \langle f - \sum_{e\in E} \sum_{J\in D} c_J^e \psi_J^e, \widetilde{\psi}_K^{e'} \rangle \Big\|_{\dot{f}^s_{p,q}}.$$

From which we get

$$\begin{aligned} \langle f, \widetilde{\psi}_K^{e'} \rangle &= \langle \sum_{e\in E} \sum_{J\in D} c_J^e \psi_J^e, \widetilde{\psi}_K^{e'} \rangle \qquad (2.22) \\ &= \sum_{I\in D} \langle \sum_{e\in E} \sum_{J\in D} c_J^e \psi_J^e, \phi_I \rangle \langle \phi_I, \widetilde{\psi}_K^{e'} \rangle \\ &= \sum_{I\in D} \sum_{e\in E} \sum_{J\in D} c_J^e \langle \psi_J^e, \phi_I \rangle \langle \phi_I, \widetilde{\psi}_K^{e'} \rangle. \end{aligned}$$

From the assumption that $(c_J^e)_{J\in D} \in \dot{f}^s_{p,q}$, and the boundedness of $\boldsymbol{A}_e$ and $\widetilde{\boldsymbol{A}}_e$ on $\dot{f}^s_{p,q}$, we have that the series converges absolutely, and we may therefore interchange the sums.

$$\begin{aligned} &= \sum_{e\in E} \sum_{J\in D} \sum_{I\in D} c_J^e \langle \psi_J^e, \phi_I \rangle \langle \phi_I, \widetilde{\psi}_K^{e'} \rangle \\ &= \sum_{e\in E} \sum_{J\in D} c_J^e \langle \sum_{I\in D} \langle \psi_J^e, \phi_I \rangle \phi_I, \widetilde{\psi}_K^{e'} \rangle \qquad (2.23) \\ &= \sum_{e\in E} \sum_{J\in D} c_J^e \langle \psi_J^e, \widetilde{\psi}_K^{e'} \rangle \qquad (2.24) \\ &= c_K^{e'} \end{aligned}$$

the equality (2.23) follows from the continuity of $\langle \cdot, \cdot \rangle$ in L^2, and (2.24) follows from (1.10) by adapting the proof of Lemma 1.3. ■

One should notice that series with only a finite number of terms from the wavelet decomposition are in $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$. This will be of interrest when we study n-term approximation, especially the Bernstein inequality.

3 Interpolation and approximation spaces

In this section we wish to characterize the space of functions which by n-term approximation from a basis from Proposition 2.6 have a certain decay in the error of approximation in $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$. The characterization will be done with an interpolation space $X_{\theta,q}$ that involves the K-functional. Proving this characterization will depend on Jackson and Bernstein inequalities for $X_{\theta,q}$. We begin by introducing the interpolation space $X_{\theta,q}$.

Let X_0, X_1 be quasi-normed complete linear spaces. They are called a pair (X_0, X_1) if each of them are continuously embedded in a linear Hausdorff topological space Y. For a pair we define the linear space $X_0 + X_1$ consisting of elements $f = f_0 + f_1$, $f_i \in X_i$, $i = 0, 1$ for which

$$\|f\|_{X_0+X_1} = \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + \|f_1\|_{X_1}) < \infty. \tag{3.1}$$

We also define the linear space $X_0 \cap X_1$ for which

$$\|f\|_{X_0 \cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\} < \infty. \tag{3.2}$$

Proposition 3.1
The spaces $X_0 \cap X_1$ and $X_0 + X_1$ are complete in the quasi-norms (3.2) and (3.1), respectively, and furthermore if X_0, X_1 are Banach spaces then so are $X_0 \cap X_1$ and $X_0 + X_1$.

Proof:

First we observe that it follows from the definition of $\|\cdot\|_{X_0 \cap X_1}$ that $X_0 \cap X_1$ is quasi-normed (normed) if the spaces X_0, X_1 are. For the completeness let x_n be a Cauchy sequence in $X_0 \cap X_1$. By the definition of $\|\cdot\|_{X_0 \cap X_1}$ this implies that x_n is also Cauchy in X_0 and X_1. Since these spaces are complete there exists element $x^0 \in X_0$ and $x^1 \in X_1$ such that

$$\|x^0 - x_n\|_{X_0} \to 0 \qquad \|x^1 - x_n\|_{X_1} \to 0, \quad \text{for } n \to \infty. \tag{3.3}$$

Since both spaces are continuously embedded in Y we have that $x_n \to x^0$ and $x_n \to x^1$ in Y. By the Hausdorff property we must have $x^0 = x^1$. The element $x = x^0 = x^1$ is in X_0 and X_1, hence in $X_0 \cap X_1$ and by (3.3) we have that x_n converges to x in $\|\cdot\|_{X_0 \cap X_1}$.

For $\|\cdot\|_{X_0+X_1}$ it easily follows that $\|af\|_{X_0+X_1} = |a|\|f\|_{X_0+X_1}$ by the same property of X_0 and X_1. For the triangle inequality notice that

$$\begin{aligned}\inf_{f+g=h_0+h_1} \{\|h_0\|_{X_0} + \|h_1\|_{X_1}\} &\le \|f_0 + g_0\|_{X_0} + \|f_1 + g_1\|_{X_1} \\ &\le C(\|f_0\|_{X_0} + \|f_1\|_{X_1} + \|g_0\|_{X_0} + \|g_1\|_{X_1}),\end{aligned} \tag{3.4}$$

where $f = f_0 + f_1$ and $g = g_0 + g_1$ with $f_0, g_0 \in X_0$ and $f_1, g_1 \in X_1$. Taking the infimum over all such decompositions of f and g in (3.4) yields the triangle inequality (with $C = 1$ if X_0, X_1 are normed). Last we check whether $\|x\|_{X_0+X_1} = 0$ implies $x = 0$. Assume that $\|x\|_{X_0+X_1} = 0$. As the $\|\cdot\|_{X_0+X_1}$-norm is an infimum, there exists $x_n^0 \in X_0$, $x_n^1 \in X_1$ for all $n \in \mathbb{N}$ such that

$$x = x_n^0 + x_n^1$$

and

$$0 = \|x\|_{X_0+X_1} \leq \|x_n^0\|_{X_0} + \|x_n^1\|_{X_1} \leq \|x\|_{X_0+X_1} + \frac{1}{n} = \frac{1}{n}.$$

This implies that the sequence x_n^0 converges to 0 in X_0, and also in Y since X_0 is continuosly embedded in Y. The same holds for x_n^1. Then the sequence $\{x_n^0 + x_n^1\}_{n=1}^{\infty}$ converges to 0 in Y. Since $x = x_n^0 + x_n^1$ this implies that $x = 0$. To establish the completeness we shall prove that every absolute convergent series in $X_0 + X_1$ is convergent in $X_0 + X_1$. Assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $X_0 + X_1$ such that $\sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1} < \infty$. As before we can write $x_n = x_n^0 + x_n^1$ where $x_n^0 \in X_0$ and $x_n^1 \in X_1$, such that

$$\|x_n^0\|_{X_0} + \|x_n^1\|_{X_1} \leq \|x_n\|_{X_0+X_1} + 2^{-n}.$$

This gives that $\sum_{n=1}^{\infty} \|x_n^0\|_{X_0} < \infty$ and $\sum_{n=1}^{\infty} \|x_n^1\|_{X_1} < \infty$, which by the completeness of X_0 and X_1 implies that there exists an element $x^0 \in X_0$ and $x^1 \in X_1$ such that

$$\lim_{N\to\infty} \Big\| \sum_{n=1}^{N} x_n^0 - x^0 \Big\|_{X_0} = 0 \qquad \lim_{N\to\infty} \Big\| \sum_{n=1}^{N} x_n^1 - x^1 \Big\|_{X_1} = 0. \tag{3.5}$$

The element $x = x^0 + x^1$ belongs to $X_0 + X_1$ as we have that $x^0 \in X_0$ and $x^1 \in X_1$. Moreover

$$\Big\| \sum_{n=1}^{N} x_n - x \Big\|_{X_0+X_1} \leq \Big\| \sum_{n=1}^{N} x_n^0 - x^0 \Big\|_{X_0} + \Big\| \sum_{n=1}^{N} x_n^1 - x^1 \Big\|_{X_1}.$$

Taking the limit for $N \to \infty$ we have by (3.5) that $\sum_{n=1}^{N} x_n$ converges to x in $X_0 + X_1$. ■

It is easy to see that $X_0 \cap X_1 \hookrightarrow X_0, X_1 \hookrightarrow X_0 + X_1$. A third quasi-normed space X is called an intermediate space for (X_0, X_1) if there are continuous embeddings

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1. \tag{3.6}$$

When $X_1 \hookrightarrow X_0$, (3.6) simplifies to $X_1 \hookrightarrow X \hookrightarrow X_0$. As mentioned earlier we shall work with the intermediate space $X_{\theta,q}$, which we will need the K-functional to define. The K-functional for $f \in X_0 + X_1$ and $t \geq 0$ is given by

$$K(f,t) = \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}).$$

Note that the K-functional as a function of t is continuous and increasing and we have

$$\min\{1,t\}\|f\|_{X_0+X_1} \le K(f,t) \le \max\{1,t\}\|f\|_{X_0+X_1}. \tag{3.7}$$

If $X_1 \hookrightarrow X_0$ then the K-functional simplifies to

$$K(f,t) = \inf_{g\in X_1} (\|f-g\|_{X_0} + t\|g\|_{X_1}) \tag{3.8}$$

and this quantity gives some approximation properties of $f \in X_0$ by X_1. When $K(f,t) < \varepsilon$ for some t then (3.8) implies that f can be approximated with error $\|f-g\|_{X_0} < \varepsilon$ by an element $g \in X_1$ with norm $\|g\|_{X_1} < \varepsilon t^{-1}$. We can now define the space $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ for $0 < \theta < 1, 0 < q \le \infty$, consisting of the functions $f \in X_0 + X_1$ for which

$$\rho(f)_{\theta,q} = \begin{Bmatrix} \Big(\int_0^\infty (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} t^{-\theta}K(f,t), & q = \infty \end{Bmatrix} < \infty. \tag{3.9}$$

Our purpose is not to study $X_{\theta,q}$ in great detail, but we give some well-known examples, show that they are complete and give a discretization of $\rho(f)_{\theta,q}$ which we will need in the charactization of the approximation spaces mentioned at the beginning of the section.

Example 3.2

i) $(L^r, L^s)_{\theta,q} = L_{p,q}$ where $1/p = (1-\theta)/r + \theta/s$ and $L_{p,q}$ is the Lorentz space.

ii) $(L^p, W_r^p)_{\theta,q} = B_{p,q}^{\theta r}$ where W_r^p is the Sobolev space and $B_{p,q}^{\theta r}$ is the inhomogenous Besov space.

For both examples see [3, p. 196]. ⋆

To show that $X_{\theta,q}$ is complete we first show that it is an intermediate space. By using (3.7) we get that $X_{\theta,q} \hookrightarrow X_0 + X_1$ which follows from

$$\rho(f)_{\theta,q}^q \ge \int_0^\infty (t^{-\theta}\min\{1,t\}\|f\|_{X_0+X_1})^q \frac{\mathrm{d}t}{t} \ge C\|f\|_{X_0+X_1}^q. \tag{3.10}$$

That $X_0 \cap X_1 \hookrightarrow X_{\theta,q}$ follows from

$$\begin{aligned}\rho(f)_{\theta,q}^q &\le \int_0^{\frac{\|f\|_{X_0}}{\|f\|_{X_1}}} (t^{-\theta}t\|f\|_{X_1})^q \frac{\mathrm{d}t}{t} + \int_{\frac{\|f\|_{X_0}}{\|f\|_{X_1}}}^\infty (t^{-\theta}\|f\|_{X_0})^q \frac{\mathrm{d}t}{t} \\ &\le C\Big(\frac{\|f\|_{X_0}}{\|f\|_{X_1}}\Big)^{-\theta q}\|f\|_{X_0}^q \le C\|f\|_{X_0\cap X_1}^q.\end{aligned}$$

Proposition 3.3
The space $X_{\theta,q}$ is a complete linear space with quasi-norm $\rho(f)_{\theta,q}$. Moreover if X_0 and X_1 are Banach spaces and $1 \le q \le \infty$ then $X_{\theta,q}$ is also a Banach space.

Proof:

First we show that $X_{\theta,q}$ is quasi-normed (normed). The triangle inequality (with $C = 1$ if X_0, X_1 are Banach spaces) and $|a|\rho(f)_{\theta,q} = \rho(af)_{\theta,q}$ follow directly from the properties of X_0 and X_1. That $f = 0$ implies $\rho(f)_{\theta,q} = 0$ follows the same way as in $X_0 + X_1$, because we have $K(f,t) = 0$. For the completeness we note two things. From (3.7) we have that as a quasi-norm on $X_0 + X_1$, the K-functional for fixed t is equivalent to $\|\cdot\|_{X_0+X_1} = K(\cdot, 1)$. Secondly from (3.10) we get that $X_{\theta,q} \hookrightarrow X_0 + X_1$. Take a Cauchy sequence $\{f_n\}_{n\in N}$ in $X_{\theta,q}$. We then have that there exists a $f \in X_0 + X_1$ such that $\lim_{n\to\infty} f_n = f$ in $X_0 + X_1$ and $K(f_n, t)$ converges pointwise to $K(f,t)$. By using Fatou's lemma we have

$$\int_0^\infty (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t} \le \liminf_{n\to\infty} \int_0^\infty (t^{-\theta}K(f_n,t))^q \frac{\mathrm{d}t}{t} < \infty,$$

as the Cauchy sequence f_n is bounded in $X_{\theta,q}$. Similary we apply Fatou's lemma to $f_m - f_n$ to get that f_n converges to f in $X_{\theta,q}$. ■

When $X_1 \hookrightarrow X_0$ then the integral in (3.9) can be taken over $[0, a]$ for any fixed $a > 0$ to get

$$\rho(f)_{\theta,q} \asymp \begin{cases} \left(\int_0^a (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{0<t\le a} t^{-\theta}K(f,t), & q = \infty \end{cases}, \tag{3.11}$$

as follows. For $t \ge a$ and $g \in X_1$ we have

$$K(f,t) \le \|f\|_{X_0} \le C(\|f-g\|_{X_0} + \|g\|_{X_0}) \le C(\|f-g\|_{X_0} + a\|g\|_{X_1}),$$

so $K(f,t) \le CK(f,a)$, $t \ge a$. This implies that

$$\int_a^\infty (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t} \le CK(f,a)^q \le CK(f,\tfrac{a}{2})^q \le C\int_{\frac{a}{2}}^a (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t}.$$

From (3.11) we can obtain discrete versions of the quasi-norm $\rho(f)_{\theta,q}$.

Proposition 3.4

Assume that $X_1 \hookrightarrow X_0$ then

$$\rho(f)_{\theta,q} \asymp \begin{cases} \left(\sum\limits_{n=N}^\infty (2^{nr\theta}K(f,2^{-nr}))^q\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{n\ge N} 2^{rn\theta}K(f,2^{-nr}), & q = \infty \end{cases} \tag{3.12}$$

and

$$\rho(f)_{\theta,q} \asymp \begin{cases} \left(\sum\limits_{n=N}^\infty (n^{r\theta}K(f,n^{-r}))^q \frac{1}{n}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{n\ge N} n^{r\theta}K(f,n^{-r}), & q = \infty, \end{cases} \tag{3.13}$$

where $N \in \mathbb{N}$ and $r > 0$ are arbitrary but fixed.

Proof:

The equation in (3.11) can be rewritten as

$$\rho(f)_{\theta,q} \asymp \Big(\sum_{n=N}^{\infty} \int_{2^{-(n+1)r}}^{2^{-nr}} (t^{-\theta}K(f,t))^q \frac{\mathrm{d}t}{t} \Big)^{1/q}. \tag{3.14}$$

Notice that

$$K(f,t) \leq \max\{1, \tfrac{t}{s}\} K(f,s),$$

which implies that $K(f,\cdot)$ satisfies

$$2^{-r}K(f,2^{-nr}) \leq K(f,t) \leq K(f,2^{-nr}), \quad 2^{-(n+1)r} \leq t \leq 2^{-nr}. \tag{3.15}$$

Consequently

$$2^{n\theta r}2^{-r}K(f,2^{nr}) \leq t^{-\theta}K(f,t) \leq 2^{\theta r}2^{n\theta r}K(f,2^{nr}), \quad 2^{-(n+1)r} \leq t \leq 2^{-nr}.$$

Which by substitution into (3.14) yields (3.12).

