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Abstract

The stationary and time-dependent von Karman equations are under consideration in this thesis.
In the first part, various preliminary tools are introduced: A product on Sobolev spaces, an existence, uniqueness and regularity theorem for elliptic boundary value problems, some results on the biharmonic operator and the Monge-Ampére form, and finaly some theory on dynamical systems and stability.
In the second part, the von Karman equations are treated: First it is shown, that a weak solution is continuous with respect to time. Then existence and uniqueness theorems for the time-dependent von Karman equations are shown. Next an existence and a regularity theorem for the stationary von Karman equations are shown. The thesis is concluded with a stabilization result for time-dependent von Karman equations with boundary feedback.

Preface

The present thesis is the result of a project in mathematical analysis on the MAT6-term, 2002 at Aalborg University.

The theme of the project is applied mathematical analysis, and we have chosen to work within the frame of the von Karman equations.

The thesis is our master thesis. In this connection it is pointed out that Bjarne Pedersen is responsible for the chapters 2, 3, 9 and 10. Henrik Vie Christensen is responsible for the chapters 6 and 11, and for the chapters 4, 5, 7 and 8 are both responsible.

Thank goes to Professor Jakob Stoustrup at Department of Control Engineering, Aalborg University, for suggestions on where to find references to the stability theory.

Aalborg, June 17th, 2002

Henrik Vie Christensen

Bjarne Pedersen

Dansk resumé/Danish Abstract

Den foreliggende rapport er resultatet af projektarbejdet på MAT6 semestret ved Aalborg Universitet. Vi har valgt at arbejde inden for området "Anvendt matematisk analyse", og emnet der arbejdes med er von Karmans ligninger.

For overskuelighedens skyld er rapporten inddelt i 2 dele.

I den første del arbejdes der med værktøjer, som anvendes i forbindelse med beviserne i anden del. Kapitlerne i dette afsnit er af meget forskellig karakter, hvad angår dybden hvormed de er behandlet.

I kapitel 2 defineres et produkt af funktioner tilhørende Sobolevrum. Det vises at under visse betingelser på ordenen af Sobolevrummene, så vil produktet af to funktioner, tilhørende Sobolevrum af reel orden, tilhøre et Sobolevrum. De første beviser i dette kapitel er en gennemregning af L. Hörmanders fremstilling i [Hörmander, 1997]. Kapitlet afrundes med en udvidelse af L. Hörmanders resultater til Sobolevrum defineret på en åben, begrænset delmængde af $\mathbb{R}^n$, samt et korollar, der giver nogle normvurderinger for Monge-Ampére formen - et resultat der specielt finder anvendelse i kapitel 9.

Kapitel 3 er en gennemregning af L. Hörmanders fremstilling af teorien for elliptiske randværdiproblemer, som det er fremstillet i [Hörmander, 1985]. Det har været ønsket at forbedre resultaterne, således at de var dækkende i forhold til deres anvendelser i kapitel 4, men i forbindelse med arbejdet med beviserne, har det vist sig at dette ikke var generelt muligt. Da formålet med kapitlet har været at give en eksistens-, entydigheds- og regularitetssætning for elliptiske randværdiproblemer, er visse resultater og beviser for pseudo-differentielle operatorer bevidst udeladt. Kapitlet er derfor ikke en rigoristisk gennemregning af teorien for elliptiske rand-

værdiproblemer, men mere en gennemgang af hovedtrækkene, således at læseren kan få et overblik over teorien.

Kapitel 4 og 5 indeholder de hovedresultater for den biharmoniske operator og for Monge-Ampére formen, som anvendes i anden halvdel af projektet. På baggrund af de i kapitel 3 viste resultater, vises en eksistens-, entydigheds- og regularitetssætning for den biharmoniske operator. Herudover vises forskellige kontinuitetsegenskaber for Dirichlet realisationen af den biharmoniske operator, for Monge-Ampére formen og for en trilineær form, der involverer Monge-Ampére formen. Endeligt indføres ved hjælp af den inverse til Dirichlet realisationen en ækvivalent norm på Sobolevrum af reel orden $-2 \leq s \leq 2$.

Den første del afsluttes med kapitel 6, der indeholder definitionen af dynamiske systemer på Banachrum og stabilitet af sådanne. Desuden vises ved hjælp af Lyapunov-funktioner nogle sætninger, der relaterer stabiliteten af et dynamisk system til en Lyapunov funktion for systemet.

I anden del er de sætninger placeret, som direkte relaterer til von Karmans ligninger. Formålet med kapitlerne i denne del har primært været at gennemregne nogle allerede kendte resultater. Beviserne i denne del er derfor hovedsageligt gennemregninger af de beviser, som er givet i hovedkilderne. Afvigende herfra er dog a priori resultatet for kontinuitet af svage løsninger. Med udgangspunkt i de i kapitel 4 fundne resultater vises det, at både funktionen u og dens tidsafledte u' er kontinuerte med hensyn til normtopologien på de vektorrum, hvori de antager deres værdier. Således vises det, at kontinuitets betingelser i definitionen af en svag løsning er overflødige. Desuden er det i kapitel 10 givne regularitetsbevis nyskrevet. Resultatet vises på baggrund af de i kapitel 2 og 4 fundne resultater, således at kun de generelle ideer fra det regularitetsbevis, der er givet i hovedkilden, er anvendt.

Kapitel 7 indeholder dels en kontinuitetssætning for vektorværdi funktioner, der finder anvendelse flere steder i de efterfølgende kapitler, og dels så vises det, at hvis (u, v) tilhører passende vektorværdi-funktionsrum og løser de tidsafhængige von Karman-ligninger som vektordistributioner, så er de kontinuerte på $[0, T]$ med hensyn til normtopologien på de rum, hvori de antager deres værdier.

I kapitel 8 vises eksistenssætningen for svage løsninger til de tidsafhængige von Karman-ligninger med homogene Dirichlet-randbetingelser. Beviset består i at vise, at der eksisterer en svag*-konvergent følge, således at grænseværdien er en svag løsning. Første del af beviset omhandler konstruktionen af denne følge, mens sidste del består i at vise, at grænseværdien af den konstruerede følge er en svag løsning.

I kapitel 9 vises det, at svage løsninger til de tidsafhængige von Karman ligninger med homogene Dirichlet randbetingelser er entydigt bestemte. På baggrund af normvurderinger af visse projektioner på et endeligt dimensionalt underrum af $L^2(\Omega)$, vises det, at differensen mellem 2 svage løsninger opfylder en given normvurdering. Denne normvurdering tillader en anvendelse af Gronwalls lemma, og det konkluderes herved, at for et lille tidsinterval er differensen mellem to svage løsninger 0. De i kapitel 7 fundne kontinuitetsegenskaber anvendes herefter til at udvide intervallet til at omfatte hele definitionsintervallet for løsningerne.

Mens der i de øvrige kapitler i anden del af projektet udelukkende arbejdes med de tidsafhængige von Karman-ligninger, adskiller kapitel 10 sig derved, at der arbejdes med de stationære von Karman ligninger. Dermed er de anvendte metoder også anderledes. Desuden så vil der for dette problem blive arbejdet med ikke-homogene Dirichlet randbetingelser, mens eksistens- og entydighedssætningerne i kapitel 8 og 9 kun omfatter homogene Dirichlet randbetingelser. I kapitlet vises det, at det at finde løsninger til de stationære von Karman-ligninger med ikke-homogene randbetingelser er ækvivalent med at minimere et givet ikke-lineært funktional, og det vises at der eksisterer en funktion der minimere funktionalet, samt at enhver funktion der minimerer det, er en løsning til de stationære von Karman-ligninger. Kapitlet afrundes med et regularitetsbevis for de stationære von Karman-ligninger, hvori det vises, at regularitetssætningen for de stationære von Karman-ligninger modsvarer den regularitetssætning, der i kapitel 4 er vist for den biharmoniske operator.

Det sidste emne der behandles i anden del adskiller sig også fra de andre hvad udgangspunkt angår. Mens synspunktet i de første kapitler i anden del vedrører eksistens, entydighed og regularitet af løsninger, omhandler kapitel 11 stabilitet af løsninger. Målet med dette kapitel er at vise, at der eksisterer en rand-tilbagekoblingsfunktion, således at lukketsløjfe-systemet bliver stabilt. Således arbejdes der i dette kapitel også med en anden klasse af randbetingelser, nemlig de skæve Neumann randbetingelser. Det vises at ved et bestemt valg af energifunktion, så eksisterer der et valg af rand-tilbagekoblingsfunktioner, således at systemet bliver eksponentielt stabilt, forudsat at løsningerne er klassiske i en i kapitlet nærmere defineret forstand.

I appendiks har vi valgt at placere nogle resultater, som ligger til grund for det der arbejdes med i hoveddelene af projektet. Det er resultater som ikke decideret vedrører det behandlede emne, men som er naturligt tilknyttet de behandlede emner. Det første afsnit omhandler homeomorfier, og det vises, at den til et givet tripel knyttede Lax-Milgram operator, er en homeomorfi mellem det vektorrum, hvorpå den er defineret, og dets antidualrum. Andet afsnit indeholder definitionerne af de Sobolevrum, der arbejdes med i rapporten samt et antal egenskaber for disse. Med henblik på behandlingen af svage løsninger er der i tredie afsnit placeret definitionen af vektorværdi-funktioner og -distributioner, og et antal egenskaber ved disse vises. Til sidst opskrives nogle sætninger for ordinære differentialligninger, som anvendes i projektet. Disse sætninger er generaliseringer af kendte sætninger, og beviserne er derfor udeladt.

Det har gennem rapporten været hensigten at homogenisere notationen, således at fremstillingen fremstår konsistent og flydende. Dette har dog ikke i været hensigtsmæssigt alle steder, da det i visse tilfælde ville medføre tab af overblik. I disse tilfælde følger notationen den i hovedkilderne anvendte notation, evt. med mindre ændringer. Desuden så skal det bemærkes at bortset fra nogle få steder, så er afbildninger generelt defineret over det reelle tal-legeme. Enkelte steder arbejdes der, af hensyn til overskueligheden i forbindelse med arbejdet med hovedkilderne, i stedet med det komplekse tal-legeme.

Contents

Chapter 1

Introduction

The present thesis is the product of the project work on the MAT6-term at Aalborg University. The aim of the project has been to investigate some known results regarding the von Karman equations.

The thesis consists of two parts. The first part consists of some preliminary results, which is used in the second part.

The second part consists of various results about the von Karman equations. The results investigated are an existence and uniqueness theorem for the time-dependent von Karman equations with homogeneous boundary conditions, an existence and regularity result for the stationary von Karman equations, and a stability result for the time-dependent von Karman equations with boundary feedback.

In more detail this thesis contains:

Part I - Preliminaries

Chapter 2 - Products in Sobolev Spaces: Introduction of a product of functions belonging to Sobolev spaces. It is shown that the product of two functions belongs to a Sobolev space, provided that the order of the Sobolev spaces fulfil certain relations. The aim of the chapter is to provide a tool to evaluate the norm of the Monge-Amére form. Thus the chapter is concluded with a corollary, which gives the necessary expressions for the norm.

Chapter 3 - Elliptic Boundary Value Problems: Provides an existence, uniqueness, and regularity theorem for elliptic boundary value problems. Using pseudo-differential methods it is shown, that an elliptic boundary value problem defines a Fredholm operator. This theorem is then used to provide the existence, uniqueness, and regularity theorem.

Chapter 4 - The Biharmonic Operator: Some results for the biharmonic operator, which will be needed in the second part of the project. It is shown that the biharmonic operator together with the first two Dirichlet traces defines an elliptic boundary value problem, and then an existence, uniqueness, and regularity statement for this problem is proved using results from chapter 3. Moreover various lemmas regarding the biharmonic operator are shown here. Especially it is proved, that the biharmonic operator is a homeomorfism between suitable Sobolev spaces, and this result is used to define an equivalent norm in the Sobolev spaces.

Chapter 5 - The Monge-Ampére Form: Definition of the Monge-Ampére form and two related forms. It is shown that in suitable Sobolev spaces these forms coincide. Moreover the chapter contains some symmetry and continuity results for the Monge-Ampére form.

Chapter 6 - Dynamical Systems and Stability: Introduction of dynamical systems in a Banach space and stability of such systems. Lyapunov functions are defined, and it is shown how stability of a dynamical system is related to the Lyapunov functions.

Part II - The von Karman Equations

Chapter 7 - Continuity of Weak Solutions: It is proved that weak solutions to the von Karman equations, belonging to certain vector valued function spaces, are norm continuous. Especially this chapter contains the result, that it is not necessary to put a continuity condition on the functions, when defining the concept of a weak solution.

Chapter 8 - Existence of Weak Solutions: Main result for the common part of this thesis. It is shown, that there exists solutions to the time-dependent von Karman equations.

Chapter 9 - Uniqueness of Weak Solutions: It is proved that there exists at most one solution to the time-dependent von Karman equations. Using the results obtained in Chapter 2 it is shown that the difference between two weak solutions fulfil a norm relation. This relation allows one to apply the Gronwall lemma to obtain, that the solutions are equal on a small closed time interval, and it is then shown, that this interval can be enlarged to cover the interval of definition.

Chapter 10 - Stationary von Karman Equations: Existence of solutions to the stationary von Karman equations with non-homogeneous boundary conditions. Using a certain non-linear functional it is shown, that solving the stationary von Karman equations is equivalent to minimizing this functional, and it is then shown, that there exists such a minimizer. Moreover, the chapter contains a regularity result for stationary von Karman equations. Using the results obtained in Chapter 2 and 4 it is proved, that the regularity statement for the stationary von Karman equations is exactly that of the biharmonic operator with Dirichlet boundary conditions.

Chapter 11 - Boundary Stabilization of von Karman Plates: Provides a stability result for the time-dependent von Karman equations with boundary feedback. It is shown, that there exists a special choice of boundary feedback, such that the closed-loop dynamical system becomes exponentially stable in the sense defined in Chapter 6.

Appendix A - Prerequisites: Some results on homeomorfisms, Sobolev spaces, vector valued distributions and -functions, and ordinary differential equations are placed here for convenient reference.

Appendix B - Nomenclature: List of symbols used in this thesis.

1.1 The von Karman Equations

In this section the equations under consideration are presented. The authors have chosen not to put an effort into the physical description of thin elastic plates, and the derivation of the equations which follows thereof. Thus the equations are only stated here.

In this thesis two sets of equations are considered. The first is the time-dependent von Karman equations:

$$\begin{aligned} u'' + a_1\Delta^2 u - [u,v] &= f \quad \text{on } \;]0,T[\; \times \Omega \\ a_2\Delta^2 v + [u,u] &= 0 \quad \text{on } \;]0,T[\; \times \Omega \end{aligned} \tag{1.1}$$

The second is the stationary von Karman equations:

$$\begin{aligned} \Delta^2 u - [u,v] &= f \\ \Delta^2 v + [u,u] &= 0 \end{aligned} \tag{1.2}$$

for $x \in \Omega$, where

$$[u,v] = (\partial_{11}^2 u)(\partial_{22}^2 v) - 2(\partial_{12}^2 u)(\partial_{12}^2 v) + (\partial_{22}^2 u)(\partial_{11}^2 v)$$

For the time-dependent problem a set of initial conditions is needed. The following set is considered

$$\begin{aligned} u(0,x) &= u_0(x) \\ u'(0,x) &= u_1(x) \end{aligned} \tag{1.3}$$

for $x \in \Omega$.

It should be noted, that only initial conditions for u is prescribed, but it will be shown in chapter 8 that the absence of time derivatives on v, implies that the time dependence of v is given through u, and thus no initial conditions for v are needed.

To solve the time-dependent problem, a set of boundary conditions are necessary. In the chapters concerning existence and uniqueness the following set of homogeneous Dirichlet boundary conditions are considered:

$$\begin{aligned} \gamma_0 u = \gamma_1 u &= 0 \\ \gamma_0 v = \gamma_1 v &= 0 \end{aligned} \tag{1.4}$$

for $(t,x) \in \;]0,T[\; \times \Gamma$, whereas in chapter 11 the following set of boundary conditions with boundary feedback is considered:

$$\begin{aligned} \gamma_0 v = \gamma_1 v &= 0 \\ \Delta u + (1-\mu)B_1 u &= -\alpha(m\cdot\nu)\frac{\partial u}{\partial \nu} \\ \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \tau} &= \lambda(m\cdot\nu)u' + \beta(m\cdot\nu)u - \alpha\frac{\partial}{\partial \tau}((m\cdot\nu)\frac{\partial u}{\partial \tau}) \\ B_1 u &= 2\nu_1\nu_2\partial_{12}^2 u - \nu_1^2\partial_{22}^2 u - \nu_2^2\partial_{11}^2 u \\ B_2 u &= (\nu_1^2 - \nu_2^2)\partial_{12}^2 u + \nu_1\nu_2(\partial_{22}^2 u - \partial_{11}^2 u) \end{aligned} \tag{1.5}$$

for $(t,x) \in \,]0,T[\, \times \Gamma$, where $0 < \mu < \frac{1}{2}$, $\alpha, \beta, \lambda > 0$, ν is the unit normal vector field and τ is the unit tangent vector field, given by a counter-clockwise orientation of Γ.

Also for the stationary von Karman equations, a set of boundary conditions are necessary. The following set of non-homogeneous Dirichlet boundary conditions is considered:

$$\begin{aligned} \gamma_0 u &= \gamma_1 u = 0 \\ \gamma_0 v &= \phi_0 \\ \gamma_1 v &= \phi_1 \end{aligned} \tag{1.6}$$

for $(t,x) \in \,]0,T[\, \times \Gamma$.

1.2 Weak solutions

In this section the concept of being a weak solution to the time-dependent von Karman equations is introduced.

In [Boutet de Monvel and Chueshov, 1998, p. 420] a weak solution to the time-dependent von Karman equations is defined as

Definition 1.1
If

$$f \in L^2([0,T] \times \Omega), \quad u_0 \in H_0^2(\overline{\Omega}), \quad u_1 \in L^2(\Omega)$$

then (u,v) *is said to be a weak solution to* (1.1), (1.3) *and* (1.4) *if*

$$\begin{aligned} u &\in L^\infty(0,T; H_0^2(\overline{\Omega})) \\ v &\in L^\infty(0,T; H_0^2(\overline{\Omega})) \\ u' &\in L^\infty(0,T; L^2(\Omega)) \end{aligned}$$

and they satisfy the following conditions:

1. (1.1) *is satisfied in* $\mathscr{D}'(0,T; H^{-2}(\bar{\Omega}))$.
2. *The vector valued function* $t \mapsto (u(t), u'(t)) \in H_0^2(\overline{\Omega}) \times L^2(\Omega)$ *is weakly continuous with* $(u(0), u'(0)) = (u_0, u_1)$.

Remark: In [Lions, 1969] the concept of weak solutions is slightly differently introduced. Here condition (2) is replaced with the following:

> $2'$. $u(t)$ and $u'(t)$ are norm continuous with values in $H_0^2(\overline{\Omega})$ respectively $L^2(\Omega)$, with $u(0) = u_0$ and $u'(0) = u_1$.

However, in chapter 7 it will be shown, that if

$$
\begin{aligned}
u &\in L^\infty(0,T;H_0^2(\overline{\Omega})) \\
v &\in L^\infty(0,T;H_0^2(\overline{\Omega})) \\
u' &\in L^\infty(0,T;L^2(\Omega))
\end{aligned}
$$

solves (1.1) in $\mathscr{D}'(0,T;H^{-2}(\bar{\Omega}))$, then (2) as well as (2′) holds. Thus the continuity condition given in definition 1.1 as well as the one given in [Lions, 1969] is contained in condition (1).

1.3 Notation

It has been one of the aims of the project to homogenize the notation, such that in the general the notation will follow that of [Grubb, 2000]. However, in some of the chapters, it has not been sensible to follow the notation in [Grubb, 2000] entirely. The main reason for this has been a wish to keep the track of things, when investigating the sources for the chapters. On these occasions the notation has been carried over from the main sources for the individual chapters.

One convention used in general is, that whenever it is obvious which norm there is referred to, the subscript, indicating the space, is left out. Especially when referring to Sobolev spaces, a norm or inner product without subscript refers to the norm or inner product in L^2.

Moreover it should be noted, that whenever nothing else is noted, the scalar field is $\mathbb{R}$.

Part I

Preliminaries

Chapter 2

Products in Sobolev Spaces

In this chapter a product in Sobolev spaces will be addressed. The chapter is mainly an investigation of the proofs given in [Hörmander, 1997, p. 189ff]. In relation hereto, only the corollaries in the end of the chapter are new.

It will be shown that under suitable restrictions on the order of the Sobolev spaces, the product of two functions $u_i \in H^{s_i}(\mathbb{R}^n)$ will be in $H^s(\mathbb{R}^n)$, with

$$\|u_1 u_2\|_s \leq c\|u_1\|_{s_1}\|u_2\|_{s_2}$$

These bounds are of special use in connection with the Monge-Ampére form, and in the end of the chapter a corollary on the bounds of the Monge-Ampére form will be shown.

Definition 2.1
Let $f, g \in \mathscr{S}'(\mathbb{R}^n)$. Then the product fg is defined as the element of $\mathscr{S}'(\mathbb{R}^n)$ such that

$$\langle fg, \phi\rangle = \langle f, g\phi\rangle$$

for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, whenever the duality on the right hand side makes sense.

The first thing to concider is, whether the product is well defined. The following lemma covers some of this issue. What remains is to check that the product of two functions $u_i \in H^{s_i}(\bar{\Omega})$ is independent of s_i. However, due to lack of time, this part has been left out.

Lemma 2.2
Assume that $u_i \in H^{s_i}(\mathbb{R}^n)$, $i = 1, 2$, *and* $s_1 + s_2 \geq 0$. *Then the product* $u_1 u_2 \in \mathscr{S}'(\mathbb{R}^n)$ *is well defined.*

Proof:
Let $u_i \in H^{s_i}(\mathbb{R}^n)$, $i = 1, 2$, and $s_1 + s_2 \geq 0$, and let $\phi \in \mathscr{S}(\mathbb{R}^n)$. It is assumed for simplicity that $s_1 \geq s_2$. The case $s_2 \geq s_1$ is treated similarly. It now follows, since $H^{s_2}(\mathbb{R}^n) \overset{\mathrm{d}}{\hookrightarrow} H^{-s_1}(\mathbb{R}^n) = (H^{s_1}(\mathbb{R}^n))^*$, that

$$\begin{aligned} |\langle u_1, u_2\phi\rangle| &\leq c_1 \|u_1\|_{s_1} \|\phi u_2\|_{-s_1} \\ &\leq c_2 \|u_1\|_{s_1} \|\phi u_2\|_{s_2} \leq c_3 \|\phi\|_M \|u_1\|_{s_1} \|u_2\|_{s_2} \end{aligned}$$

for some suitable semi-norm $\|\cdot\|_M$ in $\mathscr{S}$, which proves that $\langle u_1, u_2\phi\rangle$ depends linearly and continuously on $\phi \in \mathscr{S}(\mathbb{R}^n)$ and thus determines a unique element $u_1 u_2 \in \mathscr{S}'(\mathbb{R}^n)$. □

The next lemma is a purely technical lemma, which serves as the main tool, when showing the main theorem of this chapter. The lemma is a variant of Schur's lemma as stated in [Hörmander, 1985, p. 74].

Lemma 2.3
Let $F : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ *be a piecewise continuous function, and let*

$$T_F(f, g)(\xi) = \int_{\mathbb{R}^n} F(\xi, \eta) f(\eta) g(\xi - \eta)\, d\eta$$

for $f, g \in C_0(\mathbb{R}^n)$.

If there exists an $M \geq 0$ *such that one of the following properties holds*

$$(1) \qquad \int_{\mathbb{R}^n} |F(\xi, \eta)|^2\, d\eta \leq M^2 \quad \forall \xi \tag{2.1}$$

$$(2) \qquad \int_{\mathbb{R}^n} |F(\xi, \eta)|^2\, d\xi \leq M^2 \quad \forall \eta \tag{2.2}$$

$$(3) \qquad \int_{\mathbb{R}^n} |F(\xi, \xi - \eta)|^2\, d\xi \leq M^2 \quad \forall \eta \tag{2.3}$$

then $T_F(f, g) \in L^2(\mathbb{R}^n)$, *and*

$$\|T_F(f, g)\|_0 \leq M \|f\|_0 \|g\|_0$$

Proof:
Throughout the proof it is assumed that $f, g \in C_0(\mathbb{R}^n)$.

First assume that (2.1) holds.

Then by Cauchy-Schwarz' inequality it follows that

$$\begin{aligned}
|T_F(f,g)(\xi)|^2 &= \Big| \int_{\mathbb{R}^n} F(\xi,\eta) f(\eta) g(\xi-\eta)\, d\eta \Big|^2 \\
&\leq \Big(\int_{\mathbb{R}^n} |F(\xi,\eta) f(\eta) g(\xi-\eta)|\ d\eta \Big)^2 \\
&\leq \int_{\mathbb{R}^n} |F(\xi,\eta)|^2\, d\eta \int_{\mathbb{R}^n} |f(\eta) g(\xi-\eta)|^2\, d\eta \\
&\leq M^2 \int_{\mathbb{R}^n} |f(\eta) g(\xi-\eta)|^2\, d\eta
\end{aligned}$$

and thus by Fubini's theorem and by the translation invariance of the Lebesgue measure

$$\begin{aligned}
\|T_F(f,g)\|^2 &= \int_{\mathbb{R}^n} |T_F(f,g)|^2\, d\xi \\
&\leq M^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\eta) g(\xi-\eta)|^2\, d\eta\, d\xi \\
&= M^2 \int_{\mathbb{R}^n} |f(\eta)|^2 \int_{\mathbb{R}^n} |g(\xi-\eta)|^2\, d\xi\, d\eta \\
&= M^2 \|f\|_0^2 \|g\|_0^2
\end{aligned}$$

proving the theorem in the present case.

Next assume that (2.2) holds.

Then for all $h \in C_0(\mathbb{R}^n)$ it follows as in the first part of the proof that

$$\int_{\mathbb{R}^{2n}} |F(\xi,\eta) g(\xi-\eta) h(\xi)|^2\, d\xi\, d\eta \leq M^2 \|g\|_0^2 \|h\|_0^2$$

so

$$\Big|\int\limits_{\mathbb{R}^n} T_F(f,g)(\xi)h(\xi)\,d\xi\Big|^2 = \Big|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} F(\xi,\eta)f(\eta)g(\xi-\eta)h(\xi)\,d\eta\,d\xi\Big|^2$$

$$= \Big|\int\limits_{\mathbb{R}^n} f(\eta)\int\limits_{\mathbb{R}^n} F(\xi,\eta)g(\xi-\eta)h(\xi)\,d\xi\,d\eta\Big|^2$$

$$\leq \int\limits_{\mathbb{R}^n} |f(\eta)|^2\,d\eta \int\limits_{\mathbb{R}^{2n}} |F(\xi,\eta)g(\xi-\eta)h(\xi)|^2\,d\xi\,d\eta$$

$$\leq M^2\|f\|_0^2\|g\|_0^2\|h\|_0^2$$

by Fubini's theorem, the translation invariance of the Lebesgue measure, and the Cauchy-Schwarz inequality. This proves the second statement.

Lastly assume that (2.3) holds, and note that by a change of variables

$$|T_F(f,g)(\xi)| = \Big|\int\limits_{\mathbb{R}^n} F(\xi,\eta)f(\eta)g(\xi-\eta)\,d\eta\Big|$$

$$= \Big|\int\limits_{\mathbb{R}^n} F(\xi,\xi-\eta)f(\xi-\eta)g(\eta)\,d\eta\Big|$$

$$= \Big|\int\limits_{\mathbb{R}^n} G(\xi,\eta)f(\xi-\eta)g(\eta)\,d\eta\Big|$$

where

$$\int\limits_{\mathbb{R}^n} |G(\xi,\eta)|^2\,d\xi = \int\limits_{\mathbb{R}^n} |F(\xi,\xi-\eta)|^2 \leq M^2$$

for all $\eta \in \mathbb{R}^n$. Hence the third statement follows from the second. □

Corollary 2.4
There exists an extension $\tilde{T}_F$ of T_F, such that for all $f,g \in L^2(\mathbb{R}^n)$,

$$\|\tilde{T}_F(f,g)\|_0 \leq M\|f\|_0\|g\|_0$$

whenever (2.1), (2.2) *or* (2.3) *holds.*

Proof:
Since $C_0(\mathbb{R}^n)$ is dense in $L^2(\Omega)$, it follows that for all $f \in L^2(\mathbb{R}^n)$ there exists a sequence f_n in $C_0(\mathbb{R}^n)$, such that $f_n \to f$ in $L^2(\mathbb{R}^n)$ as $n \to \infty$.

Now define for $f, g \in L^2(\mathbb{R}^n)$

$$\tilde{T}_F(f,g) := \lim_{n,k\to\infty} T_F(f_n, g_k)$$

where $f_n, g_k \in C_0(\mathbb{R}^n)$, $f_n \to f$ and $g_k \to g$ in $L^2(\mathbb{R}^n)$ as $n, k \to \infty$.

It then follows that if $f, g \in C_0(\mathbb{R}^n)$, then $\tilde{T}_F(f,g) = T_F(f,g)$, so $\tilde{T}_F$ is an extension of T_F.

Assume now, that (2.1), (2.2) or (2.3) holds. It then follows from Lemma 2.3 that

$$\|\tilde{T}_F(f,g)\| = \lim_{n,k\to\infty} \|T_F(f_n,g_k)\| \le c_1 \lim_{n,k\to\infty} \|f_n\|_0\|g_k\|_0 = c_1\|f\|_0\|g\|_0$$

□

The following theorem is the main theorem of this chapter. It serves to provide a product in Sobolev spaces of real order, when the pointwise product no longer suffices.

Theorem 2.5
Assume that $u_i \in H^{s_i}(\mathbb{R}^n)$, $i = 1, 2$, and that $s_1 + s_2 \ge 0$. Then $u_1u_2 \in H^s(\mathbb{R}^n)$ if

$$s \le s_1 \quad s \le s_2 \quad s \le s_1 + s_2 - \frac{n}{2}$$

where the last inequality must be strict if $s_1 = \frac{n}{2}$, $s_2 = \frac{n}{2}$ or $s = -\frac{n}{2}$.

Moreover

$$\|u_1u_2\|_s \le c\|u_1\|_{s_1}\|u_2\|_{s_2}$$

Proof:
Assume that $u_i \in H^{s_i}(\mathbb{R}^n)$, with $s_1 + s_2 \ge 0$.

If $u_1, u_2 \in L^2(\mathbb{R}^n)$, i.e. if $s_1, s_2 \ge 0$, then

$$\langle\xi\rangle^s \widehat{u_1u_2}(\xi) = \int_{\mathbb{R}^n} \langle\xi\rangle^s \langle\xi-\eta\rangle^{-s_1}\langle\eta\rangle^{-s_2}\langle\xi-\eta\rangle^{s_1}\hat{u_1}(\xi-\eta)\langle\eta\rangle^{s_2}\hat{u_2}(\eta)\,d\eta \quad (2.4)$$

This motivates, that the first part of the proof consists of showing that

$$\xi \mapsto \langle\xi\rangle^s \widehat{u_1u_2}(\xi) = \int_{\mathbb{R}^n} \langle\xi\rangle^s \langle\xi-\eta\rangle^{-s_1}\langle\eta\rangle^{-s_2}\langle\xi-\eta\rangle^{s_1}\hat{u_1}(\xi-\eta)\langle\eta\rangle^{s_2}\hat{u_2}(\eta)\,d\eta \quad (2.5)$$

is in $L^2(\mathbb{R}^n)$ when $u_i \in H^{s_i}(\mathbb{R}^n)$. Afterwards (2.4) will be generalized to cover the case when for some i, $s_i < 0$.

Let the following functions be defined:

$$f_1(\xi) = \langle\xi\rangle^{s_1}\hat{u_1}(\xi)$$
$$f_2(\xi) = \langle\xi\rangle^{s_2}\hat{u_2}(\xi)$$
$$F(\xi,\eta) = \langle\xi\rangle^{s}\langle\xi-\eta\rangle^{-s_1}\langle\eta\rangle^{-s_2}$$

and note, that then equation (2.5) can be written on the form

$$\int_{\mathbb{R}^n} F(\xi,\eta) f_1(\xi-\eta) f_2(\eta)\, d\eta \in L^2(\mathbb{R}^n)$$

Now, since $u_i \in H^{s_i}(\mathbb{R}^n)$, it follows that $f_i \in L^2(\mathbb{R}^n)$.

The following part of the proof consists of partitioning $\mathbb{R}^{2n}$ into three disjoint sets, such that the union has a zero-measure complement. It will then be shown that when F is multiplied with the characteristic function of each of the sets, it fulfils one of the conditions given in Lemma 2.3.

Let $\mathbb{R}^{2n}$ be divided into the following three sets:

$$A_1 = \{(\xi,\eta) \in \mathbb{R}^{2n} \,|\, \langle\eta\rangle < \langle\xi\rangle/2\}$$
$$A_2 = \{(\xi,\eta) \in \mathbb{R}^{2n} \,|\, \langle\xi-\eta\rangle < \langle\xi\rangle/2\}$$
$$A_3 = \{(\xi,\eta) \in \mathbb{R}^{2n} \,|\, 2\langle\eta\rangle > \langle\xi\rangle, 2\langle\xi-\eta\rangle > \langle\xi\rangle\}$$

Now the following inequality holds

$$\langle\xi\rangle^2 \le \langle\xi-\eta\rangle^2 + \langle\eta\rangle^2$$

Then, using the definitions of A_1 and A_2, it follows that $A_1 \cap A_2 = \emptyset$, so the A_i are all disjoint. Moreover $\mathbb{R}^{2n} \setminus (\cup A_i)$ is a finite union of $2n-1$-dimensional surfaces in $\mathbb{R}^{2n}$, and thus it is a Lebesgue null-set.

In the following part, estimates for $\int |F|^2$ will be shown for each of the sets A_i.

A_1: Let $F(\xi,\eta)$ be multiplied by the characteristic function of A_1. Then, since $\langle\eta\rangle < \langle\xi\rangle/2$, it follows that $|\chi_{A_1} F(\xi,\eta)| \le c\langle\xi\rangle^{s-s_1}\langle\eta\rangle^{-s_2}$, which is seen in the following way:

It is noted that

$$\begin{aligned}\langle \xi - \eta \rangle^2 &= 1 + |\xi - \eta|^2 \\ &\leq 1 + (|\xi| + |\eta|)^2 \\ &\leq 1 + (\frac{3}{2}\langle \xi \rangle)^2 \\ &= 1 + \frac{9}{4} + \frac{9}{4}|\xi|^2 \\ &\leq \frac{13}{4}(1 + |\xi|^2)\end{aligned}$$

so there exists a $c > 1$, such that $\langle \xi - \eta \rangle \leq c\langle \xi \rangle$, so if $s_1 < 0$, it follows that $\langle \xi - \eta \rangle^{-s_1} \leq c_1 \langle \xi \rangle^{-s_1}$.

Similarly

$$\begin{aligned}\langle \xi - \eta \rangle^2 &= 1 + |\xi - \eta|^2 \\ &\geq 1 + (|\xi| - |\eta|)^2 \\ &\geq 1 + (\frac{1}{2}\langle \xi \rangle)^2 \\ &\geq \frac{1}{4}(1 + |\xi|^2)\end{aligned}$$

so there exists a $c < 1$, such that $\langle \xi - \eta \rangle \geq c\langle \xi \rangle$, so if $s_1 \geq 0$, it follows that $\langle \xi - \eta \rangle^{-s_1} \leq c_2 \langle \xi \rangle^{-s_1}$.

Thus there exists a $c > 0$, such that $|\chi_{A_1} F(\xi, \eta)| \leq c\langle \xi \rangle^{s-s_1} \langle \eta \rangle^{-s_2}$ as claimed.

To show the existence of a bound M for F, uniform in ξ, the following cases are considered:

$s_2 > \frac{n}{2}$: Since by assumption $s \leq s_1$, it follows that $\langle \xi \rangle^{s-s_1}$ is uniformly bounded in ξ, and since $s_2 > \frac{n}{2}$, it follows that there exists an $M < \infty$ such that

$$\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi, \eta)|^2 \, d\eta \leq c \int_{\langle \eta \rangle < \langle \xi \rangle / 2} \langle \eta \rangle^{-s_2} \, d\eta \leq M^2$$

for all $\xi \in \mathbb{R}^n$.

$s_2 = \frac{n}{2}$: Then $s < s_1 + s_2 - \frac{n}{2}$ by assumption, and thus $s < s_1$, so $\langle \xi \rangle^{s-s_1} < c\langle \eta \rangle^{s-s_1}$, since $\langle \eta \rangle < \langle \xi \rangle / 2$. Then there exists an $M < \infty$ such that for all $\xi \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi, \eta)|^2 \, d\eta \leq c \int_{\langle \eta \rangle < \langle \xi \rangle / 2} \langle \eta \rangle^{2(s-s_1)-2s_2} \, d\eta \leq M^2$$

since $2(s-s_1)-2s_2 < -n$.

$s_2 < \frac{n}{2}$: If $s_2 \leq 0$ there exists an $M < \infty$, such that for all $\xi \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi,\eta)|^2 \, d\eta \leq c \int_{\langle\eta\rangle < \langle\xi\rangle/2} \langle\xi\rangle^{2(s-s_1)-2s_2} \, d\eta \leq M^2 \langle\xi\rangle^{2(s-s_1)-2s_2+n} \leq M^2$$

since $2(s-s_1)-2s_2+n \leq 0$

Assume then that $0 < 2s_2 < n$ and $2s \leq 2s_1 + 2s_2 - n$. Then

$$\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi,\eta)|^2 \, d\eta \leq c_1 \langle\xi\rangle^{2s_2-n} \int_{2\langle\eta\rangle < \langle\xi\rangle} \langle\eta\rangle^{-2s_2} \, d\eta$$

By introducing polar coordinates and evaluating the integral it follows that there exists constants $c_1, c_2 > 0$, such that

$$\int_{2\langle\eta\rangle < \langle\xi\rangle} \langle\eta\rangle^{-2s_2} \, d\eta \leq c_1 + c_2 \langle\xi\rangle^{n-2s_2}$$

so since $2s_2 < n$, it follows that there exists an $M < \infty$, such that

$$\begin{aligned}\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi,\eta)|^2 \, d\eta &\leq c_1 \langle\xi\rangle^{2s_2-n} \int_{2\langle\eta\rangle < \langle\xi\rangle} \langle\eta\rangle^{-2s_2} \, d\eta \\ &\leq c_3 \langle\xi\rangle^{2s_2-n} + c_4 \\ &\leq M^2\end{aligned}$$

Thus it is concluded, that there exists an $M < \infty$, such that

$$\int_{\mathbb{R}^n} |\chi_{A_1} F(\xi,\eta)|^2 \, d\eta \leq M^2$$

for all $\xi \in \mathbb{R}^n$, and hence by Corollary 2.4

$$\tilde{T}_{\chi_{A_1} F}(f_1, f_2) \in L^2(\mathbb{R}^n)$$

A_2**:** It should be noted that

$$F(\xi, \xi-\eta) = \langle\xi\rangle^s \langle\eta\rangle^{-s_1} \langle\xi-\eta\rangle^{-s_2}$$

and by the same arguments as for A_1, it is seen that there exists a $c > 0$, such that $F(\xi,\xi-\eta) \leq c\langle\xi\rangle^{s-s_1}\langle\xi-\eta\rangle^{-s_2}$, so this case is treated with exactly the same arguments as for A_1, this time using (2.3), and it is concluded, that there exists an $M < \infty$, such that

$$\int_{\mathbb{R}^n} |\chi_{A_2} F(\xi, \xi - \eta)|^2 \, d\xi \leq M^2$$

for all $\eta \in \mathbb{R}^n$, and hence by Corollary 2.4

$$\tilde{T}_{\chi_{A_2} F}(f_1, f_2) \in L^2(\mathbb{R}^n)$$

A_3: First the subset of A_3, where $\langle \xi \rangle \leq 4$ is considered.

If $s_1 < 0$, then by Peetre's inequality [Grubb, 1996b, p. 9.5]

$$F(\xi, \eta) \leq c \langle \xi \rangle^{s - s_1} \langle \eta \rangle^{-(s_1 + s_2)} \leq c$$

the last inequality following since $s \leq s_1$ and $s_1 + s_2 \geq 0$. So since the set where $\langle \xi \rangle \leq 4$ is bounded, it follows that there exists an $M < \infty$ such that for all $\eta \in \mathbb{R}^n$

$$\int_{\langle \xi \rangle \leq 4} |F(\xi, \eta)|^2 \, d\xi \leq M^2$$

If $s_2 < 0$ it can be shown in a similar way that there exists an $M < \infty$ such that for all $\eta \in \mathbb{R}^n$

$$\int_{\langle \xi \rangle \leq 4} |F(\xi, \eta)|^2 \, d\xi \leq M^2$$

If $s_1 \geq 0$ and $s_2 \geq 0$. Then

$$F(\xi, \eta) \leq c \langle \xi \rangle^s$$

so since the set where $\langle \xi \rangle \leq 4$ is bounded, it follows that there exists an $M < \infty$ such that for all $\eta \in \mathbb{R}^n$

$$\int_{\langle \xi \rangle \leq 4} |F(\xi, \eta)|^2 \, d\xi \leq M^2$$

For all $\eta \in \mathbb{R}^n$ define

$$A_3' = \{\xi \in \mathbb{R}^n \mid 2\langle \eta \rangle > \langle \xi \rangle, 2\langle \xi - \eta \rangle > \langle \xi \rangle, \langle \xi \rangle < 4\}$$

It then follows, that $\langle \eta \rangle > 2$ and $\langle \xi - \eta \rangle > 2$.

Since $\langle \xi \rangle > 4$, $\langle \eta \rangle > 2$ and $\langle \xi - \eta \rangle > 2$, there exists constants c_1, c_2, c_3, such that

$$|\xi| \le \langle \xi \rangle \le c_1 |\xi|$$
$$|\eta| \le \langle \eta \rangle \le c_2 |\eta|$$
$$|\xi - \eta| \le \langle \xi - \eta \rangle \le c_3 |\xi - \eta|$$

so

$$\int_{A_3'} \langle \xi \rangle^{2s} \langle \xi - \eta \rangle^{-2s_1} \langle \eta \rangle^{-2s_2} \, d\xi \le c \int_{A_3'} |\xi|^{2s} |\xi - \eta|^{-2s_1} |\eta|^{-2s_2} \, d\xi$$

Moreover by substituting $\zeta = \frac{\xi}{|\eta|}$ it follows that the integral on the right hand side can be written as

$$c|\eta|^{n+2(s-s_1-s_2)} \int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta$$

where

$$A_3'' = \{\zeta \in \mathbb{R}^n \,|\, \frac{2}{|\eta|} < |\zeta| < 4, \, |\zeta| < 4|\zeta - \frac{\eta}{|\eta|}|\} \supset A_3'$$

Now, three different cases for the value of s_1 is considered:

$s_1 > 0$: It is first noted, that whenever $|\eta| \le 4$, the integral is bounded. To see this, note that for $\zeta \in A_3''$, $|\zeta| < 4|\zeta - \frac{\eta}{|\eta|}|$ so

$$\begin{aligned}
\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta &\le c_1 \int_{A_3''} |\zeta|^{2s} |\zeta|^{-2s_1} \, d\zeta \\
&\le c_2 \int_{2/|\eta|}^{4} \rho^{2s-2s_1} \rho^{n-1} \, d\rho \\
&\le c_2 \int_{\frac{1}{2}}^{4} \rho^{2s-2s_1+n-1} \, d\rho \\
&\le C_1
\end{aligned}$$

where (ρ, ω) denotes polar coordinates in $\mathbb{R}^n$.

Then it is assumed, that $|\eta| > 4$. And the following sets are defined:

$$O' = \{\zeta \in A_3'' | \frac{1}{2} < |\zeta| < 4\}$$
$$O'' = \{\zeta \in A_3'' | \frac{2}{|\eta|} < |\zeta| < \frac{1}{2}\}$$

and it is noted, as above, that

$$\int_{O'} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta \le C_2$$

Moreover, for $\zeta \in O''$

$$|\zeta - \frac{\eta}{|\eta|}| \ge ||\zeta| - 1| \ge 1 - |\zeta| \ge \frac{1}{2}$$

so

$$|\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \le 2^{2s_1} \le K$$

Thus

$$\int_{O''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta \le c_1 \int_{O''} |\zeta|^{2s} \, d\zeta$$
$$\le c_2 \int_{\frac{2}{|\eta|}}^{\frac{1}{2}} \rho^{2s+n-1} \, d\rho$$

so

$$\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta \le C_3 + c_2 \int_{\frac{2}{|\eta|}}^{\frac{1}{2}} \rho^{2s+n-1} \, d\rho$$

$s_1 = 0$: Using this assumption it follows that

$$\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta \le c \int_{\frac{2}{|\eta|}}^{4} \rho^{2s+n-1} \, d\rho$$
$$\le c_1 + c_2 \int_{\frac{2}{|\eta|}}^{\frac{1}{2}} \rho^{2s+n-1} \, d\rho$$

$s_1 < 0$: Using this assumption it follows that

$$\begin{aligned}\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1}\, d\zeta &\le \int_{A_3''} |\zeta|^{2s} ||\zeta| + 1|^{-2s_1}\, d\zeta \\ &\le c \int_{A_3''} |\zeta|^{2s}\, d\zeta \\ &\le c \int_{\frac{2}{|\eta|}}^{4} \rho^{2s+n-1}\, d\rho \\ &\le c_1 + c_2 \int_{\frac{2}{|\eta|}}^{\frac{1}{2}} \rho^{2s+n-1}\, d\rho\end{aligned}$$

since $||\zeta| + 1|^{-2s_1} \le C$ for $\zeta \in A_3''$.

Thus it is concluded that for all s_1, there exists constants C, K, such that

$$\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1}\, d\zeta \le C + K \int_{\frac{2}{|\eta|}}^{\frac{1}{2}} \rho^{2s+n-1}\, d\rho \tag{2.6}$$

Now, this case is proved, by observing the following:

For $s \neq -\frac{n}{2}$, it follows by (2.6) that

$$\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1}\, d\zeta \le C_1 + C_2 |\eta|^{-n-2s}$$

so since $2(s - s_1 - s_2) + n \le 0$ and $s_1 + s_2 \ge 0$, it follows that there exists an $M < \infty$ such that

$$c|\eta|^{2(s-s_1-s_2)+n} \int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1}\, d\zeta \le M^2$$

for all $\eta \in \mathbb{R}^n$.

For $s = -\frac{n}{2}$, it follows by (2.6) that

$$\int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1}\, d\zeta \le C_1 + C_2 \log(|\eta|)$$

so since then $s_1 + s_2 > 0$, it follows that there exists an $M < \infty$ such that

$$c|\eta|^{2(s-s_1-s_2)+n} \int_{A_3''} |\zeta|^{2s} |\zeta - \frac{\eta}{|\eta|}|^{-2s_1} \, d\zeta \leq M^2$$

for all $\eta \in \mathbb{R}^n$.

Thus it has been proved, that there exists an $M < \infty$, such that

$$\int_{\mathbb{R}^n} |\chi_{A_3} F(\xi, \eta)|^2 \, d\xi \leq M^2$$

so by Corollary 2.4

$$\tilde{T}_{\chi_{A_3}}(f_1, f_2) \in L^2(\mathbb{R}^n)$$

Now it is observed that for $(v, w) \in C_0(\mathbb{R}^n) \times C_0(\mathbb{R}^n)$

$$\begin{aligned} \tilde{T}_F(v, w) &= \sum_{i=1}^{3} \int \chi_{A_i} F(\xi, \eta) v(\xi - \eta) w(\eta) \, d\eta \\ &= \tilde{T}_{\chi_{A_1} F}(v, w) + \tilde{T}_{\chi_{A_2} F}(v, w) + \tilde{T}_{\chi_{A_3} F}(v, w) \end{aligned}$$

so by the preceding observations and Corollary 2.4,

$$\tilde{T}(f_1, f_2) \in L^2(\mathbb{R}^n)$$

It only remains to be shown, that

$$\langle \xi \rangle^s \widehat{u_1 u_2}(\xi) = \tilde{T}_F(f_1, f_2) \tag{2.7}$$

from which the claims of the theorem will follow.

Since $s_1 + s_2 \geq 0$, at least one of them is ≥ 0. Assume for simplicity that it is s_1. Then u_1 may be approximated in $H^{s_1}(\mathbb{R}^n)$ by a sequence of Schwartz functions $u_1^{(j)}$, so by [Grubb, 1996b, p. 8.9]

$$\langle \xi \rangle^s \widehat{u_1^{(j)} u_2}(\xi) = \tilde{T}_F(f_1^{(j)}, f_2)$$

where $f_1^{(j)}(\xi) = \langle \xi \rangle^{s_1} \hat{u}_1^{(j)}$.

Then, by the definition of the product

$$\lim_{j \to \infty} u_1^{(j)} u_2 = u_1 u_2$$

so since the Fourier transform is continuous in $\mathscr{S}'(\mathbb{R}^n)$ it follows that

$$\langle\xi\rangle^s\widehat{u_1u_2}(\xi) = \tilde{T}_F(f_1, f_2)$$

It now follows by Corollary 2.4 and (2.7) that

$$\begin{aligned}\|u_1u_2\|_s &= \|\langle\xi\rangle^s\widehat{u_1u_2}(\xi)\|_0 = \|\tilde{T}_F(f_1, f_2)\|_0 \\ &\leq c\|f_1\|_0\|f_2\|_0 \leq c\|u_1\|_{s_1}\|u_2\|_{s_2}\end{aligned}$$

which completes the proof. □

The preceding theorem covers the case of Sovolev spaces on $\mathbb{R}^n$. However, when dealing with boundary value problems, one needs a similar result for Sobolev spaces on an open, bounded subset of $\mathbb{R}^n$. The following corollary covers this case:

Corollary 2.6
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $u_i \in H^{s_i}(\bar{\Omega})$ with $s_i \geq 0$, $i = 1, 2$. Then $u_1u_2 \in H^s(\bar{\Omega})$ if

$$s \leq s_1 \quad s \leq s_2 \quad s \leq s_1 + s_2 - \frac{n}{2}$$

with the last inequality strict if $s_1 = \frac{n}{2}$, $s_2 = \frac{n}{2}$ or $s = -\frac{n}{2}$, and

$$\|u_1u_2\|_s \leq c\|u_1\|_{s_1}\|u_2\|_{s_2}$$

Remark: The corollary does not cover entirely what is needed. It can be shown, that the product defined in Definition 2.1 is independent of the order of the Sobolev spaces. This allows one to work on dense subspaces of the Sobolev spaces. When this is applied to Corollary 2.6 it can be shown that also for $\Omega \neq \mathbb{R}^n$ it is a sufficient condition that $s_1 + s_2 \geq 0$, but for lack of time this is not explored here.

Proof:
Assume that $u_i \in H^{s_i}(\bar{\Omega})$ and assume that s_1, s_2 and s fulfil the conditions of the corollary. Then

$$\begin{aligned}\|u_1u_2\|_s &= \inf\{\|w\|_s \,|\, w \in H^s(\mathbb{R}^n),\, u_1u_2 = r_\Omega w\} \\ &\leq \inf\{\|U_1U_2\|_s \,|\, U_i \in H^{s_i}(\mathbb{R}^n),\, u_i = r_\Omega U_i\} \\ &\leq c\|u_1\|_{s_1}\|u_2\|_{s_2}\end{aligned}$$

□

Corollary 2.7
Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and let $[\cdot,\cdot]$ and $M_2(\cdot,\cdot)$ be defined as in Definition 5.1.

1. *If* $u_i \in H^{2+s_i}(\bar{\Omega})$, $i = 1,2$, *with* $s_1, s_2 \geq 0$ *and if*

$$s \leq s_1 \quad s \leq s_2 \quad s \leq s_1 + s_2 - 1$$

the last inequality being strict if $-s$, s_1 *or* s_2 *equals* $\frac{n}{2}$, *then* $[u_1, u_2] \in H^s(\bar{\Omega})$, *and*

$$\|[u_1, u_2]\|_s \leq c\|u_1\|_{s_1}\|u_2\|_{s_2}$$

2. *If* $u \in H^{2-\beta}(\bar{\Omega})$ *and* $v \in H^{2+\beta}(\bar{\Omega})$ *with* $0 < \beta < 1$, *then* $M_2(u,v) \in H^{-1}(\bar{\Omega})$, *and*

$$\|M_2(u,v)\|_{-1} \leq c\|u\|_{2-\beta}\|v\|_{2+\beta}$$

3. *If* $u \in H^{2-\theta+\beta}(\bar{\Omega})$ *and* $v \in H^{3-j-\beta}(\bar{\Omega})$, $j = 0,1$ *with* $0 < \beta \leq \theta < 1$, *then* $M_2(u,v) \in H^{-j-\theta}(\bar{\Omega})$, *and*

$$\|M_2(u,v)\|_{-j-\theta} \leq c\|u\|_{2-\theta+\beta}\|v\|_{3-j-\beta}$$

Remark: It should be noted that when $j = 1$, then $M_2(u,v)$ is not defined for $u \in H^{2-\theta+\beta}(\bar{\Omega})$ and $v \in H^{3-j-\beta}(\bar{\Omega})$. This problem can easily be overcome, when Corollary 2.6 has been proved under the condition that $s_1 + s_2 \geq 0$. Then $M_2(u,v)$ can be defined also for $u \in H^{2-\theta+\beta}(\bar{\Omega})$ and $v \in H^{3-j-\beta}(\bar{\Omega})$. This construction will be used in the following proof.

Proof:
First assume that $u_i \in H^{s_i+2}(\bar{\Omega})$. Then $\partial^2_{jk}u_i \in H^{s_i}(\bar{\Omega})$ for all $j,k = 1,2$. Thus, by Corollary 2.6, $\partial^2_{jk}u_1\partial^2_{mn}u_2 \in H^s(\bar{\Omega})$, if $s \leq s_1$, $s \leq s_2$ and $s \leq s_1 + s_2 - 1$, where the last inequality must be strict if for some $i = 1,2$, $s_i = 1$ or if $s = -1$. Thus, since $[u_1, u_2]$ can be written as a sum of terms of the form $\partial^2_{jk}u_1\partial^2_{mn}u_2$ it follows that $[u_1, u_2] \in H^s(\bar{\Omega})$.

Next assume that $u \in H^{2-\beta}(\bar{\Omega})$, $v \in H^{2+\beta}(\bar{\Omega})$ and $0 < \beta < 1$. Then for $i = 1,2$, $\partial_i u \in H^{1-\beta}(\bar{\Omega})$ and for all $j,k = 1,2$ $\partial^2_{jk}v \in H^{\beta}(\bar{\Omega})$. Then by Corollary 2.6, $(\partial_i u)(\partial^2_{jk}v) \in L^2(\Omega)$ for all $i,j,k = 1,2$, since $0 \leq \beta$, $0 \leq 1-\beta$ and $0 \leq \beta + 1 - \beta - 1 = 0$, and none of the s_i and s can become 1. But then $M_2(u,v) \in H^{-1}(\bar{\Omega})$, and the expression for the norm follows by Corollary 2.6.

Lastly assume that $u \in H^{2-\theta+\beta}(\bar{\Omega})$, $v \in H^{3-j-\beta}(\bar{\Omega})$, $j = 0,1$ and $0 < \beta \leq \theta < 1$. Then $\partial_l u \in H^{1-\theta+\beta}(\bar{\Omega})$ for $l = 1,2$, and $\partial^2_{mn}v \in H^{1-j-\beta}(\bar{\Omega})$ for $m,n = 1,2$. Thus by Corollary 2.6 $(\partial_l u)(\partial^2_{mn}v) \in H^{1-\theta-j}(\bar{\Omega})$, since $1-\theta-j \leq 1-\theta+\beta$, $1-\theta-j \leq 1-j-\beta$, and $1-\theta-j \leq 1-\theta+\beta+1-\theta-j-1 = 1-\theta-j$. Thus $M_2(u,v) \in H^{-j-\theta}(\bar{\Omega})$, and the expression for the norm follows by Corollary 2.6. □

Chapter 3

Elliptic Boundary Value Problems

In this chapter the following problem will be discussed:

> Let Ω be an open and bounded subset of $\mathbb{R}^n$ with boundary Γ, let A be an elliptic differential operator of order m, and let B be boundary differential operator. To what extent does the problem
>
> $$\begin{aligned} Au &= f \quad x \text{ in } \Omega \\ Bu &= \phi \quad x \text{ in } \Gamma \end{aligned} \tag{3.1}$$
>
> admit solutions, and to what extent are these solutions unique? Furthermore it will be investigated, if improved regularity of data implies improved regularity of the solutions.

Since the main goal of this chapter is to derive an existence, uniqueness, and regularity result for the biharmonic operator, it has not been the purpose to give a complete presentation of this subject. This means in particular that most results regarding pseudo-differential operators have been left out. Moreover a few arguments in the proofs concerning pseudo-differential operators have been left out due to lack of time.

The presentation here is an investigation of the presentation of this subject as given in [Hörmander, 1985].

Since the presentation concerns elliptic boundary value problems, the first thing is to give a precise definition of an elliptic boundary value problem.

The following definition is that of [Hörmander, 1985]. It coincides with the one given in [Grubb, 2000]:

Definition 3.1
Let Ω be an open and bounded subset of $\mathbb{R}^n$ with boundary Γ, and consider the following boundary value problem:

$$\begin{aligned} Au &= f \quad & x \in \Omega \\ B_j u &= \phi_j \quad & x \in \Gamma \end{aligned} \tag{3.2}$$

for $j = 1, \ldots, J$, where A is a differential operator of order m and for each j, B_j is a boundary differential operator of transversal order $\leq m - 1$.

Then the boundary value problem (3.2) is called elliptic if

1. *$A(x, D)$ is an elliptic operator.*
2. *The B_j's form an elliptic system of boundary conditions, i.e. they fulfil the Shapiro-Lopatinskiĭ condition (cf. [Grubb, 2000, p. 7.12]).*

Remark: It should be noted, that for problems with boundary differential operators of transversal order $\geq m$, the problem can be reduced to a problem having boundary differential operators of transversal order $\leq m - 1$. A presentation of the reduction of transversal order is given in [Hörmander, 1985, p. 233].

The main theorem of this chapter is

Theorem 3.2
Let A be a differential operator of order m, and let B be a system of differential boundary operators of transversal order $\leq \mu$ with $\mu \leq m - 1$. If (3.1) is an elliptic boundary value problem, then

$$\begin{pmatrix} A \\ B \end{pmatrix} : H^s(\bar{\Omega}) \to \begin{matrix} H^{s-m}(\bar{\Omega}) \\ \times \\ \prod_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Gamma) \end{matrix}$$

is a Fredholm operator for all $s \geq \mu + 1$.

When this theorem has been proved, together with an additional theorem, it is possible to give the desired existence, uniqueness, and regularity theorem for elliptic boundary value problems.

The proofs and ideas in this chapter is essentially an investigation of section 20.1 of [Hörmander, 1985], although a few simplifications will be made. The primary simplification is, that only open, bounded subsets Ω of $\mathbb{R}^n$ will be considered. Then $\Gamma = \partial\Omega$ becomes a compact manifold. Thus all vector bundles can be assumed to be $\Omega \times \mathbb{C}$ or $\Gamma \times \mathbb{C}$.

It should be noted, that in [Hörmander, 1985] the results are only proved for$s \geq m$. Since this assumption is stronger than necessary, and in fact inadequate for proving the existence, uniqueness, and regularity result for the biharmonic operator, the proof here will be given for $s \geq \mu + 1$ as stated in Theorem 3.2. However, through the work with the present chapter, it has been found, that the tools provided in [Hörmander, 1997] are not adequate to make the enhancement that the theorem holds for $s \geq \mu + 1$. In general the continuity properties can only be proved in the form given here for $s \geq m$, but since there exists tools to overcome this problem, but the time has been too short to look into these, the presentation will still be for the case of $s \geq \mu + 1$, and it will be remarked for the lemmas, if there are problems for $\mu+1 \leq s < m$. A way which could lead to the enhanced results is to redefine the Dirichlet trace operator γ, such that it only contains the first μ traces, but in lack of time, it has not been possible to pursuit this.

As noted in the introduction to this chapter, the presentation relies on a preliminary knowledge of pseudo-differential operators. The reader, who is not familiar with those, can refer to [Hörmander, 1997, sec. 18.1] for the definitions. Moreover a few results regarding Fredholm theory and the index of an elliptic operator will be used, and the reader may refer to [Hörmander, 1985, sec. 19.1+19.5].

The following definition of a pseudo-differential operator on an open subset of $\mathbb{R}^n$ is as in [Hörmander, 1985, p.85].

Definition 3.3
Let X be a C^∞-manifold, and let (X_κ) be an atlas of X. Then the class of pseudo-differential operators $\Psi^m(X)$, $m \in \mathbb{R}$, is the set of continuous linear mappings

$$A : C_0^\infty(X) \to C^\infty(X)$$

such that for all X_κ with coordinates $x \mapsto \kappa(x) \in \tilde{X}_\kappa \subset \mathbb{R}^n$, and for all $\phi, \psi \in C_0^\infty(\tilde{X}_\kappa)$,

$$\mathscr{S}'(\mathbb{R}^n) \ni u \mapsto \phi(\kappa^{-1})^* A \kappa^* (\psi u)$$

is in $\mathrm{Op}S^m$, where $S^m = S^m_{1,0}$ is the class of symbols of degree m, as defined in [Grubb, 2000, p. 6.4].

When $m = \pm\infty$, $\Psi^m(X)$ is defined by

$$\Psi^\infty(X) = \bigcup_{d\in\mathbb{R}} \Psi^d(X) \qquad \Psi^{-\infty}(X) = \bigcap_{d\in\mathbb{R}} \Psi^d(X)$$

Through the rest of the chapter, the term "transmission property" will be widely used.

A pseudo-differential operator S is said to have the transmission property, if

$$S : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$$

The following two properties of the transmission property will be used:

If $S = \mathrm{Op}(s)$ is a pseudo-differential operator, and s is a sum of rational functions, then S has the transmission property [Grubb, 1996a, p. 24].

The adjoint of a pseudo-differential operator having the transmission property also has the transmission property.

Another term widely used is that of one pseudo-differential operator being an approximation of another pseudo-differential operator:

Definition 3.4
Let $S, T \in \Psi^m(X)$ for some C^∞-manifold X. Then S is said to be an approximation of T, written $S \equiv T$, if

$$S = T + R$$

for some $R \in \Psi^{-\infty}(X)$

A special notice shall be given about the set $\hat{\Omega}$. First a set of geodesic normal coordinates is chosen. Using these, it is possible to extend Ω with a collar, such that A is elliptic on $\Omega \cup \,]-\varepsilon, \varepsilon\,[$. The set thus constructed will be denoted $\hat{\Omega}$. An advantage of this construction is, that $\partial_n := \partial_{x_n}$ becomes the normal derivative at $(x', 0) \in \Gamma$.

Another symbol, which has been carried over from the notation of L. Hörmander, is the trace operator γ. This operator, is the one defined in [Grubb, 2000] as $\rho^+_{(m)}$, i.e. the first m Dirichlet traces taken from the interior of Ω.

The last comment about the notation in [Hörmander, 1985] is about the δ-measure used in the presentation, and about the symbol $u \otimes \delta$.

When $u \in \mathscr{D}(\hat{\Omega})$, $\delta(x_n)$ is defined by

$$\langle \delta(x_n), u \rangle = \int_\Gamma u \, d\sigma$$

where σ denotes the Lebesgue measure in Γ. It then follows that

$$\langle \delta(x_n), u \rangle = \int_\Gamma 1_\Gamma \gamma_0 u \, d\sigma = \langle 1_\Gamma, \gamma_0 u \rangle = \langle \gamma_0^* 1_\Gamma, u \rangle \tag{3.3}$$

so

$$\delta(x_n) = \gamma_0^* 1_\Gamma$$

Next, if $g \in C^\infty(\Gamma)$, $g \otimes \delta(x_n)$ is the distribution given by

$$\langle g \otimes \delta(x_n), u \rangle = \int_\Gamma gu \, d\sigma$$

It then follows, with a rewriting as in equation (3.3) that

$$g \otimes \delta(x_n) = \gamma_0^* g$$

In a similar way it can be found, that the operators $D_n^k \delta(x_n)$ and $g \otimes D_n^k \delta(x_n)$ are given by

$$D_n^k \delta = \gamma_k^* 1_\Gamma$$
$$g \otimes D_n^k \delta = \gamma_k^* g$$

Throughout the following presentation, the notation $D_n^k \delta$ as used in [Hörmander, 1985] will be kept.

Lastly it should be noted, that in the present chapter all forms will be over the scalar field $\mathbb{C}$. This has been chosen to avoid confusion when working with the primary source for this chapter.

Since $A \in \Psi^m(\hat{\Omega})$, the set of pseudo-differential operators with symbol in $S^m(\hat{\Omega})$, there exists a properly supported parametrix $\tilde{A} \in \Psi^{-m}(\hat{\Omega})$ such that

$$\tilde{A} A = I + R_1$$
$$A \tilde{A} = I + R_2$$

where R_1, R_2 are negligible operators.

Throughout the presentation the following operator, defined for $u \in \mathscr{D}(\Omega)$ is needed:

$$e_\Omega u := \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \hat{\Omega} \setminus \Omega \end{cases}$$

Since $e_\Omega : \mathscr{D}(\Omega) \to \mathscr{D}(\hat{\Omega})$ it follows that it has an adjoint $(e_\Omega)^* : \mathscr{D}'(\hat{\Omega}) \to \mathscr{D}'(\Omega)$. If now $u \in \mathscr{D}(\Omega)$ and $v \in \mathscr{D}(\hat{\Omega})$ it follows that

$$\begin{aligned} \langle e_\Omega u, v \rangle &= \int_{\hat{\Omega}} e_\Omega uv \, dx = \int_{\Omega} u r_\Omega v \, dx \\ &= \langle u, r_\Omega v \rangle = \langle (r_\Omega)^* u, v \rangle \end{aligned}$$

where r_Ω denotes the restriction to Ω, so on the dense subset $\mathscr{D}(\hat{\Omega})$, $(e_\Omega)^*$ coincides with r_Ω. Especially this shows by [Grubb, 1996b, p. 6.17] that e_Ω and r_Ω can be extended in a unique way to maps

$$\begin{aligned} e_\Omega &: \mathscr{D}'(\Omega) \to \mathscr{D}'(\hat{\Omega}) \\ r_\Omega &: \mathscr{D}'(\hat{\Omega}) \to \mathscr{D}'(\Omega) \end{aligned}$$

and that the maps so constructed are adjoints of each other.

The main part of the following construction relies on the following lemma:

Lemma 3.5
Let A be written on the form

$$A = \sum_{j=0}^{m} A_j(x_n) D_n^j$$

where $A_j(x_n)$ is a differential operator in Γ of order $m - j$.

Then for $u \in C^\infty(\overline{\Omega})$ the following formula holds in $\hat{\Omega}$

$$e_\Omega u + R_1 e_\Omega u = \tilde{A} e_\Omega (Au) + \tilde{A} A^c \gamma u \tag{3.4}$$

where $\tilde{A}$ is a parametrix of A such that $\tilde{A}A = I + R_1$ and A^c is the operator in Γ defined for $U = (U_0, \dots, U_{m-1}) \in \prod_{j=0}^{m-1} C^\infty(\Gamma)$ by

$$A^c U = i^{-1} \sum_{j=0}^{m-1} A_{j+1} \sum_{k=0}^{j} U_{j-k} \otimes D_n^k \delta(x_n)$$

Remark: Since Γ is a compact subset of $\mathbb{R}^n$, it is noted, that $C^\infty(\Gamma)^* = \mathscr{D}'(\Gamma)$. Thus A^c extends to a continuous map from $\mathscr{D}'(\Gamma)$ to $\mathscr{D}'(\hat{\Omega})$.

Moreover, it should be noted, that there is a little ambiguity in equation (3.4). u is assumed to be in $C^\infty(\overline{\Omega})$ and $A \in \Psi^m(\hat{\Omega})$. However, A is a differential operator, so Au is still well defined.

Proof:
The first part of the proof consist of showing the following identity

$$D_n^{m+1} e_\Omega u - e_\Omega(D_n^{m+1} u) = i^{-1} \sum_{k=0}^{m} \gamma_{m-k} u \otimes D_n^k \delta \tag{3.5}$$

The proof of (3.5) goes by induction on m. By the comments in the beginning of the chapter ∂_n equals the normal derivative at the boundary, so

$$D_n(e_\Omega u) = i^{-1} \gamma_0 u \otimes \delta(x_n) + e_\Omega(D_n u) \tag{3.6}$$

which shows (3.5) for $m = 0$.

Now assume that (3.5) has been shown for some $m > 0$. Then by (3.6)

$$\begin{aligned}
D_n^{m+1}(e_\Omega u) &= D_n(D_n^m(e_\Omega u)) \\
&= D_n(e_\Omega(D_n^m u) + i^{-1} \sum_{k=0}^{m-1} \gamma_{m-k-1} u \otimes D_n^k \delta) \\
&= D_n(e_\Omega(D_n^m u)) + i^{-1} D_n \sum_{k=0}^{m-1} \gamma_{m-k-1} u \otimes D_n^k \delta \\
&= e_\Omega(D_n^{m+1} u) + i^{-1} \gamma_0 D_n^m u \otimes \delta + i^{-1} \sum_{k=1}^{m} \gamma_{m-k} u \otimes D_n^k \delta \\
&= e_\Omega(D_n^{m+1} u) + i^{-1} \sum_{k=0}^{m} \gamma_{m-k} u \otimes D_n^k \delta
\end{aligned}$$

so (3.5) follows by induction on m.

Now it is noted, that

$$
\begin{aligned}
Ae_\Omega u - e_\Omega(Au) &= \sum_{j=0}^{m} A_j D_n^j e_\Omega u - \sum_{j=0}^{m} A_j e_\Omega(D_n^j u) \\
&= \sum_{j=0}^{m-1} A_{j+1}(D_n^{j+1} e_\Omega u - e_\Omega(D_n^{j+1} u)) \\
&= \sum_{j=0}^{m-1} A_{j+1} i^{-1} \sum_{k=0}^{j} \gamma_{j-k} u \otimes D_n^k \delta \\
&= A^c \gamma u
\end{aligned}
$$

by definition of A^c.

Then, since $\tilde{A}$ is a parametrix of A, it follows that

$$e_\Omega u + R_1 e_\Omega u = \tilde{A} A e_\Omega u = \tilde{A} e_\Omega(Au) + \tilde{A} A^c \gamma u$$

which completes the proof. □

The Green's formula just obtained leads to the following definition of the Calderón projectors.

Definition 3.6
Let A^c be defined as in lemma 3.5 and let $\tilde{A}$ be a parametrix of A. Then one defines the Calderón projector, P, by

$$PU = \gamma r_\Omega \tilde{A} A^c U \tag{3.7}$$

for $U = (U_0, \ldots, U_{m-1}) \in \prod_{j=0}^{m-1} C^\infty(\Gamma)$

Actually P is not a projection, but as will be shown in the following lemma, P is an approximate projection:

Lemma 3.7
The Calderón projector P defined by (3.7) is a system of pseudo-differential operators on Γ.

The system P is an approximate projection in the space of Cauchy data, $\prod_{j=0}^{m-1} C^\infty(\Gamma)$, *i.e.* $P^2 - P \equiv 0$.

Proof:
First it is noted, that PU can be rewritten to

$$(PU)_k = \sum_{l=0}^{m-1} P_{kl} U_l$$

where

$$P_{kl} U_l = \gamma_0 \sum_{j=0}^{m-l-1} i^{-1} D_n^k \tilde{A} A_{j+l+1} (U_l \otimes D_n^j \delta) \tag{3.8}$$

This is seen by the following

$$\begin{aligned}
(PU)_k &= \gamma_0 D_n^k \tilde{A} A^c U \\
&= \gamma_0 D_n^k r_\Omega \tilde{A} (i^{-1} \sum_{j'=0}^{m-1} A_{j'+1} \sum_{l=0}^{j'} (U_{j'-l} \otimes D_n^l \delta)) \\
&= \gamma_0 D_n^k (i^{-1} \sum_{j'=0}^{m-1} \sum_{l=0}^{j'} r_\Omega \tilde{A} A_{j'+1} (U_l \otimes D_n^{j'-l} \delta)) \\
&= \gamma_0 D_n^k (i^{-1} \sum_{l=0}^{m-1} \sum_{j'=l}^{m-1} r_\Omega \tilde{A} A_{j'+1} (U_l \otimes D_n^{j'-l} \delta)) \\
&= \gamma_0 D_n^k (i^{-1} \sum_{l=0}^{m-1} \sum_{j=0}^{m-l-1} r_\Omega \tilde{A} A_{j+l+1} (U_l \otimes D_n^j \delta)) \\
&= \sum_{l=0}^{m-1} \gamma_0 \sum_{j=0}^{m-l-1} i^{-1} D_n^k \tilde{A} A_{j+l+1} (U_l \otimes D_n^j \delta)
\end{aligned}$$

Now by [Hörmander, 1985, p. 235], P_{kl} is a pseudo-differential operator of order $k-l$ on Γ with principal symbol

$$p_{kl} = (2\pi i)^{-1} \int^+ \sum_{j=0}^{m-l-1} \xi_n^{k+j} a(x', 0, \xi', \xi_n)^{-1} a_{j+l+1}(x', 0, \xi') d\xi_n$$

where $\int^+$ denotes the integral over a closed curve in the half space of $\mathbb{C}$ where $\mathrm{Im}(z) > 0$, such that the curve encloses all poles of the integrand. By a discussion given in [Grubb, 2000, p. 8.15ff] it can be shown that the integral equals a sum of the residues of

$$\sum_{j=0}^{m-l-1} \xi_n^{k+j} a(x', 0, \xi', \xi_n)^{-1} a_{j+l+1}(x', 0, \xi')$$

Assume that $U \in \prod_{j=0}^{m-1} C^\infty(\Gamma)$, and $u = r_\Omega \tilde{A} A^c U$. Then $r_\Omega u \in C^\infty(\overline{\Omega})$, $\gamma u = PU$, and

$$r_\Omega A u = A r_\Omega \tilde{A} A^c u = r_\Omega (I + R_2) A^c U = r_\Omega R_2 A^c U$$

since $r_\Omega A^c U = 0$ by definition of A^c.

Applying (3.4) to this special choice of u gives

$$e_\Omega r_\Omega \tilde{A} A^c U + R_1 e_\Omega r_\Omega \tilde{A} A^c U = \tilde{A} e_\Omega r_\Omega R_2 A^c U + \tilde{A} A^c \gamma r_\Omega \tilde{A} A^c U$$

so by applying γr_Ω on both sides gives

$$PU + \gamma r_\Omega R_1 e_\Omega r_\Omega \tilde{A} A^c U = \gamma r_\Omega \tilde{A} e_\Omega r_\Omega R_2 A^c U + P^2 U$$

i.e.

$$P^2 U - PU = \gamma r_\Omega R_1 e_\Omega r_\Omega \tilde{A} A^c U - \gamma r_\Omega \tilde{A} e_\Omega r_\Omega R_2 A^c U$$

Since R_2 is negligible, $R_2 A^c U$ is continuous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\overline{\Omega})$. Moreover, from [Grubb, 2000, p. 6.18ff] it follows, since A is elliptic, that the symbol of $\tilde{A}$ is a sum of rational functions and thus, by the notes in the beginning of the chapter, $\tilde{A}$ has the transmission property, so $\gamma r_\Omega \tilde{A} e_\Omega (r_\Omega R_2 A^c U)$ is continuous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\Gamma)$, and thus it is given by a negligible operator.

Next let $\phi \in C^\infty(\overline{\Omega})$, $U \in \prod_{j=0}^{m-1} C^\infty(\Gamma)$ and let $(\phi_m) \in C^\infty(\overline{\Omega})$ be a sequence of functions vanishing at Γ, such that $\phi_m \to \phi$. Then

$$\begin{aligned} \langle r_\Omega \tilde{A} A^c U, \phi \rangle &= \lim_{m\to\infty} \langle r_\Omega \tilde{A} A^c U, \phi_m \rangle \\ &= \lim_{m\to\infty} \langle \tilde{A} A^c U, e_\Omega \phi_m \rangle \\ &= \lim_{m\to\infty} \langle A^c U, \tilde{A}^* e_\Omega \phi_m \rangle \\ &= \langle A^c U, \tilde{A}^* e_\Omega \phi \rangle \end{aligned}$$

Now, since A^c can be extended to a continuous map from $\mathscr{D}'(\Gamma)$ to $\mathscr{D}'(\hat{\Omega})$, and since $\tilde{A}^*$ also has the transmission property, by the remarks in the beginning of the chapter, it follows that $r_\Omega(\tilde{A} A^c U)$ extends to a continuous map from $\mathscr{D}'(\Gamma)$ to $\mathscr{E}'(\hat{\Omega})$. Then, since R_2 is negligible it follows that $\gamma r_\Omega R_1 e_\Omega r_\Omega \tilde{A} A^c U$ is continuous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\Gamma)$, and thus it is given by a negligible operator.

Thus it follows that $(P^2 - P)U = R_3 U$, where R_3 is negligible, so $P^2 - P \equiv 0$ as claimed. □

Corollary 3.8
The map $\tilde{P} : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$ *given by*

$$\tilde{P}(f) = P\gamma r_\Omega \tilde{A} e_\Omega f$$

is continuous, and can be extended to a continuous map from $L^2(\Omega)$ *to* $C^\infty(\Gamma)$.

Remark: This is the first time, where the tools are inadequate for the goal of this chapter. What is be needed is that $\tilde{P}$ has a continuous extension to a map from $H^{\mu+1-m}(\bar{\Omega})$ to $C^\infty(\overline{\Omega})$. However, this would in general be wrong, but at the point, where it is needed, there are ways to get around this.

Proof:
Let $f \in C^\infty(\overline{\Omega})$, and let $u = r_\Omega \tilde{A} e_\Omega f$. Then, since $\tilde{A}$ has the transmission property, $u \in C^\infty(\overline{\Omega})$. Moreover, since $\tilde{A}$ is a parametrix of A and A commutes with r_Ω, $Au = f + r_\Omega R_2 e_\Omega f$ in $\overline{\Omega}$. It then follows from the equation (3.4) by applying γr_Ω, that

$$\gamma u + \gamma r_\Omega R_1 e_\Omega u = \gamma r_\Omega \tilde{A} e_\Omega f + \gamma r_\Omega \tilde{A} e_\Omega r_\Omega R_2 e_\Omega f + P\gamma u \tag{3.9}$$

It is noted that, since R_2 is negligible, $f \mapsto R_2 e_\Omega f$ is continuous from $L^2(\Omega)$ to $C^\infty(\hat{\Omega})$. Thus, by the transmission property of $\tilde{A}$,

$$f \mapsto \gamma r_\Omega \tilde{A} e_\Omega r_\Omega R_2 e_\Omega f$$

is continuous from $L^2(\Omega)$ to $C^\infty(\Gamma)$.

Moreover, $f \mapsto u = r_\Omega \tilde{A} e_\Omega f$ is continuous from $L^2(\Omega)$ to $H^m(\bar{\Omega})$, so since R_1 is negligible, $f \mapsto \gamma r_\Omega R_1 e_\Omega u$ is continuous from $L^2(\Omega)$ to $C^\infty(\Gamma)$.

Now (3.9) can be written as

$$P\gamma r_\Omega \tilde{A} e_\Omega f = -(P^2 - P)\gamma u + P\gamma r_\Omega R_1 e_\Omega u - P\gamma r_\Omega \tilde{A} e_\Omega r_\Omega R_2 e_\Omega f$$

and since $P^2 - P \equiv 0$ it follows that $f \mapsto P\gamma r_\Omega \tilde{A} e_\Omega f$ is a negligible operator, and thus extends to a continuous map from $L^2(\Omega)$ to $C^\infty(\Gamma)$. □

The idea now is, with aid of P and B, to find suitable pseudo-differential operators in Γ, and then construct a parametrix to $\binom{A}{B}$ using these operators.

In the preceding lemmas it has been sufficient to consider the class Ψ^m of pseudo-differential operators. In the following another class will be necessary. The class Ψ^m_{phg} consists of the pseudo-differential operators of order m with polyhomogeneous symbols. For a definition of the class of polyhomogeneous symbols the reader may refer to [Grubb, 2000, p. 6.2].

Lemma 3.9
Let B be an elliptic system of boundary operators, and let P be the Calderón projector. Then there exists approximately uniquely determined systems of pseudo-differential operators, $S = S_{kj}$ and $S'' = S''_{kj}$, where

$$S_{kj} \in \Psi^{k-m_j}_{phg}(\Gamma)$$
$$S''_{kl} \in \Psi^{k-l}_{phg}(\Gamma)$$

for $k, l = 0, \dots, m-1$ and $j = 1, \dots, J$, such that

$$\begin{aligned} B^c S &\equiv I \\ PS &\equiv S \\ SB^c + S'' &\equiv I \\ S''P &\equiv 0 \end{aligned} \tag{3.10}$$

where B^c is given by $B = B^c\gamma$.

Remark: By simple means the lemma can be extended to the case of surjective and injective elliptic boundary value problems. However, since the main scope of the present chapter is to derive a result for elliptic boundary value problems, this extension has been omitted.

Proof:
It is noted, that B^c makes sense, since it has been assumed that B has transversal order $\leq m-1$.

B^c and P are written as matrices of operators satisfying

$$(B^c U)_k = \sum_{l=0}^{m-1} B^c_{kl} U_l$$
$$(PU)_k = \sum_{l=0}^{m-1} P_{kl} U_l$$

The first part of the proof then consists in finding operators S_{kj} such that

$$B^c S \equiv I \quad PS \equiv S$$

with $S = (S_{kj})$.

Since the boundary value problem is elliptic, $(B^cP)_{kl}$, is bijective, and the principal symbol $(b^cp)_{kl} \in S^{m_k-l}$, so by [Hörmander, 1985, p. 228] there exists an operator $T = (T_{kl})$ with $T_{kl} \in \Psi^{l-m_k}_{\text{phg}}(\Gamma)$, such that $B^cPT \equiv I$.

Now let $S = PT$. Then for $i = 0, \dots, m-1$ and $j = 1, \dots, J$, $S_{ij} \in \Psi^{i-m_j}_{\text{phg}}(\Gamma)$ and

$$
\begin{aligned}
B^cS &= B^cPT \equiv I \\
PS &= P^2T = (P^2 - P)T + PT \equiv PT = S
\end{aligned}
$$

The next part consists of showing the existence of operators $S' = (S'_{jk})$ and $S'' = (S''_{jk})$ such that

$$
S'B^c + S'' \equiv I \quad S''P \equiv 0
$$

Thus let

$$
(B^c \oplus (I - P))_{jk} = \begin{cases} B_{jk} & \text{for } j = 1, \dots, J \\ (I - P)_{jk} & \text{for } j = J + 1, \dots, J + m \end{cases}
$$

and let b^c_{jk}, resp. $1 - p_{jk}$, denote the principal symbols of B^c_{jk}, resp. $(I - P)_{jk}$. Then since the problem is elliptic, by [Hörmander, 1985, p. 228] there exists operators T'_{jk} and T''_{jk} such that

$$
\begin{pmatrix} (T' & T'') \end{pmatrix} (B^c \oplus (I - P)) \equiv I
$$

with $T' = (T'_{jk})$ and $T'' = (T''_{jk})$.