The other discrete version of $\rho(f)_{\theta,q}$ we get by averaging over the number of terms instead of only using the dyadic ones. Let N be fixed and observe that by the monotonicity of $K(f,t)$ we have

$$\begin{aligned} \sum_{m=2^n}^{2^{n+1}-1} \big(m^{r\theta}K(f,m^{-r})\big)^q \frac{1}{m} &\leq \sum_{m=2^n}^{2^{n+1}-1} \big(2^{(n+1)r\theta}K(f,2^{-nr})\big)^q \frac{1}{m} \\ &\leq \big(2^{(n+1)r\theta}K(f,2^{-nr})\big)^q \frac{2^n}{2^n} \\ &= C\big(2^{nr\theta}K(f,2^{-nr})\big)^q \end{aligned} \tag{3.16}$$

and also

$$\begin{aligned} \sum_{m=2^n}^{2^{n+1}-1} \big(m^{r\theta}K(f,m^{-r})\big)^q \frac{1}{m} &\geq \sum_{m=2^n}^{2^{n+1}-1} \big(2^{nr\theta}K(f,2^{-(n+1)r})\big)^q \frac{1}{m} \\ &\geq \big(2^{nr\theta}K(f,2^{-(n+1)r})\big)^q \frac{2^n}{2^{n+1}} \\ &= C\big(2^{nr\theta}K(f,2^{-(n+1)r})\big)^q. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17) we obtain

$$C \sum_{n=N+1}^{\infty} \big(2^{nr\theta}K(f,2^{-nr})\big)^q \tag{3.18}$$

$$\leq \sum_{n=N}^{\infty} \sum_{m=2^n}^{2^{n+1}-1} \big(m^{r\theta}K(f,m^{-r})\big)^q \frac{1}{m} \tag{3.19}$$

$$\leq C \sum_{n=N}^{\infty} \big(2^{nr\theta}K(f,2^{-nr})\big)^q.$$

Estimating the sum in (3.19) by the first term

$$\sum_{m=2^N}^{\infty} \left(m^{r\theta} K(f, m^{-r})\right)^q \frac{1}{m} \geq \left(2^{Nr\theta} K(f, 2^{-Nr})\right)^q \frac{1}{2^N}$$

we readily obtain the missing first term in (3.18) such that we have

$$\begin{aligned} C \sum_{n=N}^{\infty} \left(2^{nr\theta} K(f, 2^{-nr})\right)^q &\leq 2 \sum_{n=2^N}^{\infty} \left(n^{r\theta} K(f, n^{-r})\right)^q \frac{1}{n} \\ &\leq C \sum_{n=N}^{\infty} \left(2^{nr\theta} K(f, 2^{-nr})\right)^q \end{aligned}$$

which yields (3.13) for $n = 2^N$ by using (3.12). For general n we find m such that $2^m \leq n \leq 2^{m+1}$ and again apply (3.12). ■

If we have $X_1 \hookrightarrow X_0$, then equation (3.12) gives the following embedding properties of $X_{\theta,q}$

$$X_{\theta,s} \hookrightarrow X_{\alpha,r} \tag{3.20}$$

if $\theta > \alpha$ or if $\theta = \alpha$ and $q \leq r$ which we now show. The case $\theta = \alpha$ and $q \leq r$ follows directly from the embeddings of l^p and the case $\theta > \alpha$ from

$$\begin{aligned} \left(\sum_{n=N}^{\infty} (2^{nr\alpha} K(f, 2^{-nr}))^s\right)^{1/s} &\leq \left(\sum_{n=N}^{\infty} 2^{nr(\alpha-\theta)s}\right)^{1/s} \sup_{n \geq N} 2^{nr\theta} K(f, 2^{-nr}) \\ &\leq C \left(\sum_{n=N}^{\infty} (2^{nr\theta} K(f, 2^{-nr}))^q\right)^{1/q}. \end{aligned}$$

Before we introduce the approximation spaces and characterize them by $X_{\theta,q}$ we need to impose some conditions on the subspaces $\Phi_n, n \in \mathbb{N}_0$ by which we shall approximate $f \in X$. We will also introduce the Jackson and Bernstein inequalities needed.
We assume that X is a complete linear space with quasi-norm $\|\cdot\|_X$ and that the sequence of subspaces $\Phi = \{\Phi_n\}_{n=0}^{\infty}$ have the properties listed below

$$\left.\begin{array}{ll} \text{i)} & 0 \in \Phi_n;\ \Phi_0 = \{0\}, \\ \text{ii)} & \Phi_n \subset \Phi_{n+1}, \\ \text{iii)} & a\Phi_n = \Phi_n \text{ for all } a \neq 0, \\ \text{iv)} & \Phi_n + \Phi_n \subset \Phi_{cn} \text{ for some fixed constant } c \in \mathbb{N}, \\ \text{v)} & \cup \Phi_n \text{ is dense in } X. \end{array}\right\} \tag{3.21}$$

If the subspaces Φ_n are linear then one has $c = 1$ in iii). Property v) is not necessary for the characterization, but seems natural. For every $f \in X$ we define the error of approximation $\sigma_n(f)_X$ for f by the subspace Φ_n as

$$\sigma_n(f)_X = \inf_{\phi \in \Phi_n} \|f - \phi\|_X, \quad n \in \mathbb{N} \text{ and } \sigma_0(f) = \|f\|_X.$$

We also denote $g_n \in \Phi_n$ a near-best approximation to f from Φ_n if

$$\|f - g_n\|_X \leq C\sigma_n(f)_X.$$

We shall futhermore assume that there exists a complete linear space Y with quasi-norm $\|\cdot\|_Y$ that is continuously embedded in X such that the following Jackson and Bernstein inequalities hold for $n \in \mathbb{N}$

$$\sigma_n(f)_X \leq Cn^{-r}\|f\|_Y, \qquad f \in Y, \tag{J}$$

$$\|\phi\|_Y \leq Cn^{r}\|\phi\|_X, \qquad \phi \in \Phi_n. \tag{B}$$

The conditions (3.21), (J) and (B) give us the following proposition which will be the key to characterizing the approximation spaces as $X_{\theta,q}$.

Lemma 3.5
Assume that X, Y are a pair and that $Y \hookrightarrow X$. Furthermore assume that $r > 0$ and that $\{\Phi_n\}_{n=0}^{\infty}$ satisfies the conditions from (3.21).

i) If Jackson inequality (J) is satisfied for $n \in \mathbb{N}$, then

$$\sigma_n(f)_X \leq CK(f, n^{-r}), \qquad f \in X, \quad n \in \mathbb{N}. \tag{3.22}$$

ii) If Bernstein inequality (B) is satisfied for $n \in \mathbb{N}$, then with μ sufficiently small depending on Y

$$K(f, 2^{-mr}) \leq C2^{-mr}\left(\sum_{k=0}^{m}\left(2^{kr}\sigma_{2^{k-1}}(f)_X\right)^{\mu}\right)^{1/\mu}, \qquad f \in X, \quad n \in \mathbb{N}, \tag{3.23}$$

where $\sigma_{2^{-1}} = \sigma_0$.

Proof:
i) Take $g \in Y$. From the triangle inequality and Jacksons inequality (J) we have that

$$\sigma_n(f)_X \leq C(\|f-g\|_X + \sigma_n(g)_X) \leq C(\|f-g\|_X + n^{-r}\|g\|_Y).$$

Taking the infimum over all $g \in Y$ we find by (3.8) that (3.22) is fulfilled.
ii) First we intend to prove (3.23) for $n = 2^m$. Let ϕ_k be a near-best approximation to f from Φ_{2^k}, i.e. $\|f - \phi_k\|_X \leq C\sigma_{2^k}(f)_X$, $k \in \mathbb{N}_0$. Now define $\phi_k' = \phi_k - \phi_{k-1}$, $k \in \mathbb{N}$, with $\phi_{-1} = 0$. This gives

$$\|\phi_k'\|_X \leq C\|f - \phi_k\|_X + C\|f - \phi_{k-1}\|_X \leq C\sigma_{2^{k-1}}(f), \qquad \text{for } k \in \mathbb{N}_0, \tag{3.24}$$

where we use (3.21) i) in the case $k = 0$. From the definition of ϕ_k' we have that $\sum_{k=0}^{m}\phi_k' = \phi_m$, using Lemma A.7 we furthermore have that $\|\phi_m\|_Y \leq C(\sum_{k=0}^{m}\|\phi_k'\|_Y^{\mu})^{1/\mu}$ which yields

$$\begin{aligned} K(f, 2^{-mr}) &\leq \|f - \phi_m\|_X + 2^{-mr}\|\phi_m\|_Y \\ &\leq C\sigma_{2^m}(f)_X + 2^{-mr}\left(\sum_{k=0}^{m}\|\phi_k'\|_Y^{\mu}\right)^{1/\mu}. \end{aligned}$$

Then by the use of (B) and (3.24), as we have that $\phi'_k \in \Phi_{c2^k}$ by property ii)-iv) from (3.21), we find that

$$\begin{aligned} &\leq C\sigma_{2^m}(f)_X + C2^{-mr}\Big(\sum_{k=0}^{m}(2^{kr}\|\phi'_k\|_X)^\mu\Big)^{1/\mu} \\ &\leq C2^{-mr}\Big(\sum_{k=0}^{m}(2^{kr}\sigma_{2^{k-1}}(f)_X)^\mu\Big)^{1/\mu}. \end{aligned} \tag{3.25}$$

■

We are now ready to define the approximation spaces and give their charaterization as interpolation spaces. Let

$$\|f\|_{\mathscr{A}_q^\alpha} = \begin{cases} \Big(\sum_{n=1}^{\infty}\big(n^\alpha\sigma_{n-1}(f)_X\big)^q\frac{1}{n}\Big)^{1/q}, & \text{if } q<\infty \\ \sup_{n\geq 1}\big(n^\alpha\sigma_{n-1}(f)_X\big), & \text{if } q=\infty, \end{cases} \tag{3.26}$$

and define the approximation spaces $\mathscr{A}_q^\alpha(X,\Phi) = \mathscr{A}_q^\alpha$ as all $f \in X$ for which $\|f\|_{\mathscr{A}_q^\alpha}$ is finite. $\mathscr{A}_q^\alpha$ can be seen as the space of functions for which the error of approximation $\sigma_{n-1}(f)_X$ decays at the rate $n^{-\alpha}$ with q as a fine tuning parameter. Some details are worth noting. If $\|f\|_{\mathscr{A}_q^\alpha} = 0$ then in particular one has that $\sigma_0(f)_X = 0$ proving that $f = 0$, from (3.21) iii) and (3.21) iv) one has that $\sigma_{cn}(af+bg) \leq C(a\sigma_n(f)+b\sigma_n(g))$, $f,g \in \mathscr{A}_q^\alpha$, $a,b \in \mathbb{C}$ showing that $\|\cdot\|_{\mathscr{A}_q^\alpha}$ is a quasi-norm. By the same means as in the proof of Proposition 3.3 we get that $\mathscr{A}_q^\alpha$ is complete. Using the same technique as was used to achieve (3.20) we have the following continuous embeddings

$$\mathscr{A}_{q_1}^{\alpha_1} \hookrightarrow \mathscr{A}_{q_2}^{\alpha_2} \tag{3.27}$$

if $\alpha_2 < \alpha_1$ or if $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$. By omitting the term $n = 1$ in (3.26) we get the quasi-seminorm $|\cdot|_{\mathscr{A}_q^\alpha}$ and by the same technique as was used in the proof of Proposition 3.4 we have the equivalence

$$|f|_{\mathscr{A}_q^\alpha} \asymp \begin{cases} \Big(\sum_{n=1}^{\infty}\big(2^{n\alpha}\sigma_{2^{n-1}}(f)_X\big)^q\Big)^{1/q}, & \text{if } q<\infty \\ \sup_{n\geq 1}\big(2^{n\alpha}\sigma_{2^{n-1}}(f)_X\big), & \text{if } q=\infty. \end{cases} \tag{3.28}$$

Using the discrete versions of the norms we have the following proposition.

Proposition 3.6
Assume that X, Y is a pair and that $Y \hookrightarrow X$. Futhermore assume that Φ are subsets of X satisfying the conditions of (3.21). If the Jackson inequality (J) and Bernstein inequality (B) are valid for the spaces X and Y with $r > 0$, then for $0 < \alpha < r$ and $0 < q \leq \infty$ we have that

$$\mathscr{A}_q^\alpha(X,\Phi) = (X,Y)_{\alpha/r,q}. \tag{3.29}$$

Proof:

For the sake of notation define $Z = (X,Y)_{\alpha/r,q}$ and $\|f\|_Z = \rho(f)_{\alpha/r,q}$, with equivalent discrete representation (3.12) with $\theta = \alpha/r$. We first show that $Z \hookrightarrow \mathscr{A}_q^\alpha$ by using (J). The conditions of Lemma 3.5 i) are satisfied such that we have that $\sigma_{2^n}(f)_X \leq CK(f,2^{-nr})$ yielding $|f|_{\mathscr{A}_q^\alpha} \leq C\|f\|_Z$. Futhermore using that Z is an intermediate space we have $\|f\|_X \leq C\|f\|_Z$, resulting in $\|f\|_{\mathscr{A}_q^\alpha} \leq \|f\|_Z$.

Next we show that $\mathscr{A}_q^\alpha \hookrightarrow Z$ by using (B). Set $b_n = K(f,2^{-nr})$ for $n \geq 0$ and $b_n = 0$ for $n < 0$ and $a_n = \sigma_{2^{n-1}}(f)_X$ for $n \geq 0$ and $a_n = 0$ for $n < 0$, then by Lemma 3.5 ii) we have that

$$b_n \leq C2^{-nr}\Big(\sum_{j=-\infty}^{n}(2^{jr}a_j)^\mu\Big)^{1/\mu}.$$

So by Lemma A.5 we obtain

$$\Big(\sum_{n=0}^{\infty}\big(2^{n\alpha}K(f,2^{-nr})\big)^q\Big)^{1/q} \leq C\Big(\sum_{n=0}^{\infty}\big(2^{n\alpha}\sigma_{2^{n-1}}(f)_X\big)^q\Big)^{1/q}$$

for all $\alpha < r$, which implies that $\|f\|_Z \leq C\|f\|_{\mathscr{A}_q^\alpha}$. ■

n-term approximation from wavelet bases

We now apply the characterization to approximation spaces with n-term approximation by a basis B from Proposition 2.6. For appropriate indices we set X to $\dot{F}^s_{p,t}$ or $\dot{B}^s_{p,t}$ and $Y = \dot{B}^\gamma_{\tau,r}$ and let B be a basis for X and Y with the related norm equivalence (2.21). By Proposition 3.6 we need to show that (3.21) and the Jackson and Bernstein inequalities are satisfied. We define Φ_n as

$$\Phi_n = \Big\{f : f = \sum_{I\in H}\sum_{e\in E} a_I^e\psi_I^e, a_I^e \in \mathbb{C}, \psi_I^e \in B, \#H = n\Big\}$$

and note that (3.21) i)-iii) follow directly. Property iv) follows with $c = 2$ and v) is fulfilled because B is a basis for $\dot{F}^s_{p,t}$ and $\dot{B}^s_{p,t}$. To simplify the notation we denote

$$A_I(f) = \sum_{e\in E}\langle f,\widetilde{\psi}_I^e\rangle\,\psi_I^e \quad\text{and}\quad a_I(f) = \sum_{e\in E}|\langle f,\widetilde{\psi}_I^e\rangle|$$

which gives

$$\|f\|_{\dot{F}^s_{p,t}} \asymp \|a_I(f)\|_{\dot{f}^s_{p,t}}$$

and similarly for $\dot{B}^s_{p,t}$. As B is a basis in the sense that the coefficients in $\dot{f}^s_{p,t}$ or $\dot{b}^s_{p,t}$ are unique we also have

$$\Phi_n = \Big\{f : f = \sum_{I\in H} A_I(f), \#H = n\Big\},$$

which will be crucial in showing the Bernstein inequalities. Part of the setup for applying Proposition 3.6 are also the embeddings $\dot{B}^{\gamma}_{\tau,\tau} \hookrightarrow \dot{F}^{s}_{p,t}$ and $\dot{B}^{\gamma}_{\tau,r} \hookrightarrow \dot{B}^{s}_{p,t}$ which will follow by the proof of the Jackson inequalities. We begin with the Jackson inequality for the Triebel-Lizorkin space.

Proposition 3.7
Let $0 < p < \infty$, $0 < t \leq \infty$ and $s < \gamma$. If furthermore $1/\tau = (\gamma - s)/d + 1/p$ then for $f \in \dot{B}^{\gamma}_{\tau,\tau}$ we have

$$\sigma_n(f)_{\dot{F}^s_{p,t}} \leq C n^{-(\gamma-s)/d} \|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}. \tag{3.30}$$

Proof:
We set $\tilde{a}_I(f) = a_I(f)|I|^{-\gamma/d+1/\tau-1/2}$ and $M = \|\tilde{a}_I(f)\|_{\ell^\tau} = \|f\|_{B^{\gamma}_{\tau,\tau}}$. For $j \in \mathbb{Z}$, $\varepsilon > 0$ we define

$$\Lambda_{j,\varepsilon} = \{I : 2^{-j}\varepsilon < \tilde{a}_I(f) \leq 2^{-j+1}\varepsilon\}$$

and $S_{j,\varepsilon} = \sum_{I \in \Lambda_{j,\varepsilon}} A_I(f)$. We will approximate f by $T_{k,\varepsilon} = \sum_{j \leq k} S_{j,\varepsilon}$. Since $(\tilde{a}_I(f))_{I \in D} \in \ell^\tau$ it follows that for $r > 0$ we have

$$\#\{I : \tilde{a}_I(f) \geq r\} \leq M^\tau r^{-\tau}$$

which gives that $T_{k,\varepsilon} \in \Phi_N$, $N = \left\lfloor M^\tau 2^{k\tau} \varepsilon^{-\tau} \right\rfloor$. To prove (3.30) we will show that

$$\|f - T_{k,\varepsilon}\|_{\dot{F}^s_{p,t}} \leq C(M^\tau 2^{k\tau}\varepsilon^{-\tau})^{-(\gamma-s)/d} \|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}.$$

and the result for general $n \in \mathbb{N}$ will then follow from choosing $\varepsilon = Mn^{-1/\tau}$ and $k = 0$. First we take $t \geq p$ and get

$$\begin{aligned}
\|f - T_{k,\varepsilon}\|^p_{\dot{F}^s_{p,t}} &= \int_{\mathbb{R}^d} \Big(\sum_{j \geq k+1} \sum_{I \in \Lambda_{j,\varepsilon}} (a_I(f)|I|^{-s/d-1/2}\chi_I)^t \Big)^{p/t} \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \sum_{j \geq k+1} \sum_{I \in \Lambda_{j,\varepsilon}} (a_I(f)|I|^{-s/d-1/2}\chi_I)^p \mathrm{d}x \\
&\leq 2^p \varepsilon^p \sum_{j \geq k+1} 2^{-jp} \sum_{I \in \Lambda_{j,\varepsilon}} \int_{\mathbb{R}^d} (|I|^{-1/p}\chi_I)^p \mathrm{d}x \\
&= 2^p \varepsilon^p \sum_{j \geq k+1} 2^{-jp} \#\Lambda_{j,\varepsilon} \\
&\leq 2^p \varepsilon^{p-\tau} M^\tau \sum_{j \geq k+1} 2^{-j(p-\tau)} \leq C M^\tau (2^{-k}\varepsilon)^{p-\tau}.
\end{aligned}$$

Next for $t < p$ we have

$$\begin{aligned}
\|f - T_k\|^p_{\dot{F}^s_{p,t}} &= \int_{\mathbb{R}^d} \Big(\sum_{j \geq k+1} \sum_{I \in \Lambda_{j,\varepsilon}} (a_I(f)|I|^{-s/d-1/2}\chi_I)^t \Big)^{p/t} \mathrm{d}x \\
&\leq 2^p \varepsilon^p \int_{\mathbb{R}^d} \Big(\sum_{j \geq k+1} \sum_{I \in \Lambda_{j,\varepsilon}} (2^{-j}|I|^{-1/p}\chi_I)^t \Big)^{p/t} \mathrm{d}x.
\end{aligned}$$

Since $p > \tau$ we can choose $\delta > 0$ sufficiently small such that $(t-\delta)p/t > \tau$. Using Hölder's inequality with $p/(p-t)$ and p/t we get

$$
\begin{aligned}
&= 2^p \varepsilon^p \int_{\mathbb{R}^d} \Big(\sum_{j\geq k+1} 2^{-j\delta} \sum_{I\in\Lambda_{j,\varepsilon}} 2^{-j(t-\delta)} |I|^{-t/p} \chi_I \Big)^{p/t} \mathrm{d}x \\
&\leq 2^p \varepsilon^p \int_{\mathbb{R}^d} \Big(\sum_{j\geq k+1} 2^{-j\delta p/(p-t)} \Big)^{\frac{p}{t}\frac{p-t}{p}} \\
&\qquad\qquad \cdot \sum_{j\geq k+1} \Big(\sum_{I\in\Lambda_{j,\varepsilon}} 2^{-j(t-\delta)} |I|^{-t/p} \chi_I \Big)^{p/t} \mathrm{d}x \\
&\leq C\varepsilon^p 2^{-k\delta p/t} \int_{\mathbb{R}^d} \sum_{j\geq k+1} \Big(\sum_{I\in\Lambda_{j,\varepsilon}} 2^{-j(t-\delta)} |I|^{-t/p} \chi_I \Big)^{p/t} \mathrm{d}x \\
&= C\varepsilon^p 2^{-k\delta p/t} \sum_{j\geq k+1} 2^{-j(t-\delta)p/t} \int_{\mathbb{R}^d} \Big(\sum_{I\in\Lambda_{j,\varepsilon}} |I|^{-t/p} \chi_I \Big)^{p/t} \mathrm{d}x.
\end{aligned}
$$

For each finite set of dyadic cubes Λ we let $I_\Lambda(x)$ denote the smallest cube in Λ that contains x.

$$
\begin{aligned}
&\leq C\varepsilon^p 2^{-k\delta p/t} \sum_{j\geq k+1} 2^{-j(t-\delta)p/t} \int_{\mathbb{R}^d} |I_{\Lambda_{j,\varepsilon}}(x)|^{-1} \mathrm{d}x \\
&\leq C\varepsilon^p 2^{-k\delta p/t} \sum_{j\geq k+1} 2^{-j(t-\delta)p/t} \# \Lambda_{j,\varepsilon} \\
&\leq C M^\tau \varepsilon^{p-\tau} 2^{-k\delta p/t} \sum_{j\geq k+1} 2^{-j((t-\delta)/t-\tau)} \\
&\leq C M^\tau (2^{-k}\varepsilon)^{p-\tau}.
\end{aligned}
$$

In both cases we have proved that

$$
\begin{aligned}
\|f - T_k\|_{\dot{F}^s_{p,t}} &\leq C(M^\tau 2^{k\tau} \varepsilon^{-\tau})^{-(p-\tau)/(p\tau)} \|f\|_{\dot{B}^\gamma_{\tau,\tau}} \\
&= C(M^\tau 2^{k\tau} \varepsilon^{-\tau})^{-(\gamma-s)/d} \|f\|_{\dot{B}^\gamma_{\tau,\tau}}.
\end{aligned}
$$

■

We note that $\dot{B}^\gamma_{\tau,\tau} \hookrightarrow \dot{F}^s_{p,t}$ follows by choosing $M2^k\varepsilon^{-1} < 1$, so that $T_k = 0$. We proceed with the Jackson inequality for the Besov space which will use the preceding Jackson inequality for the Triebel-Lizorkin space.

Proposition 3.8
Let $0 < p < \infty$, $0 < t \leq \infty$ and $s < \gamma$. If furthermore $1/\tau - 1/p = 1/r - 1/t = (\gamma - s)/d$ then for $f \in \dot{B}^\gamma_{\tau,r}$ we have

$$
\sigma_n(f)_{\dot{B}^s_{p,t}} \leq C n^{-(\gamma-s)/d} \|f\|_{\dot{B}^\gamma_{\tau,r}}. \tag{3.31}
$$

Proof:

Let $\varepsilon > 0$ and define for every $j \in \mathbb{Z}$

$$\Lambda_{j,\varepsilon} = \{m \in \mathbb{Z} : 2^j\varepsilon < \Big(\sum_{I\in D_m} \widetilde{a}_I(f)^\tau\Big)^{1/\tau} \leq 2^{j+1}\varepsilon\},$$

where $\widetilde{a}_I(f)$ is defined as in the previous proof to be $|I|^{-\gamma/d+1/\tau-1/2}a_I(f)$. Using $1/\tau - 1/p = (\gamma - s)/d$ we note that

$$\|f\|_{\dot{B}^s_{p,t}} \asymp \Big(\sum_{m\in\mathbb{Z}} \Big(\widetilde{a}_I(f)^p\Big)^{t/p}\Big)^{1/t}.$$

We now apply Proposition 3.7 on $f_m = \sum_{I\in D_m} A_I(f)$ for $m \in \mathbb{Z}$. Then for every $n \in \mathbb{Z}_+$ we can find subsets $K_n^m \subset D_m$ with cardinality not exceding n, such that

$$\begin{aligned}\Big(\sum_{I\in D_m\setminus K_n^m} \widetilde{a}_I(f)^p\Big)^{1/p} &\leq C\|f_m - S_n^m\|_{\dot{F}^s_{p,p}}\\ &\leq Cn^{-(\gamma-s)/d}\|f_m - S_n^m\|_{\dot{B}^\gamma_{\tau,\tau}}\\ &\leq Cn^{(\gamma-s)/d}\Big(\sum_{I\in D_m} \widetilde{a}_I(f)^\tau\Big)^{1/\tau}, \end{aligned} \tag{3.32}$$

where $S_n^m = \sum_{I\in K_n^m} A_I(f) \in \Phi_n$. Define $T_\varepsilon = \sum_{j\geq 0}\sum_{m\in\Lambda_{j,\varepsilon}} S^m_{\lfloor 2^{jr}\rfloor}$. By construction we have that $T_\varepsilon \in \Phi_{N_\varepsilon}$ where $N_\varepsilon \leq \sum_{j\geq 0} \#\Lambda_{j,\varepsilon}2^{jr}$. We have by definition of $\Lambda_{j,\varepsilon}$ that

$$\|f\|^r_{\dot{B}^\gamma_{\tau,r}} = \sum_{j\in\mathbb{Z}}\sum_{m\in\Lambda_{j,\varepsilon}} \Big(\sum_{I\in D_m} \widetilde{a}_I(f)^\tau\Big)^{r/\tau} \geq \sum_{j\in\mathbb{Z}} \#\Lambda_{j,\varepsilon}(2^j\varepsilon)^r,$$

whereby the estimate on N_ε becomes $N_\varepsilon \leq \varepsilon^{-r}\|f\|^r_{\dot{B}^\gamma_{\tau,r}}$. We intend to prove that

$$\sigma_{N_\varepsilon}(f)_{\dot{B}^s_{p,t}} \leq C(\varepsilon^{-r}\|f\|^r_{\dot{B}^\gamma_{\tau,r}})^{-(\gamma-s)/d}\|f\|_{\dot{B}^\gamma_{\tau,r}}.$$

The result for $n \in \mathbb{N}$ will follow by choosing $\varepsilon = n^{-1/r}\|f\|_{\dot{B}^\gamma_{\tau,r}}$ and the monotonicity of $\sigma_n(f)$. We estimate $\sigma_{N_\varepsilon}(f)^t_{\dot{B}^s_{p,t}}$ by splitting into two cases

$$\begin{aligned}\sigma_{N_\varepsilon}(f)^t_{\dot{B}^s_{p,t}} &\leq C\Big(\Big\|\sum_{j<0}\sum_{m\in\Lambda_{j,\varepsilon}} f_m\Big\|^t_{\dot{B}^s_{p,t}} + \Big\|\sum_{j\geq 0}\sum_{m\in\Lambda_{j,\varepsilon}} (f_m - S^m_{\lfloor 2^{jr}\rfloor})\Big\|^t_{\dot{B}^s_{p,t}}\Big)\\ &= C(A_1^t + A_2^t).\end{aligned} \tag{3.33}$$

We estimate A_1 by using the definition of $\Lambda_{j,\varepsilon}$ and the assumptions that

$p > \tau$ and $t > r$

$$\begin{aligned} A_1^t &\leq C \sum_{j<0} \sum_{m \in \Lambda_{j,\varepsilon}} \Big(\sum_{I \in D_m} \widetilde{a}_I(f)^\tau \Big)^{t/\tau} \\ &\leq C \sum_{j<0} \#\Lambda_{j,\varepsilon}(2^j\varepsilon)^t \\ &\leq C\varepsilon^{t-r} \sum_{j<0} \#\Lambda_{j,\varepsilon}(2^j\varepsilon)^r \\ &\leq C\varepsilon^{t-r} \sum_{m \in \Lambda_{j,\varepsilon}} \Big(\sum_{I \in D_m} \widetilde{a}_I(f)^\tau \Big)^{r/\tau} \\ &\leq C\varepsilon^{t-r} \|f\|^r_{\dot{B}^\gamma_{\tau,r}} . \end{aligned} \tag{3.34}$$

For A_2 we use (3.32) and the fact that $1/r - 1/t = (\gamma - s)/d$ to obtain

$$\begin{aligned} A_2^t &\leq C \sum_{j \geq 0} \sum_{m \in \Lambda_{j,\varepsilon}} \Big(\sum_{I \in D_m \setminus K^m_{\lfloor 2^j r \rfloor}} \widetilde{a}_I(f)^p \Big)^{t/p} \\ &\leq C \sum_{j \geq 0} \sum_{m \in \Lambda_{j,\varepsilon}} 2^{jrt(\gamma-s)/d} \Big(\sum_{I \in D_m} \widetilde{a}_I(f)^\tau \Big)^{t/\tau} \\ &= C \sum_{j \geq 0} \sum_{m \in \Lambda_{j,\varepsilon}} \Big(\sum_{I \in D_m} \widetilde{a}_I(f)^\tau \Big)^{t/\tau}. \end{aligned}$$

Now using the same technique as in the estimate for A_1 we have that

$$\begin{aligned} &\leq C \sum_{j \geq 0} \sum_{m \in \Lambda_{j,\varepsilon}} 2^{-j(t-r)}(2^j\varepsilon)^t \\ &\leq C \sum_{j \geq 0} \#\Lambda_{j,\varepsilon} 2^{jr}\varepsilon^t \\ &\leq C\varepsilon^{t-r} \|f\|^r_{\dot{B}^\gamma_{\tau,r}}. \end{aligned} \tag{3.35}$$

Recalling again that $1/r - 1/t = (\gamma - s)/d$ we observe that the estimates (3.34) and (3.35) inserted into (3.33) yields

$$\sigma_{N_\varepsilon}(f)_{\dot{B}^s_{p,t}} \leq C(\varepsilon^{-r}\|f\|^r_{\dot{B}^\gamma_{\tau,r}})^{-(\gamma-s)/d}\|f\|_{\dot{B}^\gamma_{\tau,r}} . \tag{3.36}$$

■

As before we note that $\dot{B}^\gamma_{\tau,r} \hookrightarrow \dot{B}^s_{p,t}$ follows by choosing $\varepsilon^{-1}\|f\|_{\dot{B}^\gamma_{\tau,r}} < 1$. It only remains to establish the Bernstein inequalities. We prove it for the Triebel-Lizorkin space as the Besov space follows in a similar vein.

Proposition 3.9
Let $0 < p < \infty$, $0 < t \leq \infty$ and $s < \gamma$. Define τ by the equation $1/\tau = (\gamma - s)/d + 1/p$, then for every $S \in \Phi_n$

$$\|S\|_{\dot{B}^{\gamma}_{\tau,\tau}} \leq Cn^{(\gamma-s)/d}\|S\|_{\dot{F}^{s}_{p,t}}.$$

Proof:
Let $S = \sum_{I\in\Lambda} A_I(S) \in \Phi_n$ and consider the following

$$\begin{aligned}
\|S\|^{\tau}_{\dot{B}^{\gamma}_{\tau,\tau}} &= \sum_{I\in\Lambda} (|I|^{-\gamma/d+1/\tau-1/2} a_I(S))^{\tau} \\
&= \int_{\mathbb{R}^d} \sum_{I\in\Lambda} |I|^{\tau(s-\gamma)/d} (|I|^{-s/d-1/2} a_I(S))^{\tau} \chi_I \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \Big(\sup_{I\in D} (|I|^{-s/d-1/2} a_I(S))^{\tau} \chi_I \Big) \sum_{I\in\Lambda} |I|^{\tau(s-\gamma)/d} \chi_I \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \Big(\sum_{I\in D} (|I|^{-s/d-1/2} a_I(S)) \chi_I)^{t} \Big)^{\tau/t} \sum_{I\in\Lambda} |I|^{\tau(s-\gamma)/d} \chi_I \mathrm{d}x,
\end{aligned}$$

where we have used Hölder's inequality and the embedding properties of the $\ell^{t/\tau}$-norm. By using Hölder's inequality once more with exponents p/τ and $p/(p-\tau)$ we find

$$\begin{aligned}
&\leq \Big\| \sum_{I\in D} (|I|^{-s/d-1/2} a_I(S)) \chi_I)^{t} \Big)^{1/t} \Big\|^{\tau}_{L^p} \\
&\qquad \cdot \Big(\int_{\mathbb{R}^d} \Big(\sum_{I\in\Lambda} |I|^{\tau(s-\gamma)/d} \chi_I \Big)^{p/p-\tau} \mathrm{d}x \Big)^{p-\tau/p} \\
&\leq C\|S\|^{\tau}_{\dot{F}^{s}_{p,t}} \Big(\int_{\mathbb{R}^d} |I_{\Lambda}(x)|^{\tau p(s-\gamma)/d(p-\tau)} \mathrm{d}x \Big)^{(p-\tau)/p},
\end{aligned}$$

where $I_{\Lambda}(x)$ denotes the smallest cube in Λ that contains x. From the equation defining τ we have

$$\begin{aligned}
&= C\|S\|^{\tau}_{\dot{F}^{s}_{p,t}} \Big(\int_{\mathbb{R}^d} |I_{\Lambda}(x)|^{-1} \mathrm{d}x \Big)^{(p-\tau)/p} \\
&\leq C(\#\Lambda)^{(p-\tau)/p} \|S\|^{\tau}_{\dot{F}^{s}_{p,t}} \\
&= Cn^{\tau(\gamma-s)/d} \|S\|^{\tau}_{\dot{F}^{s}_{p,t}}.
\end{aligned}$$

■

Proposition 3.10
Let $0 < p < \infty$, $0 < t \leq \infty$ and $s < \gamma$. Define τ and r by the equation $1/\tau - 1/p = 1/r - 1/t = (\gamma - s)/d$. Then for $S \in \Phi_n$ one has

$$\|S\|_{\dot{B}^{\gamma}_{\tau,r}} \leq Cn^{(\gamma-s)/d}\|S\|_{\dot{B}^{s}_{p,t}}.$$ □

Having proved (3.21) for our choice of Φ and the Jackson and Bernstein inequalities we now have the desired characterization of the approximation spaces by interpolation spaces.

Theorem 3.11
Let $0 < p < \infty$, $0 < q,t \leq \infty$, $\gamma,s \in \mathbb{R}$. If $0 < \alpha < \gamma - s$ and τ is defined as $1/\tau = (\gamma - s)/d + 1/p$ then

$$\mathscr{A}_q^{\alpha/d}(\dot{F}^s_{p,t}) = (\dot{F}^s_{p,t}, \dot{B}^\gamma_{\tau,\tau})_{\alpha/(\gamma-s),q}.$$

□

Theorem 3.12
Let $0 < p < \infty$, $0 < q,t \leq \infty$, $\gamma,s \in \mathbb{R}$. If $0 < \alpha < \gamma - s$ and τ and r is defined as $1/\tau - 1/p = 1/r - 1/t = (\gamma - s)/d$ then

$$\mathscr{A}_q^{\alpha/d}(\dot{B}^s_{p,t}) = (\dot{B}^s_{p,t}, \dot{B}^\gamma_{\tau,r})_{\alpha/(\gamma-s),q}.$$

□

Note that if we had taken a decomposition system from Theorem 2.5 instead of a basis the Jackson inequalities would still hold, since they only require the system to be spanning and have the related norm equivalences. The proof of the Bernstein inequalities on the other hand require norm equivalence for all n-term approximations. So for decomposition systems we only have the embeddings

$$(\dot{F}^s_{p,t}, \dot{B}^\gamma_{\tau,\tau})_{\alpha/(\gamma-s),q} \hookrightarrow \mathscr{A}_q^{\alpha/d}(\dot{F}^s_{p,t})$$
$$(\dot{B}^s_{p,t}\dot{B}^\gamma_{\tau,r})_{\alpha/(\gamma-s),q} \hookrightarrow \mathscr{A}_q^{\alpha/d}(\dot{B}^s_{p,t}).$$

We end this section with a result that follows from the Jackson and Bernstein estimates in the Triebel-Lizorkin space and the following lemma.