Now define $S' = T'$ and $S'' = T''(I - P)$. Then for $i = 0, \dots, m-1$ and $j = 1, \dots, J$, $S'_{ij} \in \Psi^{i-m_j}_{\text{phg}}(\Gamma)$ and for $i, j = 0, \dots, m-1$, $S''_{ij} \in \Psi^{i-j}_{\text{phg}}(\Gamma)$. Moreover

$$
\begin{aligned}
S'B^c + S'' &= T'B^c + T''(I - P) \equiv I \\
S''P &= T''(I - P)P = T''(P - P^2) \equiv 0
\end{aligned}
$$

Thus candidates S, S', S'' have been found. It only remains to be shown, that $S \equiv S'$ and that they are approximately uniquely determined. The uniqueness of S'' follows from $S'B^c + S'' = I$ once $S \equiv S'$ has been shown. By definitions of S, S' and S''

$$
S' \equiv S'BS \equiv S'BS + S''(S - PS) \equiv S'BS + S''S \equiv S
$$

which completes the proof. □

Now with the aid of the operators S and S'' it is possible to prove, that $\binom{A}{B}$ defined in the beginning of the chapter, has a parametrix, $\begin{pmatrix} T_1 & T_2 \end{pmatrix}$, and that $\binom{A}{B} : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \times \prod_{j=1}^{J} C^\infty(\Gamma)$ is a Fredholm operator.

Lemma 3.10
Let $\binom{A}{B}$ be an elliptic boundary value problem. Then

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} : C^\infty(\overline{\Omega}) \times \prod_{j=1}^{J} C^\infty(\Gamma) \to C^\infty(\overline{\Omega})$$

given by

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega f + r_\Omega \tilde{A} A^c S g$$

is a parametrix of $\binom{A}{B}$ in the sense, that there exists operators

$$\begin{aligned} K &: H^m(\bar{\Omega}) \to C^\infty(\overline{\Omega}) \\ K_1 &: L^2(\Omega) \to C^\infty(\overline{\Omega}) \\ K_2 &: \mathscr{D}'(\Gamma) \to C^\infty(\overline{\Omega}) \\ K_3 &: L^2(\Omega) \to C^\infty(\overline{\Omega}) \\ K_4 &: \mathscr{D}'(\Gamma) \to C^\infty(\overline{\Omega}) \end{aligned}$$

such that

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = I + K$$
$$\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} T_1 & T_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$$

Moreover

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B_1 \\ \vdots \\ B_J \end{pmatrix} : C^\infty(\overline{\Omega}) \to \begin{matrix} C^\infty(\overline{\Omega}) \\ \times \\ \prod_{j=1}^{J} C^\infty(\Gamma) \end{matrix}$$

is a Fredholm operator.

Remark: In this lemma the problems from Corollary 3.8 will be carried over. Thus, also here the tools are inadequate. What would be needed is, that K_1 and K_3 extend to continuous maps from $H^{\mu+1-m}(\bar{\Omega})$ to $C^\infty(\overline{\Omega})$.

Proof:
The first step of the proof is to show that

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega f + r_\Omega \tilde{A} A^c S g$$

is an approximative left and right inverse of $\begin{pmatrix} A \\ B \end{pmatrix}$.

Let u be a solution of (3.1), and let $U = \gamma u$. Then by definition $B_j^c U = g_j$.

Note that from (3.10) there exists a negligible operator R such that

$$SB^c + S''(I - P) = I + R \tag{3.11}$$

Thus by (3.4)

$$\begin{aligned}
\begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} u &= (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega (Au) + r_\Omega \tilde{A} A^c S B u \\
&= r_\Omega \tilde{A} e_\Omega (Au) + r_\Omega \tilde{A} A^c S'' \gamma r_\Omega \tilde{A} e_\Omega (Au) \\
&\quad + r_\Omega \tilde{A} A^c S B^c \gamma u \\
&= u + r_\Omega R_1 e_\Omega u - r_\Omega \tilde{A} A^c \gamma u \\
&\quad + r_\Omega \tilde{A} A^c S'' \gamma (u + r_\Omega R_1 e_\Omega u - r_\Omega \tilde{A} A^c \gamma u) \\
&\quad + r_\Omega \tilde{A} A^c S B^c U \\
&= u + r_\Omega R_1 e_\Omega u - r_\Omega \tilde{A} A^c U + r_\Omega \tilde{A} A^c S'' U \\
&\quad + r_\Omega \tilde{A} A^c S'' \gamma r_\Omega R_1 e_\Omega u \\
&\quad - r_\Omega \tilde{A} A^c S'' \gamma r_\Omega \tilde{A} A^c U + r_\Omega \tilde{A} A^c S B^c U \\
&= u + r_\Omega \tilde{A} A^c (SB^c + S'' - S'' P - I) U \\
&\quad + (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega R_1 e_\Omega u \\
&= u + r_\Omega \tilde{A} A^c R U + (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega R_1 e_\Omega u \\
&= u + K
\end{aligned}$$

where

$$Ku = r_\Omega \tilde{A} A^c R \gamma u + (I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega R_1 e_\Omega u$$

is continuous from $H^m(\bar{\Omega})$ to $C^\infty(\overline{\Omega})$, since R and R_1 are negligible.

Thus it follows, that $\begin{pmatrix} T_1 & T_2 \end{pmatrix}$ is a left parametrix of $\begin{pmatrix} A \\ B \end{pmatrix}$.

Next it is noted, that

$$\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} T_1 & T_2 \end{pmatrix} = \begin{pmatrix} AT_1 & AT_2 \\ BT_1 & BT_2 \end{pmatrix}$$

so it must be shown that

$$\begin{pmatrix} AT_1 & AT_2 \\ BT_1 & BT_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$$

and the K_j's have the proper continuity properties.

Since A^c is supported in Γ it follows that for all $f \in C^\infty(\overline{\Omega})$, $g \in C^\infty(\Gamma)$

$$r_\Omega A^c S'' \gamma \tilde{A} e_\Omega f = 0 = r_\Omega A^c S g$$

Thus let $f \in C^\infty(\overline{\Omega})$ and $g \in C^\infty(\Gamma)$. Then for $x \in \Omega$

$$\begin{aligned} AT_1 f &= A(I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega f \\ &= r_\Omega A \tilde{A} (I + A^c S'' \gamma r_\Omega \tilde{A}) e_\Omega f \\ &= r_\Omega (I + R_2)(I + A^c S'' \gamma r_\Omega \tilde{A}) e_\Omega f \\ &= f + r_\Omega R_2 (I + A^c S'' \gamma r_\Omega \tilde{A}) e_\Omega f \\ &= f + K_1 \end{aligned}$$

where

$$K_1 = r_\Omega R_2 (I + A^c S'' \gamma r_\Omega \tilde{A}) e_\Omega f$$

is continous from $L^2(\Omega)$ to $C^\infty(\overline{\Omega})$.

Also

$$\begin{aligned} AT_2 g &= A r_\Omega \tilde{A} A^c S g \\ &= r_\Omega (I + R_2) A^c S g \\ &= r_\Omega R_2 A^c S g \\ &=: K_2 \end{aligned}$$

where K_2 is continous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\overline{\Omega})$.

Moreover, by the properties of S and S'' it follows that there exists negligible operators R_3, R_4 and R_5 such that

$$\begin{aligned} S'' &= I - SB^c + R_3 \\ PS &= S - R_4 \\ B^c S &= I - R_5 \end{aligned}$$

$$\begin{aligned}
BT_1 f &= B(I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega f \\
&= B^c (I + PS'') \gamma r_\Omega \tilde{A} e_\Omega f \\
&= B^c (I + P(I - SB^c + R_3)) \gamma r_\Omega \tilde{A} e_\Omega f \\
&= (B^c + B^c P - B^c PSB^c + B^c PR_3) \gamma r_\Omega \tilde{A} e_\Omega f \\
&= (B^c + B^c P - B^c SB^c + B^c R_4 B^c + B^c PR_3) \gamma r_\Omega \tilde{A} e_\Omega f \\
&= (B^c + B^c P - B^c + R_5 B^c + B^c R_4 B^c + B^c PR_3) \gamma r_\Omega \tilde{A} e_\Omega f \\
&= B^c P \gamma r_\Omega \tilde{A} e_\Omega f + (R_5 B^c + B^c R_4 B^c + B^c PR_3) \gamma r_\Omega \tilde{A} e_\Omega f \\
&=: K_3
\end{aligned}$$

Now by Corollary 3.8 the first term of K_3 is continuous from $L^2(\Omega)$ to $C^\infty(\bar{\Omega})$. The last term is continuous from $L^2(\Omega)$ to $C^\infty(\overline{\Omega})$ since R_3, R_4 and R_5 are negligible.

Also

$$\begin{aligned}
BT_2 g &= B^c \gamma r_\Omega \tilde{A} A^c S g \\
&= B^c P S g \\
&= B^c S g - B^c R_4 g \\
&= g + K_4
\end{aligned}$$

where

$$K_4 = -B^c R_4 g$$

is continuous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\overline{\Omega})$.

Thus it is seen, that $\begin{pmatrix} T_1 & T_2 \end{pmatrix}$ is a parametrix of $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ in the sense defined in the lemma. Moreover K and K_i, $i = 1, 2, 3, 4$ are kompact, so there exists compact operators $\mathcal{R}_1$ and $\mathcal{R}_2$ such that

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = I + \mathcal{R}_1$$

$$\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} T_1 & T_2 \end{pmatrix} = I + \mathcal{R}_2$$

and hence it follows by [Hörmander, 1985, p. 184] that

$$\begin{pmatrix} A \\ B \end{pmatrix} : C^\infty(\overline{\Omega}) \to \begin{matrix} C^\infty(\overline{\Omega}) \\ \times \\ \prod_{j=1}^{J} C^\infty(\Gamma) \end{matrix}$$

is a Fredholm operator as claimed. □

What now remains is to show that $\begin{pmatrix} T_1 & T_2 \end{pmatrix}$ has the right continuity properties. In the proof the following technical lemma will be needed:

Lemma 3.11
Let $\tilde{A}$ be a parametrix of an elliptic operator of order m and let $f \in C^\infty(\overline{\Omega})$ and $U = (U_0, \ldots, U_{m-1}) \in \prod_{j=0}^{m-1} C^\infty(\Gamma)$. If $s \geq m$ then

$$\|r_\Omega \tilde{A} e_\Omega f\|_s \leq c\|f\|_{s-m}$$
$$\|r_\Omega \tilde{A} A^c U\|_s \leq c \sum_{j=0}^{m-1} \|U_j\|_{s-j-\frac{1}{2}}$$

Remark: This is another continuity lemma, and again it is only possible to prove the result for $s \geq m$.

Proof:
First it is noted, that it can be assumed that f, U have compact support $\subset K$, where K is a subset of a coordinate patch $\Psi = \mathcal{Y} \times [0, \epsilon[$ of $\overline{\Omega}$ with $\mathcal{Y} \subset \Gamma$, since if f is not compactly supported, then one can choose a partition of unity, (ϕ_j) subordinate to the atlas, and then show the lemma for each $\phi_j f$.

Then let $\phi \in \mathscr{D}(\hat{\Omega})$ with $\phi \equiv 1$ on a neighbourhood of K.

Let $k \in \mathbb{N}$ such that $k \geq s - m$.

Now by [Grubb, 1996b, p. 484] $H^{(0,s-m)}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{(s-m-k,k)}(\bar{\Omega})$, so

$$\|e_\Omega f\|_{(s-m-k,k)} \leq \|e_\Omega f\|_{(0,s-m)} \leq \|f\|_{(0,s-m),\Omega} \leq \|f\|_{s-m} \tag{3.12}$$

Moreover there exists pseudo-differential operators $\tilde{A}_\beta \in \Psi^{-m}(\hat{\Omega})$ such that

$$D^\alpha(\phi\tilde{A}) = \sum_{|\beta| \leq |\alpha|} \tilde{A}_\beta D^\beta \tag{3.13}$$

where $\beta_n = 0$ if $\alpha_n = 0$.

Thus let $|\beta| \leq k$ with $\beta_n = 0$. Then

$$\begin{aligned} \|D^\beta e_\Omega f\|_{s-m-k} &= \|\langle\xi\rangle^{s-m-k} \widehat{D^\beta e_\Omega f}\|_0 \\ &\leq \|\langle\xi\rangle^{s-m-k} \langle\xi'\rangle^k \widehat{e_\Omega f}\|_0 \\ &= \|e_\Omega f\|_{(s-m-k,k)} \end{aligned}$$

so for $\alpha = (\alpha', 0)$ and $|\alpha| \leq k$ it follows by (3.13) that

$$\begin{aligned}\|D^\alpha(\phi\tilde{A}e_\Omega f)\|_{s-k} &\leq c_1 \sum_{|\beta|\leq|\alpha|} \|\tilde{A}_\beta D^\beta e_\Omega f\|_{s-k} \\ &\leq c_2 \sum_{|\beta|\leq|\alpha|} \|D^\beta e_\Omega f\|_{s-m-k} \\ &\leq c_3 \|e_\Omega f\|_{(s-m-k,k)} \qquad (3.14)\end{aligned}$$

Thus by (3.12) and (3.14)

$$\begin{aligned}\|\phi r_\Omega \tilde{A}e_\Omega f\|_{(s-k,k)} &\leq c_1 \sum \|D^\alpha(\phi r_\Omega \tilde{A}e_\Omega f)\|_{s-k} \\ &\leq c_2 \|e_\Omega f\|_{(s-m-k,k)} \\ &\leq c_3 \|f\|_{s-m} \qquad (3.15)\end{aligned}$$

where the sum is over all multiindices α such that $|\alpha| \leq k$ and $\alpha_n = 0$.

Now let $\psi \in \mathscr{D}(\mathcal{Y} \times [\,0, \varepsilon\,[\,)$ be such that $\psi \equiv 1$ on a neighbourhood of K and $\phi \equiv 1$ on a neighbourhood of $\operatorname{supp}\psi$.

Then, since

$$\psi P \phi r_\Omega \tilde{A}e_\Omega f = \psi f + \psi r_\Omega R_2 e_\Omega f$$

it follows that

$$\begin{aligned}\|\psi r_\Omega \tilde{A}e_\Omega f\|_s &\leq \|\psi\phi r_\Omega \tilde{A}e_\Omega f\|_s \\ &\leq \|\phi r_\Omega \tilde{A}e_\Omega f\|_s \\ &\leq c_2(\|\psi f\|_{s-m} + \|\phi r_\Omega \tilde{A}e_\Omega f\|_s) \\ &\leq c_3(\|f\|_{s-m} + \|\phi r_\Omega \tilde{A}e_\Omega f\|_{(s-k,k)}) \\ &\leq c_4 \|f\|_{s-m}\end{aligned}$$

the last inequality following by (3.15).

Now $f \mapsto (1-\psi) r_\Omega \tilde{A}e_\Omega f$ is continuous from $L^2(\Omega)$ to $C^\infty(\overline{\Omega})$. This is a consequence of Schwartz' kernel theorem for it may be shown that its distibution kernel is zero on the diagonal. However, it would lead us too far to go into detail at this point.

Finally

$$\begin{aligned}\|r_\Omega \tilde{A}e_\Omega f\|_s &\leq \|\psi r_\Omega \tilde{A}e_\Omega f\|_s + \|(1-\psi) r_\Omega \tilde{A}e_\Omega f\|_s \\ &\leq C\|f\|_{s-m}\end{aligned}$$

and thus the first part of the theorem is proved.

Next $A^c U$ is written as

$$\sum_{j=0}^{m-1} v_j \otimes D_n^j \delta$$

where

$$v_j = \sum_{l=0}^{m-j-1} i^{-1} A_{j+l+1} U_l$$

Then since $A_j \in \Psi^{m-j}(\Gamma)$,

$$\begin{aligned}
\|v_j\|_{s-m+j+\frac{1}{2}} &= \| \sum_{l=0}^{m-j-1} i^{-1} A_{j+l+1} U_l \|_{s-m+j+\frac{1}{2}} \\
&\le \sum_{l=0}^{m-j-1} \|A_{j+l+1} U_l\|_{s-m+j+\frac{1}{2}} \\
&\le \sum_{l=0}^{m-j-1} \|U_l\|_{s-l-\frac{1}{2}}
\end{aligned}$$

so

$$\begin{aligned}
\sum_{j=0}^{m-1} \|v_j\|_{s-m+j+\frac{1}{2}} &\le \sum_{j=0}^{m-1} \sum_{l=0}^{m-j-1} \|U_l\|_{s-l-\frac{1}{2}} \\
&\le C \sum_{j=0}^{m-1} \|U_j\|_{s-j-\frac{1}{2}}
\end{aligned}$$

Now it is noted, that

$$\begin{aligned}
\mathcal{F}(v_j \otimes D_j^n \delta) &= \int_{\mathbb{R}^n} e^{-i\xi\cdot x} v_j(x') D_n^j \delta(x_n)\, dx \\
&= \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} v_j(x') \int_{\mathbb{R}} e^{-i\xi_n \cdot x_n} D_n^j \delta(x_n)\, dx_n\, dx' \\
&= \xi_n^j \hat{v}(\xi')
\end{aligned}$$

and

$$\int_{\mathbb{R}} \xi_n^{2j} (1 + |\xi'|^2 + |\xi_n|^2)^{-m} d\xi_n \le c(1 + |\xi'|^2)^{j-m+\frac{1}{2}}$$

for $0 \le j < m$.

Thus, by [Grubb, 1996a, p. 484]

$$\begin{aligned}
&\|v_j \otimes D_n^j \delta\|_{(-m+s-k,k)} \\
&\le \|v_j \otimes D_n^j \delta\|_{(-m,s)} \\
&\le c_1 \|\langle\xi\rangle^{-m} \langle\xi'\rangle^s \xi_n^j \hat{v}(\xi')\|_0 \\
&\le c_2 \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}} \xi_n^{2j} (1+|\xi'|^2+|\xi_n|^2)^{-m} \langle\xi'\rangle^{2s} |\hat{v}(\xi')|^2 d\xi_n d\xi' \\
&\le c_3 \int\limits_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2(s+j-m+\frac{1}{2})} |\hat{v}(\xi')|^2 d\xi' \\
&= c_3 \|v_j\|_{s-m+j+\frac{1}{2}}
\end{aligned}$$

so

$$\begin{aligned}
\|r_\Omega \tilde{A} A^c U\|_{(s-k,k),\Omega} &\le \sum_{j=0}^{m-1} \|\tilde{A}(v_j \otimes D_n^j \delta)\|_{(s-k,k)} \\
&\le c_1 \sum_{j=0}^{m-1} \|v_j \otimes D_n^j \delta\|_{(s-m-k,k)} \\
&\le c_2 \sum_{j=0}^{m-1} \|v_j\|_{s-m+j+\frac{1}{2}} \\
&\le c_3 \sum_{j=0}^{m-1} \|U_j\|_{s-j-\frac{1}{2}}
\end{aligned}$$

Now, since $Pr_\Omega \tilde{A} A^c U = r_\Omega R_2 P^c U$ is continuous from $\mathscr{D}'(\Gamma)$ to $C^\infty(\overline{\Omega})$, it follows, by an argument as in the first part of the proof, that

$$\|r_\Omega \tilde{A} A^c U\|_s \le c \sum_{j=0}^{m-1} \|U_j\|_{s-j-\frac{1}{2}}$$

□

Lemma 3.12
The parametrix $\begin{pmatrix} T_1 & T_2 \end{pmatrix}$ *is continuous from* $H^{s-m}(\bar{\Omega}) \times \prod_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Gamma)$ *to* $H^s(\bar{\Omega})$ *for all* $s \ge m$.

Remark: As a consequence of the lack of tools to prove the preceding lemmas, this lemma also suffers from the inadequate tools.

Proof:
Using that $S_{kj} \in \Psi_{\text{phg}}^{k-m_j}(\Gamma)$ and $S''_{kl} \in \Psi_{\text{phg}}^{k-l}(\Gamma)$ it follows with the aid of Lemma 3.11 that

$$
\begin{aligned}
\| \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \|_s &= \|(I + r_\Omega \tilde{A} A^c S'' \gamma) r_\Omega \tilde{A} e_\Omega f + r_\Omega \tilde{A} A^c S g\|_s \\
&\le \|r_\Omega \tilde{A} e_\Omega f\|_s + \|r_\Omega \tilde{A} A^c S'' \gamma r_\Omega \tilde{A} e_\Omega f\|_s + \|r_\Omega \tilde{A} A^c S g\|_s \\
&\le c_1 \|f\|_{s-m} + c_2 \sum_{j=0}^{m-1} \|(S'' \gamma r_\Omega \tilde{A} e_\Omega f)_j\|_{s-j-\frac{1}{2}} \\
&\quad + c_3 \sum_{k=0}^{m-1} \|(Sg)_k\|_{s-k-\frac{1}{2}} \\
&\le c_1 \|f\|_{s-m} + c_4 \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \|S''_{ji} (\gamma r_\Omega \tilde{A} e_\Omega f)_i\|_{s-j-\frac{1}{2}} \\
&\quad + c_5 \sum_{k=0}^{m-1} \sum_{l=0}^{J} \|S_{kl} g_l\|_{s-k-\frac{1}{2}} \\
&\le c_1 \|f\|_{s-m} + c_6 \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \|(\gamma r_\Omega \tilde{A} e_\Omega f)_i\|_{s-i-\frac{1}{2}} \\
&\quad + c_7 \sum_{k=0}^{m-1} \sum_{l=0}^{J} \|g_l\|_{s-m_l-\frac{1}{2}} \\
&\le c_1 \|f\|_{s-m} + c_8 \|r_\Omega \tilde{A} e_\Omega f\|_s + c_9 \sum_{l=0}^{J} \|g_l\|_{s-m_l-\frac{1}{2}} \\
&\le c_{10} \|f\|_{s-m} + c_9 \sum_{l=0}^{J} \|g_l\|_{s-m_l-\frac{1}{2}}
\end{aligned}
$$

□

Through the preceding lemmas the main theorem of the chapter has been proved:

Theorem 3.13
Let A be an differential operator of order m, and let B be system of differential boundary operators of transversal order $\le \mu$ with $\mu \le m-1$.

If $\binom{A}{B}$ is an elliptic boundary value problem then

$$
\begin{pmatrix} A \\ B \end{pmatrix} : H^{s-m}(\bar{\Omega}) \times \prod_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Gamma) \to H^s(\bar{\Omega})
$$

is a Fredholm operator for all $s \geq \mu + 1$.

To complete the analysis of elliptic boundary value problems, as treated in the present thesis, the following theorem concerning the kernel, the cokernel and hence the index of the boundary value problem is included:

Theorem 3.14
Let $\binom{A}{B}$ be an elliptic boundary value problem. The kernel of $\binom{A}{B}$ is a finite dimensional subspace of $C^\infty(\overline{\Omega})$. Also, the defining relations for the range are independent of s.

Moreover, the index is uniquely determined by the principal terms of A and B.

Proof:
From Theorem 3.13, $\binom{A}{B}$ is a Fredholm operator, and thus both the kernel and the cokernel have finite dimension.

If A_j is a lower order term of A, then for some $t < m$ it follows, by A.23 that

$$A_j : H^s(\bar{\Omega}) \to H^{s-t}(\bar{\Omega}) \overset{\text{comp}}{\hookrightarrow} H^{s-m}(\bar{\Omega})$$

so A_j is compact. Hence by [Hörmander, 1985, p. 183]

$$\mathrm{Ind}(A) = \mathrm{Ind}(A - A_j)$$

and thus it follows, that the index is determined by the principal terms of A. The proof for B is similar.

Now assume that $u \in H^s(\bar{\Omega})$ with $u \in \mathrm{Ker}\binom{A}{B}$. Then by Lemma 3.10 and 3.12 there exists a negligible operator $\mathcal{R}$ such that

$$u - \mathcal{R}u = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} T_1 & T_2 \end{pmatrix} 0 = 0$$

so

$$u = \mathcal{R}u \in C^\infty(\overline{\Omega})$$

and hence the kernel is a subspace of $C^\infty(\overline{\Omega})$.

Next let $u \in H^t(\bar{\Omega})$ be a solution of

$$\begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} f \\ g \end{pmatrix}$$

and let $f \in H^{s-m}(\bar{\Omega})$ and $g \in \prod_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Gamma)$ with $s \geq t \geq \mu + 1$. Let

$$\tilde{u} = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

and let $v = u - \tilde{u}$. Then there exists a negligible operator $\mathcal{R}_2$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} v = \begin{pmatrix} A \\ B \end{pmatrix} u - \begin{pmatrix} A \\ B \end{pmatrix} \tilde{u} = \mathcal{R}_2 \begin{pmatrix} f \\ g \end{pmatrix} \in \begin{matrix} C^\infty(\overline{\Omega}) \\ \times \\ \prod_{j=1}^{m-1} C^\infty(\Gamma) \end{matrix}$$

but then by Lemma 3.10 $v \in C^\infty(\overline{\Omega})$, so $u \in H^s(\bar{\Omega})$.

In particular this shows, that if for some s, $(f, g) \in \mathrm{Ran}(\binom{A}{B}|^s_H(\bar{\Omega}))$, then for all $t \geq s$, $(f, g) \in \mathrm{Ran}(\binom{A}{B}|_{H^t(\bar{\Omega})})$, and hence the defining relations for the range are independent of s. □

Now Theorem 3.13 and 3.14 gives the following existence, uniqueness, and regularity statement for elliptic boundary value problems:

Theorem 3.15
Let A be a differential operator of order m, and let B be a system of differential boundary operators of transversal order $\leq \mu$ with $\mu \leq m - 1$. Assume that $\binom{A}{B}$ defines an elliptic boundary value problem. Then

1. *If for some $s \geq \mu + 1$ exists a solution in $H^s(\bar{\Omega})$ to*

$$\begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} f \\ g \end{pmatrix}$$

   then for all $t \geq s$ there exists solutions in $H^t(\bar{\Omega})$.

2. *If there exists a solution to*

$$\begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} f \\ g \end{pmatrix}$$

   then the solution is uniquely determined modulo a finite dimensional subspace of $C^\infty(\overline{\Omega})$.

3. *If $u \in H^t(\bar{\Omega})$ is a solution of*

$$\begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} f \\ g \end{pmatrix}$$

   and $f \in H^{s-m}(\bar{\Omega})$ and $g \in \prod_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Gamma)$ with $s \geq t \geq \mu + 1$, then $u \in H^s(\bar{\Omega})$.

Chapter 4

The Biharmonic Operator

The primary goal in this chapter is to prove an existence, uniqueness and regularity theorem for the biharmonic operator with Dirichlet boundary conditions. The proof will rely on the results found in Chapter 3. Moreover this chapter will contain a few lemmas related to the biharmonic operator and its Dirichlet realization.

Throughout this chapter it is assumed that $\Omega \subset \mathbb{R}^2$ is open and bounded. Even though most of the results hold also for $\Omega \subset \mathbb{R}^n$ open and bounded, it is assumed for all lemmas that $\Omega \subset \mathbb{R}^2$, since this is what is needed in the later chapters.

First it is noted that $(\cdot|\cdot)_\Delta$, defined in Definition A.30, on $H_0^2(\overline{\Omega}) \subset L^2(\Omega)$ is the symmetric sesqui-linear form defining Δ_D^2.

Using Hölders inequality and Lemma A.29 on $(\cdot|\cdot)_\Delta$ it follows that

$$|(u|v)_\Delta| \leq \|\Delta u\|_0 \|\Delta v\|_0 \leq c_1 \|u\|_2 \|v\|_2$$

where c_1 is a positive constant. Thus $(u|v)_\Delta$ is bounded on $H_0^2(\overline{\Omega})$.

From Lemma A.29 there exists a $c_2 > 0$ such that

$$\mathrm{Re}(u|u)_\Delta = \|\Delta u\|_0^2 \geq c_2 \|u\|_2^2$$

which shows that $(\cdot|\cdot)_\Delta$ is elliptic on $H_0^2(\overline{\Omega})$.

This allows one to apply the results in Section A.1 to $(\cdot|\cdot)_\Delta$.

Theorem 4.1
$\Delta_D^2 : H_0^2(\overline{\Omega}) \to H^{-2}(\bar{\Omega})$ is a homeomorphism.

Proof:
Let the the triple $(L^2(\Omega), H_0^2(\overline{\Omega}), (\cdot|\cdot)_\Delta)$ be given.

By Lemma A.7, there exists a uniquely determined $S \in \mathbb{B}(H_0^2(\overline{\Omega}), (H_0^2(\overline{\Omega}))')$, such that $(u|v)_\Delta = \langle Su, v\rangle$. So by Lemma A.14, Δ_D^2 is a linear homeomorphism from $H_0^2(\overline{\Omega})$ to $(H_0^2(\overline{\Omega}))'$, which is isometric, and anti-linearly isomorphic to $H^{-2}(\bar{\Omega})$. $\square$

From now on, G_2 will denote the inverse of Δ_D^2.

Theorem 4.2
The operator

$$\begin{pmatrix}\Delta^2\\ \gamma_0\\ \gamma_1\end{pmatrix} : H^s(\bar{\Omega}) \to \begin{matrix} H^{s-4}(\bar{\Omega}) \\ \times \\ H^{s-\frac{1}{2}}(\bar{\Omega}) \\ \times \\ H^{s-\frac{3}{2}}(\bar{\Omega})\end{matrix}$$

defines an elliptic boundary value problem for all $s \geq 2$.

Let G_2 be the inverse of Δ_D^2 and let $\begin{pmatrix}\mathcal{K}_0 & \mathcal{K}_1\end{pmatrix}$ be a right inverse of $\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}$.

For $s = -2$ a parametrix is given by

$$\begin{pmatrix}T_1 & T_2 & T_3\end{pmatrix} = \begin{pmatrix}G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1\end{pmatrix}$$

and it is an inverse of $\begin{pmatrix}\Delta^2\\ \gamma_0\\ \gamma_1\end{pmatrix}$.

Proof:
First it is noted that

$$\Delta^2 = \mathrm{OP}(|\xi|^4)$$

so Δ^2 is an elliptic operator of order 4. Moreover

$$B = \begin{pmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\end{pmatrix}$$

satisfy the Shapiro-Lopatinskiĭ condition (cf. [Grubb, 2000, p. 7.12]), so $\begin{pmatrix}\Delta^2\\ \gamma_0\\ \gamma_1\end{pmatrix}$ is an elliptic boundary value problem in the sense of Definition 3.1.

Thus by Lemma 3.10 and 3.12 $\begin{pmatrix}\Delta^2\\ \gamma_0\\ \gamma_1\end{pmatrix}$ has a parametrix $\begin{pmatrix}T_1 & T_2 & T_3\end{pmatrix}$.

Let $f \in H^{-2}(\bar{\Omega})$, $\phi_0 \in H^{\frac{3}{2}}(\Gamma)$ and $\phi_1 \in H^{\frac{1}{2}}(\Gamma)$ and consider the following boundary value problem:

$$\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} u = \begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix} \tag{4.1}$$

By [Grubb, 2000, p. 3.27] $\begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}$ has a right inverse $\begin{pmatrix} \mathcal{K}_0 & \mathcal{K}_1 \end{pmatrix}$. Let $v = u - \mathcal{K}_0\phi_0 - \mathcal{K}_1\phi_1$. Then

$$\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} v = \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} u - \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} (\mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1) = \begin{pmatrix} f - \Delta^2(\mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1) \\ 0 \\ 0 \end{pmatrix}$$

so by Theorem 4.1

$$v = G_2(f - \Delta^2(\mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1))$$

where G_2 denotes the inverse of Δ^2_D, and hence

$$u = G_2(f - \Delta^2(\mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1)) + \mathcal{K}_0\phi_0 + \mathcal{K}_1\phi_1$$

This motivates the definition of the parametrix

$$\begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix} = \begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix}$$

which by the arguments above is a left inverse. That it is also a right inverse is seen by

$$\begin{aligned} &\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix} \\ &= \begin{pmatrix} \Delta^2 G_2 & \Delta^2(I - G_2\Delta^2)\mathcal{K}_0 & \Delta^2(I - G_2\Delta^2)\mathcal{K}_1 \\ \gamma_0 G_2 & \gamma_0(I - G_2\Delta^2)\mathcal{K}_0 & \gamma_0(I - G_2\Delta^2)\mathcal{K}_1 \\ \gamma_1 G_2 & \gamma_1(I - G_2\Delta^2)\mathcal{K}_0 & \gamma_1(I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \end{aligned}$$

□

Theorem 4.3
Let $\Omega \subset \mathbb{R}^2$ be open, bounded and connected, and let

$$\begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix} \in \begin{matrix} H^{s-4}(\bar{\Omega}) \\ \times \\ H^{s-\frac{1}{2}}(\bar{\Omega}) \\ \times \\ H^{s-\frac{3}{2}}(\bar{\Omega}) \end{matrix} \tag{4.2}$$

Then

$$\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} u = \begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix} \tag{4.3}$$

has a unique solution in $H^s(\bar{\Omega})$ *for* $s \geq 2$.

Moreover, if $2 \leq t \leq s$ *and* $u \in H^t(\bar{\Omega})$ *is a solution of* (4.3) *and if* (4.2) *is fulfilled, then* $u \in H^s(\bar{\Omega})$

Proof:

Since $\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix}$ is an elliptic boundary value problem, it follows by Theorem 3.15 that there exists solutions, if $\begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix}$ is orthogonal to a finite dimensional subspace, $\mathcal{L}_0$ of $C^\infty(\overline{\Omega})$ and that the solutions are unique modulo a finite dimensional subspace, $\mathcal{L}_1$ of $C^\infty(\overline{\Omega}) \times \prod_{i=1}^{J} C^\infty(\Gamma)$. Hence it suffices to show that $\dim \operatorname{Ker} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} = 0$ and $\dim \operatorname{Coker} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} = 0$, when $s = 2$.

Since $\begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix}$ is an inverse of $\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix}$ when $s = 2$ by Theorem 4.2, it follows that if $u \in \operatorname{Ker} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix}$, then

$$\begin{aligned} u &= \begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} u \\ &= \begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

which shows that $\dim \operatorname{Ker} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} = 0$, when $s = 2$.

Let

$$\begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix} \in \begin{matrix} H^{-2}(\bar{\Omega}) \\ \times \\ H^{\frac{3}{2}}(\bar{\Omega}) \\ \times \\ H^{\frac{1}{2}}(\bar{\Omega}) \end{matrix}$$

Then by application of $\begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix}$ it follows by Theorem 4.2 that

$$u = \begin{pmatrix} G_2 & (I - G_2\Delta^2)\mathcal{K}_0 & (I - G_2\Delta^2)\mathcal{K}_1 \end{pmatrix} \begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix}$$

is a solution, and hence $\dim \operatorname{Coker} \begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} = 0$, when $s = 2$. □

Theorem 4.4
$\Delta_D^2 : H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega}) \to L_2(\Omega)$ *is a homeomorphism.*

Proof:
Let the triple $(L^2(\Omega), H_0^2(\overline{\Omega}), (\cdot|\cdot)_\Delta)$ be given.

Since Δ_D^2, the operator defined by the triple, is a selfadjoint extension of $\Delta_{\min}^2$, $\Delta_D^2 = \Delta_D^{2\,*} \subset \Delta_{\max}^2$, so $D(\Delta_D^2) \subset D(\Delta_{\max}^2)$. Moreover $D(\Delta_D^2) \subset H_0^2(\overline{\Omega})$ this shows that $D(\Delta_D^2) \subset D(\Delta_{\max}^2) \cap H_0^2(\overline{\Omega})$.

Conversely, assume that $u \in D(\Delta_{\max}^2) \cap H_0^2(\overline{\Omega})$ then

$$(\Delta^2 u|v) = (\Delta u|\Delta v), \quad \forall v \in \mathscr{D}(\Omega)$$

This extends by closure to all of $H_0^2(\overline{\Omega})$ hence $u \in D(\Delta_D^2)$, so

$$D(\Delta_D^2) = D(\Delta_{\max}^2) \cap H_0^2(\overline{\Omega})$$

Now let $u \in D(\Delta_D^2)$. Then $u \in H_0^2(\overline{\Omega})$ and there exists an $f \in L^2(\Omega)$ such that

$$\begin{pmatrix} \Delta^2 \\ \gamma_0 \\ \gamma_1 \end{pmatrix} u = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

so by Theorem 4.3 $u \in H^4(\bar{\Omega})$, and hence

$$D(\Delta_D^2) \subset H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$$

Conversely, if $u \in H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$, then there exists an $f \in L^2(\Omega)$ such that

$$(u|v)_\Delta = (\Delta^2 u|v) = (f|v)$$

for all $v \in H_0^2(\overline{\Omega})$, i.e. $u \in D(\Delta_D^2)$, and thus it follows that

$$D(\Delta_D^2) = H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$$

□

Corollary 4.5
G_2 is a compact selfadjoint operator in $L^2(\Omega)$

Proof:
By Lemma 4.4, Δ_D^2 is a homeomorphism from $H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$ to $L^2(\Omega)$, so

$$G_2 : L^2(\Omega) \to H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$$

is continuous. But $H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega}) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega)$ by Lemma A.23 so

$$G_2 : L^2(\Omega) \to L^2(\Omega)$$

is compact.

The selfadjointness follows from selfadjointness of Δ_D^2. □

Lemma 4.6
There exists a basis for $L^2(\Omega)$ consisting of eigenvectors for Δ_D^2.

Proof:
First it is noted that if e_j is an eigenvector for Δ_D^2 then

$$e_j = \lambda_j G_2(e_j)$$

so e_j is an eigenvector for G_2. Thus the set of eigenvectors for Δ_D^2 coincides with the set of eigenvectors for G_2.

Now by Corollary 4.5, G_2 is a compact, selfadjoint operator in $L^2(\Omega)$, so the eigenvectors for G_2 form an orthonormal basis for $L^2(\Omega)$ by the spectral theorem for compact selfadjoint operators [Pedersen, 2000, p. 83], and consequently also the eigenvectors for Δ_D^2 form an orthonormal basis for $L^2(\Omega)$. □

Lemma 4.7
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $s > 0$. Then $(\Delta_D^2)^{-s} := (G_2)^s$ is a selfadjoint compact operator in $L^2(\Omega)$ with dense range.

If moreover $s \leq 2$ and $s \neq \frac{1}{2}, \frac{3}{2}$, then

$$(G_2)^{\frac{s}{4}} : L^2(\Omega) \to H_0^s(\Omega)$$

is a selfadjoint homeomorphism.

Proof:
By Corollary 4.5 G_2 is a compact, selfadjoint operator in $L^2(\Omega)$, so by the spectral theorem for compact, selfadjoint operators there exists an orthonormal system $\{e_j\}$ and constants $\{\lambda_j\}$ with $\lambda_j \to 0$ for $j \to \infty$, such that if

$$u = \sum_{j=0}^{\infty} (u|e_j)e_j$$

then

$$G_2 u = \sum_{j=0}^{\infty} \lambda_j (u|e_j)e_j$$

Now the spectrum of G_2 is a subset of $[0, \infty[$, so since z^s is analytic for $s > 0$, by the functional calculus for selfadjoint operators [Reed and Simon, 1980, p. 222], $(G_2)^s$ can be defined by

$$(G_2)^s u = \sum_{j=0}^{\infty} (\lambda_j)^s (u|e_j)e_j \tag{4.4}$$

for all $s > 0$.

Moreover, since G_2 is compact and selfadjoint, and since $\lambda_j^s \to 0$ for $j \to \infty$, it follows by [Pedersen, 2000, p. 86] that $(G_2)^s$ is compact and selfadjoint.

Next it is noted, that since $\mathrm{Ker}(G_2) = \{0\}$, $\lambda_j > 0$ for all j, so by (4.4), $\mathrm{Ker}((G_2)^s) = \{0\}$.

Thus it follows by [Pedersen, 2000, p. 57], since $(G_2)^s$ is selfadjoint that

$$L^2(\Omega) = \mathrm{Ker}((G_2)^s) \oplus \overline{\mathrm{Ran}((G_2)^s)} = \overline{\mathrm{Ran}((G_2)^s)}$$

so $(G_2)^s$ has dense range, and thus there exists a densely defined operator $(\Delta_D^2)^s$, such that

$$(G_2)^s : L^2(\Omega) \to D((\Delta_D^2)^s)$$

is a homeomorphism, and $(\Delta_D^2)^s$ is the inverse of $(G_2)^s$.

The following construction is based on the theory of interpolation spaces. However, in lack of time, the presentation of the theory of interpolation has been omitted.

Assume that $0 < s \leq 2$, with $s \neq \frac{1}{2}, \frac{3}{2}$. Then by Lemma 4.4 and the discussion in [Lions and Magenes, 1972, p. 9f], $H_0^2(\overline{\Omega})$ is the domain of definition of $(\Delta_D^2)^{\frac{1}{2}}$. Then by definition of the interpolation space

$$[H_0^2(\overline{\Omega}), L^2(\Omega)]_{1-\frac{s}{2}} = D((\Delta_D^2)^{\frac{s}{4}})$$

Then, by [Lions and Magenes, 1972, p. 64]

$$[H_0^2(\overline{\Omega}), L^2(\Omega)]_{1-\frac{s}{2}} = H_0^s(\overline{\Omega})$$

and thus it follows that

$$(G_2)^{\frac{s}{4}} : L^2(\Omega) \to H_0^s(\overline{\Omega})$$

is a homeomorphism when $0 < s \leq 2$, with $s \neq \frac{1}{2}, \frac{3}{2}$. □

Lemma 4.8
If $0 < s \leq 2$ with $s \neq \frac{1}{2}, \frac{3}{2}$, then $\|(\Delta_D^2)^{s/4} \cdot \|_0$ is an equivalent norm on $H_0^s(\overline{\Omega})$, and if $-2 \leq s < 0$ with $s \neq -\frac{1}{2}, -\frac{3}{2}$, then $\|(\Delta_D^2)^{s/4} \cdot \|_0$ is an equivalent norm on $H^s(\bar{\Omega})$.

Proof:
First it is noted that since $H_0^s(\overline{\Omega})$ is the dual space of $H^{-s}(\bar{\Omega})$ and $(\Delta_D^2)^{s/4}$ is selfadjoint, it is enough to show the equivalence for $s > 0$.

Now it follows by Lemma 4.7 for all $u \in H_0^s(\overline{\Omega})$ that

$$\|u\|_s = \|(G_2)^{\frac{s}{4}} (\Delta_D^2)^{\frac{s}{4}} u\|_s \leq c_1 \|(\Delta_D^2)^{\frac{s}{4}} u\|_0$$
$$\|(\Delta_D^2)^{\frac{s}{4}} u\|_0 \leq c_2 \|u\|_s$$

for $0 < s \leq 2$, with $s \neq \frac{1}{2}, \frac{3}{2}$, which show the equivalence. □

Chapter 5

The Monge-Ampére Form

The present chapter contains a definition of the Monge-Ampére form and two related forms. It is shown that on certain Sobolev spaces they coincide. Moreover the chapter contains a few lemmas concerning continuity and symmetry of the Monge-Ampére form and of related forms involving the Monge-Ampére form.