Lemma 3.13
Let $s \in \mathbb{R}$, $0 < p, \gamma < \infty$ and $1/q = \alpha/d + 1/p$. If $0 < \alpha < \gamma$ and $1/\tau = \gamma/d + 1/p$ then

$$\left(\dot{B}^s_{p,p}, \dot{B}^{s+\gamma}_{\tau,\tau}\right)_{\alpha/\gamma,q} = \dot{B}^{s+\alpha}_{q,q}.$$

Proof:
Assume that T is a linear mapping of a distribution f to its wavelet coefficients by the form

$$T : f \to (|I|^{-s/d+1/p-1/2} a_I(f))_I.$$

Especially one has for any $f \in \dot{B}^s_{p,p}$ that

$$\begin{aligned}\|Tf\|^p_{\ell^p} &= \sum_{I\in D}(|I|^{-s/d+1/p-1/2}a_I(f))^p \\ &= \|f\|^p_{\dot{B}^s_{p,p}}.\end{aligned}$$

As we have assumed that $1/\tau = \gamma/d + 1/p$ we also have that

$$\begin{aligned}\|Tf\|^\tau_{\ell^\tau} &= \sum_{I\in D}(|I|^{-(s+\gamma)/d+1/\tau-1/2}a_I(f))^\tau \\ &= \|f\|^\tau_{\dot{B}^{s+\gamma}_{\tau,\tau}}.\end{aligned}$$

By this we have that $f \in (\dot{B}^s_{p,p}, \dot{B}^{s+\gamma}_{\tau,\tau})_{\theta,q}$ if and only if we have that $Tf \in (\ell^p, \ell^\tau)_{\theta,q}$. If $1/q = (1-\theta)/p + \theta/\tau$ then $(\ell^p, \ell^\tau)_{\theta,q} = \ell^q$, this follows by using example 3.2 with the counting measure and the fact that $\ell_{q,q} = \ell^q$. If we choose $\theta = \alpha/\gamma$ then $1/q = \alpha/d + 1/p$ and we have

$$\begin{aligned}\|Tf\|_{(\ell^p,\ell^\tau)_{\alpha/\gamma,q}} &= \|Tf\|_{\ell^q} \\ &= \Big(\sum_{I\in D}(|I|^{-s/d+1/p-1/2}a_I(f))^q\Big)^{1/q} \\ &= \|f\|_{\dot{B}^{s+\alpha}_{q,q}},\end{aligned}$$

which proves that $f \in (\dot{B}^s_{p,p}, \dot{B}^{s+\gamma}_{\tau,\tau})_{\alpha/\gamma,q}$ if and only if $f \in \dot{B}^{s+\alpha}_{q,q}$. ■

The following proposition shows that the approximation space does not depend on the fine tuning parameter t of the Triebel-Lizorkin space.

Proposition 3.14

Let $0 < p < \infty$, $0 < q \leq \infty$, $\alpha > 0$ and $s \in \mathbb{R}$. Then for $0 < t_1, t_2 \leq \infty$ we have

$$\mathscr{A}^{\alpha/d}_q(\dot{F}^s_{p,t_1}) = \mathscr{A}^{\alpha/d}_q(\dot{F}^s_{p,t_2}) \ .$$

Moreover, if $1/q = \alpha/d + 1/p$ and $0 < t \leq \infty$ then one has

$$\mathscr{A}^{\alpha/d}_q(\dot{F}^s_{p,t}) = \dot{B}^{s+\alpha}_{q,q} \ .$$

Proof:

We shall first prove that for τ with the property that $1/\tau = 1/p + \alpha/d$ and $\tilde{\tau} = \min\{1, \tau\}$ we have

$$\mathscr{A}^{\alpha/d}_{\tilde{\tau}}(\dot{F}^s_{p,t}) \hookrightarrow \dot{B}^{s+\alpha}_{\tau,\tau} \hookrightarrow \mathscr{A}^{\alpha/d}_\infty(\dot{F}^s_{p,t}), \tag{3.37}$$

where $0 < t \leq \infty$. Using the Jackson inequality from Proposition 3.7 we get that

$$\begin{aligned}|f|_{\mathscr{A}^{\alpha/d}_\infty(\dot{F}^s_{p,t})} &= \sup_{n\geq 1} 2^{n\alpha/d}\sigma_{2^{n-1}}(f)_{\dot{F}^s_{p,t}} \\ &\leq C\sup_{n\geq 1} 2^{(\alpha-(s+\alpha-s))(n-1)/d}\|f\|_{\dot{B}^{s+\alpha}_{\tau,\tau}} = C\|f\|_{\dot{B}^{s+\alpha}_{\tau,\tau}}.\end{aligned}$$

Together with the embedding $\dot{B}^{s+\alpha}_{\tau,\tau} \hookrightarrow \dot{F}^{s}_{p,t}$ obtained from Proposition 3.7 we get

$$\|f\|_{\mathscr{A}^{\alpha/d}_{\infty}(\dot{F}^{s}_{p,t})} \leq C\|f\|_{\dot{B}^{s+\alpha}_{\tau,\tau}} ,$$

thereby proving the right-hand side of (3.37). To prove the left-hand side take a near-best approximation S_k to f from $\Phi_{2^{k-1}}$ for $k \in \mathbb{N}$ and let $S_0 = 0$. Then since $\cup_{k\in\mathbb{N}}\Phi_k$ is dense in $\dot{B}^{s+\alpha}_{\tau,\tau}$ and by using the Bernstein inequality Proposition 3.9 we find

$$\begin{aligned}
\|f\|^{\tilde{\tau}}_{\dot{B}^{s+\alpha}_{\tau,\tau}} &\leq \sum_{k=1}^{\infty} \|S_k - S_{k-1}\|^{\tilde{\tau}}_{\dot{B}^{s+\alpha}_{\tau,\tau}} \\
&\leq C \sum_{k=1}^{\infty} 2^{k\tilde{\tau}\alpha/d} \|S_k - S_{k-1}\|^{\tilde{\tau}}_{\dot{F}^{s}_{p,t}} \\
&\leq C\Big(\sum_{k=1}^{\infty} 2^{k\tilde{\tau}\alpha/d} \sigma_{2^{k-1}}(f)^{\tilde{\tau}}_{\dot{F}^{s}_{p,t}} + \|f\|^{\tilde{\tau}}_{\dot{F}^{s}_{p,t}} \Big) \\
&\leq C\|f\|^{\tilde{\tau}}_{\mathscr{A}^{\alpha/d}_{\tilde{\tau}}(\dot{F}^{s}_{p,t})} ,
\end{aligned}$$

proving (3.37). Let $0 < \theta < 1$ and $q > 0$, then if $1/\tau_1 = 1/p + \alpha_1/d$ and $1/\tau_2 = 1/p + \alpha_2/d$ one has

$$\begin{aligned}
\big(\mathscr{A}^{\alpha_1/d}_{\tilde{\tau}_1}(\dot{F}^{s}_{p,t}), \mathscr{A}^{\alpha_2/d}_{\tilde{\tau}_2}(\dot{F}^{s}_{p,t})\big)_{\theta,q} &\subset \big(\dot{B}^{s+\alpha_1}_{\tau_1,\tau_1}, \dot{B}^{s+\alpha_2}_{\tau_2,\tau_2}\big)_{\theta,q} \\
&\subset \big(\mathscr{A}^{\alpha_1/d}_{\infty}(\dot{F}^{s}_{p,t}), \mathscr{A}^{\alpha_2/d}_{\infty}(\dot{F}^{s}_{p,t})\big)_{\theta,q} .
\end{aligned}$$

However by the reiteration theorem [3, Theorem 7.3, p.195] and Theorem 3.11 we also have for $1/\tau = \gamma/d + 1/p$ and $\max\{\alpha_1, \alpha_2\} < \gamma < \infty$ that

$$\begin{aligned}
\big(\mathscr{A}^{\alpha_1/d}_{\tilde{\tau}_1}(\dot{F}^{s}_{p,t}), \mathscr{A}^{\alpha_2/d}_{\tilde{\tau}_2}(\dot{F}^{s}_{p,t})\big)_{\theta,q} &= \Big(\big(\dot{F}^{s}_{p,t}, \dot{B}^{s+\gamma}_{\tau,\tau}\big)_{\alpha_1/\gamma,\tilde{\tau}_1}, \big(\dot{F}^{s}_{p,t}, \dot{B}^{s+\gamma}_{\tau,\tau}\big)_{\alpha_2/\gamma,\tilde{\tau}_2}\Big)_{\theta,q} \\
&= \big(\dot{F}^{s}_{p,t}, \dot{B}^{s+\gamma}_{\tau,\tau}\big)_{\alpha/\gamma,q} \\
&= \mathscr{A}^{\alpha/d}_{q}(\dot{F}^{s}_{p,t}) .
\end{aligned}$$

with $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$. An identical calculation with $\tilde{\tau}_1, \tilde{\tau}_2$ replaced by ∞ yields

$$\mathscr{A}^{\alpha/d}_{q}(\dot{F}^{s}_{p,t}) = \big(\mathscr{A}^{\alpha_1/d}_{\infty}(\dot{F}^{s}_{p,t}), \mathscr{A}^{\alpha_2/d}_{\infty}(\dot{F}^{s}_{p,t})\big)_{\theta,q} .$$

This shows that

$$\mathscr{A}^{\alpha/d}_{q}(\dot{F}^{s}_{p,t}) = \big(\dot{B}^{s+\alpha_1}_{\tau_1,\tau_1}, \dot{B}^{s+\alpha_2}_{\tau_2,\tau_2}\big)_{\theta,q},$$

for any $t > 0$. To prove the second part of the proposition we use Lemma

3.13 to get

$$
\begin{aligned}
\mathscr{A}_q^{\alpha/d}(\dot{F}_{p,t}^s) &= \left(\dot{B}_{\tau_1,\tau_1}^{s+\alpha_1}, \dot{B}_{\tau_2,\tau_2}^{s+\alpha_2}\right)_{\theta,q} \\
&= \left(\left(\dot{B}_{p,p}^s, \dot{B}_{\tau,\tau}^{s+\gamma}\right)_{\alpha_1/\gamma,\tau_1}, \left(\dot{B}_{p,p}^s, \dot{B}_{\tau,\tau}^{s+\gamma}\right)_{\alpha_2/\gamma,\tau_2}\right)_{\theta,q} \\
&= \left(\dot{B}_{p,p}^s, \dot{B}_{\tau,\tau}^{s+\gamma}\right)_{\alpha/\gamma,q}. \qquad (3.38)
\end{aligned}
$$

Since we have assumed that $1/q = \alpha/d + 1/p$ we have by using Lemma 3.13 again that (3.38) equals $\dot{B}_{q,q}^{s+\alpha}$. ■

As a corollary of Theorem 3.11 and Proposition 3.14 we have

Corollary 3.15
Let $0 < p < \infty$, $0 < t \leq \infty$, $\gamma \in \mathbb{R}$ and $0 < \alpha < \gamma - s$. If $1/\tau = (\gamma - s)/d + 1/p$ and $1/q = \alpha/d + 1/p$, then

$$
\left(\dot{F}_{p,t}^s, \dot{B}_{\tau,\tau}^{\gamma}\right)_{\alpha/(\gamma-s),q} = \dot{B}_{q,q}^{s+\alpha}
$$

□

4 New bases by almost diagonal matrices

In this section we will apply some of the previous techniques to a new system Θ consisting of functions that approximate one of the previous bases for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ to such a degree that they also form an unconditional wavelet basis. Afterwards we will show that such a new system can be created by a finite linear combination of shifts and dilates of a single nice function φ that has sufficient smoothness and decay and no vanishing moments.
Take Ψ from Proposition 2.6. Choose $\varepsilon > 0$ and take $\Theta = \{\theta^e\}_{e\in E}$ satisfying

$$|(\theta^e)^{(\alpha)}(x) - (\psi^e)^{(\alpha)}(x)| \leq \varepsilon(1+|x|)^{-M}, \quad |\alpha| \leq r_2 \tag{4.1}$$

$$\int x^\alpha \theta^e(x)\mathrm{d}x = 0, \quad |\alpha| \leq r_1 - 1. \tag{4.2}$$

Compared to earlier we are missing a way to expand ψ^e_I in the new system Θ, the boundedness of the matrix that results from this and the uniqueness of this expansion in $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$. All three things will follow from showing that the matrix $\boldsymbol{B} = \left(\langle \theta^{e'}_J, \tilde{\psi}^e_I\rangle\right)_{(I,e),(J,e')\in D\times E}$ has a bounded invers on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$. To prove this we introduce the term almost diagonal.

Definition 4.1
Let $s \in \mathbb{R}$ and for $\dot{f}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$ and for $\dot{b}^s_{p,q}$ let $0 < p, q \leq \infty$. The infinite matrix

$$\boldsymbol{C} = \left(c_{(I,e),(J,e')}\right)_{(I,e),(J,e')\in D\times E}$$

is called almost diagonal on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$ if there exists $\varepsilon > 0$ and $C > 0$ such that

$$|c_{(I,e),(J,e')}| \leq C\omega^s_{\varepsilon,\varepsilon}(I,J)$$

with

$$\omega^s_{\varepsilon,\tilde{\varepsilon}}(I,J) = \min\left\{ \left(\frac{\ell(I)}{\ell(J)}\right)^{(d+\tilde{\varepsilon})/2+s}, \left(\frac{\ell(J)}{\ell(I)}\right)^{(\tilde{\varepsilon}-d)/2+L-s} \right\} \cdot \left(1 + \frac{|x_I - x_J|}{\max\{\ell(I),\ell(J)\}}\right)^{-L-\varepsilon},$$

where $L = d/\min\{1,p,q\}$ for $\dot{f}^s_{p,q}$ and $L = d/\min\{1,p\}$ for $\dot{b}^s_{p,q}$. When no confusion arises we use the notation $\omega^s_{\varepsilon,\varepsilon} = \omega_\varepsilon$. ◇

Proposition 4.2
Let $s \in \mathbb{R}$ and for $\dot{f}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$ and for $\dot{b}^s_{p,q}$ let $0 < p, q \leq \infty$. Assume that $\boldsymbol{C}$ is almost diagonal on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$ then $\boldsymbol{C}$ is bounded on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$.

Proof:
This follows by repeating the proof of Proposition 2.2. ■

Proposition 4.3
Let $\varepsilon > 0$, $s \in \mathbb{R}$ and for $\dot{f}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$ and for $\dot{b}^s_{p,q}$ let $0 < p, q \leq \infty$. There exists $\delta > 0$ such that if $\boldsymbol{C}$ is almost diagonal on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$ and $\|\boldsymbol{I} - \boldsymbol{C}\|_\varepsilon < \delta$, where

$$\|\boldsymbol{C}\|_\varepsilon = \sup_{(I,e),(J,e')} \frac{|c_{(I,e),(J,e')}|}{\omega_\varepsilon(I,J)},$$

then $\boldsymbol{C}$ has an invers on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$ which is almost diagonal.

Proof:
Denote $\boldsymbol{I} - \boldsymbol{C} = \widetilde{\boldsymbol{C}}$ and assume that $|\tilde{c}_{(I,e),(J,e')}| \leq \delta\omega_\varepsilon(I,J)$ for some ε, $\delta > 0$. By the definition of $\omega_{\varepsilon,\tilde{\varepsilon}}(I,J)$ one has that it is a non-decreasing function of ε and $\tilde{\varepsilon}$. This implies that $\omega_\varepsilon(I,J) < \omega_{\tilde{\varepsilon},\tilde{\varepsilon}}(I,J)$ for fixed $\tilde{\varepsilon} < \varepsilon$. We define $a^n_{I,J} = b_{I,J}$ for $\boldsymbol{A}^n = \boldsymbol{B}$. Lemma A.9 yields the following

$$\begin{aligned}
|\tilde{c}^2_{(I,e),(J,e')}| &= \Big|\sum_{K,e''} \tilde{c}_{(I,e),(K,e'')}\tilde{c}_{(K,e''),(J,e')}\Big| \\
&\leq \delta^2 \sum_{K,e''} \omega_\varepsilon(I,K)\omega_{\tilde{\varepsilon}}(K,J) \\
&\leq \delta^2 \sum_{K,e} \omega_{\tilde{\varepsilon},\varepsilon}(I,K)\omega_{\tilde{\varepsilon},\tilde{\varepsilon}}(K,J) \\
&\leq C\delta^2 \omega_{\tilde{\varepsilon},\tilde{\varepsilon}}(I,J) \\
&= C\delta^2 \omega_{\tilde{\varepsilon}}(I,J).
\end{aligned}$$

By induction one finds that $|\tilde{c}^n_{(I,e),(J,e')}| \leq C^{n-1}\delta^n\omega_{\tilde{\varepsilon}}(I,J)$ for all $n \in \mathbb{N}$. Choosing δ such that $\delta < \min\{1, C^{-1}\}$ insures the convergence of the Neumann series $\sum_{n\geq 0} \widetilde{\boldsymbol{C}}^n$ since the space of almost diagonal matrices with finite $\|\cdot\|_{\tilde{\varepsilon}}$ is a weighted ℓ^∞-space, therefore a Banach space. The Neumann series $\sum_{n\geq 0} \widetilde{\boldsymbol{C}}^n = (I - \widetilde{\boldsymbol{C}})^{-1} = \boldsymbol{C}^{-1}$. By construction we have that $|c^{-1}_{(I,e),(J,e')}| \leq (1 - C\delta)^{-1}\omega_{\tilde{\varepsilon}}(I,J)$. ■

By using Proposition 4.3 and Proposition 4.2 we now show that $\boldsymbol{B}$ has a bounded invers on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$.

Lemma 4.4
Let $s \in \mathbb{R}$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p, q \leq \infty$, $L = d/\min\{1,p\}$. Furthermore let $k = \max\{\lfloor s - d/p \rfloor, -1\}$, $r_1 > L - d - s$, $r_2 > s$ and $M = \max\{L, d + r_1, d + r_2\}$ for $\dot{F}^s_{p,q}$ and $M = \max\{d + r_1, d + r_2\}$ for $\dot{B}^s_{p,q}$. Take Ψ from Proposition 2.6. If Θ satisfies (4.1) and (4.2) with sufficiently small δ, then $\boldsymbol{B} = \big(\langle\theta^{e'}_J, \widetilde{\psi}^e_I\rangle\big)_{(I,e),(J,e')\in D\times E}$ has a bounded invers on $\dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$.

Proof:

Choose $\varepsilon > 0$ such that $L + \varepsilon \leq M$, $s + \varepsilon/2 \leq r_2$ and $L - d - s + \varepsilon/2 \leq r_1$ from Proposition 2.6. By Proposition 4.3 and Proposition 4.2 we only need to show that $\boldsymbol{B}$ is almost diagonal and $\|\boldsymbol{I} - \boldsymbol{B}\|_\varepsilon < \delta_\varepsilon$. Take $(I,e) \neq (J,e'), |J| \leq |I|$. By the ortogonality of Ψ we have

$$|\langle \theta_J^{e'}, \widetilde{\psi}_I^e \rangle| = \Big| \int_{\mathbb{R}^d} (\theta_J^{e'}(x) - \psi_J^{e'}(x)) \overline{\widetilde{\psi}_I^e(x)} \mathrm{d}x \Big|.$$

By repeating the proof of Lemma 2.1 we get

$$\leq C\delta \Big(\frac{\ell(J)}{\ell(I)} \Big)^{r_1 + d/2} \Big(1 + \frac{|x_I - x_J|}{\ell(I)} \Big)^{-M}$$
$$\leq C\delta \omega_\varepsilon(I,J)$$

Similary for $(I,e) \neq (J,e'), |J| > |I|$. For $(I,e) = (J,e')$ we use the biorthogonality of Ψ to get

$$\langle \theta_I^e, \widetilde{\psi}_I^e \rangle = 1 + \int_{\mathbb{R}^d} (\theta_I^e(x) - \psi_I^e(x)) \widetilde{\psi}_I^e(x) \mathrm{d}x$$

which gives

$$|\langle \theta_I^e, \widetilde{\psi}_I^e \rangle - 1| = \Big| \int_{\mathbb{R}^d} (\theta_I^e(x) - \psi_I^e(x)) \widetilde{\psi}_I^e(x) \mathrm{d}x \Big| \leq C\delta,$$

where we used the technique of the proof of Lemma 2.1 again. We now have that $\boldsymbol{B}$ is almost diagonal and that $\|\boldsymbol{I} - \boldsymbol{B}\|_\varepsilon \leq C\delta$. Choosing δ sufficiently small such that $C\delta < \delta_\varepsilon$ the result follows. ∎

We will follow the steps of page 20-25 to show that Θ forms an unconditional wavelet basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$. As previously we first insure that the series converges in S'/P_k by bounding the series with the $\dot{f}^s_{p,q}, \dot{b}^s_{p,q}$-norm of the coefficients.

Lemma 4.5

Let $s \in \mathbb{R}$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p, q \leq \infty$, $L = d/\min\{1,p\}$. Furthermore let $k = \max\{\lfloor s - d/p \rfloor, -1\}$, $r_1 > L - d - s$, $r_2 > s$ and $M = \max\{L, d + r_1, d + r_2\}$ for $\dot{F}^s_{p,q}$ and $M = \max\{d + r_1, d + r_2\}$ for $\dot{B}^s_{p,q}$. Take Ψ from Proposition 2.6 and let Θ be a family of functions satisfying the conditions (4.1) and (4.2). If $d = \{d_{I,e}\} \in \dot{f}^s_{p,q}$ or $\dot{b}^s_{p,q}$ then the series

$$\sum_{(I,e) \in D \times E} d_{I,e} \theta_I^e$$

converges in S'/P_k.

Proof:

Notice that the assumption (4.1) implies that $|(\theta^e)^{(\alpha)}(x)| \leq C(1+|x|)^{-M}$. We have that θ_I^e fulfills the same demands as θ_I from Lemma 2.3. By repeating the proof of said lemma with θ_I^e instead of θ_I proves this lemma. ■

Next we need to expand the elements of Ψ in our new system Θ. Previously in Theorem 2.6 we used that Θ formed a wavelet basis for $L^2(\mathbb{R}^d)$. To prove the result for our new system we will instead use the boundedness of $\boldsymbol{B}^{-1}$ as the following lemma shows. We denote the entries of $\boldsymbol{B}^{-1} = (\langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle)$ as it seems to be the most natural notation.

Lemma 4.6
Let $s \in \mathbb{R}$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p,q \leq \infty$, $L = d/\min\{1,p\}$. Furthermore let $k = \max\{\lfloor s - d/p \rfloor, -1\}$, $r_1 > L - d - s$, $r_2 > s$ and $M = \max\{L, d+r_1, d+r_2\}$ for $\dot{F}^s_{p,q}$ and $M = \max\{d + r_1, d + r_2\}$ for $\dot{B}^s_{p,q}$. Take Ψ from Proposition 2.6 and let Θ be a family of functions satisfying the conditions (4.1) and (4.2). Then one has that

$$\psi_I^e = \sum_{(J,e') \in D \times E} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'}, \qquad (I,e) \in D \times E, \tag{4.3}$$

in S'/P_k.

Proof:

We shall prove the lemma for $\dot{F}^s_{p,q}$, since $\dot{B}^s_{p,q}$ follows in a similar manner. Using Kroneckers delta function we define the sequence $\delta^{I,e} = \left\{\delta_{(I,e),(J,e')}\right\}_{J,e' \in D \times E}$for $(I,e) \in D \times E$. We now use $\delta^{I,e}$ to write the sequence of the series in (4.3), $(\langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle)_{J,e'} = \boldsymbol{B}^{-1}\delta^{I,e}$. The conditions of Lemma 4.4 are satisfied and we therefore have that the matrix $\boldsymbol{B}^{-1}$ is bounded on $\dot{f}^s_{p,q}$. Since our sequence $\delta^{I,e} \in \dot{f}^s_{p,q}$ it follows that $(\langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle)_{J,e'}$ is in $\dot{f}^s_{p,q}$. By Lemma 4.5 we have that the series in (4.3) converges in S'/P_k. From this we have

$$\left\langle \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'}, \phi_K \right\rangle = \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \langle \theta_J^{e'}, \phi_K \rangle.$$

By Proposition 2.2 we have that the matrix $(\langle \theta_J^{e'}, \phi_K \rangle)_{K,J}$ is bounded on $\dot{f}^s_{p,q}$. Using the norm equivalence of ϕ, Proposition 1.4, we have that

$\sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'} \in \dot{F}^s_{p,q}$. Next we expand in Ψ and for this we notice that

$$\begin{aligned}\big\langle \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'}, \widetilde{\psi}_\Delta^{e''} \big\rangle &= \sum_K \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \langle \theta_J^{e'}, \phi_K \rangle \langle \phi_K, \widetilde{\psi}_\Delta^{e''} \rangle \\ &= \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \langle \theta_J^{e'}, \widetilde{\psi}_\Delta^{e''} \rangle \\ &= \begin{cases} 1, & \text{if } (I,e) = (\Delta, e''), \\ 0, & \text{if } (I,e) \neq (\Delta, e''), \end{cases}\end{aligned}$$

which gives

$$\sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'} = \sum_{\Delta,e''} \big\langle \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'}, \widetilde{\psi}_\Delta^{e''} \big\rangle \psi_\Delta^{e''} = \psi_I^e.$$

■

Using the previous lemmas we now prove that the new system Θ also forms an unconditional wavelet basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ for appropriate indeces. This theorem will be pivotal in the proof that a finite number of shifts and dilates of a single function forms a basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$.

Theorem 4.7
Let $s \in \mathbb{R}$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$, $L = d/\min\{1, p, q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p, q \leq \infty$, $L = d/\min\{1, p\}$. Furthermore let $k = \max\{\lfloor s - d/p \rfloor, -1\}$, $r_1 > L - d - s$, $r_2 > s$ and $M = \max\{L, d + r_1, d + r_2\}$ for $\dot{F}^s_{p,q}$ and $M = \max\{d + r_1, d + r_2\}$ for $\dot{B}^s_{p,q}$. Take Ψ from Proposition 2.6 and let Θ be a family of functions satisfying the conditions (4.1) and (4.2). Then for $f \in \dot{F}^s_{p,q}$ there exists unique coefficients $d = (d_{I,e})_{(I,e) \in D \times E} \in \dot{f}^s_{p,q}$ such that

$$f = \sum_{(I,e) \in D \times E} d_{I,e} \theta_I^e \tag{4.4}$$

in S'/P_k and in $\dot{F}^s_{p,q}$ if $q \neq \infty$. Furthermore one has that

$$\|f\|_{\dot{F}^s_{p,q}} \asymp \|\sum_{e \in E} d_{I,e}\|_{\dot{f}^s_{p,q}}.$$

In $\dot{B}^s_{p,q}$ a similar phrasing applies with convergens in $\dot{B}^s_{p,q}$ for $p, q \neq \infty$.

Proof:
We give the proof for $\dot{F}^s_{p,q}$ as the proof for $\dot{B}^s_{p,q}$ follows in a similar vein. By Lemma 4.6 we have that $\psi_I^e = \sum_{J,e'} \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle \theta_J^{e'}$ in S'/P_k. From Lemma 4.4 one has the boundedness of the matrix $\boldsymbol{B}^{-1}$ on $\dot{f}^s_{p,q}$. With minor alterations the proof in Theorem 2.5 can be repeated to show the norm equivalence and the convergence of (4.4) with $d_{J,e'} = \sum_{I,e} \langle f, \widetilde{\psi}_I^e \rangle \langle \psi_I^e, \widetilde{\theta}_J^{e'} \rangle$. To show the

uniqueness of the coefficients in $\dot{f}^s_{p,q}$, assume that there are two sequences d^1 and d^2 both in $\dot{f}^s_{p,q}$ such that

$$\sum_{J,e'} d^1_{J,e'} \theta^{e'}_J = f = \sum_{J,e'} d^2_{J,e'} \theta^{e'}_J \qquad \text{in } S'/P_k.$$

We expand each $\theta^{e'}_J$ in the wavelet basis formed by Ψ and get

$$\sum_{J,e'} \sum_{I,e} d^i_{J,e'} \langle \theta^{e'}_J, \widetilde{\psi}^e_I \rangle \psi^e_I = \sum_{I,e} \sum_{J,e'} d^i_{J,e'} \langle \theta^{e'}_J, \widetilde{\psi}^e_I \rangle \psi^e_I, \quad i = 1, 2$$

using the boundedness of $\boldsymbol{B}$ on $\dot{f}^s_{p,q}$ and Lemma 2.3. By the uniqueness of the wavelet coefficients in $\dot{f}^s_{p,q}$ from the wavelet basis formed by Ψ we then have that

$$\boldsymbol{B}d^1 = \boldsymbol{B}d^2.$$

Finally we apply Lemma 4.4 to get $d^1 = d^2$. ∎

Bases from shifts and dilates of a single function

We now show that a finite number of linear combinations of shifts and dilates of a single function with sufficient smoothness and decay and no vanishing moments fulfills (4.1) and (4.2) with Meyer's wavelet set [2, p.137]. Therefore we can use Theorem 4.7 to create a basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ from these. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a family of functions in $C^{r_2+1}(\mathbb{R}^d)$, which satisfy:

There exists $M' > d + r_1$, $\mu > 0$ such that

$$|\varphi^{(\alpha)}_n(x)| \le C \frac{n^{|\alpha|\mu+d}}{(1+n|x|)^{M'}}, \quad |\alpha| \le r_2 + 1, \tag{4.5}$$

$$\int_{\mathbb{R}^d} \varphi_n(x) \mathrm{d}x = 1. \tag{4.6}$$

One way of constructing φ_n is taking a function $\zeta \in C^{r_2+1}(\mathbb{R}_+)$ with $|\zeta^{(\alpha)}(t)| \le C(1+t)^{-M'}$, $|\alpha| \le r_2 + 1$, and then defining $\varphi_n(\cdot) = C_1 n^d \zeta(n|\cdot|)$ with C_1 chosen such that (4.6) is satisfied. Examples of ζ are $e^{-\cdot}$ and $(1+\cdot)^{-M''}$, $M'' \ge M'$. To approximate Meyer wavelets we will use the set $\Theta_{K,n}$,

$$\Theta_{K,n} = \left\{\theta : \theta(\cdot) = \sum_{i=1}^{K} a_i \varphi_n(\cdot + b_i),\ a_i \in \mathbb{R}, b_i \in \mathbb{R}^d\right\}$$

Proposition 4.8
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfy (4.5) and (4.6) with $M' > M > d$ and let Ψ be Meyer's

wavelet set. Then for $\varepsilon > 0$ there exists $K, n \geq 1$ such that for $\psi^e \in \Psi$ there is $\theta \in \Theta_{K,n}$ for which

$$|(\psi^e)^{(\alpha)}(x) - \theta^{(\alpha)}(x)| \leq \varepsilon(1+|x|)^{-M}, \quad |\alpha| \leq r_2, \tag{4.7}$$

$$\int_{\mathbb{R}^d} x^\alpha \theta(x) \mathrm{d}x = 0, \quad |\alpha| \leq r_1 - 1. \tag{4.8}$$

Proof:

We begin with showing that for any function $g \in C^{r_2+1}(\mathbb{R}^d)$ satisfying

$$|g^{(\alpha)}(x)| \leq C(1+|x|)^{-M'}, \quad |\alpha| \leq r_2 + 1 \tag{4.9}$$

there exists $\theta' \in \Theta_{K,n}$ for which

$$|g^{(\alpha)}(x) - (\theta')^{(\alpha)}(x)| \leq \varepsilon(1+|x|)^{-M}, \quad |\alpha| \leq r_2.$$

This we will do in three steps first approximating g by a convolution operator $\omega_n = g * \varphi_n$, then approximating ω_n by $\lambda_{q,n}$ which is the integral in ω_n taken over a dyadic cube and finally approximating $\lambda_{q,n}$ by a discretization $\theta_{m,q,n}$. We have

$$g^{(\alpha)}(x) - \omega_n^{(\alpha)}(x) = \int_{\mathbb{R}^d} (g^{(\alpha)}(x) - g^{(\alpha)}(x-y))\varphi_n(y)\mathrm{d}y, \quad |\alpha| \leq r_2, \tag{4.10}$$

by using (4.6). Define $U = n^{\eta/2M}$, where $\eta = \min\{1, M' - M\}$. For $|x| \leq U$ we have by using the mean value theorem that

$$|g^{(\alpha)}(x) - g^{(\alpha)}(x-y)| \leq C|y|, \quad |\alpha| \leq r_2.$$

Inserting this in (4.10) we get

$$\begin{aligned} |g^{(\alpha)}(x) - \omega_n^{(\alpha)}(x)| &\leq C \int_{\mathbb{R}^d} \frac{|y|n^d}{(1+n|y|)^{M'}} \mathrm{d}y \\ &\leq Cn^{-1} \leq \frac{Cn^{-\eta/2}}{U^M} \leq \frac{Cn^{-\eta/2}}{(1+|x|)^M}, \end{aligned} \tag{4.11}$$

using that $U \geq 1$ and $M' > d+1$. For $|x| > U$ we split the integral over $\Omega = \{y : |y| \leq |x|/2\}$ and Ω^c. If $y \in \Omega$ then $|x-y| \geq |x|/2$ and we have

$$|g^{(\alpha)}(x) - g^{(\alpha)}(x-y)| \leq |g^{(\alpha)}(x)| + |g^{(\alpha)}(x-y)| \leq \frac{C}{(1+|x|)^{M'}}.$$

Therefore

$$\begin{aligned} \int_\Omega |g^{(\alpha)}(x) - g^{(\alpha)}(x-y)||\varphi_n(y)|\mathrm{d}y &\leq \frac{C}{(1+|x|)^{M'}} \\ &\leq \frac{C}{(1+|U|)^\eta (1+|x|)^M} \leq \frac{Cn^{-\eta^2/2M}}{(1+|x|)^M}. \end{aligned} \tag{4.12}$$