Throughout this chapter it is assumed that $\Omega \subset \mathbb{R}^2$ is open, bounded and smooth.

Definition 5.1
For $u, v \in H^2(\bar{\Omega})$ the Monge-Ampére form $[\cdot,\cdot]$ is defined by

$$[u, v] = (\partial_{11}^2 u)(\partial_{22}^2 v) - 2(\partial_{12}^2 u)(\partial_{12}^2 v) + (\partial_{22}^2 u)(\partial_{11}^2 v)$$

For $u \in H^{1+\delta}(\bar{\Omega})$ and $v \in H^2(\bar{\Omega})$ with $\delta > 0$, $M_1(u, v)$ is defined by

$$M_1(u, v) = \partial_{11}^2 (u \partial_{22}^2 v) - 2\partial_{12}^2 (u \partial_{12}^2 v) + \partial_{22}^2 (u \partial_{11}^2 v)$$

For $u \in H^{1+\delta}(\bar{\Omega})$ and $v \in H^2(\bar{\Omega})$ with $\delta > 0$, $M_2(u, v)$ is defined by

$$M_2(u, v) = \partial_1((\partial_1 u)(\partial_{22}^2 v) - (\partial_2 u)(\partial_{12}^2 v)) + \partial_2((\partial_2 u)(\partial_{11}^2 v) - (\partial_1 u)(\partial_{12}^2 v))$$

Lemma 5.2
$[\cdot,\cdot] : H^2(\bar{\Omega}) \times H^2(\bar{\Omega}) \mapsto L^1(\Omega)$ *is continuous.*

Proof:
By the definition of the Monge-Ampére form, it can be written as a sum

of terms of the form $\partial^2_{ij}u\partial^2_{kl}v$, so by Hölder's inequality

$$\|[u,v]\|_{0,1} \leq \sum \|\partial^2_{ij}u\partial^2_{kl}v\|_{0,1} \leq c\|u\|_2\|v\|_2$$

□

Lemma 5.3
$[\cdot,\cdot] : H^2(\bar{\Omega}) \times H^2(\bar{\Omega}) \mapsto H^{-2}(\bar{\Omega})$ is bilinear and continuous.

$M_1(u,v) : H^{1+\delta}(\bar{\Omega} \times H^2(\bar{\Omega}) \to H^{-2}(\bar{\Omega})$ is bilinear and continuous for all $\delta > 0$.

$M_2(u,v) : H^{1+\delta}(\bar{\Omega}) \times H^2(\bar{\Omega}) \to H^{-2}(\bar{\Omega})$ is bilinear and continuous for all $\delta > 0$.

Proof:
First it is noted, that $[\cdot,\cdot]$, $M_1(\cdot,\cdot)$, and $M_2(\cdot,\cdot)$ are bilinear by definition.

Assume next that $u, v \in H^2(\bar{\Omega})$. Then for all $i, j = 1, 2$, $\partial^2_{ij}u, \partial^2_{ij}v \in L^2\Omega$, so by Corollary 2.7 with $s_1 = s_2 = 0$ and $s = -2$, $[u,v] \in H^{-2}(\bar{\Omega})$, the continuity following from the continuity of the product.

Assume next that $u \in H^{1+\delta}(\bar{\Omega})$ and $v \in H^2(\bar{\Omega})$. Then for all $i, j = 1, 2$, $\partial_{ij}v \in L^2(\Omega)$, and for all $k = 1, 2$, $\partial_k u \in H^\delta(\bar{\Omega})$. By Corollary 2.6

$$u\partial^2_{ij}v \in H^\delta(\bar{\Omega})$$

so

$$M_1(u,v) \in H^{-2+\delta}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{-2}(\bar{\Omega})$$

Moreover

$$\partial_k u\partial^2_{ij}v \in H^{-1+\delta}(\bar{\Omega})$$

and thus

$$M_2(u,v) \in H^{-2+\delta}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{-2}(\bar{\Omega})$$

□

Lemma 5.4
Let $u, v \in H^2(\bar{\Omega})$ one of them being in $H^2_0(\overline{\Omega})$. Then

$$[u,v] = M_1(u,v) = M_2(u,v)$$

Proof:
Let $u, v \in H^2(\bar{\Omega})$ one of them being in $H^2_0(\overline{\Omega})$. Assume for simplicity that

$v \in H_0^2(\overline{\Omega})$. Then there exists a sequence, (v_j), in $\mathscr{D}(\Omega)$ such that $v_j \to v$ in $H_0^2(\overline{\Omega})$. Then by Lemma 5.3

$$\begin{aligned}
[u,v] &= \lim_{j\to\infty} [u,v_j] \\
&= \lim_{j\to\infty} ((\partial_{11}^2 u)(\partial_{22}^2 v_j) - 2(\partial_{12}^2 u)(\partial_{12}^2 v_j) + (\partial_{22}^2 u)(\partial_{11}^2 v_j)) \\
&= \lim_{j\to\infty} (\partial_1((\partial_1 u)(\partial_{22}^2 v_j) - (\partial_2 u)(\partial_{12}^2 v_j)) \\
&\qquad\qquad + \partial_2((\partial_2 u)(\partial_{11}^2 v_j) - (\partial_1 u)(\partial_{12}^2 v_j))) \\
&= \lim_{j\to\infty} M_2(u,v_j) \\
&= M_2(u,v)
\end{aligned}$$

Moreover by Lemma 5.3

$$\begin{aligned}
[u,v] &= \lim_{j\to\infty} [u,v_j] \\
&= \lim_{j\to\infty} ((\partial_{11}^2 u)(\partial_{22}^2 v_j) - 2(\partial_{12}^2 u)(\partial_{12}^2 v_j) + (\partial_{22}^2 u)(\partial_{11}^2 v_j)) \\
&= \lim_{j\to\infty} (\partial_{11}^2(u\partial_{22}^2 v_j) - 2\partial_{12}^2(u\partial_{12}^2 v_j) + \partial_{22}^2(u\partial_{11}^2 v_j)) \\
&= \lim_{j\to\infty} M_1(u,v_j) \\
&= M_1(u,v)
\end{aligned}$$

□

Lemma 5.5
The map $T : H^2(\bar{\Omega}) \times H^2(\bar{\Omega}) \times H^2(\bar{\Omega}) \mapsto \mathbb{R}$ *defined by*

$$T(u,v,w) = \int_{\Omega} [u,v] w \, dx$$

is trilinear and continuous. Moreover T *is invariant under all permutations of* (u,v,w) *if at least one of the functions is in* $H_0^2(\overline{\Omega})$*, and in this case, there exists a* $C > 0$ *such that*

$$|T(u,v,w)| \leq C\|\Delta u\|_0 \|v\|_{1,4} \|w\|_{1,4} \tag{5.1}$$

Proof:
Trilinearity of T follows directly from the bilinearity of $[u,v]$.

Now, since $[u,v] \in L^1(\Omega)$, by Lemma 5.2, and $H^2(\bar{\Omega}) \overset{\text{d}}{\hookrightarrow} L^\infty(\Omega)$ it follows by Hölder's inequality that

$$\Big| \int_{\Omega} [u,v] w \, dx \Big| \leq \|[u,v]\|_{0,1} \|w\|_{0,\infty} \leq c_1 \|[u,v]\|_{0,1} \|w\|_2$$

so the continuity of T on $H^2(\bar{\Omega}) \times H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$ follows from Lemma 5.2.

Furthermore, it is noted that $T(u,v,w) = T(v,u,w)$, since $[u,v] = [v,u]$, so it only remains to be checked, that $T(u,v,w) = T(u,w,v)$, and that (5.1) holds.

Now assume that $(u,v,w) \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ one of them, say v for simplicity, being in $\mathscr{D}(\Omega)$. Then, by Lemma 5.4, T can be written as

$$\int_\Omega [\partial_1(\partial_1 v \partial_{22}^2 u - \partial_2 u \partial_{12}^2 v) + \partial_2(\partial_2 v \partial_{11}^2 u - \partial_1 u \partial_{12}^2 v)] w \, dx$$

By Green's formulae it follows, that

$$\begin{aligned} T(u,v,w) &= \int_\Omega \partial_1 w(\partial_1 v \partial_{22}^2 u - \partial_2 u \partial_{12}^2 v) + \partial_2 w(\partial_2 v \partial_{11}^2 u - \partial_1 u \partial_{12}^2 v) \, dx \\ &= \int_\Omega [\partial_1(\partial_1 w \partial_{22}^2 u - \partial_2 u \partial_{12}^2 w) + \partial_2(\partial_2 w \partial_{11}^2 u - \partial_1 u \partial_{12}^2 w)] v \, dx \\ &= T(u,w,v) \end{aligned} \tag{5.2}$$

since $v \in C_0^\infty(\Omega)$.

Also from (5.2) it follows, that there exists a $c > 0$, such that

$$|T(u,v,w)| \leq c\|\Delta u\|_0 \|v\|_{1,4} \|w\|_{1,4}$$

which completes the proof, since by Lemma A.27, $C^\infty(\overline{\Omega})$ is dense in $H^2(\bar{\Omega})$ and since $\mathscr{D}(\Omega)$ is dense in $H_0^2(\overline{\Omega})$. □

Lemma 5.6
If $u \in H_0^2(\overline{\Omega})$, then $[u,u] = 0$ if and only if $u = 0$.

Proof:
It is obvious, from the definition, that $[u,u] = 0$ if $u = 0$.

Assume that $u \in H_0^2(\overline{\Omega})$ and $[u,u] = 0$. Then let $w(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2)$. Then

$$\begin{aligned} [u,w] &= (\partial_{11}^2 u)(\partial_{22}^2 w) - 2(\partial_{12}^2 u)(\partial_{12}^2 w) + (\partial_{22}^2 u)(\partial_{11}^2 w) \\ &= \partial_{11}^2 u + \partial_{22}^2 u \\ &= \Delta u \end{aligned}$$

Thus it follows, by Lemma 5.5, that

$$0 = \int_{\Omega} [u,u]w \, dx = \int_{\Omega} [u,w]u \, dx = \int_{\Omega} u\Delta u \, dx = \sum_{j=1}^{2} \|\partial_j u\|_0^2$$

Thus $\partial_j u = 0$ for $j = 1, 2$, but then, since $u \in H_0^2(\overline{\Omega})$, $u = 0$. □

Chapter 6

Dynamical Systems and Stability

This chapter describes some basic things about dynamical systems on a Banach space and stability. Especially Lyapunov functions are defined and their connection to stability is investigated. The presentation here follows mainly [Walker, 1976, p. 193ff] and [Khalil, 1996, p. 97ff].

Definition 6.1
A dynamical system on a Banach space X is a map $U : \overline{\mathbb{R}}_+ \times X \to X$ such that $U(t, \cdot) : X \to X$ is continuous for each $t \geq 0$, $U(\cdot, x) : \overline{\mathbb{R}}_+ \to X$ is continuous for each $x \in X$, and $U(0, x) = x$, $U(t+s, x) = U(t, U(s, x))$, for all $x \in X$.

An equilibrium point for the dynamical system U is a point x_0 where $U(t, x_0) = x_0$ for all $t \geq 0$. Without loss of generality the equilibrium point can be assumed to be at the origin, $x = 0$. This can always be achieved by a change of variables. Suppose $x_0 \neq 0$, and consider the change of variables $y = x - x_0$. Then

$$U(t, x) = U(t, y + x_0) \equiv H(t, y)$$

where $H(t, 0) = 0$ in the y-coordinates.

Now having defined a dynamical system and an equilibrium point for that system, stability is defined as

Definition 6.2
Let U be a dynamical system on a Banach space X, with equilibrium point $x = 0$. The equilibrium is

- *stable if, for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $\|U(t_0, x)\| < \delta$ implies $\|U(t, x)\| < \epsilon$ for all $t \geq t_0 \geq 0$.*
- *uniformly stable if, for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x\| < \delta$ implies $\|U(t, x)\| < \epsilon$ for all $t \geq 0$.*
- *asymptotically stable if it is stable and there exists $\gamma = \gamma(t_0) > 0$ such that $\|U(t_0, x)\| < \gamma$ implies $\|U(t, x)\| \to 0$ as $t \to \infty$.*
- *uniformly asymptotically stable if it is uniformly stable and there exists $\gamma > 0$ such that $\|x\| < \gamma$ implies $\|U(t, x)\| \to 0$ as $t \to \infty$.*
- *unstable if it is not stable.*

Here it is noted that asymptotic stability is a stronger property than stability, even though the intuition at first tells that it might be weaker.

Investigation of the stability of an equilibrium point utilizes certain functions known as Lyapunov functions, defined as

Definition 6.3
Let U be a dynamical system on a Banach space X. A continuous function $V : X \to \mathbb{R}$ is said to be a Lyapunov function for U if $V(0) = 0$ and $V'(x) \leq 0$ for all $x \in X$, where

$$V'(x) = \limsup_{t \to 0^+} \frac{1}{t}(V(U(t, x)) - V(x)), \quad x \in X$$

A continuous function $W : X \to \overline{\mathbb{R}}_+$ is called associated to the Lyapunov function V if $V'(x) \leq -W(x) \leq 0$ for all $x \in X$.

The following lemma is essential for proving the theorems of stability using Lyapunov functions.

Lemma 6.4
Let $V : X \to \mathbb{R}$ be a Lyapunov function for a dynamical system U on a Banach space X. If $x \in X$, then $V(U(\cdot, x))$ is monotonic decreasing on $\overline{\mathbb{R}}_+$, $V(U(\cdot, x))$ is differentiable a.e. on bounded subsets of $\overline{\mathbb{R}}_+$, and for every function W associated to V

$$V(U(t, x)) - V(x) \leq -\int_0^t W(U(s, x))\, ds \quad \forall t \geq 0$$

Proof:
Proof given in [Walker, 1976, p. 194]. □

Definition 6.5
Let $h > 0$. A scalar function $f : [0,h] \to \overline{\mathbb{R}}_+$ is of class M_h if $f(0) = 0$ and

$$0 \leq r_1 < r_2 \leq h \quad \Rightarrow \quad f(r_2) > f(r_1)$$

At this point the main theorems of this chapter can be stated and proved. The proofs are generalizations from finite dimension systems to systems on Banach spaces, of proofs found in [Betounes, 2001, p. 295f] and [Khalil, 1996, p. 102] respectively.

Theorem 6.6
Let U be a dynamical system on a Banach space X, with equilibrium point $x = 0$. If $V : X \to \mathbb{R}$ is a Lyapunov function for U and

$$V(x) \geq f(\|x\|) \quad \text{for } \|x\| \leq h \tag{6.1}$$

where f is of class M_h, then the equilibrium $x = 0$ is uniformly stable.

Proof:
Let $\epsilon > 0$ be given. It can be assumed that $\epsilon < h$. The closed ball $\overline{B}(0, \epsilon)$ is bounded in X. Define

$$\mu = \inf_{z \in \partial B(0,\epsilon)} V(z)$$

then $\mu \geq f(\epsilon) > 0$ by (6.1).

V is continuous so there exists $\delta > 0$ such that

$$B(0, \delta) \subseteq \{x \in B(0, \epsilon) | V(x) < f(\epsilon)\} = B(0, \epsilon) \cap V^{-1}(-\infty, f(\epsilon))$$

It remains to be shown that for each $x \in B(0, \delta)$ the trajectory $t \mapsto U(t, x)$ remains in $B(0, \epsilon)$ for all $t \in \overline{\mathbb{R}}_+$. For the rest of the proof, x will be fixed.

Now $V(x) < \mu$, by definition of μ and choice of x. Since V is a Lyapunov function, Lemma 6.4 gives that $V(U(t, x)) \leq V(x)$, for all $t \in \overline{\mathbb{R}}_+$. Thus

$$V(U(t, x)) < \mu \qquad \forall t \in \overline{\mathbb{R}}_+ \tag{6.2}$$

For convenience, define a real-valued function $g : \overline{\mathbb{R}}_+ \to \mathbb{R}$ by

$$g(t) = \|U(t, x)\|$$

In this notation, what is to be shown is the following claim: $g(t) < \epsilon$ for all $t \in \overline{\mathbb{R}}_+$.

Assume the contrary. Then there exists a $t_1 \in \overline{\mathbb{R}}_+$ so that $g(t_1) \geq \epsilon$. Then by continuity of g, the Intermediate Value Theorem guarantees the existence of a $t_0 \in [0, t_1]$ such that $g(t_0) = \epsilon$ i.e. $U(t_0, x) \in \partial B(0, \epsilon)$. So by definition of μ

$$\mu \leq V(U(t_0, x))$$

But this contradicts inequality (6.2), thus the claim must be true.

Writing the claim out explicitly gives

$$\|U(t,x)\| < \epsilon \qquad \forall t \in \overline{\mathbb{R}}_+$$

Which shows that the equilibrium point $x = 0$ is uniformly stable. □

Theorem 6.7
Let U be a dynamical system on a Banach space X, with equilibrium point $x = 0$. If $V : X \to \overline{\mathbb{R}}_+$ is a Lyapunov function, $W : X \to \overline{\mathbb{R}}_+$ is a function associated to V, and

$$V(x) \geq f(\|x\|), \quad W(x) \geq f_1(\|x\|) \quad \text{for } \|x\| \leq h \tag{6.3}$$

where f and f_1 are of class M_h, then the equilibrium $x = 0$ is uniformly asymptotically stable.

Proof:
From Theorem 6.6 the equilibrium point is uniformly stable.

It has to be shown that $U(t,x) \to 0$ as $t \to \infty$, that is, for every $a > 0$ there is $T > 0$ such that $\|U(t,x)\| < a$, for all $t > T$. By repeating the arguments of the proof of Theorem 6.6, it is seen that for every $a > 0$, some $b > 0$ can be chosen such that for $x \in B(0,b)$ then $U(t,x) \in B(0,a)$ for all $t \geq 0$. Therefore, it is sufficient to show that $V(U(t,x)) \to 0$ as $t \to \infty$ due to (6.3).

The fact that W is associated to the Lyapunov function V and fulfils (6.3), assures that V is strictly monotonic decreasing and V is bounded from below by zero, so

$$V(U(t,x)) \to c \quad \text{as } t \to \infty$$

To show $c = 0$, assume $c > 0$. By the continuity of V there exists a $d > 0$ such that $B(0,d) \subseteq \{x \in B(0,c) | V(x) < f(c)\}$. The limit $V(U(t,x)) \to c$ implies that the solution trajectory $U(t,x)$ lies outside $B(0,d)$ for all $t \geq 0$.

W is continuous so let $\gamma = \max_{d \leq \|x\| \leq \epsilon} W(x)$ then $\gamma > 0$ from (6.3). By Lemma 6.4 it follows that

$$V(U(t,x)) \leq V(x) - \int_0^t W(U(s,x))\,ds \leq V(x) - \gamma t$$

Since the right hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. ☐

Having defined stability, Lyapunov functions and how Lyapunov functions can be used to show stability, some comments are in place. First, Lyapunov functions are not unique, i.e. to a given dynamical system there might be several choices for a Lyapunov function. There is no systematic procedure for finding Lyapunov functions. However, energy functions are Lyapunov functions (with the reference frame of the potential energy chosen such that $E(0) = 0$), and would be natural candidates in some situations e.g. electrical and mechanical systems.

In [Lagnese, 1989, p. 5] several notions of stability are defined using the energy:

Definition 6.8
Let U be a dynamical system on a Banach space X, with equilibrium point $x = 0$, and let $E(t) = E(U(t,x))$ be the energy of the system. Then the system is

- *strongly stable if*
$$E(t) \to 0 \quad \text{as } t \to \infty \tag{6.4}$$

- *uniformly stable if*
$$E(t) \leq f(t)E(0) \quad f(t) \to 0 \text{ as } t \to \infty \tag{6.5}$$

- *exponentially stable if*
$$E(t) \leq Ce^{-\omega t}E(0) \quad \omega > 0 \tag{6.6}$$

Remark: From this definition it is seen, that exponential stability is a special case of uniform stability, where the decay rate of the solution, the function $f(t)$ in (6.5), is exponentially decreasing.

If a dynamical system is defined on a Banach space X as in Definition 6.1, and the energy of the solution to that system is denoted $E(t) = E(U(t,x_0))$ where x_0 is the initial state. Then, if $E(t) \geq c\|U(t,x_0)\|_X$

for all t, strong stability of Definition 6.8 implies asymptotic stability as defined in Definition 6.2. Uniform stability and exponential stability of Definition 6.8 implies uniform asymptotic stability in the sense of Definition 6.2.

On the other hand, if $E(t) \leq c\|U(t,x_0)\|_X$ for all t, then asymptotic stability of Definition 6.2 implies strong stability in the sense of Definition 6.8, where as uniform asymptotic stability in the sense of Definition 6.2 implies uniform stability of Definition 6.8. Here it is noted that a uniform asymptotically stable system in the sense of Definition 6.2 has to fulfil the additional requirement (6.6) to be exponentially stable as defined in Definition 6.8.

Part II

The von Karman Equations

Chapter 7

Continuity of Weak Solutions

In this chapter continuity of weak solutions to the evolutionary von Karman equations are shown.

Before stating the main theorem, two lemmas are proved:

Lemma 7.1
Let (x_n) be a weakly convergent sequence i a Hilbert space H, such that $x_n \rightharpoonup x$. Then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$$

Further more if $\|x_n\| \to \|x\|$ then $x_n \to x$.

Proof:
Consider a sequence (x_n) in H, such that $x_n \rightharpoonup x$. Then

$$0 \leq \|x_n - x\|^2 = \|x_n\|^2 - 2\text{Re}(x_n|x) + \|x\|^2 \tag{7.1}$$

so

$$2\text{Re}(x_n|x) - \|x\|^2 \leq \|x_n\|^2$$

Then, by taking limes inferior on both sides, and noting that $\liminf(x_n|x) = \|x\|^2$, it follows that

$$\|x\|^2 \leq \liminf_{n\to\infty} \|x_n\|^2$$

If it furthermore is assumed that $\|x_n\| \to \|x\|$ as $n \to \infty$, then by (7.1)

$$\lim_{n\to\infty} \|x_n - x\|^2 = \lim_{n\to\infty} \|x_n\|^2 - \lim_{n\to\infty} 2\text{Re}(x_n|x) + \|x\|^2 = 0$$

which shows that (x_n) is strongly convergent. □

Lemma 7.2
Let X and Y be Banach spaces. If $X \overset{d}{\hookrightarrow} Y$, then $Y^* \overset{d}{\hookrightarrow} X^*$.

Proof:
Since $X \overset{d}{\hookrightarrow} Y$, the injection $I \in \mathbb{B}(X,Y)$ is injective, has dense range and for all $x \in X$

$$\|x\|_Y = \|Ix\|_Y \leq c\|x\|_X$$

Moreover, since I has dense range, the adjoint I^* exists, is densely defined, injective, and bounded [Pedersen, 1995, p. 59]. Thus I^* can be extended by continuity to $I' \in \mathbb{B}(Y^*, X^*)$, and for all $\phi \in Y^*$

$$\|\phi\|_{X^*} = \|I'\phi\|_{X^*} \leq c\|\phi\|_{Y^*}$$

Thus it only remains to be shown that I' has dense range.

First it is noted, that $\overline{\text{Ran}(I')}$ is a norm closed subspace of X^*, and consequently a weak star closed subspace.

Assume that $\overline{\text{Ran}(I')} \neq X^*$, and let $\phi \in X^* \backslash \overline{\text{Ran}(I')}$. Then, by [Pedersen, 1995, p. 66], there exists an $x \in \overline{\text{Ran}(I')}^{\perp}$, such that

$$\langle x, \phi \rangle \neq 0 \tag{7.2}$$

Now for all $\psi \in Y^*$

$$\langle Ix, \psi \rangle = \langle x, I'\psi \rangle = 0$$

since $x \in \overline{\text{Ran}(I')}^{\perp}$. But then $x = 0$, since I is injective, which contradicts (7.2), and hence $\phi \in \overline{\text{Ran}(I')}$. □

The main theorem of this chapter is now stated. The proof follows from the lemmas following it.

Theorem 7.3
Let V and H be Hilbert spaces, with $V \overset{d}{\hookrightarrow} H$, and let V^* denote the dual of V. If $u \in L^2(a,b;V)$ and $u' \in L^2(a,b;V^*)$, then u is almost everywhere equal to a strongly continuous function from $[a,b]$ into H. Furthermore the following equality holds in the scalar distributional sense on $]\,a,b\,[$:

$$\frac{d}{dt}\|u\|_H^2 = 2\langle u', u \rangle$$

Definition 7.4
Let $h_j \in \mathscr{D}(\mathbb{R})$ be a sequence of positive functions such that $\int h_j dx = 1$ and $\operatorname{supp} h_j \subset \overline{B(0,\frac{1}{j})}$ *for all $j \in \mathbb{N}$. The sequence (h_j) is denoted a* regularizing sequence.

Lemma 7.5
Let X be a reflexive Banach space with dual X^* and let u and g be two functions in $L^1(a,b;X)$. Then the following conditions are equivalent:

1. *u is a.e. equal to a primitive function of g, i.e.*

$$u(t) = \xi + \int_a^t g(s)\,ds \tag{7.3}$$

   for almost all $t \in [a,b]$ and for some $\xi \in X$

2. *For each $\phi \in \mathscr{D}(\,]\,a,b\,[\,)$*

$$\int_a^b u(t)\phi'(t)\,dt = -\int_a^b g(t)\phi(t)\,dt \tag{7.4}$$

3. *For each $\eta \in X^*$*

$$\frac{d}{dt}\langle u,\eta\rangle = \langle g,\eta\rangle \tag{7.5}$$

Furthermore, if 1-3 are satisfied, u is almost everywhere equal to a strongly continuous function from $[a,b]$ to X.

Proof:
First it is noted from (7.3) that u is almost everywhere continuous on $[a,b]$ with respect to the norm topology on X.

Next, by partial integration, (7.4) follows from (7.3). Moreover (7.5) follows from (7.3), since

$$\frac{\partial}{\partial t}\langle u,\eta\rangle = \frac{\partial}{\partial t}\langle \xi,\eta\rangle + \frac{\partial}{\partial t}\langle \int_a^t g(s)\,ds,\eta\rangle = \frac{\partial}{\partial t}\int_a^t \langle g(s),\eta\rangle\,ds = \langle g,\eta\rangle$$

by Bochner's identity, (A.11).

Now assume that (7.5) holds, and that $\phi \in \mathscr{D}(]\,a,b\,[)$. By definition, it then follows for all $\eta \in X^*$ that

$$\int_a^b \langle u(t),\eta\rangle\phi'(t)\,dt = -\int_a^b \langle g(t),\eta\rangle\phi(t)\,dt$$

which implies by application of Bochner's identity (A.11) that for all $\eta \in X^*$

$$\Big\langle \int_a^b [u(t)\phi'(t) + g(t)\phi(t)]\,dt, \eta \Big\rangle = 0$$

and thus (7.4) follows.

Now assume that (7.4) holds. Define u_0 by

$$u_0(t) = \int_a^t g(s)\,ds$$

Then by [Hartman and Mikusiński, 1961, p. 99] u_0 is absolutely continuous and $u_0' = g$. Now let $w = u - u_0$. It then follows from (7.4) that

$$-\int_a^b g(t)\phi(t)\,dt = \int_a^b (w(t)+u_0(t))\phi'(t)\,dt = \int_a^b w(t)\phi'(t)\,dt - \int_a^b g(t)\phi(t)\,dt$$

so

$$\int_a^b w(t)\phi'(t)\,dt = 0 \tag{7.6}$$

for all $\phi \in \mathscr{D}(]\,a,b\,[)$.

Now let $\phi_0 \in \mathscr{D}(]\,a,b\,[)$ be such that

$$\int_a^b \phi_0(t)\,dt = 1$$

For any function $\phi \in \mathscr{D}(]\,a,b\,[)$ there exists a $\psi \in \mathscr{D}(]\,a,b\,[)$ such that

$$\phi = \lambda\phi_0 + \psi'$$

where

$$\lambda = \int_a^b \phi(t)\,dt$$

Let ξ be defined by

$$\xi = \int_a^b w(s)\phi_0(s)\,ds$$

Then it follows from (7.6) that

$$\begin{aligned}\int_a^b w(t)\phi(t)\,dt &= \int_a^b w(t)(\lambda\phi_0(t) + \psi'(t))\,dt \\ &= \int_a^b w(t)\lambda\phi_0(t)\,dt + \int_a^b w(t)\psi'(t)\,dt \\ &= \lambda\int_a^b w(t)\phi_0(t)\,dt \\ &= \lambda\xi = \int_a^b \xi\phi(t)\,dt\end{aligned}$$

which shows that

$$\int_a^b (w(t) - \xi)\phi(t)\,dt = 0$$

for all $\phi \in \mathscr{D}(]\,a,b\,[)$, so by a variant of Du Bois-Reymond's lemma [Grubb, 1996b, p. 6.1], adapted for vector valued functions, $w(t) = \xi$ a.e., and so it follows that (7.4) implies (7.3). □

Lemma 7.6
Let X and Y be two Banach spaces, such that $X \hookrightarrow Y$. If $\phi \in L^\infty(a,b;X)$ and $\phi \in C([a,b];Y_w)$, then $\phi \in C([a,b];X_w)$.

Proof:
First it should be noted, that if X is not dense in Y, then Y may be replaced with the closure of X in Y, and thus it can be assumed that X is dense in Y.

Now, since $X \overset{d}{\hookrightarrow} Y$, $Y^* \overset{d}{\hookrightarrow} X^*$ by lemma 7.2. Furthermore, by assumption, for all $\eta \in Y^*$ and all $t_0 \in \,]a,b[$

$$\langle \phi(t), \eta \rangle \to \langle \phi(t_0), \eta \rangle \tag{7.7}$$

as $t \to t_0$.

It should be proved, that $\phi(t) \in X$ for all $t \in [a,b]$, and that (7.7) holds for all $\eta \in X^*$.

So let $\tilde{\phi}$ be the function defined on $\mathbb{R}$, which is equal to ϕ on $[a,b]$ and equal to 0 elsewhere. Then using a regularizing sequence, a sequence, (ϕ_m), is constructed, such that for all $\eta \in Y^*$

$$\langle \phi_m(t), \eta \rangle \to \langle \phi(t), \eta \rangle \tag{7.8}$$

and for all $t \in [a,b]$ and all m it follows from [Grubb, 1996b, p. 5.9] that

$$\|\phi_m(t)\|_X \leq \operatorname*{ess\,sup}_{t \in [a,b]} \|\phi_m(t)\|_X \leq \|\phi\|_{L^\infty(I;X)} \tag{7.9}$$

From (7.9) it follows, that for all $\eta \in Y^*$

$$|\langle \phi_m(t), \eta \rangle| \leq \|\phi\|_{L^\infty(I;X)} \|\eta\|_{X^*}$$

and thus in the limit, for all $\eta \in Y^*$

$$|\langle \phi(t), \eta \rangle| \leq \|\phi\|_{L^\infty(I;X)} \|\eta\|_{X^*} \tag{7.10}$$

But this is also valid for all $\eta \in X^*$ by continuity, since $Y^* \overset{d}{\hookrightarrow} X^*$.

This shows that $\phi(t) \in X^{**}$ for all $t \in [a,b]$, so if X is reflexive, $\phi(t) \in X$ for all $t \in [a,b]$.

Assume then that X is not reflexive, and note that X equals the closure of X with the $\sigma(X^{**}, X^*)$ topology, which follows since the $\sigma(X^{**}, X^*)$ topology is weaker than the norm topology. Let Z denote X with the $\sigma(X^{**}, X^*)$ topology, and assume that $\phi(t) \in X^{**} \setminus Z$. Then by [Pedersen, 1995, p. 66] there exists an $x \in Z^\perp$, such that $\langle x, \phi(t) \rangle \neq 0$. But then, since $\phi_m \in Z$ for all m by (7.9), and $\langle \phi_m(t) - \phi(t), \eta \rangle \to 0$ for $j \to \infty$ and for all $\eta \in X^*$, it follows that

$$0 = \langle \phi_j(t), x \rangle \to \langle \phi(t), x \rangle \neq 0$$

Consequently $\phi(t) \in Z = X$ for all $t \in [a,b]$, and by (7.10)

$$\|\phi(t)\|_X \leq \|\phi\|_{L^\infty(I;X)}$$

It thus only remains to be shown, that (7.8) also holds for all $\eta \in X^*$. So let $\eta \in X^*$ be arbitrary. Since Y^* is dense in X^*, it follows, that for all $\varepsilon > 0$, there exists a $\eta_\varepsilon \in Y^*$, such that

$$\|\eta - \eta_\varepsilon\|_{X^*} < \varepsilon$$

Now it follows that

$$\begin{aligned} |\langle \phi(t) - \phi(t_0), \eta \rangle| &\leq |\langle \phi(t) - \phi(t_0), \eta - \eta_\varepsilon \rangle| + |\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle| \\ &\leq 2\varepsilon \|\phi\|_{L^\infty(I;X)} + |\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle| \end{aligned}$$

and since $\eta_\varepsilon \in Y^*$ and ϕ is weakly continuous with values in Y it follows for $t \to t_0$ that

$$|\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle| \to 0$$

and thus

$$\limsup_{t \to t_0} |\langle \phi(t) - \phi(t_0) \rangle| \leq 2\varepsilon \|\phi\|_{L^\infty(I;X)} \tag{7.11}$$

and since (7.11) holds for arbitrary ε the limit is zero, thus proving the weak continuity with values in X. □

Lemma 7.7

Let V and H be Hilbert spaces, with $V \overset{d}{\hookrightarrow} H$. If $u \in L^2(a, b; V)$ and $u' \in L^2(a, b; V^*)$, then the following equality holds in the scalar distributional sense on I:

$$\frac{d}{dt} \|u\|_H^2 = 2\langle u', u \rangle \tag{7.12}$$

Proof:

First it is noted, that (7.12) makes sense, since $\frac{d}{dt}\|u\|_H^2$ and $\langle u', u \rangle$ are both integrable.

Next it is claimed, that there exists a sequence (u_m), each u_m being infinitely differentiable from $[a, b]$ to V, such that

$$u_m \to u \quad \text{in } L^2_{\text{loc}}(a, b; V) \tag{7.13}$$

$$u'_m \to u' \quad \text{in } L^2_{\text{loc}}(a, b; V^*) \tag{7.14}$$

To see this, let (h_m) be a regularizing sequence, and let $\tilde{u}$ be the function on $\mathbb{R}$, which equals u on $[a, b]$ and 0 elsewhere. Then let $u_m = h_m * \tilde{u}$. It is then easily seen [Grubb, 1996b, p. 5.9], that u_m is infinitely differentiable on $[a, b]$ with values in V. By a similar argument as in [Grubb, 1996b, p. 7.3f] it follows that $u_m \to u$ in $L^2_{\text{loc}}(a, b; V)$ and $u'_m \to u'$ in $L^2_{\text{loc}}(a, b; V^*)$, and thus the claim follows.

Now, since the u_m's are infinitely differentiable, it follows for each m that

$$\frac{d}{dt}\|u_m\|_H^2 = \frac{d}{dt}(u_m|u_m)_H = 2(u_m'|u_m)_H = 2\langle u_m', u_m\rangle \tag{7.15}$$

pointwise.

By (7.13) and (7.14) it follows, that

$$\|u_m\|_H^2 \to \|u\|_H^2 \quad \text{in } L^1_{\text{loc}}(]\,a,b\,[) \tag{7.16}$$

$$\langle u_m', u_m\rangle \to \langle u', u\rangle \quad \text{in } L^1_{\text{loc}}(]\,a,b\,[) \tag{7.17}$$

Since convergence in L^1_{loc} implies convergence in $\mathscr{D}'$, the limit of (7.15) as $m \to \infty$ exists in distributional sense, and

$$\frac{d}{dt}\|u\|_H^2 = 2\langle u', u\rangle$$

□

Lemma 7.8
Let V and H be Hilbert spaces, with $V \overset{\text{d}}{\hookrightarrow} H$. If $u \in L^2(a,b;V)$ and $u' \in L^2(a,b;V^*)$, then u is almost everywhere equal to a strongly continuous function from $[a,b]$ into H.

Proof:
First it is noted, that the function $t \mapsto \langle u'(t), u(t)\rangle$ is integrable, so by Lemma 7.7 and 7.5

$$\|u(t)\|_H^2 = \|u(a)\|_H^2 + \int_a^t 2\langle u'(s), u(s)\rangle\, ds$$

and thus

$$\operatorname*{ess\,sup}_{t\in[a,b]}\|u(t)\|_H^2 \le \|u(a)\|_H^2 + \int_a^b 2|\langle u'(s), u(s)\rangle|\, ds < \infty$$

so $u \in L^\infty(a,b;H)$. Since $u \in L^2(a,b;V) \overset{\text{d}}{\hookrightarrow} L^2(a,b;V^*)$, u and u' fulfil the requirements of Lemma 7.5, so u is almost everywhere equal to a function, $\tilde{u} \in C([a,b];V^*)$. It then follows that $\tilde{u} \in C([a,b];V^*_{\text{w}})$ and $\tilde{u} \in L^\infty(a,b;H)$, so by Lemma 7.6, $\tilde{u} \in C([a,b];H_{\text{w}})$.

Now from (7.12) of Lemma 7.7 it follows that

$$\|\tilde{u}(t)\|_H^2 = \|\tilde{u}(t_0)\|_H^2 + \int_{t_o}^t 2\langle \tilde{u}'(t), \tilde{u}(t)\rangle\, dt \tag{7.18}$$

so $\|\tilde{u}(t)\|_H^2 \to \|\tilde{u}(t_0)\|_H^2$ as $t \to t_0$.

It then follows, from Lemma 7.1, that $\tilde{u} \in C([a,b];H)$. $\square$

The following theorem is an a priori result on continuity properties of weak solutions to the von Karman equations. In particular it will be shown, that the continuity condition, stated in Definition 1.1 is unnecessary.

Theorem 7.9

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth, and let $u(t)$ and $v(t)$ be measurable on $]\,0,T\,[$ with values in $H_0^2(\overline{\Omega})$.

If

$$
\begin{aligned}
u &\in L^\infty(0,T;H_0^2(\overline{\Omega})) \\
v &\in L^\infty(0,T;H_0^2(\overline{\Omega})) \\
u' &\in L^\infty(0,T;L^2(\Omega))
\end{aligned}
$$

satisfy (1.1) *in* $\mathscr{D}'(0,T;H^{-2}(\bar{\Omega}))$, *then*

1. $u'' \in L^\infty(0,T;H^{-2}(\bar{\Omega}))$

2. $t \mapsto u(t)$ *is continuous with respect to the norm topology on* $H_0^2(\overline{\Omega})$.

3. $t \mapsto u'(t)$ *is continuous with respect to the norm topology on* $L^2(\Omega)$.

Proof:

1: Since $u,v \in L^\infty(0,T;H_0^2(\overline{\Omega}))$ by assumption, it follows from Lemma 5.2 that $[u,v] \in H^{-2}(\bar{\Omega})$ for almost all $t \in [0,T]$. Moreover by Theorem 4.1, $\Delta^2 u \in H^{-2}(\bar{\Omega})$ for almost all $t \in [0,T]$. Then, since $f \in L^2([0,T] \times \Omega)$ and (u,v) is a weak solution

$$u'' \in L^\infty(0,T;H^{-2}(\bar{\Omega}))$$

2: Since $u' \in L^\infty(0,T;L^2(\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^\infty(0,T;H^{-2}(\bar{\Omega}))$ and since $H^{-2}(\bar{\Omega})^* = H_0^2(\overline{\Omega})$ by Lemma A.20, it follows from Theorem 7.3 with $V = H = H_0^2(\overline{\Omega})$, that u as element of $L^\infty(0,T;H_0^2(\overline{\Omega}))$ is equal to a function $\tilde{u}$ such that

$$\tilde{u} \in C([0,T];H_0^2(\overline{\Omega}))$$

as claimed.

3: First it is noted, that by Lemma 4.8,

$$\|(\Delta_D^2)^{-\frac{1}{2}}u'\|_2 \leq c_1\|(\Delta_D^2)^{\frac{1}{2}}(\Delta_D^2)^{-\frac{1}{2}}u'\|_0 = c_1\|u'\|_0$$
$$\|(\Delta_D^2)^{-\frac{1}{2}}u''\|_{-2} \leq c_2\|(\Delta_D^2)^{-\frac{1}{2}}u''\|_0 \leq c_3\|u''\|_{-2}$$

so

$$(\Delta_D^2)^{-\frac{1}{2}}u' \in L^\infty(0,T;H_0^2(\overline{\Omega}))$$
$$(\Delta_D^2)^{-\frac{1}{2}}u'' \in L^\infty(0,T;H^{-2}(\bar{\Omega}))$$

Now Theorem 7.3 can be applied with $V = H = H_0^2(\overline{\Omega})$ to give

$$(\Delta_D^2)^{-\frac{1}{2}}u' \in C([0,T];H_0^2(\overline{\Omega}))$$

so by another application of Lemma 4.8 it follows that

$$u' \in C([0,T];L^2(\Omega))$$

□

Chapter 8

Existence of Weak Solutions

Theorem 8.1
Let $\Omega \subset \mathbb{R}^2$ be open, bounded, smooth and connected. If

$$f \in L^2([0,T] \times \Omega), \quad u_0 \in H_0^2(\overline{\Omega}), \quad u_1 \in L^2(\Omega)$$

then there exists a weak solution, (u, v), in the sense of Definition 1.1 to (1.1), (1.3) *and* (1.4).

Proof:
By Lemma A.22 there exists an orthonormal basis $\{w_1, \ldots, w_m, \ldots\}$ for $H_0^2(\overline{\Omega})$ where $w_j \in \mathscr{D}(\Omega)$. Then by [Pedersen, 2000, p. 40] $\{w_j\}$ is also an orthonormal basis for $L^2(\Omega)$.

Since $\text{span}\{w_j\}$ is dense in $H_0^2(\overline{\Omega})$ and $L^2(\Omega)$, there exists sequences (u_{0m}) and (u_{1m}) such that

$$\begin{aligned} &u_{0m} \in \text{span}\{w_1, \ldots, w_m\},\ u_{0m} \to u_0 \text{ in } H_0^2(\overline{\Omega}) \\ &u_{1m} \in \text{span}\{w_1, \ldots, w_m\},\ u_{1m} \to u_1 \text{ in } L^2(\Omega) \end{aligned} \tag{8.1}$$

For $u_m(t) \in H_0^2(\overline{\Omega})$ the following equation is defined:

$$u_m(t) \in \operatorname{span}\{w_1, \dots, w_m\}, \quad \text{i.e. } u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \tag{8.2}$$

$$(u_m''(t)|w_j) + a_1 (u_m(t)|w_j)_\Delta + \frac{1}{a_2}\Big(\big[u_m(t), G_2\big([u_m(t), u_m(t)]\big)\big]\,|w_j\Big) = (f(t)|w_j) \tag{8.3}$$

for $1 \leq j \leq m$. It makes sense, since $[u_m(t), u_m(t)] \in H^{-2}(\bar{\Omega})$ by Lemma 5.3 and $G_2 : H^{-2}(\bar{\Omega}) \to H_0^2(\overline{\Omega})$ by Theorem 4.1.

Let the following initial values be given:

$$g_{jm}(0) = (u_{0m}|w_j) \tag{8.4}$$

$$g_{jm}'(0) = (u_{1m}|w_j) \tag{8.5}$$

Then for each $m \in \mathbb{N}$ and $1 \leq j \leq m$, a $t_m > 0$ and a solution $g_{im} \in C^1([0, t_m])$ to (8.2) and (8.3) is sought.

To show existence of such a solution $(g_{jm}(t))$, equation (8.2) is used in (8.3) to rewrite it as a system of m second order ODEs:

$$\begin{bmatrix} g_{1m}''(t) \\ \vdots \\ g_{mm}''(t) \end{bmatrix} = \begin{bmatrix} F_1(t, g_{1m}(t), \dots, g_{mm}(t)) \\ \vdots \\ F_m(t, g_{1m}(t), \dots, g_{mm}(t)) \end{bmatrix} \tag{8.6}$$

where $F_i(t, \cdot)$ is continuous and $F_i(\cdot, x) \in L^1([0, T])$.

In order to apply the theorems on ordinary differential equations, each F_j must be shown to be Lipschitz continuous. Let $x, y \in \mathbb{R}^m$. Then by the mean value theorem there exists a $0 \leq \theta \leq 1$ such that

$$|F_j(t, x) - F_j(t, y)| = |\nabla F_j(t, x + \theta(y - x)) \cdot (x - y)| \leq C\|x - y\|$$

where

$$C = \sup_{\|z\| \leq R} \|\nabla F_j(t, z)\|_{0,1}$$

Now the system (8.6) of m second order equations with initial conditions (8.4) and (8.5) can be rewritten as a system of $2m$ first order ODEs, so by Theorem A.41 and A.42 there exists for each m a $t_m > 0$ and solutions

$$g_{jm} \in C^1([0, t_m])$$

for $j = 1, \dots, m$.

Then

$$u_m(t) \in C([0,t_m], \mathscr{D}(\Omega))$$
$$u_m'(t) \in C([0,t_m], \mathscr{D}(\Omega))$$

are solutions to (8.3) so $u_m''(t) \in L^2([0,t_m], \mathscr{D}(\Omega))$.

Define for $t \in [0,t_m]$ a sequence

$$v_m(t) = -\frac{1}{a_2} G_2\big([u_m(t), u_m(t)]\big) \tag{8.7}$$

or equivalently

$$a_2 \Delta^2 v_m(t) + [u_m(t), u_m(t)] = 0, \quad v_m(t) \in H_0^2(\overline{\Omega}) \tag{8.8}$$

Since $u_m(t) \in \mathscr{D}(\Omega)$, $[u_m(t), u_m(t)] \in \mathscr{D}(\Omega)$ so $v_m(t) \in C^\infty(\overline{\Omega})$ by Theorem 4.3. Inserting (8.7) in (8.3) gives for $1 \leq j \leq m$

$$(u_m''(t)|w_j) + a_1(\Delta u_m(t)|\Delta w_j) - \big(\big[u_m(t), v_m(t)\big]\big|w_j\big) = (f(t)|w_j) \tag{8.9}$$

Now multiplying (8.9) with $g_{jm}'(t)$ and summing over j results in

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\|u_m'(t)\|^2 + a_1\|\Delta u_m(t)\|^2 + \frac{a_2}{2}\|\Delta v_m(t)\|^2\Big) = \big(f(t)|u_m'(t)\big) \tag{8.10}$$

because the first term in (8.10) is obtained from

$$\begin{aligned}
\sum_{j=1}^m g_{jm}'(t)(u_m''(t)|w_j) &= \sum_{j=1}^m g_{jm}'(t)\langle u_m''(t), w_j\rangle \\
&= \langle u_m''(t)|\sum_{j=1}^m g_{jm}'(t)w_j\rangle \\
&= \langle u_m''(t)|u_m'(t)\rangle \\
&= \frac{1}{2}\frac{\partial}{\partial t}\|u_m'(t)\|^2
\end{aligned}$$

where Theorem 7.3 has been used.

The second term in (8.10) and the right hand side is obtained similarly.

The third term in (8.10) is obtained using Lemma 5.5, (8.8) and Theorem 7.3

$$
\begin{aligned}
-\sum_{j=1}^{m} g'_{jm}(t)\big([u_m(t), v_m(t)]\big|w_j\big) &= -\sum_{j=1}^{m} g'_{jm}(t)\big\langle [u_m(t), v_m(t)], w_j\big\rangle \\
&= -\big\langle [u_m(t), v_m(t)], u'_m(t)\big\rangle \\
&= -\big\langle [u_m(t), u'_m(t)], v_m(t)\big\rangle \\
&= -\frac{1}{2}\Big\langle \frac{\partial}{\partial t}[u_m(t), u_m(t)], v_m(t)\Big\rangle \\
&= \frac{a_2}{2}\big\langle \frac{\partial}{\partial t}\Delta^2 v_m(t), v_m(t)\big\rangle \\
&= \frac{a_2}{2}\big\langle \Delta v'_m(t), \Delta v_m(t)\big\rangle \\
&= \frac{a_2}{2}\big\langle \frac{\partial}{\partial t}\Delta v_m(t), \Delta v_m(t)\big\rangle \\
&= \frac{a_2}{4}\frac{\partial}{\partial t}\|\Delta v_m(t)\|^2
\end{aligned}
$$

Integrating (8.10) with respect to t gives

$$
\begin{aligned}
&\Big(\|u'_m(t)\|^2 + a_1\|\Delta u_m(t)\|^2 + \frac{a_2}{2}\|\Delta v_m(t)\|^2\Big) \\
&= \Big(\|u_{1m}\|^2 + a_1\|\Delta u_{0m}\|^2 + \frac{a_2}{2}\|\Delta v_m(0)\|^2\Big) + 2\int_0^t \big(f(\tau)|u'_m(\tau)\big)\,d\tau
\end{aligned}
$$

According to (8.1) and Lemma A.29 there exists a constant c_1 such that

$$\|u_{1m}\|^2 + a_1\|\Delta u_{0m}\|^2 \le c_1$$

From (8.7) and the continuity of G_2:

$$v_m(0) = -\frac{1}{a_2}G_2\big([u_{0m}, u_{0m}]\big)$$

where the right hand side is a composition of continuous functions, by Lemma 5.3 and Theorem 4.1, and since u_{0m} is bounded, $v_m(0)$ is bounded in $H_0^2(\overline{\Omega})$.

Using that $2ab \leq a^2 + b^2$ it is seen that

$$2\left|\int_0^t \big(f(\tau)|u'_m(\tau)\big)\,d\tau\right| \leq 2\int_0^t \|f(\tau)\|\|u'_m(\tau)\|\,d\tau$$
$$\leq \int_0^T \|f(\tau)\|^2\,d\tau + \int_0^t \|u'_m(\tau)\|^2\,d\tau$$

Thus there exists a constant C such that

$$\|u'_m(t)\|^2 \leq C + \int_0^t \|u'_m(\tau)\|^2\,d\tau$$

so by Lemma 9.1

$$\|u'_m(t)\|^2 \leq Ce^t \leq C_1(I)$$

for any bounded interval $I \subset \overline{\mathbb{R}}_+$. Since (w_j) is an orthonormal system in $L^2(\Omega)$ it follows that

$$\|u'_m(t)\|^2 = (u'_m(t)|u'_m(t)) = \sum_{j=1}^m |g'_{jm}(t)|^2$$

so $g'_{jm}(t)$ and consequently g_{jm} are bounded on any bounded interval $I \subset \overline{\mathbb{R}}_+$, so by Theorem A.45 there exists extensions of g_{jm} to $[0,T]$.

Now there exists a constant c_3 such that

$$\|u'_m(t)\|^2 + a_1\|\Delta u_m(t)\|^2 + \frac{a_2}{2}\|\Delta v_m(t)\|^2 \leq c_3$$

so since by Lemma A.29 there exists $c_4, c_5 > 0$ such that

$$\|u'_m(t)\|^2 + a_1 c_4\|u_m(t)\|_2^2 + \frac{a_2 c_5}{2}\|v_m(t)\|_2^2 \leq c_3$$

it follows that

$$\begin{aligned} (u_m), (v_m) \quad &\text{is bounded in } L^\infty(0,T;H_0^2(\overline{\Omega})) \\ (u'_m) \quad &\text{is bounded in } L^\infty(0,T;L^2(\Omega)) \end{aligned}$$

Since $L^\infty(0,T;X)$ is dual space to the Banach space $L^1(0,T;X^*)$ when X is reflexive, there exists by Banach-Alaoglu's theorem [Reed and Simon, 1980, p. 115] a subsequence (u_ζ) of (u_m) such that

$$u_\zeta \xrightarrow{w^*} u \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega}))$$

Extract subsequences (v_ς) of (v_m) and (u'_ς) of (u'_m) with the same index set as (u_ς). Then by another application of Banach-Alauglo's theorem there is a subsequence (v_ν) of (v_ς) such that

$$v_\nu \xrightarrow{w^*} v \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega}))$$

Again extracting subsequences (u_ν) of (u_ς) and (u'_ν) of (u'_ς) with the same index set as (v_ν), it follows by yet another application of Banach-Alauglo's theorem that there exists a subsequence (u'_μ) of (u'_ν) such that

$$u'_\mu \xrightarrow{w^*} u' \text{ in } L^\infty(0,T;L^2(\Omega))$$

Once more extracting subsequences (u_μ) of (u_ν) and (v_μ) of (v_ν) with the same index set as (u'_μ), it follows that

$$u_\mu \xrightarrow{w^*} u \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega}))$$
$$v_\mu \xrightarrow{w^*} v \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega}))$$
$$u'_\mu \xrightarrow{w^*} u' \text{ in } L^\infty(0,T;L^2(\Omega))$$

Moreover, (u_μ) is a bounded sequence in $W(0,T;1;H_0^2(\overline{\Omega}),L^2(\Omega))$, by the continuous injections $L^\infty(0,T;H_0^2(\overline{\Omega})) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H_0^2(\overline{\Omega}))$ and $L^\infty(0,T;L^2(\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$, so by Corollary A.26, there exists a subsequence (u_k) of (u_μ), such that

$$u_k \to u \text{ in } L^2(0,T;L^2(\Omega)) = L^2([0,T]\times\Omega) \tag{8.11}$$

and for subsequences (v_k) of (u_μ) and (u'_k) of (u'_μ) with the same index set as (u_k), then

$$u_k \xrightarrow{w^*} u \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega})) \tag{8.12}$$
$$v_k \xrightarrow{w^*} v \text{ in } L^\infty(0,T;H_0^2(\overline{\Omega})) \tag{8.13}$$
$$u'_k \xrightarrow{w^*} u' \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{8.14}$$

still holds. Furthermore for every $\phi \in \mathscr{D}(I)$

$$\langle \partial_t u, \phi\rangle = -\langle u, \partial_t\phi\rangle = -\lim_{k\to\infty}\langle u_k, \partial_t\phi\rangle = \lim_{k\to\infty}\langle u'_k, \phi\rangle = \langle u', \phi\rangle$$

so $\partial_t u = u'$.

What now remains, is to show that (u, v) so defined is a weak solution to (1.1), (1.3) and (1.4).

Since $u, v \in L^\infty(0,T;H_0^2(\overline{\Omega}))$, the boundary conditions

$$\gamma_0 u = \gamma_1 u = 0$$
$$\gamma_0 v = \gamma_1 v = 0$$

are automatically satisfied.

Moreover, it follows from the definition of (u_m) that

$$\begin{aligned} u_k(t) &\to u_k(0) && \text{for } t \to 0 \\ u_k(0) &\to u_0 && \text{for } k \to \infty \\ u_k'(t) &\to u_k'(0) && \text{for } t \to 0 \\ u_k'(0) &\to u_1 && \text{for } k \to \infty \end{aligned}$$

so by an $\epsilon/3$-argument, it follows that

$$(u(0)|w_j)_2 = \lim_{t\to 0}(u(t)|w_j)_2 = (u_0|w_j)_2$$
$$(u'(0)|w_j) = \lim_{t\to 0}(u'(t)|w_j) = (u_1|w_j)$$

for all $j \in \mathbb{N}$, and thus also for each finite linear combination of the $w_j's$, so since (w_j) is a basis for $H_0^2(\overline{\Omega})$, it follows that $u(0) = u_0$ in the weak sense. Also, (w_j) is dense in $L^2(\Omega)$, it follows that $u'(0) = u_1$.

The remaining part of the proof consists of showing that (u, v) satisfies (1.1) in $\mathscr{D}'(0,T;H^{-2}(\bar{\Omega}))$.

So let $\phi \in C^1([0,T])$ with $\phi_j(T) = 0$ be given, and define

$$\psi_j(t) = \phi(t) w_j \tag{8.15}$$

Then it follows from (8.9) by integrating from 0 to T, that in terms of duality

$$\int_0^T \langle u_k''(t), \psi_j(t)\rangle\, dt + a_1 \int_0^T \langle \Delta u_k(t), \Delta\psi_j(t)\rangle\, dt - \int_0^T \langle [u_k(t), v_k(t)], \psi_j(t)\rangle\, dt$$
$$= \int_0^T \langle f(t), \psi_j(t)\rangle\, dt \tag{8.16}$$

for all $j \in \mathbb{N}$.

Each of the terms will now for an arbitrary $j \in \mathbb{N}$ be evaluated separately as $k \to \infty$:

First it follows, by an application of Bochner's identity (A.11), that

$$\begin{aligned}\int_0^T \langle u_k''(t), \psi_j(t)\rangle\, dt &= \int_0^T \langle u_k''(t)\phi(t), w_j\rangle\, dt \\ &= \langle \int_0^T u_k''(t)\phi(t)\, dt, w_j\rangle \\ &= -\langle u_k'(0)\phi(0), w_j\rangle - \langle \int_0^T u_k'(t)\phi'(t)\, dt, w_j\rangle \\ &= -\langle u_k'(0)\psi_j(0)\rangle - \int_0^T \langle u_k'(t)\phi'(t), w_j\rangle\, dt \\ &= -\langle u_k'(0), \psi_j(0)\rangle - \int_0^T \langle u_k'(t), \psi_j'(t)\rangle\, dt\end{aligned}$$

since $\phi(T) = 0$. Thus as $k \to \infty$,

$$\int_0^T \langle u_k''(t), \psi_j(t)\rangle\, dt \to -\langle u_1, \psi_j(0)\rangle - \int_0^T \langle u'(t), \psi_j'(t)\rangle\, dt$$

by the theorem of dominated convergence, and by (8.5) and (8.14).

Next, it follows from the theorem of dominated convergence and from (8.12), that

$$\int_0^T \langle \Delta u_k(t), \Delta\psi_j(t)\rangle\, dt \to \int_0^T \langle \Delta u(t), \Delta\psi_j(t)\rangle\, dt$$

Moreover, by Lemma 5.5, it follows by another application of the theorem of dominated convergence and by (8.13) and (8.11), that

$$\begin{aligned}&\int_0^T \langle [u_k(t), v_k(t)], \psi_j(t)\rangle\, dt = \int_0^T \langle [\psi_j(t), v_k(t)], u_k(t)\rangle\, dt \\ &\to \int_0^T \langle [\psi_j(t), v(t)], u(t)\rangle\, dt = \int_0^T \langle [u(t), v(t)], \psi_j(t)\rangle\, dt\end{aligned}$$

Altogether it follows, by letting $k \to \infty$, that

$$-\int_0^T \langle u'(t), \psi_j'(t)\rangle\, dt + a_1 \int_0^T \langle \Delta u(t), \Delta\psi_j(t)\rangle\, dt - \int_0^T \langle [u(t), v(t)], \psi_j(t)\rangle\, dt$$
$$= \int_0^T \langle f(t), \psi_j(t)\rangle\, dt + \langle u_1, \psi_j(0)\rangle \tag{8.17}$$

for all ψ as in (8.15).

What is left to prove is, that for each $\phi \in \mathscr{D}(]\,0, T\,[)$, then as elements of $H^{-2}(\bar{\Omega})$

$$\langle u''(t) + a_1\Delta^2 u(t) - [u(t), v(t)], \phi(t)\rangle = \langle f(t), \phi(t)\rangle$$

but since the w_j's form a basis for $H_0^2(\overline{\Omega})$ it is enough to show, that

$$\langle\langle u''(t) + a_1\Delta^2 u(t) - [u(t), v(t)], \phi(t)\rangle, w_j\rangle = \langle\langle f(t), \phi(t)\rangle, w_j\rangle \tag{8.18}$$

for all $j \in \mathbb{N}$ and all $\phi \in \mathscr{D}(]\,0, T\,[)$.

Now let $\phi \in \mathscr{D}(]\,0, T\,[)$. Then

$$\langle u''(t), \phi(t)\rangle = -\langle u'(t), \phi'(t)\rangle$$

as elements of $H^{-2}(\bar{\Omega})$, and thus

$$\langle\langle u''(t), \phi(t)\rangle, w_j\rangle = -\langle\langle u'(t), \phi'(t)\rangle, w_j\rangle$$

for each j. Then by Bochner's identity, (A.11)

$$\begin{aligned}\langle\langle u''(t), \phi(t)\rangle, w_j\rangle &= -\Big\langle \int_0^T u'(t)\phi'(t)\, dt, w_j\Big\rangle \\ &= -\int_0^T \langle u'(t)\phi'(t), w_j\rangle\, dt \\ &= -\int_0^T \langle u'(t), \phi'(t)w_j\rangle\, dt\end{aligned}$$

It is noted that $\phi(0) = 0$, so by another application of Bochner's identity, (8.17) gives that

$$\begin{aligned}\langle\langle u''(t), \phi(t)\rangle, w_j\rangle &= \int_0^T \langle -a_1\Delta^2 u(t) + [u(t), v(t)] + f, \phi(t) w_j\rangle\, dt \\ &= \int_0^T \langle(-a_1\Delta^2 u(t) + [u(t), v(t)] + f)\phi(t), w_j\rangle\, dt \\ &= \Big\langle \int_0^T (-a_1\Delta^2 u(t) + [u(t), v(t)] + f)\phi(t)\, dt, w_j\Big\rangle \\ &= \langle\langle -a_1\Delta^2 u(t) + [u(t), v(t)] + f, \phi(t)\rangle, w_j\rangle\end{aligned}$$

but then by linearity

$$\langle\langle u''(t) + a_1\Delta^2 u(t) - [u(t), v(t)], \phi(t)\rangle, w_j\rangle = \langle\langle f(t), \phi(t)\rangle, w_j\rangle$$

which shows (8.18). Thus

$$u''(t) + a_1\Delta^2 u(t) - [u(t), v(t)] = f(t)$$

as elements of $\mathscr{D}'(0, T; H^{-2}(\bar{\Omega}))$.

Moreover, by Lemma 5.5,

$$\begin{aligned}\langle a_2\Delta^2 v, w_j\rangle &= \lim_{k\to\infty} \langle a_2\Delta^2 v_k, w_j\rangle \\ &= -\lim_{k\to\infty} \langle [u_k, u_k], w_j\rangle \\ &= -\lim_{k\to\infty} \langle [u_k, w_j], u_k\rangle \\ &= -\langle [u, w_j], u\rangle \\ &= -\langle [u, u], w_j\rangle\end{aligned}$$

for almost all $t \in [0, T]$, and thus, since the w_j's are a basis for $H_0^2(\overline{\Omega})$

$$a_2\Delta^2 v + [u, u] = 0$$

as elements of $\mathscr{D}'(0, T; H^{-2}(\bar{\Omega}))$. □

Chapter 9

Uniqueness of Weak Solutions

In this chapter a uniqueness theorem for weak solutions to the von Karman equations with homogeneous boundary values will be proven. The chapter is an investigation of the uniqueness theorem given by A. Boutet de Monvel and I. Chueshov in 1998, [Boutet de Monvel and Chueshov, 1998].

It will be assumed throughout the chapter that $\Omega \subset \mathbb{R}^2$ is open, bounded, smooth and connected.

Throughout the chapter, let $\{\lambda_i\}$, $i \in \mathbb{N}$ denote the set of eigenvalues of Δ_D^2. Then the corresponding set of eigenvectors $\{e_i\}$ is a basis for $L^2(\Omega)$ by Lemma 4.6. Furthermore let P_N denote the projector in $L^2(\Omega)$ on the subspace spanned by $\{e_1, \dots, e_N\}$.

Repeatedly through the chapter the following facts will be used:

There exists an $N_0 \in \mathbb{N}$ such that $\ln(1+\lambda_N)^{-1} < \frac{1}{2}$ for all $N \geq N_0$. This follows from the fact that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and λ_N is increasing to ∞ for $N \to \infty$, since Δ_D^2 is unbounded in $L^2(\Omega)$.

There exists a $C > 0$ and a $N_0 \in \mathbb{N}$ such that $\lambda_N^{\frac{1}{4\ln(1+\lambda_N)}} \leq C$ for $N \geq N_0$, which is seen by

$$\lim_{N\to\infty} \lambda_N^{\frac{1}{4\ln(1+\lambda_N)}} = e^{\frac{1}{4}}$$

$$\lim_{N\to 0} \lambda_N^{\frac{1}{4\ln(1+\lambda_N)}} = 0$$

Before the main theorem is stated, some lemmas, which are needed in the proof of the main theorem, will be proved.

Lemma 9.1 (Gronwall)
Let $E : \overline{\mathbb{R}}_+ \to \mathbb{R}$ *be non-negative, increasing, and continuous, and let* $\phi \in L^p(\overline{\mathbb{R}}_+)$ *and* $k \in L^q(\overline{\mathbb{R}}_+)$*, with* $p \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} \leq 1$*, be non-negative. If*

$$\phi(t) \leq E(t) + \int_0^t \phi(\tau)k(\tau)\,d\tau$$

then

$$\phi(t) \leq E(t)e^{\int_0^t k(\tau)\,d\tau} \tag{9.1}$$

Proof:
First it is noted, that it is enough to show (9.1) when $t = T$, and then it can be assumed, that $E(t)$ is a constant.

Now define

$$F(t) = E + \int_0^t \phi(\tau)k(\tau)\,d\tau$$

and note that since $\phi \in L^p(\overline{\mathbb{R}}_+)$ and $k \in L^q(\overline{\mathbb{R}}_+)$ with $p \geq 1$ and $\frac{1}{p}+\frac{1}{q} \leq 1$, $\int_0^t \phi(s)k(s)\,ds$ is differentiable a.e., and

$$F'(t) = \phi(t)k(t) \leq F(t)k(t)$$

since $\phi \leq F$.

By multiplying by an appropriate integration factor, it follows that

$$e^{-\int_0^t k(\tau)\,d\tau}F'(t) - F(t)k(t)e^{-\int_0^t k(\tau)\,d\tau} \leq 0 \tag{9.2}$$

Then there exists a sequence (k_j) such that $k_j \in C_0^\infty(\overline{\mathbb{R}}_+)$ and $k_j \to k$ in $L^q(\overline{\mathbb{R}}_+)$. Thus since $\int_0^T$ is continuous from $L^1([0,T])$ to $AC([0,T])$, the absolutely continuous functions on $[0,T]$, and since $\frac{\partial}{\partial t}$ is continuous from $AC([0,T])$ to $L^1([0,T])$ it follows that

$$\begin{aligned}
\frac{\partial}{\partial t}e^{-\int_0^t k(\tau)\,d\tau} &= \lim_{k\to\infty}\frac{\partial}{\partial t}e^{-\int_0^t k_j(\tau)\,d\tau} \\
&= -\lim_{k\to\infty} k_j(t)e^{-\int_0^t k_j(\tau)\,d\tau} \\
&= k(t)e^{-\int_0^t k(\tau)\,d\tau}
\end{aligned}$$

It then follows by (9.2) that

$$\frac{\partial}{\partial t}(e^{-\int_0^t k(\tau)\,d\tau}F(t)) \leq 0$$

and thus

$$e^{-\int_0^T k(\tau)\,d\tau}F(T) \leq F(0)$$

so

$$\phi(T) \leq F(T) \leq F(0)e^{\int_0^T k(\tau)\,d\tau} = Ee^{\int_0^T k(\tau)\,d\tau}$$

whereby the lemma follows. □

Lemma 9.2
Let $f \in H_0^1(\overline{\Omega})$. Then there exists $N_0 \in \mathbb{N}$, and $C > 0$, such that for all $N \geq N_0$

$$\max_{x\in\Omega}|(P_N f)(x)| \leq C\sqrt{\ln(1+\lambda_N)}\|f\|_1$$

and C is independent of N.

Proof:
First let $\phi \in \mathscr{D}(\mathbb{R}^2)$, with $\operatorname{supp}\phi \subset \overline{\Omega}$. Then

$$\max_{x\in\Omega}|\phi(x)| \leq \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}|\hat{\phi}(k)|\,dk$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ. Then by the Cauchy-Schwarz inequality it follows for $s > 1$ that

$$\begin{aligned}\max_{x\in\Omega}|\phi(x)| &\leq \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\langle k\rangle^{s-s}|\hat{\phi}(k)|\,dk \\ &\leq \frac{1}{2\pi}\Big(\int\limits_{\mathbb{R}^2}\langle k\rangle^{2s}|\hat{\phi}(k)|^2\,dk\Big)^{\frac{1}{2}}\Big(\int\limits_{\mathbb{R}^2}\langle k\rangle^{-2s}\,dk\Big)^{\frac{1}{2}}\end{aligned}$$

Now, since the subset of $\mathscr{D}(\mathbb{R}^2)$ consisting of functions with support in $\overline{\Omega}$ is dense in $H^{1+\sigma}(\bar{\Omega}) \cap H_0^1(\overline{\Omega})$ for $0 < \sigma < \frac{1}{2}$, it follows that for all $g \in H^{1+\sigma}(\bar{\Omega}) \cap H_0^1(\overline{\Omega})$ with $0 < \sigma < \frac{1}{2}$

$$\max_{x\in\Omega}|g(x)| \leq C\sigma^{-\frac{1}{2}}\|g\|_{1+\sigma}$$

Then, since $P_N f \in C^\infty(\overline{\Omega}) \cap H_0^1(\overline{\Omega})$ it follows by Lemma 4.8, that

$$\begin{aligned}\max_{x\in\Omega}|(P_N f)(x)| &\le C\sigma^{-\frac{1}{2}}\|P_N f\|_{1+\sigma}\\ &\le C\sigma^{-\frac{1}{2}}\|(\Delta_D^2)^{\frac{1+\sigma}{4}}(P_N f)\|\\ &\le C\sigma^{-\frac{1}{2}}(\lambda_N)^{\frac{\sigma}{4}}\|f\|_1\\ &\le C_1\sigma^{-\frac{1}{2}}\|f\|_1\end{aligned}$$

since there exists a $C > 0$ such that $(\lambda_N)^{\frac{\sigma}{4}} < C$ for all $N \in \mathbb{N}$.

Also, there exists a $N \in \mathbb{N}$ such that $\ln(1+\lambda_N)^{-1} < \frac{1}{2}$ for all $N \ge N_0$, and consequently for $N \ge N_0$

$$\max_{x\in\Omega}|(P_N f)(x)| \le C\sqrt{\ln(1+\lambda_N)}\|f\|_1$$

where C is independent of N, as claimed. □

Lemma 9.3
Let $f \in H^\sigma(\bar{\Omega})$ for $0 < \sigma \le 1$. Then for all $1 < p < (1-\sigma)^{-1}$

$$\|f\|_{0,2p} \le C\left[\pi\frac{p-1}{\sigma p - p + 1}\right]^{\frac{p-1}{2p}}\|f\|_\sigma$$

Proof:
Since

$$\|f\|_\sigma = \inf\{\|g\|_\sigma | g \in H^\sigma(\mathbb{R}^2),\ f = r_\Omega g\}$$

there exists a $g \in H^\sigma(\mathbb{R}^2)$ and $c_1, c_2 > 0$ such that

$$c_1\|f\|_\sigma \le \|g\|_\sigma \le c_2\|f\|_\sigma$$

Next it is noted, that for $1 < p < (1-\sigma)^{-1}$, $H^\sigma(\mathbb{R}^2) \overset{\mathrm{d}}{\hookrightarrow} L^{2p}(\mathbb{R}^2)$ by Lemma A.23, so

$$\|f\|_{0,2p} \le \|g\|_{0,2p} \tag{9.3}$$

and by the Hausdorff-Young inequality [Bergh and Löfström, 1976, p. 6]

$$\|g\|_{0,2p} = c_3\|\hat{g}\|_{0,2p} \le c_4\|\hat{g}\|_{0,\tilde{p}} \tag{9.4}$$

where

$$1 < \tilde{p} = \frac{2p}{2p-1} < 2$$

since $1 < p < (1-\sigma)^{-1}$.

Finally Hölder's inequality gives that

$$\|\hat{g}\|_{0,\tilde{p}} \leq \Big(\int_{\mathbb{R}^2} \langle k \rangle^{2\sigma} |\hat{g}(k)|^2 \, dk \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} \langle k \rangle^{-2\tilde{\sigma}} \, dk \Big)^{\frac{2-\tilde{p}}{2\tilde{p}}}$$

$$\leq \left(\pi \frac{p-1}{\sigma p - p + 1} \right)^{\frac{p-1}{2p}} \|g\|_{\sigma} \tag{9.5}$$

where $\tilde{\sigma} = \tilde{p}\sigma(2-\tilde{p})$.

The lemma now follows by combining (9.3), (9.4), and (9.5). □

Lemma 9.4
Let $f \in L^2(\Omega)$ and $g \in H^1(\bar{\Omega})$. Then there exists $N_0 \in \mathbb{N}$ and $C > 0$, such that for all $N \geq N_0$

$$\|(P_N f) g\|_0 \leq C \sqrt{\ln(1+\lambda_N)} \|f\|_0 \|g\|_1$$

and C is independent of N.

Proof:
First it is noted, that by applying Hölder's inequality with $p = \frac{1}{1-\theta}$ and $q = \frac{1}{\theta}$, it follows for $0 < \theta < 1$ that

$$\|(P_N f) g\|_0 \leq \|P_N f\|_{0,\frac{2}{1-\theta}} \|g\|_{0,\frac{2}{\theta}} \tag{9.6}$$

Then by applying Lemma 9.3 with $p = (1-\theta)^{-1}$ and $\sigma = 2\theta$, it follows for $0 < \theta < \frac{1}{2}$ that

$$\|P_N f\|_{0,\frac{2}{1-\theta}} \leq c_1 \|P_N f\|_{2\theta} \leq c_2 (\lambda_N)^{2\theta} \|f\|_0 \tag{9.7}$$

and with $p = \theta^{-1}$ and $\sigma = 1$ it follows for $0 < \theta < 1$ that

$$\|g\|_{0,\frac{2}{\theta}} \leq c_3 (\pi \frac{1-\theta}{\theta})^{\frac{1-\theta}{2}} \|g\|_1 \leq c_4 \theta^{-\frac{1}{2}} \|g\|_1 \tag{9.8}$$

and thus by combining (9.6), (9.7), and (9.8), it follows that

$$\|(P_N f) g\|_0 \leq c_5 (\lambda_N)^{2\theta} \theta^{-\frac{1}{2}} \|g\|_1 \|f\|_0$$

and thus the lemma follows by choosing $N_0 \in \mathbb{N}$ such that $\theta = \ln(1+\lambda_{N_0})^{-1} < \frac{1}{2}$. □

Lemma 9.5
Let $u \in H^{\beta}(\bar{\Omega})$ *and* $v \in H^{1-\beta}(\bar{\Omega})$*, where* $0 < \beta < 1$*. Then*

$$\begin{aligned} \|uv\|_0 &\leq C\|u\|_\beta \|v\|_{1-\beta} \\ \|uv\|_{-1+\beta} &\leq C\|u\|_\beta \|v\|_0 \end{aligned}$$

Proof:
The lemma follows by applying Corollary 2.6. □

The next two lemmas are the main tools in the proof of the main theorem. They serve to give a valuation of the difference between two solutions to the von Karman equations.

Lemma 9.6
Let $u_1, u_2 \in H_0^2(\overline{\Omega})$ *and assume that there exists an* $R > 0$ *such that* $\|u_j\|_2 \leq R$*. Then for all* $0 < \beta_2 < \frac{1}{16}$ *it follows that there exists a* $N_0 \in \mathbb{N}$*, such that for all* $N \geq N_0$

$$\|[u_1, v(u_1) - v(u_2)]\|_{-1} \leq C_1[\ln(1 + \lambda_N)]\|u_1 - u_2\|_1 + C_2(\lambda_{N+1})^{-\beta_2}$$

where $C_1, C_2 > 0$ *and only depends on* R, β_2*, and where*

$$v(u_i) = -G_2([u_i, u_i])$$

Proof:
Let $v_i := v(u_i) = -G_2([u_i, u_i])$ and $\tilde{v} = v_1 - v_2$. Then by Corollary 2.7

$$\begin{aligned} \|v_i\|_2 + \|v_i\|_{2+\delta} &\leq c_1(\|\Delta_D^2 v_i\|_{-2} + \|\Delta_D^2 v_i\|_{\delta-2}) \\ &\leq c_1(\|[u_i, u_i]\|_{-2} + \|[u_i, u_i]\|_{\delta-2}) \\ &\leq c_2(\|u_i\|_2\|u_i\|_2 + \|u_i\|_{1+\delta+\beta_1}\|u_i\|_{2-\beta_1}) \\ &\leq c_3\|u_i\|_2^2 \\ &\leq c_4 \end{aligned}$$

for all $0 < \beta_1 \leq \delta < 1$, so $v_i \in H^{2+\delta}(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$.

Now by Lemma 5.4

$$[u, \tilde{v}] = \partial_1((\partial_1\tilde{v})(\partial_{22}^2 u - (\partial_2 u)(\partial_{12}^2\tilde{v})) + \partial_2((\partial_2\tilde{v})(\partial_{11}^2 u) - (\partial_1 u)(\partial_{12}^2\tilde{v}))$$

since $u, \tilde{v} \in H_0^2(\overline{\Omega})$, so $[u, \tilde{v}]$ can be written as a sum of terms of the form

$$w = \partial_i((\partial_j\tilde{v})(\partial_{kl}^2 u))$$

Then, since $\tilde{v} \in H^{2+\delta}(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$, $\partial_j \tilde{v} \in H_0^{1+\delta}(\overline{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} C(\overline{\Omega})$ for all $0 < \delta < 1$, so by Hölder's inequality it follows that

$$\begin{aligned} \|w\|_{-1} &\leq \|(\partial_j \tilde{v})(\partial_{kl}^2 u)\|_0 \\ &\leq c_1 \|\partial_{kl}^2 u\|_0 \max_{x\in\Omega} |(\partial_j \tilde{v})(x)| \\ &\leq c_2 (\max_{x\in\Omega} |(P_N \partial_j \tilde{v})(x)| + \max_{x\in\Omega} |(Q_N \partial_j \tilde{v})(x)|) \end{aligned} \tag{9.9}$$

where $Q_N = 1 - P_N$.

Also, since $\partial_j \tilde{v} \in H_0^{1+\delta}(\overline{\Omega})$, it follows by Lemma 9.2 that there exists a $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\begin{aligned} \max_{x\in\Omega} |(P_N \partial_j \tilde{v})(x)| &\leq c_1 \sqrt{\ln(1+\lambda_N)} \|\partial_j \tilde{v}\|_1 \\ &\leq c_2 \sqrt{\ln(1+\lambda_N)} \|\tilde{v}\|_2 \end{aligned} \tag{9.10}$$

Moreover, since $H_0^{1+4\beta_2}(\overline{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} C^0(\Omega)$ for all $0 < \beta_2 < \frac{1}{4}$ it follows for all $0 < \beta_2 < \frac{1}{16}$ using Lemma 4.8 that

$$\begin{aligned} \max_{x\in\Omega} |(Q_N \partial_j \tilde{v})(x)| &\leq c_1 \|Q_N \partial_j \tilde{v}\|_{1+4\beta_2} \\ &\leq c_1 \|(\Delta_D^2)^{\frac{1}{4}+\beta_2} Q_N \partial_j \tilde{v}\|_0 \\ &\leq c_1 \|(\Delta_D^2)^{\frac{1}{4}+2\beta_2} (\Delta_D^2)^{-\beta_2} (1-P_N) \partial_j \tilde{v}\|_0 \\ &\leq c_2 (\lambda_{N+1})^{-\beta_2} \|(\Delta_D^2)^{\frac{1}{4}+2\beta_2} \partial_j \tilde{v}\|_0 \\ &\leq c_2 (\lambda_{N+1})^{-\beta_2} \|\partial_j \tilde{v}\|_{1+8\beta_2} \\ &\leq c_3 (\lambda_{N+1})^{-\beta_2} \|\tilde{v}\|_{2+8\beta_2} \end{aligned}$$

Then, since

$$\tilde{v} = v_1 - v_2 = -G_2([u_1, u_1] - [u_2, u_2]) = -G_2([u_1 - u_2, u_1 + u_2])$$

it follows that

$$\begin{aligned} \|\tilde{v}\|_{2+8\beta_2} &\leq c_1 \|\Delta_D^2 \tilde{v}\|_{8\beta_2 - 2} \\ &\leq c_1 \|[u_1 - u_2, u_1 + u_2]\|_{8\beta_2 - 2} \\ &\leq c_2 \|u_1 - u_2\|_2 \|u_1 + u_2\|_2 \\ &\leq c_3 \end{aligned}$$

so consequently

$$\max_{x\in\Omega} |(Q_N \partial_j \tilde{v})(x)| \leq c_4 (\lambda_{N+1})^{-\beta_2} \tag{9.11}$$

so by (9.9), (9.10), and (9.11)

$$\|[u, v_1 - v_2]\|_{-1} \leq C(\sqrt{\ln(1+\lambda_N)}\|v_1 - v_2\|_2 + (\lambda_{N+1})^{-\beta_2}) \tag{9.12}$$

Next it is proved, that there exists a $N_0 \in \mathbb{N}$ such that for all $0 < \beta_3 < \frac{1}{4}$ and $N \geq N_0$

$$\|v_1 - v_2\|_2 \leq c_1\sqrt{\ln(1+\lambda_N)}\|u_1 - u_2\|_1 + c_2(\lambda_{N+1})^{-\beta_3} \tag{9.13}$$

To see this let $u = u_1 - u_2$. Then

$$\begin{aligned}\|v_1 - v_2\|_2 &\leq c_1\|\Delta_D^2 v_1 - \Delta_D^2 v_2\|_{-2} \\ &\leq c_1\|[P_N u, u_1 + u_2] + [Q_N u, u_1 + u_2]\|_{-2} \\ &\leq c_1(\|[P_N u, u_1 + u_2]\|_{-2} + \|[Q_N u, u_1 + u_2]\|_{-2})\end{aligned} \tag{9.14}$$

Now it should be noted, that by Lemma 5.4

$$\begin{aligned}[P_N u, u_1 + u_2] = {}& \partial_{11}^2(P_N u \partial_{22}^2(u_1 + u_2)) - 2\partial_{12}^2(P_N u \partial_{12}^2(u_1 + u_2)) \\ &+ \partial_{22}^2(P_N u \partial_{11}^2(u_1 + u_2))\end{aligned}$$

so since $P_N u \in C^\infty(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$, it follows by Lemma 9.2 that there exists a $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\begin{aligned}\|[P_N u, u_1 + u_2]\|_{-2} &\leq c_1 \sum_{i,j} \|P_N u \partial_{ij}^2(u_1 + u_2)\|_0 \\ &\leq c_1\|P_N u\|_{0,\infty} \sum_{i,j} \|\partial_{ij}^2(u_1 + u_2)\|_0 \\ &\leq c_2\|u_1 + u_2\|_2\sqrt{\ln(1+\lambda_N)}\|u\|_1 \\ &\leq c_3\sqrt{\ln(1+\lambda_N)}\|u\|_1\end{aligned} \tag{9.15}$$

Moreover

$$\begin{aligned}[Q_N u, u_1 + u_2] = {}& \partial_{11}^2(Q_N u \partial_{22}^2(u_1 + u_2)) - 2\partial_{12}^2(Q_N u \partial_{12}^2(u_1 + u_2)) \\ &+ \partial_{22}^2(Q_N u \partial_{11}^2(u_1 + u_2))\end{aligned}$$

so

$$\begin{aligned}\|[Q_N u, u_1 + u_2]\|_{-2} &\leq c_1 \sum_{i,j} \|Q_N u \partial_{ij}^2(u_1 + u_2)\|_0 \\ &\leq c_1\|Q_N u\|_{0,\infty} \sum_{i,j} \|\partial_{ij}^2(u_1 + u_2)\|_0 \\ &\leq c_2\|Q_N u\|_{0,\infty}\end{aligned}$$

and since for all $0 < \beta_3 < \frac{1}{4}$, $H_0^2(\overline{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H_0^{2-4\beta_3}(\overline{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} L^\infty(\Omega)$

$$\begin{aligned}
\|Q_N u\|_{0,\infty} &\leq c_1 \|Q_N u\|_{2-4\beta_3} \\
&\leq c_2 \|(\Delta_D^2)^{\frac{1}{2}-\beta_3} Q_N u\|_0 \\
&\leq c_2 (\lambda_{N+1})^{-\beta_3} \|(\Delta_D^2)^{\frac{1}{2}} Q_N u\|_0 \\
&\leq c_3 (\lambda_{N+1})^{-\beta_3} \|u\|_2 \\
&\leq c_4 (\lambda_{N+1})^{-\beta_3}
\end{aligned}$$

so

$$\|[Q_N u, u_1 + u_2]\|_{-2} \leq c_2 (\lambda_{N+1})^{-\beta_3} \tag{9.16}$$

Thus, by (9.14), (9.15), and (9.16) it follows that there exists a $N_0 \in \mathbb{N}$ such that for all $0 < \beta_3 < \frac{1}{4}$ and all $N \geq N_0$,

$$\|v_1 - v_2\|_2 \leq c_1 \sqrt{\ln(1+\lambda_N)} \|u_1 - u_2\|_1 + c_2 (\lambda_{N+1})^{-\beta_3}$$

Inserting (9.13) in (9.12) gives

$$\begin{aligned}
\|[u, v_1 - v_2]\|_{-1} &\leq c_1 \ln(1+\lambda_N) \|u_1 - u_2\|_1 + c_2 (\lambda_{N+1})^{-\beta_2} \\
&\quad + c_3 \sqrt{\ln(1+\lambda_N)} (\lambda_{N+1})^{-\beta_3} \\
&\leq c_1 \ln(1+\lambda_N) \|u_1 - u_2\|_1 + c_2 (\lambda_{N+1})^{-\beta_2} \\
&\quad + c_3 \sqrt{\ln(1+\lambda_N)} (\lambda_{N+1})^{-\frac{\beta_3}{2}} (\lambda_{N+1})^{-\frac{\beta_3}{2}}
\end{aligned}$$

Now it is noted, that there exists a constant C such that

$$\sqrt{\ln(1+\lambda_N)} (\lambda_{N+1})^{-\frac{\beta_3}{2}} \leq C$$

for all $N \geq N_0$. Then, by letting $\beta_3 = 2\beta_2$, it follows that there exists a $N_0 \in \mathbb{N}$ such that

$$\|[u, v_1 - v_2]\|_{-1} \leq c_1 \ln(1+\lambda_N) \|u_1 - u_2\|_1 + c_4 (\lambda_{N+1})^{-\beta_2}$$

for all $0 < \beta < \frac{1}{16}$ and all $N \geq N_0$. □

Lemma 9.7
Let $u_1, u_2 \in H_0^2(\overline{\Omega})$ and assume that there exists an $R > 0$ such that $\|u_j\|_2 \leq R$. Then for all $0 < \beta_4 < \frac{1}{4}$ it follows that there exists a $N_0 \in \mathbb{N}$ such that for $N \geq N_0$

$$\|[u_1 - u_2, v(u_2)]\|_{-1} \leq C_1 [\ln(1+\lambda_N)] \|u_1 - u_2\|_1 + C_2 (\lambda_{N+1})^{-\beta_4}$$

where $C_1, C_2 > 0$ and only depends on R, β, and where

$$v(u_2) = -G_2([u_2, u_2])$$

Proof:
Throughout the proof let $u = u_1 - u_2$.