Integrating over $\Omega^c = \{y : y > |x|/2\}$ for $|x| > U$ we get

$$\begin{aligned}
&\int_{\Omega^c} |g^{(\alpha)}(x) - g^{(\alpha)}(x-y)||\varphi_n(y)|\mathrm{d}y \\
&\leq \int_{\Omega^c} |g^{(\alpha)}(x)||\varphi_n(y)|\,\mathrm{d}y + \int_{\Omega^c} |g^{(\alpha)}(x-y)||\varphi_n(y)|\mathrm{d}y \\
&\leq \frac{C}{(1+|x|)^{M'}} + \int_{\Omega^c} \frac{Cn^d}{(1+|x-y|)^{M'}(1+n|y|)^{M'}}\,\mathrm{d}y \\
&\leq \frac{C}{(1+|x|)^{M'}} + \frac{Cn^d}{(1+n|x|)^{M'}} \leq \frac{C(1+n^{d-M'})}{(1+|x|)^{M'}} \leq \frac{C(n^{-\eta^2/2M} + n^{-\eta^3/2M})}{(1+|x|)^M},
\end{aligned} \tag{4.13}$$

where we used (2.7) and the same estimate as in (4.12). So by choosing n sufficiently large in (4.11), (4.12) and (4.13) we get

$$|g^{(\alpha)}(x) - \omega_n^{(\alpha)}(x)| \leq \varepsilon(1+|x|)^{-M}, \quad |\alpha| \leq r_2. \tag{4.14}$$

For the next step we fix n and approximate ω_n by $\lambda_{q,n}$ defined as

$$\lambda_{q,n}(\cdot) = \int_Q g(y)\varphi_n(\cdot - y)\mathrm{d}y,$$

where $Q = [-2^q, 2^q]^d$, $q \in \mathbb{N}$. Obviously we have

$$\omega_n^{(\alpha)}(x) - \lambda_{q,n}^{(\alpha)}(x) = \int_{\mathbb{R}^d \setminus Q} g(y)\varphi_n^{(\alpha)}(x-y)\mathrm{d}y, \quad |\alpha| \leq r_2,$$

from which we get

$$|\omega_n^{(\alpha)}(x) - \lambda_{q,n}^{(\alpha)}(x)| \leq C\int_{\mathbb{R}^d \setminus Q} \frac{n^{r_2\mu+d}}{(1+|y|)^{M'}(1+n|x-y|)^{M'}}\mathrm{d}y = L.$$

We first estimate the integral for $|x| \leq 2^{q-1}$, which gives that $|y| \geq 2|x|$ and $|x-y| \geq 2^{q-1}$ and hence

$$\begin{aligned}
L &\leq \frac{Cn^{r_2\mu+d}}{(1+|x|)^{M'}} \int_{\mathbb{R}^d \setminus Q} \frac{1}{(1+n|x-y|)^{M'}}\mathrm{d}y \\
&\leq \frac{Cn^{r_2\mu+d}}{(1+|x|)^{M'}} \int_{|u| \geq 2^{q-1}} \frac{1}{(1+n|u|)^{M'}}\mathrm{d}u \leq \frac{Cn^{r_2\mu+d-M'}2^{(d-M')(q-1)}}{(1+|x|)^{M'}}.
\end{aligned}$$

For $|x| > 2^{q-1}$ we split the integral over $\Omega = (\mathbb{R}^d \setminus Q) \cap \{y : |y| \leq |x|/2\}$ and $\Omega' = (\mathbb{R}^d \setminus Q) \setminus \Omega$. If $y \in \Omega$ then $|x-y| \geq |x|/2$ so we get

$$\begin{aligned}
\int_\Omega \frac{n^{r_2\mu+d}}{(1+|y|)^{M'}(1+n|x-y|)^{M'}}\mathrm{d}y &\leq \frac{Cn^{r_2\mu+d}}{(1+n|x|)^{M'}} \int_{\mathbb{R}^d} \frac{1}{(1+|y|)^{M'}}\mathrm{d}y \\
&\leq \frac{Cn^{r_2\mu+d-M'}}{(1+|x|)^{M'}} \leq \frac{Cn^{r_2\mu+d-M'}2^{-\eta(q-1)}}{(1+|x|)^M}.
\end{aligned}$$

If $y \in \Omega'$ then $|y| > |x|/2$ and hence

$$\int_{\Omega'} \frac{n^{r_2\mu+d}}{(1+|y|)^{M'}(1+n|x-y|)^{M'}}\mathrm{d}y \le \frac{Cn^{r_2\mu}}{(1+|x|)^{M'}} \int_{\mathbb{R}^d} \frac{n^d}{(1+n|x-y|)^{M'}}\mathrm{d}y$$
$$\le \frac{Cn^{r_2\mu}}{(1+|x|)^{M'}} \le \frac{Cn^{r_2\mu}2^{-\eta(q-1)}}{(1+|x|)^{M}}.$$

Therefore in both cases we have

$$L \le \frac{Cn^{r_2\mu}2^{-\eta q}}{(1+|x|)^{M}}.$$

Choosing q sufficiently large we obtain

$$|\omega_n^{(\alpha)}(x) - \lambda_{q,n}^{(\alpha)}(x)| \le \varepsilon(1+|x|)^{-M}, \quad |\alpha| \le r_2. \tag{4.15}$$

For the final step we fix q and approximate $\lambda_{q,n}$ by a discretization $\theta_{m,q,n}$. Let $H_{m,q}$ denote the set of all dyadic subcubes of Q of sidelength 2^{-m}, $m \in \mathbb{N}$. We define

$$\theta_{m,q,n}(\cdot) = \sum_{I \in H_{m,q}} |I| g(x_I)\varphi_n(\cdot - x_I)$$

and note that $\theta_{m,q,n} \in \Theta_{2^{d(q+m+1)},n}$. We have

$$\lambda_{q,n}^{(\alpha)}(x) - \theta_{m,q,n}^{(\alpha)}(x) = \sum_{I \in H_{m,q}} \int_I g(y)\varphi_n^{(\alpha)}(x-y) - g(x_I)\varphi_n^{(\alpha)}(x-x_I)\mathrm{d}y$$
$$= \sum_{I \in H_{m,q}} \int_I F(y) - F(x_I)\mathrm{d}y, \quad |\alpha| \le r_2,$$

where $F(\cdot) = g(\cdot)\varphi^{(\alpha)}(x - \cdot)$. By using the mean value theorem we get

$$|\lambda_{q,n}^{(\alpha)}(x) - \theta_{m,q,n}^{(\alpha)}(x)| \le C \sum_{I \in H_{m,q}} \int_I |y - x_I| \max_{\substack{z \in l(x_I,y) \\ |\beta|=1}} |F^{(\beta)}(z)|\mathrm{d}y$$
$$\le C2^{qd-m} \max_{\substack{z \in Q \\ |\beta|=1}} |F^{(\beta)}(z)|$$
$$\le C2^{qd-m} \max_{\substack{z \in Q \\ |\alpha| \le r_2+1}} |\varphi_n^{(\alpha)}(x-z)|$$

where $l(x_I, y)$ is the line-segment between x_I and y. If $|x| \le \sqrt{d}2^{q+1}$, $z \in Q$ and $|\alpha| \le r_2 + 1$ then

$$|\varphi_n^{(\alpha)}(x-z)| \le Cn^{(r_2+1)\mu+d} \le \frac{Cn^{(r_2+1)\mu+d}2^{qM}}{(1+|x|)^{M}}.$$

If $|x| > \sqrt{d}2^{q+1}$ and $z \in Q$ then $|x - z| \geq |x|/2$ and hence for $|\alpha| \leq r_2 + 1$ we have

$$|\varphi_n^{(\alpha)}(x - z)| \leq \frac{Cn^{(r_2+1)\mu+d}}{(1 + n|x|)^{M'}} \leq \frac{Cn^{(r_2+1)\mu+d-M'}}{(1 + |x|)^M}.$$

In both cases by choosing m sufficiently large we obtain

$$|\lambda_{q,n}^{(\alpha)}(x) - \theta_{m,q,n}^{(\alpha)}(x)| \leq \varepsilon(1 + |x|)^{-M}, \quad |\alpha| \leq r_2. \tag{4.16}$$

From (4.14)-(4.16) we conclude that for any $\varepsilon > 0$ there exists $K, n \geq 1$ such that for any g satisfying (4.9) there exists $\theta' \in \Theta_{K,n}$ ($K = 2^{d(q+m+1)}$, $\theta' = \theta_{m,q,n}$) such that

$$|g^{(\alpha)}(x) - (\theta')^{(\alpha)}(x)| \leq 3\varepsilon(1 + |x|)^{-M}, \quad |\alpha| \leq r_2. \tag{4.17}$$

Next we will use the first part of the proof to show that there exists $\theta \in \Theta_{K,n}$ which satisfies both (4.7) and (4.8). We shall do this using some of the specific properties of Meyers wavelets and the operator $\Delta_{hv_j}^r$

$$(\Delta_{hv_j}^r f)(x) = \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x + khv_j),$$

where v_j is the unitvector in the j'te direction. By using the binomial formula we get that

$$\begin{aligned}(\Delta_{hv_j}^r f)\widehat{\ }(\xi) &= \int_{\mathbb{R}^d} \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x) e^{-i(x-khv_j)\xi} \mathrm{d}x \\ &= (e^{ih\xi_j} - 1)^r \widehat{f}(\xi).\end{aligned}$$

If $|f(x)| \leq C(1 + |x)^{-M'}$ we have that

$$\begin{aligned}\int_{\mathbb{R}^d} x^\alpha (\Delta_{hv_j}^{r_1} f)(x) \mathrm{d}x &= i^{|\alpha|} (\Delta_{hv_j}^{r_1} f)^{\widehat{\ }(\alpha)}(0) \\ &= i^{|\alpha|}((e^{ih\xi_j} - 1)_1^r \widehat{f}(\xi))^{(\alpha)}(0) = 0, \quad |\alpha| \leq r_1 - 1.\end{aligned}$$

Take $\psi^e \in \Psi$ and define g by $\hat{g}(\xi) = \widehat{\psi}^e(\xi)(e^{i\xi_j/2} - 1)^{-r}$, where j is chosen such that $e_j = 1$ (see (1.2)). Observe that $e^{i\xi_j/2} - 1$ vanishes only at the integer multiples of 4π which are not in the support of $\widehat{\psi}^e$ and hence $g \in S$. Now using the first part of the proof (see (4.17)) we have that there exists $\theta' \in \Theta_{K,n}$ such that

$$|g^{(\alpha)}(x) - (\theta')^{(\alpha)}(x)| \leq 3\varepsilon(1 + |x|)^{-M}, \quad |\alpha| \leq r_2. \tag{4.18}$$

Furthermore we have that

$$|\theta'(x)| \leq C(1 + |x|)^{-M'}. \tag{4.19}$$

which follows from $|\varphi_n(x+b)| \leq Cn^d(1+|b|)(1+|x|)^{-M'}$. We define $\theta = \Delta^{r_1}_{v_j/2}\theta'$, so $\theta \in \Theta_{(r_1+1)K,n}$. We have $\psi^e - \theta = \Delta^{r_1}_{v_j/2}(g-\theta')$ and using (4.18) we get

$$|(\psi^e)^{(\alpha)}(x) - \theta^{(\alpha)}(x)| \leq C\varepsilon(1+|x|)^{-M}, \quad |\alpha| \leq r_2.$$

by using the same estimates as for (4.19). Moreover from $\theta = \Delta^{r_1}_{v_j/2}\theta'$ and (4.19) it follows that

$$\int_{\mathbb{R}^d} x^\alpha \theta(x)\mathrm{d}x = 0, \quad |\alpha| \leq r_1 - 1.$$

■

By combining Proposition 4.8 and Theorem 4.7 we have a that a finite number of linear combinations of shifts and dilates of a single nice function forms an unconditional wavelet basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$.

Proposition 4.9
Let $s \in \mathbb{R}$ and for $\dot{F}^s_{p,q}$ let $0 < p < \infty$, $0 < q \leq \infty$, $L = d/\min\{1,p,q\}$ and for $\dot{B}^s_{p,q}$ let $0 < p,q \leq \infty$, $L = d/\min\{1,p\}$. Furthermore let $k = \max\{\lfloor s - d/p \rfloor, -1\}$, $r_1 > L - d - s$, $r_2 > s$ and $M = \max\{L, d+r_1, d+r_2\}$ for $\dot{F}^s_{p,q}$ and $M = \max\{d+r_1, d+r_2\}$ for $\dot{B}^s_{p,q}$. If $\{\varphi_n\}_{n\in N}$ satisfies (4.5) and (4.6) with $M' > M$ then there exists $K, n \geq 1$ and a family of functions $\Theta \subset \Theta_{K,n}$ such that Θ forms an unconditional basis for $\dot{F}^s_{p,q}$ or $\dot{B}^s_{p,q}$ and we have the norm equivalence as in Theorem 4.7. □

We conclude this section with considering n-term approximation from a single nice function in light of the previous proposition. Especially whether the Jackson and Bernstein inequalities are satisfied, so we can characterize the approxmation space by an interpolation space by Proposition 3.6. We define

$$G(\varphi) = \{\varphi(a\cdot+b) : a \in \mathbb{R}, b \in \mathbb{R}^d\}$$
$$G_n(\varphi) = \{S : S = \sum_{j=1}^{n} a_j\varphi_j, \varphi_j \in G(\varphi)\}$$

and $\sigma^\varphi_n(f)_{\dot{F}^s_{p,t}} = \inf_{S\in G_n(\varphi)}\|f-S\|_{\dot{F}^s_{p,t}}$. By Proposition 4.9 we have that there exists K, m such that there exists $\Theta \subset \Theta_{K,m}$ which forms an unconditional wavelet basis for $\dot{F}^s_{p,t}$. As on page 34 define the error of approximation and let us here denote it $\sigma^\theta_n(f)_{\dot{F}^s_{p,t}}$. Then from Proposition 3.7 we have the Jackson inequality

$$\sigma^\theta_n(f)_{\dot{F}^s_{p,t}} \leq Cn^{-(\gamma-s)/d}\|f\|_{B^\gamma_{\tau,\tau}}.$$

Next we use the fact that $\sigma^{\varphi}_{(2^d-1)Kn}(f)_{\dot{F}^s_{p,t}} \leq \sigma^{\theta}_n(f)_{\dot{F}^s_{p,t}}$ to show the Jackson inequality for n-term approximation by the single function φ. Let $n > (2^d-1)K$ and find $n_1 \in \mathbb{N}$ such that $(2^d-1)Kn_1 < n \leq (2^d-1)K(n_1+1)$ we then have

$$\begin{aligned}\sigma^{\varphi}_n(f)_{\dot{F}^s_{p,t}} &\leq \sigma^{\varphi}_{(2^d-1)Kn_1}(f)_{\dot{F}^s_{p,t}}\\ &\leq \sigma^{\theta}_{n_1}(f)_{\dot{F}^s_{p,t}}\\ &\leq Cn_1^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}\\ &\leq C((2^d-1)K(n_1+1))^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}\\ &\leq Cn^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}.\end{aligned}$$

For $0 < n \leq (2^d-1)K$ we use the embedding $\dot{B}^{\gamma}_{\tau,\tau} \hookrightarrow \dot{F}^s_{p,t}$ which was achieved by the proof of the Jackson inequality in Proposition 3.7

$$\begin{aligned}\sigma^{\varphi}_n(f)_{\dot{F}^s_{p,t}} &\leq \|f\|_{\dot{F}^s_{p,t}}\\ &\leq C\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}\\ &\leq C((2^d-1)K)^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}\\ &\leq Cn^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,\tau}}.\end{aligned}$$

In a similar manner one also finds that

$$\sigma^{\varphi}_n(f)_{\dot{B}^s_{p,t}} \leq Cn^{-(\gamma-s)/d}\|f\|_{\dot{B}^{\gamma}_{\tau,r}}$$

holds in the Besov case. From these Jackson inequalities and Proposition 3.6 we have

$$\begin{aligned}(\dot{F}^s_{p,t}, \dot{B}^{\gamma}_{\tau,\tau})_{\alpha/(\gamma-s),q} &\hookrightarrow \mathscr{A}^{\alpha/d}_q(\dot{F}^s_{p,t}, \{G_n(\varphi)\}_{n=0}^{\infty})\\ (\dot{B}^s_{p,t}\dot{B}^{\gamma}_{\tau,r})_{\alpha/(\gamma-s),q} &\hookrightarrow \mathscr{A}^{\alpha/d}_q(\dot{B}^s_{p,t}, \{G_n(\varphi)\}_{n=0}^{\infty}).\end{aligned}$$

However it remains an unanswered question whether there also exists Bernstein inequalities for $G_n(\varphi)$, so that the approximation spaces can be entirely characterized by interpolation spaces as in Theorem 3.11 and Theorem 3.12. At the moment the only known function for which one can obtain this, is when $d = 1$ and the function is $(1+x^2)^{-N}$ [9, p.773]. The Bernstein inequality is here a result of the inverse estimate of Pekarskii ([9] refers to [13] and the references therein).

A for Appendix

We start the appendix with the following result that is used extensively in $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$ for example to justify the norm on both spaces defined in (1.6) and (1.8). We first introduce some notation. Let $\mathcal{E}(\mathbb{R}^d)$ be the space of all infinitely differential functions on $\mathbb{R}^d$ and with convergence of f_k to f in $\mathcal{E}(\mathbb{R}^d)$ if and only if $f_k, f \in \mathcal{E}(\mathbb{R}^d)$ and

$$\lim_{k\to\infty} \widetilde{\rho}_{N,\beta}(f_k - f) = \lim_{k\to\infty} \sup_{|x|\leq N} |(f_k - f)^{(\beta)}| = 0$$

for $\beta \in \mathbb{N}_0^d$ and $N \in \mathbb{N}$. By [6, p.115] one sees that $\mathcal{E}'(\mathbb{R}^d)$ is the space of distributions with compact support.

Proposition A.1
If f is a distribution with compact support then $\widehat{f}$ is a C^∞ function with derivatives that have polynomial growth.