First it is noted, since $u, v \in H_0^2(\overline{\Omega})$, that by lemma 5.4, $[u, v(u_2)]$ can be written as a sum of terms of the form

$$\begin{aligned} w &= \partial_i((\partial_j u)(\partial_{kl}^2 G_2(\partial_n((\partial_m u_2)(\partial_{pq}^2 u_2))))) \\ &= \partial_i((Q_N\partial_j u)(\partial_{kl}^2 G_2(\partial_n((\partial_m u_2)(\partial_{pq}^2 u_2))))) \\ &\quad + \partial_i((P_N\partial_j u)(\partial_{kl}^2 G_2(\partial_n((Q_N\partial_m u_2)(\partial_{pq}^2 u_2))))) \\ &\quad + \partial_i((P_N\partial_j u)(\partial_{kl}^2 G_2(\partial_n((P_N\partial_m u_2)(\partial_{pq}^2 u_2))))) \\ &=: w_1 + w_2 + w_3 \end{aligned}$$

for $i, j, k, l, m, n, p, q \in \{1, 2\}$.

Each of the functions w_i will now be evaluated separately.

By Lemma 9.5 it follows that for all $0 < \beta_4 < 1$

$$\begin{aligned} \|w_1\|_{-1} &\leq \|\partial_i((Q_N\partial_j u)(\partial_{kl}^2 G_2(\partial_n((\partial_m u_2)(\partial_{pq}^2 u_2)))))\|_{-1} \\ &\leq c_1\|(Q_N\partial_j u)(\partial_{kl}^2 G_2(\partial_n((\partial_m u_2)(\partial_{pq}^2 u_2))))\|_0 \\ &\leq c_2\|Q_N\partial_j u\|_{1-\beta_4}\|\partial_{kl}^2 G_2(\partial_n((\partial_m u_2)(\partial_{pq}^2 u_2)))\|_{\beta_4} \\ &\leq c_3\|Q_N\partial_j u\|_{1-\beta_4}\|(\partial_m u_2)(\partial_{pq}^2 u_2)\|_{-1+\beta_4} \\ &\leq c_4\|Q_N\partial_j u\|_{1-\beta_4}\|\partial_m u_2\|_{\beta_4}\|\partial_{pq}^2 u_2\|_0 \\ &\leq c_5\|u_2\|_2^2\|Q_N\partial_j u\|_{1-\beta_4} \\ &\leq c_6\|Q_N\partial_j u\|_{1-\beta_4} \end{aligned}$$

and by Lemma 4.8

$$\begin{aligned} \|Q_N\partial_j u\|_{1-\beta_1} &\leq c_1\|(\Delta_D^2)^{\frac{1}{4}-\frac{\beta_4}{4}}Q_N\partial_j u\|_0 \\ &\leq c_2(\lambda_{N+1})^{-\frac{\beta_4}{4}}\|(\Delta_D^2)^{\frac{1}{4}}\partial_j u\|_0 \\ &\leq c_3(\lambda_{N+1})^{-\frac{\beta_4}{4}}\|\partial_j u\|_1 \\ &\leq c_4(\lambda_{N+1})^{-\frac{\beta_4}{4}} \end{aligned}$$

so

$$\|w_1\|_{-1} \leq c_1(\lambda_{N+1})^{-\frac{\beta_4}{4}} \tag{9.17}$$

Also by Lemma 9.5 it follows by setting $\beta_5 = 1 - \beta_4$ that

$$\begin{aligned}
\|w_2\|_{-1} &\le \|\partial_i((P_N\partial_j u)(\partial^2_{kl}G_2(\partial_n((Q_N\partial_m u_2)(\partial^2_{pq}u_2)))))\|_{-1} \\
&\le c_1\|(P_N\partial_j u)(\partial^2_{kl}G_2(\partial_n((Q_N\partial_m u_2)(\partial^2_{pq}u_2))))\|_0 \\
&\le c_2\|P_N\partial_j u\|_{1-\beta_5}\|\partial^2_{kl}G_2(\partial_n((Q_N\partial_m u_2)(\partial^2_{pq}u_2)))\|_{\beta_5} \\
&\le c_3\|P_N\partial_j u\|_{1-\beta_5}\|(Q_N\partial_m u_2)(\partial^2_{pq}u_2)\|_{-1+\beta_5} \\
&\le c_4\|P_N\partial_j u\|_{1-\beta_5}\|Q_N\partial_m u_2\|_{\beta_5}\|\partial^2_{pq}u_2\|_0 \\
&\le c_4\|P_N\partial_j u\|_{\beta_4}\|Q_N\partial_m u_2\|_{1-\beta_4}\|\partial^2_{pq}u_2\|_0 \\
&\le c_5\|Q_N\partial_j u_2\|_{1-\beta_4}
\end{aligned}$$

and by Lemma 4.8

$$\begin{aligned}
\|Q_N\partial_j u_2\|_{1-\beta_4} &\le c_1\|(\Delta^2_D)^{\frac{1}{4}-\frac{\beta_4}{4}}Q_N\partial_j u_2\|_0 \\
&\le c_2(\lambda_{N+1})^{-\frac{\beta_4}{4}}\|(\Delta^2_D)^{\frac{1}{4}}\partial_j u_2\|_0 \\
&\le c_3(\lambda_{N+1})^{-\frac{\beta_4}{4}}\|\partial_j u_2\|_1 \\
&\le c_4(\lambda_{N+1})^{-\frac{\beta_4}{4}}
\end{aligned}$$

so

$$\|w_2\|_{-1} \le c_1(\lambda_{N+1})^{-\frac{\beta_4}{4}} \tag{9.18}$$

Lastly it follows for w_3 that

$$\|w_3\|_{-1} \le c\|(P_N\partial_j u)(\partial^2_{kl}G_2(\partial_n((P_N\partial_m u_2)(\partial^2_{pq}u_2))))\|_0$$

and it is noted that

$$\begin{aligned}
P_N\partial_j u &\in C^\infty(\overline{\Omega}) \cap H^1_0(\overline{\Omega}) \\
\partial^2_{kl}G_2(\partial_n((P_N\partial_m u_2)(\partial^2_{pq}u_2))) &\in H^1_0(\overline{\Omega})
\end{aligned}$$

so by Lemma 9.2 and 9.4 it follows that

$$\begin{aligned}
\|w_3\|_{-1} &\le c_1\sqrt{\ln(1+\lambda_N)}\|\partial_j u\|_0\|\partial^2_{kl}G_2(\partial_n((P_N\partial_m u_2)(\partial^2_{pq}u_2)))\|_1 \\
&\le c_2\sqrt{\ln(1+\lambda_N)}\|u\|_1\|(P_N\partial_m u_2)(\partial^2_{pq}u_2)\|_0 \\
&\le c_3\sqrt{\ln(1+\lambda_N)}\|u\|_1\|P_N\partial_m u_2\|_{0,\infty}\|\partial^2_{pq}u_2\|_0 \\
&\le c_3\ln(1+\lambda_N)\|u\|_1\|\partial_m u_2\|_1\|\partial^2_{pq}u_2\|_0 \\
&\le c_4\ln(1+\lambda_N)\|u\|_1
\end{aligned} \tag{9.19}$$

Now (9.17), (9.18), and (9.19) imply that there exists a $N_0 \in \mathbb{N}$ such that

$$\|[u, v(u_2)]\| \leq c_1 \ln(1+\lambda_N)\|u\|_1 + c_2(\lambda_{N+1})^{-\beta_4}$$

for all $N \geq N_0$ and all $0 < \beta_4 < \frac{1}{4}$. □

Theorem 9.8
There exists at most one solution to (1.1), (1.3) *and* (1.4).

Proof:
Let u_1, u_2 be weak solutions to (1.1) with boundary and initial conditions (1.3) and (1.4), and define $u = u_1 - u_2$. Furthermore let the following linear problem be defined:

$$\begin{aligned} \partial_{tt} w + \Delta^2 w &= P_N M \\ \gamma_0 w = \gamma_1 w &= 0a \\ w(0) = w'(0) &= 0 \end{aligned}$$

where

$$M := M(t) = [u_1, v(u_1)] - [u_2, v(u_2)]$$

Then, since the projector P_N commutes with the differential operators, it follows that $P_N u$ is a solution to the linear problem.

Next it is noted that since

$$\Delta^2 u, \partial_{tt} u \in L^\infty(0, T; H^{-2}(\bar{\Omega}))$$

it follows that

$$(\Delta_D^2)^{-\frac{1}{2}} \Delta^2 u, (\Delta_D^2)^{-\frac{1}{2}} \partial_{tt} u \in L^\infty(0, T; L^2(\Omega))$$

so since $P_N u$ solves the linear problem, the following identity holds in $L^2(\Omega)$:

$$(\Delta_D^2)^{-\frac{1}{2}} \partial_{tt} P_N u + (\Delta_D^2)^{-\frac{1}{2}} \Delta^2 P_N u = (\Delta_D^2)^{-\frac{1}{2}} P_N M$$

Moreover $\partial_t u \in L^\infty(0, T; L^2(\Omega))$, so by making the inner product in $L^2(\Omega)$ with $P_N \partial_t u$ one obtains

$$\begin{aligned} &\langle (\Delta_D^2)^{-\frac{1}{2}} \partial_{tt} P_N u, P_N \partial_t u \rangle + \langle (\Delta_D^2)^{-\frac{1}{2}} \Delta^2 P_N u, P_N \partial_t u \rangle \\ &= \langle (\Delta_D^2)^{-\frac{1}{2}} P_N M, P_N \partial_t u \rangle \end{aligned}$$

which is rewritten to

$$\begin{aligned}
&\langle \partial_t (\Delta_D^2)^{-\frac{1}{4}} \partial_t P_N u, (\Delta_D^2)^{-\frac{1}{4}} P_N \partial_t u\rangle + \langle \partial_t (\Delta_D^2)^{\frac{1}{4}} P_N u, (\Delta_D^2)^{\frac{1}{4}} P_N u\rangle \\
= &\langle (\Delta_D^2)^{-\frac{1}{4}} P_N M, (\Delta_D^2)^{-\frac{1}{4}} P_N \partial_t u\rangle
\end{aligned}$$

Now Theorem 7.3 can be applied to give

$$\begin{aligned}
&\partial_t \|(\Delta_D^2)^{-\frac{1}{4}} \partial_t P_N u\|_0^2 + \partial_t \|(\Delta_D^2)^{\frac{1}{4}} P_N u\|_0^2 \\
\leq &\, 2|\langle (\Delta_D^2)^{-\frac{1}{4}} P_N M, (\Delta_D^2)^{-\frac{1}{4}} P_N \partial_t u\rangle| \\
\leq &\, 2\|(\Delta_D^2)^{-\frac{1}{4}} P_N M\|_0 \|(\Delta_D^2)^{-\frac{1}{4}} P_N \partial_t u\|_0
\end{aligned}$$

so by Lemma 4.8

$$\partial_t \|P_N \partial_t u\|_{-1}^2 + \partial_t \|P_N u\|_1^2 \leq c_1 \|P_N M\|_{-1} \|P_N \partial_t u\|_{-1}$$

Integration from 0 to t gives that

$$\|P_N \partial_t u\|_{-1}^2 + \|P_N u\|_1^2 \leq c_1 \int_0^t \|P_N M\|_{-1} \|P_N \partial_t u\|_{-1} \, d\tau$$

so using that P_N is norm-decreasing it follows that

$$\|P_N \partial_t u\|_{-1}^2 + \|P_N u\|_1^2 \leq c_1 \int_0^t \|M\|_{-1} \|\partial_t u\|_{-1} \, d\tau$$

and thus for $N \to \infty$

$$\|\partial_t u\|_{-1}^2 + \|u\|_1^2 \leq c_1 \int_0^t \|M\|_{-1} \|\partial_t u\|_{-1} \, d\tau \tag{9.20}$$

Now it is noted that

$$M = [u_1, v(u_1) - v(u_2)] + [u, v(u_2)]$$

so by Lemma 9.6 and 9.7 it follows that there exists a $N_0 \in \mathbb{N}$ such that

$$\|M\|_{-1} \leq C_1 [\ln(1 + \lambda_N)] \|u\|_1 + C_2 (\lambda_{N+1})^{-\beta} \tag{9.21}$$

for all $N \geq N_0$ and all $0 < \beta < \frac{1}{16}$, where C_1, C_2 only depend on the norms of u_1, u_2 in $H_0^2(\overline{\Omega})$ and on β.

Moreover, by Theorem 7.9

$$u \in C([0,T]; H_0^1(\overline{\Omega}))$$
$$\partial_t u \in C([0,T]; H^{-1}(\bar{\Omega}))$$

Inserting (9.21) in (9.20) gives

$$\begin{aligned}
\|\partial_t u\|_{-1}^2 + \|u\|_1^2 &\leq c_1 \int_0^t (C_1[\ln(1+\lambda_N)]\|u\|_1 + C_2(\lambda_{N+1})^{-\beta})\|\partial_t u\|_{-1}\, d\tau \\
&\leq c_3 \int_0^t [\ln(1+\lambda_N)]\|u\|_1 \|\partial_t u\|_{-1}\, d\tau \\
&\quad + c_4 \int_0^t (\lambda_{N+1})^{-\beta}\|\partial_t u\|_{-1}\, d\tau
\end{aligned}$$

so since $\partial_t u \in C([0,T]; H^{-1}(\bar{\Omega}))$ there exists a constant c_5 such that

$$\begin{aligned}
\|\partial_t u\|_{-1}^2 + \|u\|_1^2 &\leq c_3 \int_0^t [\ln(1+\lambda_N)]\|u\|_1 \|\partial_t u\|_{-1}\, d\tau + c_5 T(\lambda_{N+1})^{-\beta} \\
&\leq c_6 \int_0^t [\ln(1+\lambda_N)]\|u\|_1^2 + \|\partial_t u\|_{-1}^2\, d\tau + c_5 T(\lambda_{N+1})^{-\beta}
\end{aligned}$$

Now let $\psi(t) = \|\partial_t u\|_{-1}^2 + \|u\|_1^2$. Then

$$\psi(t) \leq c_6[\ln(1+\lambda_N)] \int_0^t \psi(\tau)\, d\tau + c_5 T(\lambda_{N+1})^{-\beta}$$

for all $t \in [0,T]$, so by Lemma 9.1

$$\psi(t) \leq c_5 T(\lambda_{N+1})^{-\beta}(1+\lambda_N)^{c_6 t} \tag{9.22}$$

Now let a $0 < \beta < \frac{1}{16}$ be given, and choose c_6 such that (9.22) holds. Then for $0 \leq t < \frac{\beta}{C_3}$ the right hand side of (9.22) tends to zero as $N \to \infty$, so $\psi(t) = 0$ for $0 \leq t < \frac{\beta}{C_3}$.

Then, since

$$u \in C([0,T]; H_0^1(\overline{\Omega}))$$
$$\partial_t u \in C([0,T]; H^{-1}(\bar{\Omega}))$$

it follows for $0 \leq t \leq \frac{\beta}{C_3}$, that $\psi(t) = 0$. Now

$$u(\frac{\beta}{C_3}) = 0$$
$$u'(\frac{\beta}{C_3}) = 0$$

can be used as initial conditions for the linear problem, and thus it is concluded that for $\frac{\beta}{C_3} \leq t \leq 2\frac{\beta}{C_3}$, $\psi(t) = 0$. By repeatingly replacing the initial value by

$$u(n\frac{\beta}{C_3}) = 0$$
$$u'(n\frac{\beta}{C_3}) = 0$$

it is concluded that for $0 \leq t \leq T$, $\psi(t) = 0$.

Now by definition of ψ it follows that $u_1 \equiv u_2$, which completes the proof.

□

Chapter 10

Stationary von Karman Equations

In this chapter the stationary von Karman equations are addressed. In [Ciarlet, 1997, chapter 5] it is shown, that when the equations for an elastic 3-dimensional plate are reduced to a two dimensional problem, and when this problem is scaled, one is left with the von Karman equations with non homogeneous boundary conditions, thus motivating the treatment. In [Lions, 1969, section 4.3] the problem with homogeneous boundary values is treated using the Brouwer fixed point theorem. In this section, the problem with non homogeneous boundary values is treated following the approach of [Ciarlet, 1997, sec. 5.8].

With suitable restrictions on the function spaces to which ϕ_0 and ϕ_1 can belong, existence of solutions to (1.2) with boundary conditions given by (1.6) is established.

It will be assumed throughout the chapter that $\Omega \subset \mathbb{R}^2$ is open, bounded, smooth and connected, and that

$$\begin{aligned} f &\in H^{-2}(\bar{\Omega}) \\ \phi_0 &\in H^{3/2}(\Gamma) \\ \phi_1 &\in H^{1/2}(\Gamma) \end{aligned}$$

In Theorem 10.8, 10.9 and 10.10 it will be shown, that finding a solution to (1.2) with boundary conditions given by (1.6) is equivalent to minimizing the non-linear functional

$$j(u) = \frac{1}{4}(G_2([G_2([u,u]),u])|u)_\Delta + \frac{1}{2}(u - G_2([T_2\phi_0 + T_3\phi_1, u]), u)_\Delta - (G_2(f), u)_\Delta$$

where G_2 is the inverse of Δ_D^2 and $\begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix}$ is the parametrix given in Theorem 4.2.

However, to lighten the notation a few operators will be defined in the following. Moreover, before the main theorems are proved, the needed properties of the operators defined will be proved.

In defining the needed operators, the auxillary operator

$$B(u,v) = G_2([u,v])$$

is needed. For $u, v \in H^2(\bar{\Omega})$ the definition makes sense, since then by Lemma 5.3 $[u,v] \in H^{-2}(\bar{\Omega})$ and hence by Theorem 4.1, $B(u,v) \in H_0^2(\overline{\Omega})$.

The following properties of B is essential in the following:

Lemma 10.1

1. *B is symmetric on $H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$*
2. *B is bilinear.*
3. *For all $(u,v,w) \in H^2(\bar{\Omega}) \times H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$*

   $$(B(u,v)|w)_\Delta = (B(u,w)|v)_\Delta$$

4. *B is sequentially compact, i.e. for $(u_n, v_n) \rightharpoonup (u,v)$ in $H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$*

   $$B(u_n, v_n) \to B(u,v) \quad \text{in } H_0^2(\overline{\Omega})$$

5. *B is continuous.*

Proof:
1: Let $u, v \in H^2(\bar{\Omega})$, then

$$B(u,v) = G_2([u,v]) = G_2([v,u]) = B(v,u)$$

2: Since B is symmetric, it is enough to show that

$$B(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 B(u_1, v) + \alpha_2 B(u_2, v)$$

This follows from Theorem 4.1 and from the bilinearity of $[\cdot,\cdot]$.

3: Let $(u,v,w) \in H^2(\bar{\Omega}) \times H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$. Then it follows that

$$(B(u,v)|w)_\Delta = \int_\Omega \Delta B(u,v)\Delta w\,dx = \int_\Omega [u,v]w\,dx$$

since $w \in H_0^2(\overline{\Omega})$ and $\Delta^2 B(u,v) = [u,v]$.

But then it follows from Lemma 5.5, that

$$(B(u,v)|w)_\Delta = \int_\Omega [u,w]v\,dx = (B(u,w)|v)_\Delta$$

since $v \in H_0^2(\overline{\Omega})$.

4: Let (u_k, v_k) be a weakly convergent sequence in $H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$, with limit $(u,v) \in H^2(\bar{\Omega}) \times H^2(\bar{\Omega})$.

Since $\|u\|_\Delta^2 := (u|u)_\Delta = \|\Delta u\|^2$, it follows from Lemma A.29 that $\|\cdot\|_\Delta$ induced by the inner product $(\cdot|\cdot)_\Delta$ is a norm on $H_0^2(\overline{\Omega})$, equivalent to $\|\cdot\|_2$. By the definition of B and by Lemma 5.5 it then follows for all $w \in H_0^2(\overline{\Omega})$, that

$$|(B(u,v)|w)_\Delta| = \Big|\int_\Omega [u,v]w\,d\omega\Big| = \Big|\int_\Omega [w,v]u\,d\omega\Big| \le c\|w\|_\Delta \|v\|_{1,4}\|u\|_{1,4}$$

Thus

$$\|B(u,v)\|_\Delta = \sup_{w \in H_0^2(\overline{\Omega})\setminus\{0\}} \frac{|(B(u,v)|w)_\Delta|}{\|w\|_\Delta} \le c\|v\|_{1,4}\|u\|_{1,4} \tag{10.1}$$

for all $u, v \in H^2(\bar{\Omega})$.

Now by the bilinearity of B,

$$B(u_k,v_k) - B(u,v) = B(u_k - u, v) + B(u, v_k - v) + B(u_k - u, v_k - v)$$

so by (10.1)

$$\begin{aligned}
&\|B(u_k,v_k) - B(u,v)\|_\Delta \\
&\le c(\|B(u_k-u,v)\|_\Delta + \|B(u,v_k-v)\|_\Delta + \|B(u_k-u,v_k-v)\|_\Delta) \\
&\le C(\|u_k-u\|_{1,4}\|v\|_{1,4} + \|u\|_{1,4}\|v-v_k\|_{1,4} + \|u_k-u\|_{1,4}\|v-v_k\|_{1,4})
\end{aligned}$$

It then follows, since by Lemma A.23, $H^2(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} W^{1,4}(\Omega)$ compactly, that $B(u_k,v_k) \to B(u,v)$ in the Δ-norm, and thus by Lemma A.29 in H^2-norm.

5: Let $(u_k, v_k) \to (u, v)$ in $H^2(\bar{\Omega})$, Then $(u_k, v_k) \rightharpoonup (u, v)$, so by (4), $B(u_n, v_n) \to B(u, v)$, showing that B is continuous. $\square$

Definition 10.2
$C : H_0^2(\overline{\Omega}) \to H_0^2(\Omega)$ is defined by

$$C(u) = B(B(u, u), u)$$

Lemma 10.3
Let C be defined as in Definition 10.2. Then

1. *For all $u \in H_0^2(\overline{\Omega})$, $C(u)$ is well defined as element of $H_0^2(\overline{\Omega})$.*
2. *For all $u \in H_0^2(\overline{\Omega})$*

$$(C(u)|u)_\Delta = (B(u, u), B(u, u))_\Delta \geq 0$$

   and

$$(C(u)|u)_\Delta = 0 \quad \Leftrightarrow \quad u = 0$$

3. *$C(\alpha u) = \alpha^3 C(u)$.*
4. *C is sequentially compact, i.e. for $u_n \rightharpoonup u$ in $H_0^2(\overline{\Omega})$*

$$C(u_n) \to C(u) \quad \text{in } H_0^2(\overline{\Omega})$$

5. *C is continuous.*

Proof:
1: Since B is well defined as an element of $H_0^2(\overline{\Omega})$, it follows that C is well defined as the unique element of $H_0^2(\overline{\Omega})$ given by

$$C(u) = G_2([B(u, u), u])$$

2: Assume that $u \in H_0^2(\overline{\Omega})$. Then, by Lemma 10.1,(1) and (3),

$$(C(u)|u)_\Delta = (B(B(u, u), u)|u)_\Delta = (B(u, u)|B(u, u))_\Delta \geq 0 \qquad (10.2)$$

Moreover if $u = 0$, then $(C(u), u)_\Delta = 0$. Conversely if $(C(u)|u)_\Delta = 0$, then by (10.2) $B(u, u) = 0$. But then $\Delta^2 B(u, u) = [u, u] = 0$, so by Lemma 5.6, $u = 0$.

3: Let $u \in H_0^2(\overline{\Omega})$ and $\alpha \in \mathbb{R}$. Then, by the bilinearity of B, it follows that

$$C(\alpha u) = B(B(\alpha u, \alpha u), \alpha u) = \alpha^3 B(B(u,u),u) = \alpha^3 C(u)$$

4: The sequential compactness of C follows from the sequential compactness of B, since for $u_k \rightharpoonup u$ in $H_0^2(\overline{\Omega})$, it follows from Lemma 10.1,(4), that

$$C(u_k) = B(B(u_k, u_k), u_k) \to B(B(u,u),u) = C(u)$$

since $B(u_k, u_k) \to B(u,u)$ implies $B(u_k, u_k) \rightharpoonup B(u,u)$.

5: That C is continuous, follows directly from (4), since every convergent sequence is weakly convergent. □

Let $\begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix}$ be the parametrix given in Theorem 4.2. Then the following function is defined:

$$\theta_0 = T_2 \phi_0 + T_3 \phi_1$$

It is then noted that θ_0 is a solution to the boundary value problem

$$\begin{aligned} \Delta_D^2 \theta_0 &= 0 \\ \gamma_0 \theta_0 &= \phi_0 \\ \gamma_1 \theta_0 &= \phi_1 \end{aligned}$$

Using θ_0, the operator Λ can be defined:

Definition 10.4
$\Lambda : H_0^2(\overline{\Omega}) \to H_0^2(\overline{\Omega})$ is defined by

$$\Lambda(u) = B(u, \theta_0)$$

Lemma 10.5
Let Λ be defined as in Definition 10.4. Then

1. *For all $u \in H_0^2(\overline{\Omega})$, $\Lambda(u)$ is well defined as element of $H_0^2(\overline{\Omega})$.*
2. *Λ is symmetric with respect to $(\cdot|\cdot)_\Delta$.*
3. *Λ is compact on $H_0^2(\overline{\Omega})$.*
4. *Λ is continuous.*

Proof:
1: Since θ_0 is well defined as an element of $H^2(\bar{\Omega})$, it follows that $\Lambda(u)$ is well defined as an element of $H_0^2(\overline{\Omega})$.

2: Let $u, v \in H_0^2(\overline{\Omega})$. Then by Lemma 10.1,(1) and (3)

$$(\Lambda(u)|v)_\Delta = (B(u,\theta_0)|v)_\Delta = (B(v,\theta_0)|u)_\Delta = (\Lambda(v)|u)_\Delta$$

3: Let (u_n) be a bounded sequence in $H_0^2(\overline{\Omega})$. Then $[u_n, \theta_0]$ is bounded in $L^1(\Omega)$, so since $L^1(\Omega) \overset{\text{comp}}{\hookrightarrow} H^{-2}(\bar{\Omega})$, $[u_n, \theta_0]$ has a convergent subsequence (u_l) in $H^{-2}(\bar{\Omega})$, so by Theorem 4.1, $\Lambda(u_l) = G_2([u_l, \theta_0])$ is convergent in $H_0^2(\overline{\Omega})$. Thus Λ is compact.

4: This follows, as in the proof of Lemma 10.3,(5). □

Now the functional j introduced in the beginning of the chapter can be written, using C and Λ:

Definition 10.6
Let C and Λ be defined as above. The functional $j : H_0^2(\overline{\Omega}) \to \mathbb{R}$ is defined by

$$j(u) = \frac{1}{4}(C(u)|u)_\Delta + \frac{1}{2}((I-\Lambda)u|u)_\Delta - (G_2(f)|u)_\Delta$$

In the following lemma it will among other things be shown, that j is differentiable. The differentiability will be in the sense of [Reed and Simon, 1980, p. 366], i.e. a map $f : X \to Y$, where X and Y are Banach spaces, is differentiable at x if there exists a linear map T, such that

$$\lim_{\|v\|\to 0} \frac{1}{\|v\|_X} \|f(x+v) - f(x) - Tv\|_Y = 0$$

Lemma 10.7
Let j be defined as in Definition 10.6. Then

1. *j is well defined.*
2. *j is continuous.*
3. *j is differentiable with*

$$j'(u)v = (C(u) + (I-\Lambda)(u) - G_2(f)|v)_\Delta$$

4. *j is sequentially weakly lower semi-continuous on $H_0^2(\overline{\Omega})$, i.e. for all* sequences (x_n) *for which* $x_n \rightharpoonup x$

$$j(x) \leq \liminf_{n\to\infty} j(x_n)$$

5. *j is coercive on $H_0^2(\overline{\Omega})$ in the sense that if $v_n \in H_0^2(\overline{\Omega})$ with $\|v_n\|_\Delta \to \infty$, then $j(v_n) \to \infty$.*

Proof:

1: Since $(\cdot|\cdot)_\Delta$ is an inner product on $H_0^2(\overline{\Omega})$ (Lemma A.30) and since C and Λ are well defined, j is well defined.

2: This follows from the continuity of $(\cdot|\cdot)_\Delta$ and of C and Λ.

3: Let the following functionals be defined:

$$\begin{aligned} j_4(u) &= \frac{1}{4}(C(u)|u)_\Delta \\ j_2(u) &= \frac{1}{2}((I-\Lambda)u|u)_\Delta \\ j_1(u) &= (G_2(f)|u)_\Delta \end{aligned}$$

Then j is differentiable if each of the j_i's are.

First it is claimed that j_4 is differentiable with $j_4'(u)v = (C(u)|v)_\Delta$. From Lemma 10.3,(2), it follows that

$$\begin{aligned} \frac{1}{4}(C(u+v)|u+v)_\Delta &= \frac{1}{4}(B(u+v,u+v)|B(u+v,u+v))_\Delta \\ &= \frac{1}{4}(B(u,u)|B(u,u))_\Delta + (B(u,u)|B(u,v))_\Delta \\ &+ \frac{1}{2}(B(u,u)|B(v,v))_\Delta + (B(u,v)|B(u,v))_\Delta \\ &+ (B(u,v)|B(v,v))_\Delta + \frac{1}{4}(B(v,v)|B(v,v))_\Delta \end{aligned}$$

since B by Lemma 10.1,(1) is symmetric.

It is noted, that $(B(u,u)|B(u,v))_\Delta = (C(u)|v)_\Delta$ is linear in v, so it only remains to show, that $j_4'(u)v = (C(u)|v)_\Delta$ is actually the differential of j_4.

Now, since $(\cdot|\cdot)_\Delta$ and B are continuous (Lemma A.30 and 10.1,(5)), it follows that

$$
\begin{aligned}
&|\frac{1}{4}(C(u+v)|u+v)_\Delta - \frac{1}{4}(C(u)|u)_\Delta - (B(u,u)|B(u,v))_\Delta| \\
&\le |\frac{1}{2}(B(u,u)|B(v,v))_\Delta| + |(B(u,v)|B(u,v))_\Delta| \\
&+ |(B(u,v)|B(v,v))_\Delta| + |\frac{1}{4}(B(v,v)|B(v,v))_\Delta| \\
&\le c_1\|u\|_2^2\|v\|_2^2 + c_2\|u\|_2^2\|v\|_2^2 + c_3\|u\|_2\|v\|_2^3 + c_4\|v\|_2^4
\end{aligned}
$$

and thus

$$\lim_{v\to 0}\|v\|_2^{-1}|\frac{1}{4}(C(u+v)|u+v)_\Delta - \frac{1}{4}(C(u)|u)_\Delta - (B(u,u)|B(u,v))_\Delta| = 0$$

so j_4 is differentiable with $j_4'(u)v = (C(u)|v)_\Delta$ (by the definition of C).

In a similar way it can be shown, that $j_2(u)$ and $j_1(u)$ are differentiable with $j_2'(u)v = ((I-\Lambda)u|v)_\Delta$ and $j_1'(u)v = (G_2(f)|v)_\Delta$.

Thus it is shown that j is differentiable, and it follows that the differential is given by

$$j'(u)v = ((C(u)|v)_\Delta + ((I-\Lambda)u|v)_\Delta - (G_2(f)|v)_\Delta$$

4: Assume that (v_k) is a weakly convergent sequence in $H_0^2(\overline{\Omega})$ with $v_k \rightharpoonup v$. Since C and Λ are sequentially compact, it follows that

$$
\begin{aligned}
C(v_k) &\to C(v) \\
\Lambda(v_k) &\to \Lambda(v)
\end{aligned}
$$

Now $v_k \rightharpoonup v$ in $H_0^2(\overline{\Omega})$, so $\Delta v_k \rightharpoonup \Delta v$ in $L^2(\Omega)$ by continuity of Δ. But then it follows from Lemma 7.1, that

$$(v|v)_\Delta = \|\Delta v\|_0^2 \le \liminf_{k\to\infty}\|\Delta v_k\|_0^2 = \liminf_{k\to\infty}(v_k|v_k)_\Delta$$

It then follows, that

$$
\begin{aligned}
j_4(v) &= \lim_{k\to\infty}\frac{1}{4}(C(v_k)|v_k)_\Delta = \lim_{k\to\infty} j_4(v_k) \\
j_2(v) &= \frac{1}{2}(v|v)_\Delta - \frac{1}{2}(\Lambda(v)|v)_\Delta \le \liminf_{k\to\infty} j_2(v_k) \\
j_1(v) &= \lim_{k\to\infty}(G_2(f)|v_k)_\Delta = \lim_{k\to\infty} j_1(v_k)
\end{aligned}
$$

the last limit following from the definition of weak convergence, and thus

$$j(v_k) \leq \liminf_{k\to\infty} j(v_k)$$

5: The proof is given by contradiction: Assume that j is not coercive. Then there exists a sequence (v_k) and an $M < \infty$ such that $\|v_k\|_\Delta \to \infty$ and $j(v_k) \leq M$, where by Lemma A.29 $\|\cdot\|_\Delta$ is an equivalent norm on $H_0^2(\overline{\Omega})$. It may be assumed, that $v_k \neq 0$ for all k, since if for some k, $v_k = 0$, (v_k) can be replaced by the subsequence consisting of all non-zero elements of (v_k). Now define

$$\psi_k = \frac{v_k}{\|v_k\|_\Delta}$$

so $\|\psi_k\|_\Delta = 1$ for all k.

Now, by dividing the inequality $j(v_k) \leq M$ by $\|v_k\|_\Delta$, it follows after a rewriting that

$$\frac{1}{2} - \frac{1}{2}(\Lambda(\psi_k)|\psi_k)_\Delta + \|v_k\|_\Delta^2 j_4(\psi_k) \leq \frac{M}{\|v_k\|_\Delta^2} + \frac{1}{\|v_k\|_\Delta}(G_2(f)|\psi_k)_\Delta \quad (10.3)$$

Then, since $\|\psi_k\|_\Delta = 1$, there exists a weakly convergent subsequence (ψ_l), $\psi_l \rightharpoonup \psi$ in $H_0^2(\overline{\Omega})$. Then, by Lemma 10.3,(4), $j_4(\psi_l) \to j_4(\psi)$. But since the right hand side tends to 0 as $l \to \infty$, it follows, that $j_4(\psi) = 0$. Then, by Lemma 10.3,(2), $\psi = 0$.

Thus it follows, for the left hand side of (10.3), that

$$\frac{1}{2} = \lim_{l\to\infty}\left[\frac{1}{2} - \frac{1}{2}(\Lambda(\theta_l)|\theta_l)_\Delta\right] \leq \liminf_{l\to\infty}\left[\frac{1}{2} - \frac{1}{2}(\Lambda(\theta_l)|\theta_l)_\Delta + \|v_l\|_\Delta^2 j_4(\theta_l)\right]$$

giving the contradiction, since by (10.3) the right hand side tends to zero. □

Theorem 10.8
Let B, C, θ_0 and Λ be defined as above for

$$\begin{pmatrix} f \\ \phi_0 \\ \phi_1 \end{pmatrix} \in \begin{matrix} H_0^2(\overline{\Omega}) \\ \times \\ H^{\frac{3}{2}}(\Gamma) \\ \times \\ H^{\frac{1}{2}}(\Gamma) \end{matrix}$$

Then solving (1.2) *with boundary conditions given by* (1.6) *is equivalent to solving*

$$u \in H_0^2(\overline{\Omega}), \quad C(u) + (I - \Lambda)(u) - G_2(f) = 0 \tag{10.4}$$

$$v \in H_0^2(\overline{\Omega}), \quad v = \theta_0 - B(u, u) \tag{10.5}$$

Proof:
First assume that u fulfils (10.4), and let $v = \theta_0 - B(u, u)$. Then $v \in H^2(\bar{\Omega})$ and by definition of B and θ_0,

$$\Delta^2 v = \Delta^2 \theta_0 - \Delta^2 B(u, u) = -[u, u]$$

and since $B(u, u) \in H_0^2(\overline{\Omega})$,

$$\begin{aligned} \gamma_0 v &= \gamma_0(\theta_0 - B(u, u)) = \phi_0 \\ \gamma_1 v &= \gamma_1(\theta_0 - B(u, u)) = \phi_1 \end{aligned}$$

Moreover, from the definition of B, C, Λ and F it follows that

$$\begin{aligned} 0 &= \Delta^2(C(u) + (I - \Lambda)u - F) \\ &= \Delta^2 B(B(u, u), u) + \Delta^2 u - \Delta^2 \Lambda(u) - \Delta^2 F \\ &= [B(u, u), u] + \Delta^2 u - [\theta_0, u] - f \\ &= \Delta^2 u - [\theta_0 - B(u, u), u] - f \\ &= \Delta^2 u - [v, u] - f \end{aligned}$$

and since $u \in H_0^2(\overline{\Omega})$,

$$\begin{aligned} \gamma_0 u &= 0 \\ \gamma_1 u &= 0 \end{aligned}$$

Thus it is seen that, if u fulfils (10.4), then $(u, v) \in H_0^2(\overline{\Omega}) \times H^2(\bar{\Omega})$ with $v = \theta_0 - B(u, u)$ is a solution to (1.2) with boundary conditions given by (1.6).

Now assume that $(u, v) \in H_0^2(\overline{\Omega}) \times H^2(\bar{\Omega})$ is a solution to (1.2) with boundary conditions given by (1.6), and define $\tilde{v} = v - \theta_0$. Then

$$\begin{aligned} \gamma_0 \tilde{v} &= \gamma_0 v - \gamma_0 \theta_0 = 0 \\ \gamma_1 \tilde{v} &= \gamma_1 v - \gamma_1 \theta_0 = 0 \end{aligned}$$

so $\tilde{v} \in H_0^2(\overline{\Omega})$. Moreover

$$\begin{aligned} \Delta^2(u - F) &= [v, u] + f - f = [\tilde{v} + \theta_0, u] \\ \Delta^2 \tilde{v} &= -[u, u] \end{aligned}$$

so by definition of B

$$\begin{aligned} u - F &= B(\tilde{v} + \theta_0, u) \\ \tilde{v} &= -B(u, u) \end{aligned}$$

but then it follows by the linearity of B that

$$u - F = B(-B(u, u) + \theta_0, u) = -B(B(u, u), u) + B(\theta_0, u)$$

and thus

$$C(u) + (I - \Lambda)u - F = 0$$

$\square$

Theorem 10.9
Let C, Λ, F and j be defined as above. Then for $u \in H_0^2(\overline{\Omega})$

$$C(u) + (I - \Lambda)(u) - F = 0$$

if and only if

$$j'(u) = 0$$

Proof:
First assume that $u \in H_0^2(\overline{\Omega})$ and $j'(u) = 0$. Then $j'(u)v = 0$ for all $v \in H_0^2(\overline{\Omega})$, so by Lemma 10.7,(3),

$$(C(u) - (I - \Lambda)(u) + F|v)_\Delta = 0$$

for all $v \in H_0^2(\overline{\Omega})$, but by Lemma A.30 $(\cdot|\cdot)_\Delta$ is an inner product on $H_0^2(\overline{\Omega})$, so

$$C(u) - (I - \Lambda)u + F = 0$$

On the other hand, if $u \in H_0^2(\overline{\Omega})$ and $C(u) - (I - \Lambda)(u) + F = 0$, then

$$j'(u)v = (C(u) - (I - \Lambda)(u) + F|v)_\Delta = 0$$

for all $v \in H_0^2(\overline{\Omega})$, so $j'(u) = 0$. $\square$

Theorem 10.10
Let j be defined as above. Then there exists at least one $u \in H_0^2(\overline{\Omega})$ such that

$$j(u) = \inf_{v \in H_0^2(\overline{\Omega})} j(v) \tag{10.6}$$

Moreover any such $u \in H_0^2(\overline{\Omega})$ is a solution to (1.2) *with boundary conditions given by* (1.6).

Proof:
First it is noted, that if $u \in H_0^2(\overline{\Omega})$ and $j(u) = \inf_{v \in H_0^2(\overline{\Omega})} j(v)$, then $j'(u) = 0$, so by Theorem 10.8 and 10.9, u is a solution.

Then it is noted, that the infimum is finite. This is seen in the following way:

Since $j(0) = 0$, $\inf_{v \in H_0^2(\overline{\Omega})} j(v) < \infty$. From the lemmas proven above, it follows, that

$$j(u) \geq -c_1 \|\Lambda u\|_\Delta \|u\|_\Delta - c_2 \|u\|_\Delta^2$$

so whenever $\|u\|_\Delta$ is bounded, $j(u)$ is bounded below. Moreover, since j is coercive, there exists an $M > 0$, such that $j(u) \geq 0$, whenever $\|u\|_\Delta \geq M$. Thus j is bounded below, so the infimum is finite.

Next let (u_k) be an infimizing sequence, i.e. $u_k \in H_0^2(\overline{\Omega})$ and $j(u_k) \to \inf_{u \in H_0^2(\overline{\Omega})} j(u)$. Such a sequence exists, since the infimum is finite, as noted above. Then $j(u_k)$ is bounded, so since j is coercive (by Lemma 10.7,(5)), (u_k) is bounded in $H_0^2(\overline{\Omega})$. Then there exists a weakly convergent subsequence (u_l) of (u_k). But j is sequentially weakly lower semi-continuous, so

$$j(u) \leq \liminf_{l \to \infty} j(u_l) = \inf_{v \in H_0^2(\overline{\Omega})} j(v)$$

the last equality following by assumption. Thus u is an minimizer of j. □

Theorem 10.11
If $f \in H^{s-4}(\bar{\Omega})$, $\phi_0 \in H^{s-\frac{1}{2}}(\Gamma)$, $\phi_1 \in H^{s-\frac{3}{2}}(\Gamma)$, and if $(u, v) \in H_0^2(\overline{\Omega}) \times H^2(\bar{\Omega})$ is a solution to (1.2) *with boundary conditions given by* (1.6)*, then*

$$(u, v) \in H^s(\bar{\Omega}) \cap H_0^2(\overline{\Omega}) \times H^s(\bar{\Omega})$$

Proof:
The following proof differs somewhat from the proof given in [Ciarlet, 1997]. It shares the ideas of the proof given in [Ciarlet, 1997], but it is based on the product defined in Chapter 2 and the elliptic regularity

result in Chapter 3, whereas the original proof in [Ciarlet, 1997] is based on a regularity result for L^p spaces together with continuous embeddings of L^p-based Sobolev spaces.

By Theorem 4.3 there exists a unique $w \in H^s(\bar{\Omega})$, such that

$$\Delta^2 w = 0 \quad \gamma_0 w = \phi_0 \quad \gamma_1 w = \phi_1$$

Assume that $(u, v) \in H_0^2(\overline{\Omega}) \times H^2(\bar{\Omega})$ is a solution to (1.2) with boundary conditions given by (1.6), and define $\tilde{v} = v - w$. Then $(u, \tilde{v}) \in H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$ is a solution to

$$\Delta^2 u - [\tilde{v}, u] - [w, u] = f \tag{10.7}$$

$$\Delta^2 \tilde{v} - [u, u] = 0 \tag{10.8}$$

Now the proof is an induction on n, where $n \in \mathbb{N}$ is such that $2n < s \leq 2(n+1)$.

First assume that $n = 1$:

It is noted that for all $0 < \delta \leq 1$, $[\tilde{v}, u] \in H^{-1-\delta}(\bar{\Omega})$ by Corollary 2.7. Also $[w, u] \in H^{s-3-\delta}(\bar{\Omega})$ by Corollary 2.7.

If $s < 3$ there exists a $0 < \delta < 1$ such that $s - 4 = -1 - \delta$. Then $\Delta^2 u \in H^{s-4}(\bar{\Omega})$ since u solves (10.7) and $f \in H^{s-4}(\bar{\Omega})$, so by Theorem 4.3

$$u \in H^s(\bar{\Omega})$$

Consequently by Corollary 2.7

$$[u, u] \in H^{2s-5}(\bar{\Omega})$$

and thus since $\tilde{v}$ solves (10.8) it follows by Theorem 4.3 that

$$\tilde{v} \in H^{2s-1}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^s(\bar{\Omega})$$

and

$$v = w + \tilde{v} \in H^s(\bar{\Omega})$$

If $3 \leq s \leq 4$, $\Delta^2 u \in H^{-1-\delta}(\bar{\Omega})$, so by Theorem 4.3 $u \in H^{3-\delta}(\bar{\Omega})$. Then for $0 < \delta < \frac{1}{2}$ by Corollary 2.7,

$$[u, u] \in H^{1-2\delta}(\bar{\Omega})$$

so since $\tilde{v}$ solves (10.8) it follows by Theorem 4.3 that

$$\tilde{v} \in H^{5-2\delta}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^s(\bar{\Omega})$$

and

$$v = w + \tilde{v} \in H^s(\bar{\Omega})$$

Then by Corollary 2.7

$$[\tilde{v}, u] \in H^{s-2\delta}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{s-4}(\bar{\Omega})$$
$$[w, u] \in H^{s-2\delta}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{s-4}(\bar{\Omega})$$

so since u solves (10.7) and $f \in H^{s-4}(\bar{\Omega})$ it follows by Theorem 4.3 that

$$u \in H^s(\bar{\Omega})$$

which completes the proof for $n = 1$.

Now assume that the theorem has been proved for some $N \in \mathbb{N}$ and that $2(N+1) < s \leq 2(N+2)$.

Since the theorem holds for N and $s > N+1$,

$$u \in H^{2(N+1)}(\bar{\Omega}) \cap H_0^2(\overline{\Omega})$$
$$\tilde{v} \in H^{2(N+1)}(\bar{\Omega})$$

Then

$$[\tilde{v}, u] \in H^{4N-1}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{s-4}(\bar{\Omega})$$
$$[w, u] \in H^{s+2N-3}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{s-4}(\bar{\Omega})$$
$$[u, u] \in H^{4N-1}(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} H^{s-4}(\bar{\Omega})$$

so since u solves (10.7) and $f \in H^{s-4}(\bar{\Omega})$ it follows by Theorem 4.3 that

$$u \in H^s(\bar{\Omega})$$

and since $\tilde{v}$ solves (10.8) it follows by Theorem 4.3 that

$$\tilde{v} \in H^s(\bar{\Omega})$$

and

$$v = w + \tilde{v} \in H^s(\bar{\Omega})$$

which completes the induction step. □

Chapter 11

Boundary Stabilization of von Karman Plates

The intention with this chapter was to collect what was known about control theory of the von Karman equations. Great effort has been made to find literature about this subject. Many authors has written about this subject and I have found [Puel and Tucsnak, 1995], [Favini et al., 1996], [Favini et al., 1997], [Puel and Tucsnak, 1996], [Lagnese, 1988] and [Lagnese, 1989]. Most of these papers deal with the so called "Full von Karman System" where there is an additional term compared to the von Karman equations (1.1). So due to limited time, I have chosen to concentrate on boundary stabilization of the original von Karman equations of the form (1.1), that is to concentrate on [Lagnese, 1988] and [Lagnese, 1989].

In this chapter stability of the von Karman equations is investigated. It will be shown that the von Karman equations can be stabilized by boundary feedback. The presentation here follows the presentation in [Lagnese, 1988] and [Lagnese, 1989]. The reader should be aware that in these two references there is some confusion about when and where the factor $m \cdot \nu$ should be present in integrals over the boundary (with numerous misprints), and also restriction of functions to the boundary before integrating over the boundary is a neglected issue.

Throughout this chapter it is assumed that $\Omega \subseteq \mathbb{R}^2$ is open, bounded and smooth. Γ denotes the boundary of Ω. Near Γ, $\nu = (\nu_1, \nu_2)$ is the unit exterior normal vector to Ω, chosen in the following way: For $x_0 \in \Gamma$, then

$\nu(x_0)$ is the exterior unit normal vector to Γ, and

$$\nu(x) = \nu(x_0) \text{ for } x \text{ of the form } x = x_0 + s\nu(x_0) \equiv h(x_0, s)$$
$$\text{where } x_0 \in \Gamma, s \in \,]-\delta, \delta[$$

Here $\delta > 0$ is chosen so small that x, when represented by $x_0 \in \Gamma$ and $s \in \,]-\delta, \delta[$ is unique and smooth i.e. h is bijective and is a C^∞-diffeomorphism from $\Gamma \times \,]-\delta, \delta[$ to the set

$$\Sigma = h(\Gamma \times \,]-\delta, \delta[) \subset \mathbb{R}^2$$

Note that the mapping $h^{-1} : \Sigma \to \Gamma \times \,]-\delta, \delta[$ defines a function g from Σ to $]-\delta, \delta[$ which has level curves $\{x \,|\, g(x) = s\}$ that are "parallel" to Γ, and that $\nu = \nabla g$ so the normal vector field ν is a gradient vector field.

A tangential vector field τ is chosen such that $\tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$. Tangential derivative along Γ is defined (following [Grubb and Solonnikov, 1991, p. 279]) as

$$\partial_\tau f = \sum_{j=1}^{2} \tau_j \partial_j f$$

and $\gamma_\tau = \gamma_0 \partial_\tau f \equiv \partial_\tau f|_\Gamma$.

Furthermore, star-shapedness of Ω is necessary in some of the proofs, and is defined as:

Definition 11.1
Ω is called star-shaped if there exists $x_0 \in \Omega$ such that for $m(x) = x - x_0$ the relation

$$m(x) \cdot \nu = (x - x_0) \cdot \nu > 0$$

holds on Γ.

Remark: Since Ω is bounded, Γ is compact so the condition for star-shaped is equivalent to $m \cdot \nu \geq \delta_0 > 0$ for all $x \in \Gamma$.

Let the operators B_1 and B_2 be given by

$$B_1 u = 2\nu_1 \nu_2 \partial_{12}^2 u - \nu_1^2 \partial_{22}^2 u - \nu_2^2 \partial_{11}^2 u$$
$$B_2 u = (\nu_1^2 - \nu_2^2) \partial_{12}^2 u + \nu_1 \nu_2 (\partial_{22}^2 u - \partial_{11}^2 u)$$

The von Karman equations, for which stability is considered here, is the von Karman equations considered in [Lagnese, 1989, section 5.2] and [Lagnese, 1988] as stated below:

$$\begin{aligned} u'' + \Delta^2 u - [u, v] &= 0 \quad \text{in } \,]0, \infty[\, \times \Omega \\ b\Delta^2 v + [u, u] &= 0 \quad \text{in } \,]0, \infty[\, \times \Omega \end{aligned} \tag{11.1}$$

where $b > 0$ is a constant.

Boundary conditions

$$\begin{aligned}\gamma_0\Delta u + (1-\mu)\gamma_0 B_1 u &= g_1 \quad \text{on} \ \ ]0,\infty[\ \times \Gamma\\ \gamma_1\Delta u + (1-\mu)\gamma_\tau B_2 u &= g_2 \quad \text{on} \ \ ]0,\infty[\ \times \Gamma\\ \gamma_0 v = \gamma_1 v &= 0 \quad \text{on} \ \ ]0,\infty[\ \times \Gamma\end{aligned} \tag{11.2}$$

where μ is Poisson's ratio ($0 < \mu < \frac{1}{2}$).

Initial conditions

$$u(0,x) = u_0(x), \qquad u'(0,x) = u_1(x) \quad \text{on } \Omega \tag{11.3}$$

No initial conditions are prescribed for v, since these are given implicitly trough u_0 by $v(0) = -\frac{1}{b}G_2([u_0,u_0])$.

g_1 and g_2 are the control inputs used to stabilize the system.

Boundary stabilization of the von Karman equations given above, adds up to finding operators F_1 and F_2 acting on the states $u(t)$ and $u'(t)$ such that if g_1 and g_2 are defined by the feedback laws

$$\begin{aligned}g_1(t) &= F_1(u(t),u'(t))\\ g_2(t) &= F_2(u(t),u'(t))\end{aligned} \tag{11.4}$$

then the closed-loop system is stable. It will be shown that the operators F_1 and F_2 can be chosen such that the energy has exponential decay rate.

Before a solution to the von Karman equations above is defined, a bilinear form a and a Green's formula for it is needed.

Define the bilinear form a on $H^2(\bar{\Omega})$ by

$$\begin{aligned}a(u,v) = \int_\Omega (\partial^2_{11}u\partial^2_{11}v + \partial^2_{22}u\partial^2_{22}v + \mu(\partial^2_{11}u\partial^2_{22}v + \partial^2_{22}u\partial^2_{11}v)\\ + 2(1-\mu)\partial^2_{12}u\partial^2_{12}v)\,dx\end{aligned} \tag{11.5}$$

For the bilinear form a there is the following Green's formula.

Lemma 11.2
Let $\phi,\psi \in H^4(\bar{\Omega})$. Then for every $\theta \in \mathbb{R}$

$$\begin{aligned}\int_\Omega \psi\Delta^2\phi\,dx = a(\psi,\phi) + \int_\Gamma ((\gamma_1\Delta\phi + (1-\theta)\gamma_\tau B_2\phi)\gamma_0\psi\\ - \gamma_0(\Delta\phi + (1-\theta)B_1\phi)\gamma_1\psi)\,d\sigma\end{aligned} \tag{11.6}$$

Proof:
Assume $\phi, \psi \in H^4(\bar{\Omega})$.

It is noted that the symbol of Δ^2 is $|\xi|^4 = \xi_1^4 + \xi_2^4 + 2\xi_1^2\xi_2^2$, so $\Delta^2\phi = \partial_{1111}^4\phi + \partial_{2222}^4\phi + 2\partial_{1122}^4\phi$.

For the first term in a (11.5) the differentiations on ψ is moved to ϕ using Green's formula [Grubb, 1996b, p. 1.8]

$$\begin{aligned}\int_\Omega \partial_{11}^2\psi\partial_{11}^2\phi\,dx &= -\int_\Omega \partial_1\psi\partial_{111}^3\phi\,dx + \int_\Gamma \gamma_0(\nu_1\partial_1\psi\partial_{11}^2\phi)\,d\sigma \\ &= \int_\Omega \psi\partial_{1111}^4\phi\,dx - \int_\Gamma \gamma_0(\nu_1\psi\partial_{111}^3\phi)\,d\sigma \\ &\quad + \int_\Gamma \gamma_0(\nu_1\partial_1\psi\partial_{11}^2\phi)\,d\sigma\end{aligned}$$

Repeating this calculation on the remaining terms of a results in

$$\begin{aligned}a(\psi,\phi) &= \int_\Omega \psi(\partial_{1111}^4\phi + \partial_{2222}^4\phi + \theta(\partial_{1122}^4\phi + \partial_{2211}^4\phi) \\ &\qquad + (1-\theta)(\partial_{1212}^4\phi + \partial_{2112}^4\phi))\,dx \\ &\quad + \int_\Gamma \gamma_0(\nu_1\partial_1\psi\partial_{11}^2\phi + \nu_2\partial_2\psi\partial_{22}^2\phi \\ &\qquad + \theta(\nu_1\partial_1\psi\partial_{22}^2\phi + \nu_2\partial_2\psi\partial_{11}^2\phi) \\ &\qquad + (1-\theta)(\nu_2\partial_1\psi\partial_{12}^2\phi + \nu_1\partial_2\psi\partial_{12}^2\phi))\gamma_1\psi\,d\sigma \\ &\quad - \int_\Gamma \gamma_0(\nu_1\partial_{111}^3\phi + \nu_2\partial_{222}^3\phi + \theta(\nu_1\partial_{122}^3\phi + \nu_2\partial_{211}^3\phi) \\ &\qquad + (1-\theta)(\nu_1\partial_{212}^3\phi + \nu_2\partial_{112}^3\phi))\gamma_0\psi\,d\sigma \\ &= \int_\Omega \psi\Delta^2\phi\,dx + \int_\Gamma (\gamma_0\Delta\phi\gamma_1\psi - \gamma_0\psi\gamma_1\Delta\phi)\,d\sigma \\ &\quad + (1-\theta)\int_\Gamma \gamma_0(-\nu_1\partial_1\psi\partial_{22}^2\phi - \nu_2\partial_2\psi\partial_{11}^2\phi \\ &\qquad + \nu_2\partial_1\psi\partial_{12}^2\phi + \nu_1\partial_2\psi\partial_{12}^2\phi)\,d\sigma\end{aligned}$$

Now what is left to be shown is that the integral over Γ of $\gamma_0 B_1\phi\gamma_1\psi - \gamma_\tau B_2\phi\gamma_0\psi$ is equal to the last integral on the right hand side of the above equation.

$$\int_\Gamma (\gamma_0 B_1\phi\gamma_1\psi - \gamma_\tau(B_2\phi)\gamma_0\psi)\,d\sigma$$

$$= \int_\Gamma (\gamma_0 B_1\phi\gamma_1\psi - \gamma_0(-\nu_2\partial_1(B_2\phi)\psi + \nu_1\partial_2(B_2\phi)\psi))\,d\sigma$$

$$= \int_\Gamma (\gamma_0 B_1\phi\gamma_1\psi - \gamma_0(\partial_1(\nu_2\psi)B_2\phi - \partial_2(\nu_1\psi)B_2\phi))\,d\sigma$$

$$= \int_\Gamma \gamma_0(-\nu_1\partial_1\psi\partial_{22}^2\phi - \nu_2\partial_2\psi\partial_{11}^2\phi + \nu_2\partial_1\psi\partial_{12}^2\phi + \nu_1\partial_2\psi\partial_{12}^2\phi)\,d\sigma$$

$$+ \int_\Gamma \gamma_0\psi\gamma_0((\partial_2\nu_1 - \partial_1\nu_2)((\nu_1^2 - \nu_2^2)\partial_{12}^2\phi + \nu_1\nu_2(\partial_{22}^2\phi - \partial_{11}^2\phi)))\,d\sigma$$

The factor $\partial_2\nu_1 - \partial_1\nu_2$ in the last integral is $\operatorname{curl}\nu$. Since Γ is smooth and the normal vector field ν is constructed as it is, there exists a function g such that $\nu = \nabla g$, then $\operatorname{curl}\nu = \operatorname{curl}\nabla g = \partial_2\partial_1 g - \partial_1\partial_2 g = 0$, which ends the proof. □

To define weak solutions to the von Karman equations (11.1), (11.2), and (11.3) with $(u, u', v) \in H^2(\bar\Omega) \times L^2(\Omega) \times H_0^2(\overline{\Omega}))$ for every $t \in \,]0, T[$, some meaning has to be given to the boundary operators in (11.3) for $u \in H^2(\bar\Omega)$. From (11.3), χ_1 and χ_2 has well defined meaning for $u \in H^4(\bar\Omega)$.

Define trace operators

$$\begin{aligned} \chi_1 u &= \gamma_0\Delta u + (1-\mu)\gamma_0 B_1 u \\ \chi_2 u &= \gamma_1\Delta u + (1-\mu)\gamma_\tau B_2 u \end{aligned} \tag{11.7}$$

Then for $u \in H^2(\bar\Omega) \cap D(\Delta^2_{\max})$, the trace operators in (11.7) is given a generalized meaning as the elements

$$\begin{pmatrix}\chi_1 u \\ \chi_2 u\end{pmatrix} \in \begin{matrix} H^{-\frac{1}{2}}(\Gamma) \\ \times \\ H^{-\frac{3}{2}}(\Gamma) \end{matrix}$$

so that for $(\eta_1, \eta_2) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)$ and with $w = \mathcal{K}_0\eta_1 + \mathcal{K}_1\eta_2 \in H^2(\bar\Omega)$, where $(\mathcal{K}_0\ \mathcal{K}_1)$ is a right inverse to $\begin{pmatrix}\gamma_0 \\ \gamma_1\end{pmatrix}$ [Grubb, 2000, p. 3.27], the following identity holds

$$(\Delta^2 u|w) = a(u, w) + \langle\chi_1 u, \eta_1\rangle_\Gamma + \langle\chi_2 u, \eta_2\rangle \tag{11.8}$$

This is well defined, which is seen in the following way: For $\eta_1 = 0$ then as $\eta_2 \to 0$ in $H^{\frac{3}{2}}(\Gamma)$, $w \to 0$ in $H^2(\bar{\Omega})$ since $\mathcal{K}_1$ is continuous, so $(\Delta^2 u|w) \to 0$ and $a(u,w) \to 0$ hence from (11.8), $\langle \chi_2 u, \eta_2 \rangle \to 0$. Therefore $\chi_2 u$ is a continuous functional on $H^{\frac{3}{2}}(\Gamma)$ whence $\chi_2 u \in H^{-\frac{3}{2}}(\Gamma)$. And for $\eta_2 = 0$ then as $\eta_1 \to 0$ in $H^{\frac{1}{2}}(\Gamma)$, $w \to 0$ in $H^2(\bar{\Omega})$ since $\mathcal{K}_0$ is continuous, so $(\Delta^2 u|w) \to 0$ and $a(u,w) \to 0$ hence from (11.8), $\langle \chi_1 u, \eta_1 \rangle \to 0$, so that $\chi_1 u \in H^{-\frac{1}{2}}(\Gamma)$.

Also

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} : H^2(\bar{\Omega}) \cap D(\Delta^2_{\max}) \to H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{3}{2}}(\Gamma)$$

is continuous since

$$\begin{aligned} |\langle \chi_1 u, \eta_1 \rangle_\Gamma + \langle \chi_2 u, \eta_2 \rangle_\Gamma| &\le |a(u,w)| + |(\Delta^2 u|w)| \\ &\le \|u\|_2 \|w\|_2 + \|\Delta^2 u\| \, \|w\| \\ &\le \sqrt{\|u\|_2^2 + \|\Delta^2 u\|^2} \sqrt{2} \|w\|_2^2 \\ &\le \|u\|_{H^2(\bar{\Omega}) \cap D(\Delta^2_{\max})} c \|(\eta_1, \eta_2)\|_{H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)} \end{aligned}$$

so

$$\sup_{(\eta_1,\eta_2) \in H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)} \{ |\langle \chi_1 u, \eta_1 \rangle_\Gamma + \langle \chi_2 u, \eta_2 \rangle_\Gamma| \mid \|(\eta_1, \eta_2)\| \le 1 \} \le \|u\|_{H^2(\bar{\Omega}) \cap D(\Delta^2_{\max})} c$$

From this discussion the trace operators in (11.7), and hence (11.2), has a generalized meaning for $u \in H^2(\bar{\Omega}) \cap D(\Delta^2_{\max})$.

Remark: A definition of a solution to the von Karman equations could be the following though it is not quite satisfactory, because u has not been shown to be in $D(\Delta^2_{\max})$ by the end of the project period. In [Lagnese, 1989] no general definition of solutions is given. There solutions are only discussed for a variational formulation of the problem, where u, u', v is valued in $H^2(\bar{\Omega}), L^2(\Omega), H^2_0(\overline{\Omega})$ respectively.

Definition 11.3
If

$$u_0 \in H^2(\bar{\Omega}), \quad u_1 \in L^2(\Omega), \quad g_1 \in (0,T; H^{\frac{1}{2}}(\Gamma)), \quad g_2 \in (0,T; H^{\frac{3}{2}}(\Gamma))$$

then $u, v \in \mathscr{D}'(\mathbb{R}_+; L^2(\Omega))$ *is said to be a solution to* (11.1), (11.2), (11.3), *if for every* $T > 0$,

$$\begin{aligned} u &\in L^\infty(0,T; H^2(\bar{\Omega})) \\ u' &\in L^\infty(0,T; L^2(\Omega)) \\ v &\in L^\infty(0,T; H^2_0(\overline{\Omega})) \end{aligned}$$

and if they satisfy (11.1) *in* $\mathscr{D}'(0,T;H^{-2}(\bar{\Omega}))$ *and* (11.2) *in* $H^{-\frac{1}{2}}(\Gamma)$ *and* $H^{-\frac{3}{2}}(\Gamma)$ *respectively for every* $t \in\ ]0,T[$, *and the vector valued function* $t \mapsto (u(t),u'(t)) \in H^2(\bar{\Omega}) \times L^2(\Omega)$ *is weakly continuous with* $(u(0),u'(0)) = (u_0,u_1)$.

To define a dynamical system as in Definition 6.1, for the von Karman equations (11.1), (11.2), and (11.4) with a particular choice of F_1 and F_2, let

$$X = H^2(\bar{\Omega}) \times L^2(\Omega) \times H_0^2(\overline{\Omega}) \tag{11.9}$$

Equip X with the norm

$$\|(w_1,w_2,w_3)\|_X = \sqrt{\|w_1\|_2^2 + \|w_2\|^2 + \|w_3\|_2^2} \tag{11.10}$$

and let

$$U(t,(u_0,u_1,v_0)) = (u(t),u'(t),v(t)) \tag{11.11}$$

Then if it can be shown, that U is norm-continuous in t, and continuous in (w_1,w_2,w_3) with respect to the norm on X, then U satisfies the conditions in Definition 6.1.

Remark: The continuity properties of U has not been shown due to lack of time and the remark just before Definition 11.3. To show the t-continuity an argument like that in the proof of Theorem 7.9 might be carried out. The difference is the boundary conditions, and an argument about these has to be found. To show continuity of $U(t,\cdot)$ first note that v_0 is given implicit from u_0. So it is necessary to shown, that given two sets of initial conditions (u_0,u_1) and $(\tilde{u}_0,\tilde{u}_1)$, then $\|(u(t),u'(t),v(t)) - (\tilde{u}(t),\tilde{u}'(t),\tilde{v}(t))\|_X \leq c\sqrt{\|u_0-\tilde{u}_0\|_2^2 + \|u_1-\tilde{u}_1\|^2}$.

The dynamical system U would obviously have an equilibrium point in $(w_1,w_2,w_3) = (0,0,0)$, since this is the trivial solution to the von Karman equations.

Having defined what is meant by a solution to the problem, the next natural step would be to show existence of such solutions. But this requires a choice of the operators F_1 and F_2, which has not been made yet. Therefore, for a while let us just assume existence of a solution to the von Karman equations (11.1), (11.2) and (11.3), in order to get some points about stability for the system.

In the results about stability the solution is assumed to be classical, defined as

Definition 11.4
A classical solution to (11.1)*,* (11.2) *and* (11.3) *is a solution where*

$$\begin{aligned} u &\in L^\infty(0,T;H^4(\bar\Omega)) \\ u' &\in L^\infty(0,T;H^2(\bar\Omega)) \\ u'' &\in L^\infty(0,T;L^2(\Omega)) \\ v &\in L^\infty(0,T;H^4(\bar\Omega)\cap H_0^2(\overline\Omega)) \end{aligned} \tag{11.12}$$

Remark: There is no regularity theorem for solutions to this problem, i.e. conditions on the initial data (u_0, u_1) such that this holds [Lagnese, 1989, p. 113]. Lagnese does not discuss the existence of classical solutions, and due to limited time this discussion has not been carried out here either. By restricting the discussion of stability to classical solutions, the boundary conditions (11.2) makes sense without the generalization prior to Definition 11.3.

Assuming that there exists a classical solution (u, v) to (11.1), (11.2) and (11.3) then v is even more regular. Since $u \in L^\infty(0,T;H^4(\bar\Omega))$ then from Corollary 2.7 $[u,u] \in L^\infty(0,T;H^3(\bar\Omega))$. Now since $v \in L^\infty(0,T;H^4(\bar\Omega))$ is a solution to

$$\begin{pmatrix}\Delta^2\\ \gamma_0\\ \gamma_1\end{pmatrix} v = \begin{pmatrix} f\\ 0\\ 0\end{pmatrix}$$

where $f = -[u,u] \in L^\infty(0,T;H^3(\bar\Omega))$ Theorem 4.3 gives that

$$v \in L^\infty(0,T;H^7(\bar\Omega)\cap H_0^2(\overline\Omega)) \tag{11.13}$$

since $v \in L^\infty(0,T;H^4(\bar\Omega)\cap H_0^2(\overline\Omega))$ by assumption. Moreover,

$$b\Delta^2 v' = -2[u',u] \in L^\infty(0,T;H^1(\bar\Omega))$$

by Corollary 2.7, hence another application of Theorem 4.3 gives that

$$v' \in L^\infty(0,T;H^5(\bar\Omega)\cap H_0^2(\overline\Omega)) \tag{11.14}$$

The motivation for using the word "classic" in Definition 11.4 is that u, u' and v can be shown to be continuous functions:

First, $u \in L^\infty(0,T;H^4(\bar\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H^4(\bar\Omega))$ and since $H^2(\bar\Omega) \overset{\mathrm{d}}{\hookrightarrow} L^2(\Omega) = H_0^0(\overline\Omega) \overset{\mathrm{d}}{\hookrightarrow} H_0^{-4}(\overline\Omega)$, (11.12) gives that $u' \in L^\infty(0,T;H_0^{-4}(\overline\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H_0^{-4}(\overline\Omega))$. Now Theorem 7.3 gives that u is almost everywhere equal to a strongly continuous function, hence $u \in C([0,T];H^4(\bar\Omega))$. By the Sobolev embedding theorem [Grubb, 1996b, p. 9.9] $H^4(\bar\Omega) \hookrightarrow C(\overline\Omega)$ so $u \in C([0,T];C(\overline\Omega))$.

$u' \in L^\infty(0,T;H^2(\bar\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H^2(\bar\Omega))$ and $L^2(\Omega) \overset{\mathrm{d}}{\hookrightarrow} H_0^{-2}(\overline{\Omega})$ so $u'' \in L^\infty(0,T;H_0^{-2}(\overline{\Omega})) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H_0^{-2}(\overline{\Omega}))$. Using Theorem 7.3 it follows that $u' \in C([0,T];H^2(\bar\Omega))$. Again by the Sobolev embedding theorem $u' \in C([0,T];C(\overline{\Omega}))$.

From (11.13) and (11.14) $v \in L^\infty(0,T;H_0^2(\overline{\Omega})) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H_0^2(\overline{\Omega}))$ and $v' \in L^\infty(0,T;H^{-2}(\bar\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;H^{-2}(\bar\Omega))$. Thus using Theorem 7.3, $v \in C([0,T];H_0^2(\overline{\Omega}))$. $v \in C([0,T];H_0^2(\overline{\Omega})) \hookrightarrow C([0,T];C(\overline{\Omega}))$ using the Sobolev embedding theorem once more.

Now an application of the Green's formula in Lemma 11.2, which will be necessary later, is shown.

Lemma 11.5
Let Ω be star-shaped, and let $u \in H^4(\bar\Omega)$. Then for every $\theta \in \mathbb{R}$

$$\begin{aligned}\int_\Omega (m\cdot\nabla u)\Delta^2 u\,dx = a(u,u) &+ \frac{1}{2}\int_\Gamma (m\cdot\nu)\gamma_0((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 \\ &+ 2\theta\partial_{11}^2 u\partial_{22}^2 u + 2(1-\theta)(\partial_{12}^2 u)^2)\,d\sigma \\ &+ \int_\Gamma ((\gamma_1\Delta u + (1-\theta)\gamma_\tau B_2 u)\gamma_0(m\cdot\nabla u) \\ &- \gamma_0(\Delta u + (1-\theta)B_1 u)\gamma_1(m\cdot\nabla u))\,d\sigma\end{aligned}$$

Proof:
Assume $u \in C^\infty(\overline{\Omega})$. Lemma 11.2 yields

$$\begin{aligned}\int_\Omega (m\cdot\nabla u)\Delta^2 u\,dx = a(m\cdot\nabla u, u) & \\ + \int_\Gamma ((\gamma_1\Delta u + (1-\theta)\gamma_\tau B_2 u)\gamma_0(m\cdot\nabla u) & \\ - \gamma_0(\Delta u + (1-\theta)B_1 u)\gamma_1(m\cdot\nabla u))\,d\sigma &\end{aligned} \tag{11.15}$$

where $a(m\cdot\nabla u, u)$ can be calculated to be

$$\begin{aligned}a(m\cdot\nabla u, u) = \int_\Omega &(\partial_{11}^2 u\partial_{11}^2(m\cdot\nabla u) + \partial_{22}^2 u\partial_{22}^2(m\cdot\nabla u) \\ &+ \theta(\partial_{11}^2 u\partial_{22}^2(m\cdot\nabla u) + \partial_{22}^2 u\partial_{11}^2(m\cdot\nabla u)) \\ &+ 2(1-\theta)\partial_{12}^2 u\partial_{12}^2(m\cdot\nabla u)\,dx\end{aligned}$$

Calculation of the derivatives of $(m \cdot \nabla u)$ gives for the first term on the right hand side

$$\begin{aligned}\partial_{11}^2 u \partial_{11}^2 (m \cdot \nabla u) &= \partial_{11}^2 u \partial_{11}^2 (m_1 \partial_1 u + m_2 \partial_2 u) \\ &= \partial_{11}^2 u \partial_1 (\partial_1 u + m_1 \partial_{11}^2 u + m_2 \partial_{12}^2 u) \\ &= \partial_{11}^2 u (\partial_{11}^2 u + \partial_{11}^2 u + m_1 \partial_{111}^3 u + m_2 \partial_{112}^3 u) \\ &= 2(\partial_{11}^2 u)^2 + \frac{1}{2}(m \cdot \nabla(\partial_{11}^2 u)^2)\end{aligned}$$

doing the same calculations on the remaining terms gives

$$\begin{aligned}a(m \cdot \nabla u, u) = 2a(u,u) + \frac{1}{2}\int_\Omega & m \cdot \nabla((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 \\ & + 2\theta \partial_{11}^2 u \partial_{22}^2 + 2(1-\theta)(\partial_{12}^2 u)^2)\, dx \\ = a(u,u) + \frac{1}{2}\int_\Omega & \operatorname{div}(m((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 \\ & + 2\theta \partial_{11}^2 u \partial_{22}^2 + 2(1-\theta)(\partial_{12}^2 u)^2))\, dx \\ = a(u,u) + \frac{1}{2}\int_\Gamma & m \cdot \nu \gamma_0((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 \\ & + 2\theta \partial_{11}^2 u \partial_{22}^2 + 2(1-\theta)(\partial_{12}^2 u)^2)\, d\sigma\end{aligned}$$

where Gauss' divergence theorem [Williamson et al., 1972, p. 551] has been used in the last equality. Inserting this in (11.15) gives the desired result for $u \in C^\infty(\overline{\Omega})$, which extends to $H^4(\bar{\Omega})$ by continuity, using Lemma A.27. □

The energy for the system in (11.1), (11.2) is defined as [Lagnese, 1988, p. 359f]

Definition 11.6
Let (u, v) be a classical solution to (11.1), (11.2), (11.3). Then the energy is defined to be

$$E(t) = \frac{1}{2}(\|u'(t)\|^2 + a(u(t), u(t)) + \frac{b}{2}\|\Delta v(t)\|^2) \tag{11.16}$$

The energy has the following physical interpretation [Lagnese, 1988, p. 360] of the right hand side: The first two terms are respectively, kinetic and strain energies in bending in linear plate theory, and the last term is result a of the coupling between bending and in-plane stretching that occurs in the strain energy in the von Karman theory.

Lemma 11.7
Let (u, v) be a classical solution to (11.1), (11.2), (11.3). *Then the energy $E(t)$ is differentiable with*

$$E'(t) = (u''(t)|u'(t)) + a(u'(t), u(t)) + \frac{b}{2}(\Delta v'(t)|\Delta v(t)) \tag{11.17}$$

Proof:
For the first term on the right hand side of (11.17) note, that since $H^2(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} L^2(\Omega)$ then $u' \in L^\infty(0,T;L^2(\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$ and since $u'' \in L^\infty(0,T;L^2(\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$, Theorem 7.3 with $V = H = L^2(\Omega)$ gives that

$$\frac{1}{2}\frac{d}{dt}\|u'(t)\|^2 = \langle u''(t), u'(t)\rangle = (u''(t)|u'(t)) \tag{11.18}$$

For the second term, rewrite $a(u, u)$ as

$$\begin{aligned} a(u,u) &= \int\limits_\Omega (\mu(\partial_{11}^2 u + \partial_{22}^2 u)^2 + (1-\mu)((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2) + 2(\partial_{12}^2 u)^2))\,dx \\ &= \mu\|\partial_{11}^2 u + \partial_{22}^2 u\|^2 + (1-\mu)(\|\partial_{11}^2 u\|^2 + \|\partial_{22}^2 u\|^2 + 2\|\partial_{12}^2 u\|^2) \end{aligned} \tag{11.19}$$

Since $H^2(\bar{\Omega}) \overset{\mathrm{d}}{\hookrightarrow} L^2(\Omega)$, then $\partial_{jk}^2 u \in L^\infty(0,T;H^2(\bar{\Omega})) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$ for all $j,k = 1,2$ and since $u' \in L^\infty(0,T;H^2(\bar{\Omega})) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$, Theorem 7.3 with $V = H = L^2(\Omega)$ can be used on each term in (11.19) to give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}a(u(t),u(t)) &= \mu(\partial_{11}^2 u' + \partial_{22}^2 u' | \partial_{11}^2 u + \partial_{22}^2 u) \\ &\quad + (1-\mu)((\partial_{11}^2 u'|\partial_{11}^2 u) + (\partial_{22}^2 u'|\partial_{22}^2 u) + 2(\partial_{12}^2 u'|\partial_{12}^2 u)) \\ &= \int\limits_\Omega (\partial_{11}^2 u' \partial_{11}^2 u + \partial_{22}^2 u' \partial_{22}^2 u + \mu(\partial_{11}^2 u' \partial_{22}^2 u' + \partial_{22}^2 u' \partial_{11}^2 u) \\ &\qquad\qquad + 2(1-\mu)\partial_{12}^2 u' \partial_{12}^2 u)\,dx \\ &= a(u'(t), u(t)) \end{aligned} \tag{11.20}$$

For the last term on the right hand side of (11.17), from (11.12) $v \in L^\infty(0,T;H^4(\bar{\Omega}) \cap H_0^2(\overline{\Omega}))$ so $\Delta v \in L^\infty(0,T;L^2(\Omega)) \overset{\mathrm{d}}{\hookrightarrow} L^2(0,T;L^2(\Omega))$.

By (11.14), $\Delta v' \in L^\infty(0,T;L^2(\Omega)) \overset{d}{\hookrightarrow} L^2(0,T;L^2(\Omega))$ so by Theorem 7.3

$$\frac{1}{2}\frac{d}{dt}\|\Delta v(t)\|^2 = \langle \Delta v'(t), \Delta v(t)\rangle = (\Delta v'(t)|\Delta v(t)) \tag{11.21}$$

Adding (11.18), (11.20), (11.21) gives the desired result. □

For a classical solution u, v to (11.1), (11.2), (11.3), equation (11.17) can, since $\Delta^2 v' \in L^2(\Omega)$ from (11.14) by use of Lemma 11.2, (11.1) and (11.2), be rewritten as

$$\begin{aligned} E'(t) &= (u'' + \Delta^2 u|u') - \int_\Gamma (g_2\gamma_0 u' - g_1\gamma_1 u')\, d\sigma + \frac{\gamma}{2}(\Delta^2 v'|v) \\ &= ([u,v]|u') - ([u',u]|v) - \int_\Gamma (g_2\gamma_0 u' - g_1\gamma_1 u')\, d\sigma \\ &= -\int_\Gamma (g_2\gamma_0 u' - g_1\gamma_1 u')\, d\sigma \end{aligned}$$

where Lemma 5.5 is used to give the last equation.

Now if we let

$$g_1 = -\lambda_1\gamma_1 u', \quad g_2 = \lambda_2\gamma_0 u', \qquad \lambda_1, \lambda_2 > 0$$

this would lead to

$$E'(t) = -\int_\Gamma (\lambda_2\gamma_0 u'^2 + \lambda_1(\gamma_1 u')^2)\, d\sigma \le 0$$

which stabilizes the system. But a feedback law that only involves u' will not lead to uniform decay rate (or even strong stability), because $a(u,u)$ vanishes on the set of rigid motions $u = ax_1 + bx_2 + c$.

Instead let

$$\begin{aligned} g_1 &= -\alpha(m\cdot\nu)\gamma_1 u \\ g_2 &= \lambda(m\cdot\nu)\gamma_0 u' + \beta(m\cdot\nu)\gamma_0 u - \alpha\gamma_\tau((m\cdot\nu)\partial_\tau u) \end{aligned} \tag{11.22}$$

where $\alpha, \beta, \lambda > 0$ and Γ is assumed star-shaped with respect to some point $x_0 \in \Omega$.