Proof:

In this proof we use $f(\eta)$ instead of $\langle f, \eta\rangle$ to emphasize the fact that f is a continuous functional on $\mathcal{E}(\mathbb{R}^d)$. This also gives that the function $f(e^{-2\pi i x \cdot})$ is well defined. Let $\eta \in S$ we then have

$$\langle \widehat{f}, \eta\rangle = \langle f, \widehat{\eta}\rangle = f\left(\int_{\mathbb{R}^n} \eta(x) e^{-2\pi i x \cdot} \mathrm{d}x\right) = \int_{\mathbb{R}^n} \eta(x) f(e^{-2\pi i x\cdot})\mathrm{d}x,$$

provided we justify the last inequality. We wish to show that the Riemann sum converges in the topology of $\mathcal{E}(\mathbb{R}^d)$ to the integral, such that we can use the continuity of the distribution f to move the Riemann sum out of f. Choose $\widetilde{\rho}_{N,0}$, the result for $\widetilde{\rho}_{N,\beta}$ follows from the fact that $(2\pi i x)^\beta \eta(x) \in S$. Take a partition of $\mathbb{R}^d$ into cubes Q_j with sidelength $\frac{1}{2\sqrt{d}N}$, and select $t_j \in Q_j$. We examine the convergence of the Riemann sum and the integral.

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \eta(x) e^{-2\pi i x\xi} \mathrm{d}x - \sum_j \eta(t_j) e^{-2\pi i t_j \xi} |Q_j| \Big| \\
=&\Big| \sum_j \int_{Q_j} \eta(x) e^{-2\pi i x \xi} - \eta(t_j) e^{-2\pi i t_j \xi} \mathrm{d}x \Big| \\
\leq&\sum_j \Big| \int_{Q_j} \eta(x)(e^{-2\pi i x\xi} - e^{-2\pi i t_j \xi})\mathrm{d}x \Big| + \Big| \int_{Q_j} (\eta(x) - \eta(t_j)) e^{-2\pi i t_j \xi} \mathrm{d}x \Big| \\
\leq&\sum_j \int_{Q_j} |\eta(x)||1 - e^{2\pi i (x - t_j)\xi}|\mathrm{d}x + \int_{Q_j} |\eta(x) - \eta(t_j)|\mathrm{d}x, \\
\leq&\sum_j \int_{Q_j} |\eta(x)||1 - e^{2\pi i |x - t_j| N}|\mathrm{d}x + \int_{Q_j} |\eta(x) - \eta(t_j)|\mathrm{d}x,
\end{aligned}$$

where $|\xi| \leq N$ and the last inequality follows from the fact that $2\pi|x - t_j|N \leq \pi$. Now by the use of the dominated convergence theorem on the first and the second term the entire expressions tends to zero as $|Q_j| \to 0$. Next to prove that $f(e^{-2\pi i x \cdot})$ is in C^∞ we show that

$$f_{[\xi]}(-2\pi i \xi^{v_i} e^{-2\pi i x \xi}) = f((e^{-2\pi i x \cdot})^{(v_i)}) = f(e^{-2\pi i x \cdot})^{(v_i)}, \tag{A.1}$$

where v_i is the unitvector in the i'te direction. The result for general α then follows by repeated use of this since $-2\pi i \xi^\alpha e^{-2\pi i x \xi} \in C^\infty$. We need that $(e^{-2\pi i (x+v_i\delta)\xi} - e^{-2\pi i x \xi})\delta^{-1}$ converges to $-2\pi i \xi^{v_i} e^{-2\pi i x \xi}$ in $\mathcal{E}(\mathbb{R}^d)$ for $\delta \to 0$. Choose $\widetilde{\rho}_{N,0}$ and take δ small enough such that $\delta N \leq \frac{1}{2}$. The result for generel β follows from similar calculations with terms that impose lesser conditions. We then have

$$\begin{aligned}
&\left| \left(\frac{e^{-2\pi i (x+v_i\delta)\xi} - e^{-2\pi i x \xi}}{\delta} + 2\pi i \xi_i e^{-2\pi i x \xi} \right) \right| \\
&= \left| \frac{e^{-2\pi i \delta \xi_i} - 1}{\delta} + 2\pi i \xi_i \right| \\
&\leq \left(\frac{|\cos(2\pi\delta\xi_i) - 1|}{\delta} + \left| \frac{\sin(2\pi\delta\xi_i)}{\delta} - 2\pi\xi_i \right| \right) \\
&\leq \left(\frac{|\cos(2\pi\delta N) - 1|}{\delta} + \left| \frac{\sin(2\pi\delta N)}{\delta} - 2\pi N \right| \right)
\end{aligned}$$

for $|\xi| \leq N$. [6, Proposition 2.3.4, p.110] gives that there exists N, m such that

$$|\langle f, \theta \rangle| \leq C \sum_{|\beta| \leq m} \sup_{|\xi| \leq N} |\theta^{(\beta)}(\xi)|,$$

for $\theta \in C^\infty$. By applying this to $f(e^{-2\pi i x \cdot})$ we get the polynomial growth.

■

This lemma is used in showing the norm equivalence in Proposition 1.4.

Lemma A.2
Let $x_j, x_m \in \mathbb{R}^d$, $M, N > 0$ and L be a non-negative integer. Assume that ϕ_m and ϕ_j are two functions on $\mathbb{R}^d$ that satisfy

$$\begin{aligned}
|\partial^\alpha \phi_m(x)| &\leq \frac{A_\alpha 2^{md} 2^{mL}}{(1 + 2^m|x - x_m|)^M}, \quad \text{for all } |\alpha| = L, \\
|\phi_j(x)| &\leq \frac{B 2^{jd}}{(1 + 2^j|x - x_j|)^N},
\end{aligned}$$

for constants A_α, B and

$$\int_{\mathbb{R}^d} \phi_j(x) x^\beta \mathrm{d}x = 0, \quad \text{for all } |\beta| \leq L - 1, \text{ which is void if } L = 0.$$

If $N > M + L + d$ and $j \geq m$, then

$$\left| \int_{\mathbb{R}^d} \phi_m(x)\phi_j(x)\mathrm{d}x \right| \leq C \frac{2^{md}2^{-(j-m)L}}{(1+2^m|x_m - x_j|)^M}. \tag{A.2}$$

Proof:

As in the proof of Lemma 2.1 we introduce a Taylor polynomial of order $L-1$ at the point x_j in the integral on the left-hand side of (A.2). Using the remainder theorem we have by the assumed estimates that

$$\left| \int_{\mathbb{R}^d} \phi_m(x)\phi_j(x)\mathrm{d}x \right|$$
$$\leq B \sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!} \int_{\mathbb{R}^d} \frac{|x-x_j|^L 2^{md}2^{mL}}{(1+2^m|x^0 - x_m|)^M} \frac{2^{jd}}{(1+2^j|x-x_j|)^N}\mathrm{d}x, \tag{A.3}$$

where $x^0 \in l(x, x_j)$. Since $j \geq m$ we have that

$$(1+2^m|x^0 - x_m|)(1+2^j|x-x_j|) \geq 1 + 2^m|x_m - x_j|.$$

Using this estimate in (A.3) we get

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \phi_m(x)\phi_j(x)\mathrm{d}x \right| &\leq C \int_{\mathbb{R}^d} \frac{|x-x_j|^L 2^{md}2^{mL}}{(1+2^m|x_m - x_j|)^M} \frac{2^{jd}}{(1+2^j|x-x_j|)^{N-M}}\mathrm{d}x \\ &\leq C \frac{2^{md}2^{(m-j)L}}{(1+2^m|x_m - x_j|)^M} \int_0^\infty (1+r)^{-N+M+L+d-1}\mathrm{d}r. \end{aligned}$$

Because we have $N > M + L + d$ the integral is finite. ■

Next we present Fefferman Stein's maximal inequality which is a deep result in analysis so therefore we omit the proof, [6, Theorem 4.6.6., p.331]. It was used in showing the norm equivalence of ϕ (Proposition 1.4) and the boundedness of operators on $\dot{f}^s_{p,q}$ and $\dot{b}^s_{p,q}$ (Proposition 2.2).

Proposition A.3

For a locally integrabel function f and $t > 0$ we recall the maximal operator M_t as

$$M_t(f)(x) = \Big(\sup_{\{Q : x \in Q\}} |Q|^{-1} \int_Q |f(y)|^t \mathrm{d}y \Big)^{1/t},$$

where the supremum is taken over all cubes with sidelengths parallel to the axes. If $0 < t < \min\{p, q\}$ then for any sequence of functions f_j and $0 < p < \infty$, $0 < q \leq \infty$ one has

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |M_t(f_j)|^q \Big)^{1/q} \Big\|_{L^p} \leq C \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p}. \tag{A.4}$$

□

Now follows a lemma about the maximal operator which was used in Proposition 2.2 and Lemma 2.3.

Lemma A.4
If $0 < t \leq 1$ and $M > d/t$, then for any $m \in \mathbb{Z}$, any sequence of complex numbers $(h_J)_{I \in D}$, and $x \in I \in D$ one has

$$\sum_{J \in D_m} |h_J| \Big(1 + \frac{|x_I - x_J|}{\max(\ell(I), \ell(J))}\Big)^{-M} \leq C \max((|I|/|J|)^{1/t}, 1) M_t \Big(\sum_{J \in D_m} |h_J| \chi_J\Big)(x).$$

Proof:
We consider two cases. First $|I| \leq 2^{-md}$. Set $\delta = M/d - 1/t$, and define for all $j \in \mathbb{N}$ the set $Q_j = \{J \in D_m : 2^m(x_I - x_J) \in [-2^j, 2^j]\}$. We also define the disjoint sets $\Omega_j = Q_j \setminus Q_{j-1}$ where $\Omega_0 = Q_0$. Then for $x \in I$, we have by the use of Hölder's inequality that

$$\begin{aligned}
&\sum_{J \in D_m} |h_J|(1 + 2^m |x_I - x_J|)^{-M} \\
\leq\, &C \sum_{j=0}^{\infty} \sum_{J \in \Omega_j} |h_J| 2^{-jM} \\
=\, &C \sum_{j=0}^{\infty} 2^{-jd/t - j\delta d} \sum_{J \in \Omega_j} |h_J| \\
\leq\, &C \sup_{j \geq 0} \Big(2^{-jd/t} \sum_{J \in \Omega_j} |h_J|\Big) \Big(\sum_{j=0}^{\infty} 2^{-j\delta d}\Big),
\end{aligned}$$

where the last factor in the second inequality is a constant since $\delta > 0$. We use $0 < t \leq 1$ to take a ℓ^t-norm and introduce an integral in the sum, which is possible due to the disjointness of the $J \in D_m$

$$\begin{aligned}
\leq\, &C \sup_{j \geq 0} \Big(2^{-jd} \sum_{J \in \Omega_j} |h_J|^t\Big)^{1/t} \\
=\, &C \Big(\sup_{j \geq 0} 2^{-jd} 2^{md} \int_{\mathbb{R}^d} \Big(\sum_{J \in \Omega_j} |h_J| \chi_J\Big)^t \mathrm{d}x\Big)^{1/t}.
\end{aligned}$$

We now turn our attention to the size of $|\cup_{J \in Q_j} J|$. It can be estimated by 2^{-md} multiplied with the number of integers in the cube centred in x_I (or the origin) with sidelength 2^{j+1}. So $|\cup_{J \in Q_j} J| \leq 2^{(j+2)d} 2^{-md}$ which yields

$$\begin{aligned}
\leq\, &C \Big(\sup_{j \geq 0} \frac{1}{|\cup_{J \in Q_j} J|} \int_{\cup_{J \in Q_j} J} \Big(\sum_{J \in Q_j} |h_J| \chi_J\Big)^t \mathrm{d}x\Big)^{1/t} \\
\leq\, &C M_t \Big(\sum_{J \in D_m} |h_J| \chi_J\Big)(x).
\end{aligned}$$

If we instead have $|I| > 2^{-md}$, assume that $\ell(I) = 2^{-n}$, where $n < m$. Define the sets $Q_j = \{J \in D_m : 2^n(x_I - x_J) \in [-2^j, 2^j]\}$, and $\Omega_j = Q_j \setminus Q_{j-1}$ where $\Omega_0 = Q_0$. Now in the exactly same fashion as before we find for $x \in I$

$$
\begin{aligned}
&\sum_{J\in D_m} |h_J|(1+2^n|x_I - x_J|)^{-M} \\
&\leq C\sum_{j=0}^{\infty}\sum_{J\in\Omega} |h_J| 2^{-jM} \\
&\leq C\Big(\sup_{j\geq 0} 2^{-jd}\sum_{J\in\Omega_j} |h_J|^t\Big)^{1/t} \\
&= C\Big(\sup_{j\geq 0} 2^{-jd}2^{md}\int_{\mathbb{R}^d}\Big(\sum_{J\in\Omega_j}|h_J|\chi_J\Big)^t \mathrm{d}x\Big)^{1/t}.
\end{aligned}
$$

We estimate the set $\cup_{J\in Q_j} J$ in the same manner as before. To do this we need to estimate the number of integers in cube with sidelengths 2^{j+1}, and multiply this figure with the area of a cubes in D_m. This yields the estimate $|\cup_{J\in Q_j} J| \leq 2^{(j+2-n)d}$

$$
\begin{aligned}
&\leq C2^{(m-n)d/t}\Big(\sup_{j\geq 0}\frac{1}{|\cup_{J\in Q_j} J|}\int_{\cup_{J\in Q_j} J}\Big(\sum_{J\in Q_j}|h_J|\chi_J\Big)^t \mathrm{d}x\Big)^{1/t} \\
&\leq C2^{(m-n)d/t} M_t\Big(\sum_{J\in D_m}|h_J|\chi_J\Big)(x).
\end{aligned}
$$

■

The following Hardy type inequality finds its use among other things in Lemma 3.5.

Lemma A.5
Let $0 \leq \lambda < \theta$, $0 < q \leq \infty$ and let a_n, b_n be two positive sequences. If sequences a_n and b_n satisfy

$$
b_n \leq C\Big(\sum_{m\leq n}(2^{(m-n)\theta}a_m)^{\mu}\Big)^{1/\mu}, \tag{A.5}
$$

then

$$
\Big(\sum_{n\in\mathbb{Z}}(2^{\lambda n}b_n)^q\Big)^{1/q} \leq C\Big(\sum_{n\in\mathbb{Z}}(2^{\lambda n}a_n)^q\Big)^{1/q} \tag{A.6}
$$

Proof:
From (A.5) and the embedding properties of the ℓ^q-norms we have

$$
b_n \leq C\Big(\sum_{m\leq n}\big(2^{(m-n)\theta}a_m\big)^{\mu'}\Big)^{1/\mu'},
$$

for all $0 < \mu' \leq \mu$, especially we will assume that it holds for $\mu' < q$. Choose r so that $\mu'/q + \mu'/r = 1$ and β such that $\lambda < \beta < \theta$. Then by writing $a_m = 2^{m(\beta-\lambda)} a_m 2^{-m(\beta-\lambda)}$ and using Hölder's inequality with exponents q/μ' and r/μ' we have that

$$b_n \leq C 2^{-n\theta} \Big(\sum_{m \leq n} \big(2^{m\theta} 2^{-m(\beta-\lambda)} a_m\big)^q \Big)^{1/q} \Big(\sum_{m \leq n} 2^{m(\beta-\lambda)r} \Big)^{1/r}$$
$$\leq C 2^{-n\theta} 2^{n(\beta-\lambda)} \Big(\sum_{m \leq n} \big(2^{m\theta} 2^{-m(\beta-\lambda)} a_m\big)^q \Big)^{1/q}.$$

This implies that

$$\Big(\sum_{n \in \mathbb{Z}} (2^{n\lambda} b_n)^q \Big)^{1/q} \leq C \Big(\sum_{n \in \mathbb{Z}} 2^{-n(\theta-\beta)q} \sum_{m \leq n} 2^{m(\theta+\lambda-\beta)q} a_m^q \Big)^{1/q}$$
$$= C \Big(\sum_{m \in \mathbb{Z}} 2^{m(\theta+\lambda-\beta)q} a_m^q \sum_{n \geq m} 2^{-n(\theta-\beta)q} \Big)^{1/q}$$
$$\leq C \Big(\sum_{m \in \mathbb{Z}} 2^{m(\theta+\lambda-\beta)q} a_m^q 2^{-m(\theta-\beta)q} \Big)^{1/q}$$
$$= C \Big(\sum_{m \in \mathbb{Z}} (2^{m\lambda} a_m)^q \Big)^{1/q}.$$

■

The next lemma is used in Lemma 2.3.

Lemma A.6
Let $m, n \in \mathbb{Z}$ with $m \geq n$, if $J \in D_n$ and $M > d$ then

$$\sum_{I \in D_m} \left(1 + \frac{|x_I - x_J|}{\ell(J)}\right)^{-M} \leq C 2^{(m-n)d}.$$

Proof:
We have

$$\sum_{I \in D_m} \left(1 + \frac{|x_I - x_J|}{\ell(J)}\right)^{-M} = 2^{(m-n)M} \sum_{j \in \mathbb{Z}^d} (2^{m-n} + |2^m x_J - j|)^{-M}.$$

We wish to use

$$\sum_{j \in \mathbb{Z}^d} (\rho + |j|)^{-M} \leq C \rho^{d-M}, \tag{A.7}$$

$\rho \geq 1$, by which the lemma follows. We will show (A.7) for $d = 1$ the generel result follows by induction (see the proof of [11, Lemma 2.7, p.9]). Choose $a \in \mathbb{Z}$ closest to ρ and estimate the sum by an integral except for $j = a$

$$\sum_{j \in \mathbb{Z} \setminus \{a\}} (\rho + |j|)^{-M} \leq \int_{\mathbb{R}} (\rho + |x|)^{-M} \mathrm{d}x \leq C \rho^{1-M},$$

then estimate the term $j = a$ by ρ^{-M}. ■

This lemma was also used in Lemma 3.5.

Lemma A.7
Let X be a quasi-normed linear space with quasi-norm $\|\cdot\|$. Then there exists an equivalent quasi-norm $\|\cdot\|_0$ and scalar $\mu > 0$ such that $\|\cdot\|_0^\mu$ is subadditiv i.e. for $f, g \in X$ one has

$$\|f+g\|_0^\mu \leq \|f\|_0^\mu + \|g\|_0^\mu. \tag{A.8}$$

Proof:

Let C denote the constant used with the quasi-norm $\|\cdot\|$. Let $C_0 = 2C$ and note that $C \geq 1$ can be chosen. For $f, g \in X$ we immediately have that

$$\|f+g\| \leq \max\{C_0\|f\|, C_0\|g\|\}. \tag{A.9}$$

By repetion one has that

$$\|f_1 + \cdots + f_n\| \leq \max_{1\leq j\leq n}\{C_0^j\|f_j\|\}. \tag{A.10}$$

Take μ such that it satisfies $C_0^\mu = 2$ and define $\|\cdot\|_0$ as

$$\|f\|_0 = \inf_{f=f_1+\cdots+f_n} \left(\|f_1\|^\mu + \cdots + \|f_n\|^\mu\right)^{1/\mu},$$

with the infimum being over all decompositions of $f \in X$. We then have that

$$\|f+g\|_0^\mu \leq \left(\|f_1\|^\mu + \cdots + \|f_m\|^\mu\right) + \left(\|g_1\|^\mu + \cdots + \|g_n\|^\mu\right), \tag{A.11}$$

where $\sum_{j=1}^m f_j = f$ and $\sum_{j=1}^n g_j = g$. Taking the infimum over all decompositions for f and g on the right-hand side of (A.11), we find that (A.8) is fulfilled. Taking the decomposition $f_1 = f$ and $f_j = 0$ for $i \geq 2$ one clearly sees that $\|f\|_0 \leq \|f\|$. For the other inequality define

$$N(f) = \begin{cases} 0, & \text{if } f = 0 \\ C_0^k, & \text{if } C_0^{k-1} < \|f\| \leq C_0^k, \text{ for } k \in \mathbb{Z}. \end{cases}$$

If we can establish that

$$\|f_1 + \cdots + f_n\| \leq C_0\Big(\sum_{i=1}^n N(f_i)^\mu\Big)^{1/\mu}, \tag{A.12}$$

then the equivalence will be proved since $C_0^{-1}N(f) \leq \|f\| \leq N(f)$. We prove (A.12) by induction. For $n = 1$ the equation holds by definition of $N(f)$. Now assume that (A.12) holds for $n-1$, and we shall prove that it also holds for n. Take $f_j \in X$ for $1 \le j \le n$ and without loss of generality we shall assume that $\|f_1\| \ge \cdots \ge \|f_n\|$. First we consider the case were the values of all $N(f_j)$ are distict. This implies that for every $1 \le j < n-1$ one has that

$$N(f_j) \ge C_0 N(f_{j+1}).$$

From this we have

$$C_0^j \|f_j\| \le C_0^j N(f_j) \le C_0 N(f_1) \le C_0 \Big(\sum_{i=1}^n N(f_i)^\mu \Big)^{1/\mu}$$

which together with (A.10) proves (A.12) for this case. If we instead have $N(f_j) = N(f_{j+1}) = C_0^l$ for some $1 \le j < n$ and $l \in \mathbb{Z}$ then one has that $\|f_j + f_{j+1}\| \le C_0 \|f_j\| \le C_0^{l+1}$ by (A.9). This in turn implies that

$$N(f_j + f_{j+1})^\mu \le C_0^{\mu(l+1)} = 2^{l+1} = N(f_j)^\mu + N(f_{j+1})^\mu. \tag{A.13}$$

Using (A.13) and the induction hypothesis we have

$$\begin{aligned} \|f_1 + \cdots + f_n\| &\le C_0 \{ N(f_1)^\mu + \cdots + N(f_j + f_{j+1})^\mu + \cdots + N(f_n)^\mu \}^{1/\mu} \\ &\le C_0 \Big(\sum_{i=1}^n N(f_i)^\mu \Big)^{1/\mu}. \end{aligned}$$

That the triangel inequality holds for $\|\cdot\|_0$ follows from the equivalence with the $\|\cdot\|$-norm. Same for $\|f\|_0 \ge 0$ and $\|f\|_0 = 0$ if and only if $f = 0$. The property $\|af\|_0 = |a| \|f\|_0$ for all scalars a and $f \in X$ follows from the definition of the $\|\cdot\|_0$-norm. ■

To prove that some of the almost diagonal matrices have inverses that are also almost diagonal (Proposition 4.3) we shall need the following two lemmas.

Lemma A.8
Assume that $\ell(J) \le \ell(I)$, $r \in \mathbb{Z}$ and $M > d$. For $x \in \mathbb{R}^d$, let

$$g_{I,J,M,r}(x) = \sum_{K \in D_r} \Big(1 + \frac{|x_K - x_I|}{\max\{\ell(K), \ell(I)\}} \Big)^{-M} \Big(1 + \frac{|x - x_K|}{\max\{\ell(K), \ell(J)\}} \Big)^{-M}. \tag{A.14}$$

Then one has that

$$g_{I,J,M,r}(x) \le C \Big(1 + \frac{|x - x_I|}{\max\{\ell(K), \ell(I)\}} \Big)^{-M} \max \Big\{ 1, \frac{\ell(J)}{\ell(K)} \Big\}^d.$$

Proof:

Note that from the proof of Lemma A.4 with $h_K = 1$ for all $K \in D_r$ and $t = 1$ we have the following inequality for $x \in J$

$$\sum_{K \in D_r} \Big(1 + \frac{|x - x_K|}{\max\{\ell(K), \ell(J)\}} \Big)^{-M} \le C \max \Big\{ 1, \frac{\ell(J)}{\ell(K)} \Big\}^d. \tag{A.15}$$

We consider the case were $|x - x_I| \leq \max\{\ell(K), \ell(I)\}$. Together with the estimate $(1 + |x_K - x_I| / \max\{\ell(K), \ell(I)\})^{-M} \leq 1$ we have that (A.14) can be estimated as

$$\begin{aligned}
g_{I,J,M,r}(x) &\leq \sum_{K \in D_r} \Big(1 + \frac{|x - x_K|}{\max\{\ell(K), \ell(J)\}}\Big)^{-M} \\
&\leq C \max\Big\{1, \frac{\ell(J)}{\ell(K)}\Big\}^d \\
&\leq C \Big(1 + \frac{|x - x_I|}{\max\{\ell(K), \ell(I)\}}\Big)^{-M} \max\Big\{1, \frac{\ell(J)}{\ell(K)}\Big\}^d,
\end{aligned}$$

where we in the last line used the fact that $|x - x_I| \leq \max\{\ell(K), \ell(I)\}$. If $|x - x_I| \geq \max\{\ell(K), \ell(I)\}$ define the sets

$$\begin{aligned}
A_r &= \{K \in D_r : |x_K - x_I| < \tfrac{1}{2}|x - x_I|\} \\
A_r^c &= \{K \in D_r : |x_K - x_I| \geq \tfrac{1}{2}|x - x_I|\}.
\end{aligned}$$

Consider splitting the sum in two

$$g_{I,J,M,r}(x) = \sum_{A_r} + \sum_{A_r^c} = \mathrm{I} + \mathrm{II}.$$

By the properties of A_r^c we have that

$$\Big(1 + \frac{|x_K - x_I|}{\max\{\ell(K), \ell(I)\}}\Big)^{-M} \leq C \Big(1 + \frac{|x - x_I|}{\max\{\ell(K), \ell(I)\}}\Big)^{-M}$$

such that (A.14) for II with the use of (A.15) yields the desired estimate. For $K \in A_r$ notice that $|x - x_K| > \frac{1}{2}|x - x_I|$. By this and the same kind of estimation as (2.7) because we have $|x - x_I| \geq \max\{\ell(K), \ell(I)\}$ and $\ell(J) \leq \ell(I)$ we find for I that

$$\begin{aligned}
&\Big(1 + \frac{|x - x_K|}{\max\{\ell(K), \ell(J)\}}\Big)^{-M} \\
&\leq \Big(1 + \frac{\frac{1}{2}|x - x_I|}{\max\{\ell(K), \ell(I)\}} \frac{\max\{\ell(K), \ell(I)\}}{\max\{\ell(K), \ell(J)\}}\Big)^{-M} \\
&\leq C \Big(\frac{\max\{\ell(K), \ell(J)\}}{\max\{\ell(K), \ell(I)\}}\Big)^d \Big(1 + \frac{|x - x_I|}{\max\{\ell(K), \ell(I)\}}\Big)^{-M}. \qquad \text{(A.16)}
\end{aligned}$$

Using (A.15) with $x = x_I$ we obtain

$$\sum_{K \in D_r} \Big(1 + \frac{|x_K - x_I|}{\max\{\ell(K), \ell(I)\}}\Big)^{-M} \leq C \ell(K)^{-d} (\max\{\ell(K), \ell(I)\})^d.$$

From this and the estimate in (A.16) the lemma follows. ■

Before we give the next lemma we will introduce some new notation. Take $L = d/\min\{1,p,q\}$ for $\dot{f}^s_{p,q}$ and $L = d/\min\{1,p\}$ for $\dot{b}^s_{p,q}$, and recall that

$$\omega^{\alpha}_{\beta,\gamma}(I,J) = \Big(\frac{\ell(I)}{\ell(J)}\Big)^{\alpha}\Big(1+\frac{|x_I - x_J|}{\max\{\ell(I),\ell(J)\}}\Big)^{-L-\beta} \cdot \min\Big\{\Big(\frac{\ell(I)}{\ell(J)}\Big)^{(d+\gamma)/2},\Big(\frac{\ell(J)}{\ell(I)}\Big)^{(d+\gamma)/2+L-d}\Big\},$$

and define

$$W^{\alpha}_{\beta,\gamma_1,\gamma_2}(I,J) = \sum_{K\in D}\omega^{\alpha}_{\beta,\gamma_1}(I,K)\omega^{\alpha}_{\beta,\gamma_2}(K,J).$$

Lemma A.9
Let $\alpha \in \mathbb{R}$. If $\beta,\gamma_1,\gamma_2 > 0$ where $\gamma_1 \neq \gamma_2$ and $\gamma_1+\gamma_2 > 2\beta$ then

$$W^{\alpha}_{\beta,\gamma_1,\gamma_2}(I,J) \leq C\omega^{\alpha}_{\beta,\min\{\gamma_1,\gamma_2\}}(I,J).$$

Proof:
Choose $\gamma = \min\{\gamma_1,\gamma_2\}$. First we consider the case $\ell(J) \leq \ell(I)$. Notice that the factor $\ell(K)^{\alpha}$ in $\omega^{\alpha}_{\beta,\gamma_1}(I,K)$ and $\omega^{\alpha}_{\beta,\gamma_2}(K,J)$ cancel eachother out leaving the factor $(\ell(I)/\ell(J))^{\alpha}$ which can be moved out of the sum. Therefore we only need to deal with the two last factors of $\omega^{\alpha}_{\beta,\gamma_1}(I,K)$ and $\omega^{\alpha}_{\beta,\gamma_2}(K,J)$. We split into three subcases

$$\begin{aligned} W^{\alpha}_{\beta,\gamma_1,\gamma_2}(I,J) &= \Big(\frac{\ell(I)}{\ell(J)}\Big)^{\alpha}\Big(\sum_{\ell(K)<\ell(J)\leq\ell(I)} + \sum_{\ell(J)\leq\ell(K)\leq\ell(I)} + \sum_{\ell(J)\leq\ell(I)<\ell(K)}\Big) \\ &= \Big(\frac{\ell(I)}{\ell(J)}\Big)^{\alpha}\Big(\mathrm{I}+\mathrm{II}+\mathrm{III}\Big). \end{aligned}$$

Assume that $K \in D_r$, $J \in D_m$ and $I \in D_n$. Now for I we have

$$\begin{aligned} \mathrm{I} &= \sum_{r=m+1}^{\infty}\sum_{K\in D_r}\Big(1+\frac{|x_I-x_K|}{\ell(I)}\Big)^{-L-\beta}\Big(\frac{\ell(K)}{\ell(I)}\Big)^{(d+\gamma_1)/2+L-d} \\ &\qquad\cdot\Big(1+\frac{|x_K-x_J|}{\ell(J)}\Big)^{-L-\beta}\Big(\frac{\ell(K)}{\ell(J)}\Big)^{(d+\gamma_2)/2} \\ &= 2^{n((d+\gamma_1)/2+L-d)}2^{m(d+\gamma_2)/2}\sum_{r=m+1}^{\infty}2^{-r((\gamma_1+\gamma_2)/2+L)}g_{I,J,L+\beta,r}(x_J). \end{aligned}$$

We now employ Lemma A.8 to find

$$
\begin{aligned}
&\le C2^{n((d+\gamma_1)/2+L-d)}2^{-m(d-\gamma_2)/2}\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta} \\
&\qquad\qquad\qquad \cdot \sum_{r=m+1}^{\infty} 2^{-r((\gamma_1+\gamma_2)/2+L-d)} \\
&= C2^{(n-m)((d+\gamma_1)/2+L-d)}\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta}\sum_{r=1}^{\infty}2^{-r((\gamma_1+\gamma_2)/2+L-d)} \\
&\le C\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{(d+\gamma)/2+L-d}.
\end{aligned}
$$

For II we first consider $\gamma_1 > \gamma_2$ and find in a similar way as earlier using Lemma A.8 that

$$
\begin{aligned}
\mathrm{II} = {}& 2^{n((d+\gamma_1)/2+L-d)}2^{-m((d+\gamma_2)/2+L-d)}\sum_{r=n}^{m}2^{-r(\gamma_1-\gamma_2)/2}g_{I,J,L+\beta,r}(x_J) \\
\le {}& C2^{n((d+\gamma_1)/2+L-d)}2^{-m((d+\gamma_2)/2+L-d)}2^{-n(\gamma_1-\gamma_2)/2} \\
&\qquad \cdot\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-M}\sum_{r=0}^{\infty}2^{-r(\gamma_1-\gamma_2)/2} \\
= {}& C\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-M}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{(d+\gamma)/2+L-d}.
\end{aligned}
$$

Following the same technique one finds for $\gamma_1 < \gamma_2$ that

$$
\begin{aligned}
\mathrm{II} = {}& 2^{n((d+\gamma_1)/2+L-d)}2^{-m((d+\gamma_2)/2+L-d)}\sum_{r=n}^{m}2^{-r(\gamma_1-\gamma_2)/2}g_{I,J,L+\beta,r}(x_J) \\
\le {}& C2^{n((d+\gamma_1)/2+L-d)}2^{-m((d+\gamma_2)/2+L-d)}2^{-m(\gamma_1-\gamma_2)/2} \\
&\qquad \cdot\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-M}\sum_{r=0}^{\infty}2^{-r(\gamma_2-\gamma_1)/2} \\
= {}& C\Big(1+\frac{|x_J-x_I|}{\ell(I)}\Big)^{-M}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{(d+\gamma)/2+L-d},
\end{aligned}
$$

yielding the estimate for II. We now turn to III and have the following

$$
\mathrm{III} = 2^{-n(d+\gamma_1)/2}2^{-m((d+\gamma_2)/2+L-d)}\sum_{r=-\infty}^{n-1}2^{r((\gamma_1+\gamma_2)/2+L)}g_{I,J,L+\beta,r}(x_J). \tag{A.17}
$$

Using Lemma A.8 we have that

$$
\begin{aligned}
g_{I,J,L+\beta,r}(x_J) &\le C\Big(1+\frac{|x_I-x_J|}{\ell(K)}\Big)^{-L-\beta} \\
&\le C\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta}\Big(\frac{\ell(I)}{\ell(K)}\Big)^{-L-\beta}.
\end{aligned}
$$

Inserting this into (A.17) we find that

$$
\begin{aligned}
\mathrm{III} &\leq C 2^{-n((d+\gamma_1)/2-L-\beta)} 2^{-m((d+\gamma_2)/2+L-d)} \\
&\quad \cdot \Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta} \sum_{r=-\infty}^{n-1} 2^{r((\gamma_1+\gamma_2)/2-\beta)} \\
&= C 2^{-n((d+\gamma_1)/2-L-\beta)} 2^{-m((d+\gamma_2)/2+L-d)} \Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta} \\
&\qquad \cdot 2^{n((\gamma_1+\gamma_2)/2-\beta)} \sum_{r=1}^{\infty} 2^{-r((\gamma_1+\gamma_2)/2-\beta)} \\
&= C\Big(1+\frac{|x_I-x_J|}{\ell(I)}\Big)^{-L-\beta} \Big(\frac{\ell(J)}{\ell(I)}\Big)^{(d+\gamma)/2+L-d},
\end{aligned}
$$

where we in the last equation used that $\gamma_1+\gamma_2>2\beta$ such that the sum is finite. This proves the lemma when $\ell(J)\leq\ell(I)$.

For the case when $\ell(I)\leq\ell(J)$ we observe that $\omega^{\alpha}_{\beta,\gamma}(I,J)=\omega^{L-\alpha-d}_{\beta,\gamma}(J,I)$. From this observation one also has that $W^{\alpha}_{\beta,\gamma_1,\gamma_2}(I,J) = W^{L-\alpha-d}_{\beta,\gamma_2,\gamma_1}(J,I)$. Applying the first case to $W^{L-\alpha-d}_{\beta,\gamma_2,\gamma_1}(J,I)$ proves the lemma for the case $\ell(I)\leq\ell(J)$. ■

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