With feedback law (11.22) the derivative of the energy is

$$
\begin{aligned}
E'(t) &= -\int_\Gamma (\lambda(m\cdot\nu)\gamma_0 u'^2 + \beta(m\cdot\nu)\gamma_0(uu') \\
&\qquad\qquad - \alpha\gamma_\tau((m\cdot\nu)\gamma_\tau u)\gamma_0 u' + \alpha(m\cdot\nu)\gamma_1 u\gamma_1 u')\,d\sigma \\
&= -\int_\Gamma (\lambda(m\cdot\nu)\gamma_0 u'^2 + \beta(m\cdot\nu)\gamma_0(uu') \\
&\qquad\qquad + \alpha(m\cdot\nu)(\gamma_\tau u\gamma_\tau u' + \gamma_1 u\gamma_1 u'))\,d\sigma
\end{aligned}
$$

From this it is seen that

$$
\begin{aligned}
&\frac{d}{dt}\Big(E(t) + \frac{1}{2}\int_\Gamma (m\cdot\nu)\gamma_0(\beta u^2 + \alpha|\nabla u|^2)\,d\sigma\Big) \\
&\qquad\qquad = -\lambda\int_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma \le 0
\end{aligned}
\tag{11.23}
$$

What has been done here is augmenting the strain energy a so that it is positive definite:

$$
\tilde{a}(u,v) = a(u,v) + \int_\Gamma m\cdot\nu\gamma_0(\beta uv + \alpha\nabla u\cdot\nabla v)\,d\sigma \tag{11.24}
$$

where a is given by (11.5) and $\alpha, \beta > 0$. Since a is bilinear and symmetric so is $\tilde{a}$.

Thus define an augmented energy as

$$
\begin{aligned}
\tilde{E}(t) &= E(t) + \frac{1}{2}\int_\Gamma (m\cdot\nu)\gamma_0(\beta u^2 + \alpha|\nabla u|^2)\,d\sigma \\
&= \frac{1}{2}(\|u'(t)\|^2 + \tilde{a}(u(t),u(t)) + \frac{b}{2}\|\Delta v(t)\|^2)
\end{aligned}
\tag{11.25}
$$

Notice here that if $\sqrt{\tilde{a}}$ is a norm on $H^2(\bar{\Omega})$ equivalent to $\|\cdot\|_2$ then stability of the dynamical system U defined in (11.11) can be shown with results from Chapter 6.

Lemma 11.8
Let $\|\!|\cdot|\!\| = \sqrt{\tilde{a}(\,\cdot\,,\,\cdot\,)}$. *Then* $\|\!|\cdot|\!\|$ *is a norm on* $H^2(\bar{\Omega})$.

Proof:
First note that

$$a(u,u) = \int_\Omega ((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 + 2\mu \partial_{11}^2 u \partial_{22}^2 u + 2(1-\mu)(\partial_{12}^2 u)^2)\, dx$$
$$\geq \int_\Omega (\mu(\partial_{11}^2 u + \partial_{22}^2 u)^2 + 2(1-\mu)(\partial_{12}^2 u)^2)\, dx$$
$$\geq 0$$

for all $u \in H^2(\bar{\Omega})$. So $\tilde{a}(u,u) \geq 0$ since the boundary integral in (11.24) is positive for $u = v$.

Then assume that

$$\tilde{a}(u,u) = a(u,u) + \int_\Gamma m \cdot \nu(\beta(\gamma_0 u)^2 + \alpha\gamma_0|\nabla u|^2)\, d\sigma = 0 \qquad (11.26)$$

Then, both $a(u,u) = 0$ and the integral over the boundary in (11.26) is zero, which implies that $\gamma_0 u = 0$ and $|\gamma_1 u| = \gamma_0|\nu \cdot \nabla u| \leq \gamma_0|\nabla u| = 0$, hence $u \in H_0^2(\overline{\Omega})$. For $u \in \mathscr{D}(\Omega)$, Poincaré's inequality [Grubb, 2000, p. 1.6] gives that $\|u\| \leq c_1\|\partial_1 u\| \leq c_2\|\partial_{12}^2 u\|$, $c_1, c_2 > 0$, which extends to $H_0^2(\overline{\Omega})$ by continuity.

Now $u \in H_0^2(\overline{\Omega})$ so

$$0 = a(u,u) \geq \int_\Omega (\mu(\partial_{11}^2 u + \partial_{22}^2 u)^2 + 2(1-\mu)(\partial_{12}^2 u)^2)\, dx$$
$$\geq 2(1-\mu)\|\partial_{12}^2 u\| \geq C\|u\|^2 \geq 0$$

where $C > 0$, so $\tilde{a}(u,u) = 0$ implies $u = 0$.

Cauchy-Schwarz's inequality for $\tilde{a}$:

$$|\tilde{a}(u,v)|^2 \leq \tilde{a}(u,u)\tilde{a}(v,v) \qquad (11.27)$$

holds for all $u, v \in H^2(\bar{\Omega})$, which can be seen as follows: if $\tilde{a}(v,v) = 0$ then $v = 0$ and the inequality is obviously true, so assume that $\tilde{a}(v,v) \neq 0$. Then for $c \in \mathbb{R}$

$$0 \leq \tilde{a}(u - cv, u - cv) = \tilde{a}(u,u) + c^2\tilde{a}(v,v) - 2c\tilde{a}(u,v)$$

Now choose $c = \frac{\tilde{a}(u,v)}{\tilde{a}(v,v)}$. Then

$$0 \leq \tilde{a}(u,u) - \frac{|\tilde{a}(u,v)|^2}{\tilde{a}(v,v)}$$

from which the inequality follows.

Define the function $|||u||| = \sqrt{\tilde{a}(u,u)}$. Then for $u \in H^2(\bar{\Omega})$

$$|||u||| = 0 \quad \Rightarrow \quad u = 0$$

as seen above. For $c \in \mathbb{R}$

$$|||cu||| = \sqrt{\tilde{a}(cu,cu)} = \sqrt{c^2\tilde{a}(u,u)} = |c|\sqrt{\tilde{a}(u,u)}$$

and

$$\begin{aligned} |||u+v|||^2 &= |||u||| + |||v||| + 2\tilde{a}(u,v) \\ &\le |||u||| + |||v||| + 2|\tilde{a}(u,v)| \\ &\le |||u||| + |||v||| + 2|||u||| \, |||v||| \\ &= (|||u||| + |||v|||)^2 \end{aligned}$$

where (11.27) has been used. Hence $|||\cdot|||$ is a norm on $H^2(\bar{\Omega})$. □

The norm $|||\cdot|||$ just defined on $H^2(\bar{\Omega})$ can be shown to be equivalent to $\|\cdot\|_2$.

Lemma 11.9
$|||\cdot|||$ is equivalent to $\|\cdot\|_2$ on $H^2(\bar{\Omega})$.

Proof:
First $|\tilde{a}(u,u)| \le c\|u\|_2^2$, for some $c > 0$ since from (11.24)

$$\begin{aligned} |\tilde{a}(u,u)| \le{}& \|\partial_{11}^2 u\|^2 + \|\partial_{22}^2 u\|^2 + 2\mu|(\partial_{11}^2 u|\partial_{22}^2 u)| + (2-2\mu)\|\partial_{12}^2 u\|^2 \\ &+ \int_\Gamma m\cdot\nu(\beta(\gamma_0 u)^2 + \alpha((\gamma_0\partial_1 u)^2 + (\gamma_0\partial_2 u)^2))\,d\sigma \\ \le{}& c_1\|u\|_2^2 + c_2(\|\gamma_0 u\|_{L^2(\Gamma)}^2 + \|\gamma_0\partial_1 u\|_{L^2(\Gamma)}^2 + \|\gamma_0\partial_2 u\|_{L^2(\Gamma)}^2) \\ \le{}& c_1\|u\|_2^2 + c_2(\|\gamma_0 u\|_{H^{1/2}(\Gamma)}^2 + \|\gamma_0\partial_1 u\|_{H^{1/2}(\Gamma)}^2 + \|\gamma_0\partial_2 u\|_{H^{1/2}(\Gamma)}^2) \end{aligned}$$

Now since $\gamma_0 : H^1(\bar{\Omega}) \to H^{\frac{1}{2}(\Gamma)}$ is continuous

$$|\tilde{a}(u,u)| \le c_1\|u\|_2^2 + c_3(\|u\|_1^2 + \|u\|_2^2) \le c\|u\|_2^2$$

Also $|\tilde{a}(u,u)| \ge c_0\|u\|_2^2$, for some $c_0 > 0$, which can be seen by the following.

From (11.24), since $0 < \mu < \frac{1}{2}$,

$$\begin{aligned}
|\tilde{a}(u,u)| &\geq \|\partial_{11}^2 u\|^2 + \|\partial_{22}^2 u\|^2 + 2\mu(\partial_{11}^2 u|\partial_{22}^2 u) + (2-2\mu)\|\partial_{12}^2 u\|^2 \\
&\geq \sum_{|\alpha|=2} \|\partial^\alpha u\|^2 - 2\mu\|\partial_{11}^2\|\,\|\partial_{22}^2 u\| \\
&\geq \sum_{|\alpha|=2} \|\partial^\alpha u\|^2 - \mu(\|\partial_{11}^2\|^2 + \|\partial_{22}^2 u\|^2) \\
&\geq (1-\mu) \sum_{|\alpha|=2} \|\partial^\alpha u\|^2
\end{aligned}$$

Thus the proof is finished if it can be shown that

$$\int_\Gamma m \cdot \nu \gamma_0(\beta u^2 + \alpha|\nabla u|^2)\, d\sigma \geq c_0 \|u\|_1^2$$

but at the end of the project period no argument for this has been found. According to [Lagnese, 1989, p. 112] $\tilde{a}$ is elliptic, no proof given there. □

Now, having shown that $\|\!|\cdot|\!\|$ is equivalent to $\|\cdot\|_2$ on $H^2(\bar{\Omega})$, Theorem 6.6 can be used to give uniform stability of the dynamical system U defined in (11.11), provided it follows a solution trajectory to a classical solution. This is seen in the following way: Denote the initial conditions (u_0, u_1, v_0) by x_0, and let

$$V(x) = \tilde{E}(U(t, x_0)) = \tilde{E}(t)$$

Now, since $\tilde{E}$ is continuous by Lemma 11.7 and $\tilde{E}' \leq 0$ by (11.23), V is a Lyapunov function. The function f in Theorem 6.6 of class M_h can be the square root function, and the requirements for the theorem is fulfilled.

Knowing from (11.24) that $\tilde{a}$ is bilinear and symmetric, and elliptic from Lemma 11.9, the operator associated with $\tilde{a}$ acts like Δ^2:

Assume $f \in L^2(\Omega)$ and $u \in V \subset H^4(\bar{\Omega})$ (V is identified later) such that

$$\tilde{a}(u, g) = (f|g) \qquad \forall g \in C^\infty(\overline{\Omega})$$

Then for all $\phi \in \mathscr{D}(\Omega)$

$$\begin{aligned}
\tilde{a}(u, \phi) &= \langle \partial_1^4 u + \partial_2^4 u + 2\mu\partial_1^2 u \partial_2^2 u + (2-2\mu)(\partial_{12}^2 u)^2, \phi\rangle + 0 \\
&= \langle \Delta^2 u, \phi\rangle = (f|\phi) = (\Delta^2 u|\phi)
\end{aligned}$$

Hence $\tilde{A}u = f = \Delta^2 u$.

Identification of the space V following [Grubb, 2000, p. 4.14ff]: Using Lemma 11.2

$$\begin{aligned}
\langle \tilde{A}u, g\rangle - \tilde{a}(u,g) &= (\Delta^2 u|g) - \tilde{a}(u,g) \\
&= a(g,u) + \int_\Gamma ((\gamma_1 \Delta u + (1-\mu)\gamma_\tau B_2 u)\gamma_0 g \\
&\qquad - \gamma_0(\Delta u + (1-\mu)B_1 u)\gamma_1 g)\, d\sigma - \tilde{a}(u,g) \\
&= a(g,u) + \int_\Gamma ((\gamma_1 \Delta u + (1-\mu)\gamma_\tau B_2 u)\gamma_0 g \\
&\qquad - \gamma_0(\Delta u + (1-\mu)B_1 u)\gamma_1 g)\, d\sigma - a(g,u) \\
&\qquad - \int_\Gamma (m\cdot\nu)\gamma_0(\beta u g + \alpha \nabla u \cdot \nabla g)\, d\sigma \\
&= \int_\Gamma ((\gamma_1 \Delta u + (1-\mu)\gamma_\tau B_2 u)\gamma_0 g \\
&\quad - \gamma_0(\Delta u + (1-\mu)B_1 u)\gamma_1 g \\
&\quad - (m\cdot\nu)\gamma_0(\beta u g + \alpha \nabla u \cdot \nabla g))\, d\sigma \\
&= \int_\Gamma ((\gamma_1 \Delta u + (1-\mu)\gamma_\tau B_2 u)\gamma_0 g \\
&\quad - \gamma_0(\Delta u + (1-\mu)B_1 u)\gamma_1 g - (m\cdot\nu)\beta\gamma_0(ug) \\
&\quad - \alpha(m\cdot\nu)\gamma_1 u \gamma_1 g + \alpha\gamma_\tau((m\cdot\nu)\partial_\tau u)\gamma_0 g)\, d\sigma
\end{aligned}$$

So for the triplet $(L^2(\Omega), V, \tilde{a})$ and

$$D(\tilde{A}) = \{u \in V | \exists f \in L^2(\Omega)\ \forall g \in V : (f|g) = \tilde{a}(u,g)\}$$

then $(f|g) = (\tilde{A}u|g) = \tilde{a}(u,g)$.

$$\begin{aligned}
V = \{u \in H^4(\bar{\Omega}) | &\gamma_1 \Delta u + (1-\mu)\gamma_\tau B_2 u - m\cdot\nu\beta\gamma_0 u + \gamma_\tau(m\cdot\nu\gamma_\tau u) = 0, \\
&\gamma_0\Delta u + (1-\mu)\gamma_0 B_1 u + \alpha(m\cdot\nu)\gamma_1 u = 0\}
\end{aligned}$$

Here the discussion ends due to lack of time.

Remark: To continue as Lagnese does in [Lagnese, 1989] several problems comes up at this time. To give meaning to the following variational formulation of the von Karman equations (11.1), (11.2), (11.3), and (11.22). Meaning has to be given to the boundary conditions of (11.22) for u, u', v valued in $H^2(\bar{\Omega}), L^2(\Omega), H_0^2(\overline{\Omega})$ respectively. Here there are two critical points, the first is that $\gamma_0 u'$ does not make sense, and the second is that

$\gamma_\tau((m\cdot\nu)\partial_\tau u) = \gamma_0\partial_\tau((m\cdot\nu)\partial_\tau u)$ does not have an obvious meaning because there are two differentiations on u before restriction to the boundary. The second problem might be solved by means described in [Grubb and Solonnikov, 1991, p. 278ff]. An other undiscussed problem for the variational formulation to follow is if the boundary conditions satisfied are those of (11.2) and (11.22).

The variational form of (11.1), (11.2), (11.3), and (11.22) is [Lagnese, 1989, p. 112]:

Given $u_0 \in H^2(\bar\Omega)$, $u_1 \in L^2(\Omega)$, find (u,v) such that for every $T \geq 0$

$$\begin{gathered} u \in L^\infty(0,T;H^2(\bar\Omega)), \quad u' \in L^\infty(0,T;L^2(\Omega)) \\ v \in L^\infty(0,T;H_0^2(\overline{\Omega})) \end{gathered} \tag{11.28}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{11.29}$$

$$\begin{aligned} (u''(t)|w) + \lambda \int_\Gamma (m\cdot\nu)\gamma_0(u'(t)w)\,d\sigma \\ + \tilde a(u,w) - ([u(t),v(t)]|w) = 0 \quad \forall w \in H^2(\bar\Omega) \end{aligned} \tag{11.30}$$

$$b(\Delta v(t)|\Delta w_1) + ([u(t),u(t)]|w_1) = 0 \quad \forall w_1 \in H_0^2(\overline{\Omega}) \tag{11.31}$$

Now there is the following existence theorem [Lagnese, 1989, p. 113]

Theorem 11.10
If $(u_0,u_1) \in H^2(\bar\Omega)\times L^2(\Omega)$ *then there exists a solution to* (11.28) - (11.31).

Due to lack of time and the remark above this theorem has not been proved.

Remark: It is still classical solutions as defined in Definition 11.4 for which the uniform stabilization will be shown.

Now an identity for the energy of a classical solution to the problem is established.

Lemma 11.11
Let (u,v) *be a classical solution to* (11.1), (11.2), (11.3), (11.22)*. Then with*

$$Y = (u'(t)|m\cdot\nabla u(t))\Big|_0^T + \frac{1}{2}\Big[(u'(t)|u(t)) + \frac{\lambda}{2}\int_\Gamma (m\cdot\nu)\gamma_0(u^2(t))\,d\sigma\Big]_0^T \tag{11.32}$$

and

$$a_\Gamma(u) = \int_\Gamma m\cdot\nu\gamma_0((\partial_{11}^2 u)^2 + (\partial_{22}^2 u)^2 + 2\mu\partial_{11}^2 u\partial_{22}^2 u + 2(1-\mu)(\partial_{12}^2 u)^2)\,d\sigma$$

the following identity holds for all $T \geq 0$

$$\begin{aligned} Y + \int_0^T \tilde{E}\,dt + \frac{3b}{4}\int_0^T\int_\Omega (\Delta v)^2\,dx\,dt + \int_0^T a(u,u)\,dt \\ = \frac{1}{2}\int_0^T\int_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma\,dt \\ - \frac{1}{2}\int_0^T a_\Gamma(u)\,dt - \frac{b}{2}\int_0^T\int_\gamma m\cdot\nu\gamma_0(\Delta v)^2\,d\sigma\,dt \\ - \lambda\int_0^T\int_\Gamma (m\cdot\nu)\gamma_0(u'(m\cdot\nabla u))\,d\sigma\,dt \\ - \int_0^T\int_\Gamma m\cdot\nu\gamma_0(\beta u(m\cdot\nabla u) + \alpha\nabla u\cdot\nabla(m\cdot\nabla u))\,d\sigma\,dt \end{aligned}$$

Proof:
Let (u,v) be a solution to (11.1), (11.2), (11.3), (11.22) with the regularity (11.12). Application of Lemma 11.5 to u with $\theta = \mu$ and integrating from 0 to T gives

$$\begin{aligned} \int_0^T\int_\Omega (m\cdot\nabla u)(u'' - [u,v])\,dx\,dt + \int_0^T a(u,u)\,dt \\ = -\frac{1}{2}\int_0^T a_\Gamma(u)\,dt - \lambda\int_0^T\int_\Gamma (m\cdot\nu)\gamma_0(u'(m\cdot\nabla u))\,d\sigma\,dt \\ - \int_0^T\int_\Gamma ((\beta(m\cdot\nu)\gamma_0 u - \alpha\gamma_\tau((m\cdot\nu)\partial_\tau u)\gamma_0(m\cdot\nabla u) \\ + \alpha(m\cdot\nu)\gamma_1 u\gamma_1(m\cdot\nabla u))\,d\sigma\,dt \end{aligned} \tag{11.33}$$

where (11.1), (11.2) and (11.22) have been used.

The first term in (11.33) is divided in two and with

$$Y_1 = (u'(t)|m \cdot \nabla u(t))\Big|_0^T$$

the first one is rewritten as

$$\begin{aligned}
\int_0^T\int_\Omega (m\cdot\nabla u)u''\,dx\,dt &= -\int_0^T\int_\Omega (m\cdot\nabla u')u'\,dx\,dt + Y_1\\
&= \int_0^T\int_\Omega u'^2\,dx\,dt - \int_0^T\int_\Omega u'^2 + (m\cdot\nabla u')u'\,dx\,dt + Y_1\\
&= \int_0^T\int_\Omega u'^2\,dx\,dt - \frac{1}{2}\int_0^T\int_\Omega \mathrm{div}(mu'^2)\,dx\,dt + Y_1\\
&= \int_0^T\int_\Omega u'^2\,dx\,dt - \frac{1}{2}\int_0^T\int_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma\,dt + Y_1
\end{aligned} \tag{11.34}$$

where Gauss' divergence theorem [Williamson et al., 1972, p. 551] has been used in the last equality.

The expression $([u,v]|m\cdot\nabla u)$ from the first term in (11.33) can, using Lemma 5.5, be rewritten as

$$([u,v]|m\cdot\nabla u) = ([m\cdot\nabla u, u]|v) \tag{11.35}$$

where

$$\begin{aligned}
[m\cdot\nabla u, u] &= \partial_{11}^2(m\cdot\nabla u)\partial_{11}^2 u + \partial_{22}^2(m\cdot\nabla u)\partial_{22}^2 u - 2\partial_{12}^2(m\cdot\nabla u)\partial_{12}^2 u\\
&= 2[u,u] + \nabla[u,u]\cdot m\\
&= \frac{1}{2}\mathrm{div}([u,u]m) + [u,u]
\end{aligned} \tag{11.36}$$

so using (11.36) and (11.1) in (11.35) gives

$$\begin{aligned}
([u,v]|m\cdot\nabla u) &= \frac{1}{2}(\mathrm{div}([u,u]m)|v) + ([u,u]|v)\\
&= -\frac{1}{2}([u,u]|m\cdot\nabla v) + ([u,u]|v)\\
&= \frac{b}{2}(\Delta^2 v|m\cdot\nabla v) - b(\Delta^2 v|v)
\end{aligned} \tag{11.37}$$

Since $v = \frac{\partial v}{\partial \nu} = 0$ on Γ, $\nabla v = 0$ on Γ because ∇v is orthogonal to both ν and τ which are orthogonal to each other. Then $\forall\, l = 1, 2 : (\tau \cdot \nabla)\partial_l v = 0$ on Γ that is $-\nu_2 \partial_1 \partial_l u + \nu_1 \partial_2 \partial_l v = 0$ on Γ, so $\nu_1 \partial_2 \partial_l v = \nu_2 \partial_1 \partial_l v$ on Γ. This is used to give that the following identity holds on Γ

$$
\begin{aligned}
\gamma_1(m \cdot \nabla v) &= \gamma_1 v + \sum_{j,k=1,2} m_k \nu_j \partial^2_{jk} v \\
&= m_1 \nu_1 \partial^2_{11} v + m_2 \nu_2 \partial^2_{22} v + m_1 \nu_2 \partial^2_{12} v + m_2 \nu_1 \partial^2_{12} v \\
&= m_1 \nu_1 \partial^2_{11} v + m_2 \nu_2 \partial^2_{22} v + m_1 \nu_1 \partial^2_{22} v + m_2 \nu_2 \partial^2_{11} v \\
&= (m \cdot \nu)\gamma_0(\Delta v)
\end{aligned}
$$

Using Lemma 11.5 with $\theta = 1$ on the first term on the right hand side of (11.37), it follows by the above discussion that

$$
(\Delta^2 v | m \cdot \nabla v) = \|\Delta v\|^2 - \frac{1}{2} \int_\Gamma m \cdot \nu \gamma_0(\Delta v)^2 \, d\sigma \tag{11.38}
$$

But since $v \in H^2_0(\overline{\Omega})$ the variational definition of Δ^2_D yields that

$$
(\Delta^2 v | v) = \|\Delta v\|^2 \tag{11.39}
$$

So substitution of (11.38) and (11.39) into (11.37) gives

$$
([u, v] | m \cdot \nabla u) = -\frac{b}{2} \|\Delta v\|^2 - \frac{b}{2} \int_\Gamma m \cdot \nu \gamma_0(\Delta v)^2 \, d\sigma \tag{11.40}
$$

From the last term in (11.33)

$$
\begin{aligned}
&\int_\Gamma \alpha\Big((m \cdot \nu)\gamma_1 u \gamma_1(m \cdot \nabla u) - \gamma_\tau((m \ \cdot \nu)\gamma_\tau u)\gamma_0(m \cdot \nabla u)\Big)\, d\sigma \\
&\qquad = \int_\Gamma \alpha\Big((m \cdot \nu)\gamma_1 u \gamma_1(m \cdot \nabla u) + (m \cdot \nu)\gamma_\tau u \gamma_\tau(m \cdot \nabla u)\Big)\, d\sigma \\
&\qquad = \int_\Gamma \alpha(m \cdot \nu)\gamma_0(\nabla u \cdot \nabla(m \cdot \nabla u))\, d\sigma
\end{aligned}
$$

which is inserted in (11.33) together with (11.34) and (11.40), and this results in

$$
\begin{aligned}
Y_1 + \int_0^T \int_\Omega u'^2 \, dx \, dt + \int_0^T a(u,u) \, dt + \frac{b}{2} \int_0^T \int_\Omega (\Delta v)^2 \, dx \, dt \\
= \frac{1}{2} \int_0^T \int_\Gamma (m \cdot \nu) \gamma_0 u'^2 \, d\sigma \, dt - \frac{1}{2} \int_0^T a_\Gamma(u) \, dt \\
- \frac{b}{2} \int_0^T \int_\Gamma m \cdot \nu \gamma_0 (\Delta v)^2 \, d\sigma \, dt \\
- \lambda \int_0^T \int_\Gamma (m \cdot \nu) \gamma_0 (u'(m \cdot \nabla u)) \, d\sigma \, dt \\
- \int_0^T \int_\Gamma m \cdot \nu \gamma_0 (\beta u (m \cdot \nabla u) + \alpha \nabla u \cdot \nabla (m \cdot \nabla u)) \, d\sigma \, dt
\end{aligned} \tag{11.41}
$$

Next applying Lemma 11.2 to $\phi = \psi = u$ with $\theta = \mu$. After integration from 0 to T and use of (11.1), (11.2) and (11.22) this may be written as

$$
\begin{aligned}
\int_0^T \int_\Omega (u'' - [u,v]) u \, dx \, dt + \int_0^T a(u,u) \, dt \\
= - \lambda \int_0^T \int_\Gamma (m \cdot \nu) \gamma_0 (u u') \, d\sigma \, dt \\
- \int_0^T \int_\Gamma (\beta (m \cdot \nu) \gamma_0 u^2 - \alpha \gamma_0 u \gamma_\tau ((m \cdot \nu) \partial_\tau u) + \alpha (m \cdot \nu)(\gamma_1 u)^2) \, d\sigma \, dt \\
= - \frac{\lambda}{2} \int_\Gamma (m \cdot \nu) \gamma_0 u^2 \, d\sigma \Big|_0^T - \int_0^T \int_\Gamma m \cdot \nu \gamma_0 (\beta u^2 + \alpha |\nabla u|^2) \, d\sigma \, dt
\end{aligned} \tag{11.42}
$$

where the first term on the left hand side can be rewritten using integration by parts, Lemma 5.5 and (11.1) as

$$\int_0^T \int_\Omega (u'' - [u,v])u \, dx \, dt$$

$$= (u'(t)|u(t))\Big|_0^T - \int_0^T \int_\Omega u'^2 \, dx \, dt - \int_0^T \int_\Omega [u,u]v \, dx \, dt \qquad (11.43)$$

$$= (u'(t)|u(t))\Big|_0^T - \int_0^T \int_\Omega u'^2 \, dx \, dt + b \int_0^T \int_\Omega (\Delta v)^2 \, dx \, dt$$

Inserting (11.43) in (11.42) then with

$$Y_2 = \Big[(u'(t)|u(t)) + \frac{\lambda}{2} \int_\Gamma (m \cdot \nu)\gamma_0 u^2 \, d\sigma\Big]_0^T$$

gives

$$Y_2 - \int_0^T \int_\Omega u'^2 \, dx \, dt + b \int_0^T \int_\Omega (\Delta v)^2 \, dx \, dt + \int_0^T a(u,u) \, dt$$
$$= - \int_0^T \int_\Gamma m \cdot \nu \gamma_0 (\beta u^2 + \alpha |\nabla u|^2) \, d\sigma \, dt \qquad (11.44)$$

Multiplying (11.44) by $\frac{1}{2}$ and adding the product to (11.41) gives the stated identity. □

Lemma 11.12
Let $u \in H^4(\bar{\Omega})$ and $u' \in H^2(\bar{\Omega})$. Then for arbitrary $\delta > 0$, $\eta > 0$ and

$\zeta > 0$ the following three inequalities hold:

$$\left|\lambda \int_\Gamma (m\cdot\nu)\gamma_0(u'(m\cdot\nabla u))\,d\sigma\right|$$
$$\leq \frac{\lambda C_1}{\delta}\int_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma + \lambda\delta\int_\Gamma (m\cdot\nu)\gamma_0|\nabla u|^2\,d\sigma\ ,$$

$$\left|\beta \int_\Gamma (m\cdot\nu)\gamma_0(u(m\cdot\nabla u))\,d\sigma\right|$$
$$\leq \frac{\beta C_2}{\eta}\int_\Gamma (m\cdot\nu)\gamma_0|\nabla u|^2\,d\sigma + \beta\eta\int_\Gamma (m\cdot\nu)\gamma_0 u^2\,d\sigma\ ,$$

$$\left|\alpha \int_\Gamma (m\cdot\nu)\gamma_0(\sum_{j,k=1,2} m_j\partial_k u\partial^2_{jk}u)\,d\sigma\right|$$
$$\leq \alpha\zeta\int_\Gamma (m\cdot\nu)\gamma_0|\nabla u|^2\,d\sigma + \frac{\alpha C_3}{\zeta}a_\Gamma(u)$$

where the C_i's are independent of the parameters δ, η and ζ.

Remark: In the proof of these inequalities Cauchy's inequality if $\epsilon > 0$, $a, b \in \mathbb{R}$ then $|ab| \leq \frac{\epsilon}{2}|a|^2 + \frac{1}{2\epsilon}|b|^2$ [Ziemer, 1989, p. 19] is used.

Proof:
The first inequality, using Cauchy's inequality gives

$$\left|\lambda \int_\Gamma (m\cdot\nu)\gamma_0(u'(m\cdot\nabla u))\,d\sigma\right|$$
$$\leq \lambda\int_\Gamma (m\cdot\nu)(\frac{1}{2\epsilon}\gamma_0 u'^2 + \frac{\epsilon}{2}\gamma_0|m\cdot\nabla u|^2)\,d\sigma \qquad \epsilon > 0$$
$$\leq \frac{\lambda C_1}{\delta}\int_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma + \lambda\delta\int_\Gamma (m\cdot\nu)\gamma_0|\nabla u|^2)\,d\sigma$$

where $\delta = \frac{\epsilon}{2}\max_{x\in\Gamma}|m(x)|$ and $C_1 \geq 4\max_{x\in\Gamma}|m(x)|$.

The second inequality is shown in the same manner.

The third inequality is also obtained using Cauchy's inequality

$$\Big|\alpha \int\limits_{\Gamma} (m\cdot\nu)\gamma_0 \Big(\sum_{j,k=1,2} m_j \partial_k u \partial_{jk}^2 u\Big)\, d\sigma\Big|$$

$$\le \alpha\frac{\epsilon c}{2}\int\limits_{\Gamma}(m\cdot\nu)\gamma_0|\nabla u|^2\, d\sigma + \frac{\alpha}{2\epsilon}\sum_{j,k=1,2}\int\limits_{\Gamma}(m\cdot\nu)\gamma_0(\partial_{ij}^2 u)^2\, d\sigma \quad \epsilon>0$$

$$\le \alpha\zeta\int\limits_{\Gamma}(m\cdot\nu)\gamma_0|\nabla u|^2\, d\sigma + \frac{\alpha C_3}{\zeta}a_\Gamma(u)$$

where $\zeta = \frac{\epsilon}{2}c = \frac{\epsilon}{2}4\max_{x\in\Gamma}|m(x)|$ and $C_3 \ge 16\max_{x\in\Gamma}|m(x)|$. □

Now the main theorem of this chapter about stability of the von Karman equations can be stated and proved.

Theorem 11.13
Assume Ω is bounded, smooth and star-shaped with respect to some point in Ω. Let u, v be a solution to (11.1), (11.2), (11.22) *with regularity* (11.12). *Then there is a number $\alpha_0 > 0$ and, for each α, a number $\beta_0(\alpha) > 0$ such that, if $0 < \alpha \le \alpha_0$, $0 < \beta \le \beta_0$ and $\lambda > 0$, the following estimates hold:*

$$\int\limits_t^\infty \tilde{E}(s)\, ds + \int\limits_t^\infty E_\Gamma(s)\, ds \le \frac{1}{\omega}\tilde{E}(t), \qquad t\ge 0 \tag{11.45}$$

$$\int\limits_t^\infty \tilde{E}(s)\, ds \le e^{-\omega t}\int\limits_0^\infty \tilde{E}(s)\, ds, \qquad t\ge 0 \tag{11.46}$$

$$\tilde{E}(t) \le e\cdot e^{-\omega t}\tilde{E}(0), \qquad t \ge \frac{1}{\omega} \tag{11.47}$$

for some $\omega > 0$, $\tilde{E}$ given by (11.25) *and*

$$E_\Gamma(t) = \frac{1}{2}\int\limits_{\Gamma}(m\cdot\nu)\gamma_0 u'^2\, d\sigma + \frac{1}{2}a_\Gamma(u) + b\int\limits_{\Gamma} m\cdot\nu\gamma_0(\Delta v)^2\, d\sigma$$

where a_Γ is as in Lemma 11.11.

Remark: The assumption regarding the smallness of α and β is for technical reasons and is probably inessential to the conclusions of the theorem [Lagnese, 1989, p. 118].

Proof:
Introduce $\rho(t)$ as

$$\rho(t) = (u'(t)|m \cdot \nabla u(t) + \frac{1}{2}u(t)) + \frac{\lambda}{2}\int_\Gamma (m \cdot \nu)\gamma_0(u^2(t))\,d\sigma \tag{11.48}$$

Then for Y as in (11.32) it is observed that

$$Y = \rho(T) - \rho(0)$$

Let

$$F_\epsilon(t) = \tilde{E}(t) + \epsilon\rho(t) \tag{11.49}$$

where $\epsilon > 0$ is suitable small and $\tilde{E}$ is defined in (11.25). This function is a Lyapunov function for the dynamical system defined in (11.11) provided it follows a solution trajectory for a classical solution. It is not obvious at this point, but will be clear later on in the proof.

From (11.23)

$$F'_\epsilon(t) = -\lambda\int_\Gamma (m \cdot \nu)\gamma_0 u'^2\,d\sigma + \epsilon\rho'(t) \tag{11.50}$$

and from the identity in Lemma 11.11 it follows that

$$\begin{aligned}
\rho'(t) = &-\frac{1}{2}\int_\Omega u'^2\,dx - \frac{3}{2}a(u,u) - b\int_\Omega (\Delta v)^2\,dx + \frac{1}{2}\int_\Gamma (m \cdot \nu)\gamma_0 u'^2\,d\sigma \\
&- \frac{1}{2}a_\Gamma(u) - \frac{\beta}{2}\int_\Gamma (m \cdot \nu)\gamma_0 u^2\,d\sigma - \frac{3\alpha}{2}\int_\Gamma m \cdot \nu\gamma_0|\nabla u|^2\,d\sigma \\
&- \frac{b}{2}\int_\Gamma (m \cdot \nu)\gamma_0(\Delta v)^2\,d\sigma - \lambda\int_\Gamma (m \cdot \nu)\gamma_0(u'(m \cdot \nabla u))\,d\sigma \\
&- \beta\int_\Gamma (m \cdot \nu)\gamma_0(u(m \cdot \nabla u))\,d\sigma \\
&- \alpha\int_\Gamma (m \cdot \nu)\gamma_0(\sum_{j,k=1,2} m_j\partial_k u\partial^2_{jk}u)\,d\sigma
\end{aligned}$$

The last three terms can be bounded from above using Lemma 11.12, which gives the estimate

$$\rho'(t) \le -\frac{1}{2}\int\limits_\Omega u'^2\,dx - \frac{3}{2}a(u,u) - b\int\limits_\Omega (\Delta v)^2\,dx - \frac{\beta}{4}\int\limits_\Gamma (m\cdot\nu)\gamma_0 u^2\,d\sigma$$
$$- \Big(\alpha\Big(\frac{3}{2}-\zeta\Big) - \frac{\beta C_2}{\eta} - \lambda\delta\Big)\int\limits_\Gamma m\cdot\nu\gamma_0|\nabla u|^2\,d\sigma - \Big(\frac{1}{2} - \frac{\alpha C_3}{\zeta}\Big)a_\Gamma(u)$$
$$- \frac{b}{2}\int\limits_\Gamma (m\cdot\nu)\gamma_0(\Delta v)^2\,d\sigma + \Big(\frac{1}{2} + \frac{\lambda C_1}{\delta}\Big)\int\limits_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma$$

where η, ζ, δ are arbitrary constants and the C_i's are independent of them.

Now let

$$\alpha_0 = \frac{1}{8C_3}$$

and choose

$$0 < \alpha \le \alpha_0, \quad 0 < \beta \le \frac{\alpha}{8C_2}, \quad \eta = \frac{1}{4}, \quad \zeta = \frac{1}{2}, \quad \delta = \frac{\alpha}{4\lambda}$$

Then

$$\beta\Big(\frac{1}{2}-\eta\Big) = \frac{\beta}{4}, \quad \alpha\Big(\frac{3}{2}-\zeta\Big) - \frac{\beta C_2}{\eta} - \lambda\delta \le \frac{\alpha}{4}, \quad \frac{1}{2} - \frac{\alpha C_3}{\zeta} \le \frac{1}{4}$$

which gives

$$\rho'(t) \le -\frac{1}{2}\int\limits_\Omega u'^2\,dx - \frac{3}{2}a(u,u) - b\int\limits_\Omega (\Delta v)^2\,dx$$
$$- \frac{1}{4}\int\limits_\Gamma (m\cdot\nu)\gamma_0(\beta u^2 + \alpha|\nabla u|^2)\,d\sigma - \frac{1}{4}a_\Gamma(u)$$
$$- \frac{b}{2}\int\limits_\Gamma (m\cdot\nu)\gamma_0(\Delta v)^2\,d\sigma + \Big(\frac{1}{2} + \frac{4\lambda^2 C_1}{\alpha}\Big)\int\limits_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma$$
$$\le -\frac{1}{2}\tilde{E}(t) - \frac{1}{4}a_\Gamma(u) + \Big(\frac{1}{2} + \frac{4\lambda^2 C_1}{\alpha}\Big)\int\limits_\Gamma (m\cdot\nu)\gamma_0 u'^2\,d\sigma$$

Inserting this in (11.50) gives

$$F'_\epsilon(t) \le -\Big(\lambda - \epsilon\Big(\frac{1}{2} + \frac{4\lambda^2 C_1}{\alpha}\Big)\Big) \int_\Gamma (m\cdot\nu)\gamma_0 u'^2\, d\sigma$$

$$- \frac{\epsilon b}{2}\int_\Gamma (m\cdot\nu)\gamma_0(\Delta v)^2\, d\sigma - \frac{\epsilon}{2}\tilde{E}(t) - \frac{\epsilon}{4}a_\Gamma(u)$$

Choose $\epsilon > 0$ such that

$$\lambda - \epsilon\Big(\frac{1}{2} + \frac{4\lambda^2 C_1}{\alpha}\Big) \ge \frac{\epsilon}{4} \tag{11.51}$$

Then

$$F'_\epsilon(t) \le -\frac{\epsilon}{2}(\tilde{E}(t) + E_\Gamma(t)) \tag{11.52}$$

for every $\epsilon > 0$, satisfying (11.51).

Now consider

$$|\rho(t)| = \big|(u'|m\cdot\nabla u + \frac{1}{2}u) + \frac{\lambda}{2}\int_\Gamma (m\cdot\nu)u^2\, d\sigma\big|$$

$$\le \|u'\|\|m\cdot\nabla u\| + \|u'\|\|u\| + \frac{\lambda}{2}\int_\Gamma (m\cdot\nu)u^2\, d\sigma$$

$$\le 2\|u'\|^2 + \|m\cdot\nabla u\|^2 + \|u\|^2 + \frac{\lambda}{2}\int_\Gamma (m\cdot\nu)u^2\, d\sigma$$

The second term on the right hand side is bounded by

$$\|m\cdot\nabla u\|^2 \le \sum_{j=1}^{2}\|m_j\partial_j u\|^2 \le c\sum_{j=1}^{2}\|\partial_j u\|^2 \le c\|u\|_2^2$$

where $c > \max_{x\in\overline{\Omega}}|m(x)|^2$.

So because $\tilde{a}$ is elliptic, by Lemma 11.9, there exists a constant $C > 0$ such that

$$|\rho(t)| \le C\tilde{E}(t)$$

Using this and (11.49) the following inequality holds

$$(1-\epsilon C)\tilde{E}(t) \le F_\epsilon(t) \le (1+\epsilon C)\tilde{E}(t)$$

where $\epsilon C < 1$.

Now F_ϵ for ϵ satisfying (11.51) is clearly a Lyapunov function since $V(x) = F_\epsilon(U(t,x_0)) = F_\epsilon(t)$ is continuous by (11.52), which also gives that $F'_\epsilon(t) \le 0$ whence $V'(x) \le 0$. As an associated function to V can $W(x) = \frac{\epsilon}{2}\tilde{E}(U(t,x_0)) = \frac{\epsilon}{2}\tilde{E}(t)$ be chosen since then $W(x) \ge 0$ by (11.25) and Lemma 11.9. From (11.52), $V'(x) \le W(x)$. Now Theorem 6.7 can be used with $f(z) = \sqrt{z}$ and $f_1(z) = \sqrt{\frac{\epsilon}{2}z}$ which both are of class M_h. Then $x = 0$ is an uniformly asymptotically stable equilibrium point, in the sense of Definition 6.2.

Further calculations are necessary to show that the solutions has exponential decay rate.

Multiplication of (11.52) by $e^{-\theta t}$ $(\theta > 0)$, followed by integration from t to ∞ gives, after an integration by parts,

$$\frac{\epsilon}{2}\int_t^\infty e^{-\theta s}\tilde{E}(s)\,ds + \frac{\epsilon}{2}\int_t^\infty e^{-\theta s}E_\Gamma(s)\,ds + \theta\int_t^\infty e^{-\theta s}F_\epsilon(s)\,ds \le e^{-\theta t}F_\epsilon(t) \le e^{-\theta t}(1+\epsilon C)\tilde{E}(t)$$

Since $F_\epsilon \ge 0$, the last term on the left hand side is dropped. Then passing to the limit as $\theta \to 0^+$ results in

$$\int_t^\infty (\tilde{E}(s) + E_\Gamma(s))\,ds \le \frac{1}{\omega}\tilde{E}(t) \tag{11.53}$$

where

$$\omega = \frac{\epsilon}{2(1+\epsilon C)}$$

This shows (11.45).

(11.46) follows from (11.53) by arguments as in the proof Gronwall's Lemma (Lemma 9.1): Let

$$c = \int_0^\infty \tilde{E}(s)\,ds \tag{11.54}$$

Then from (11.53)

$$\omega c - \omega\int_0^t \tilde{E}(s)\,ds \le \tilde{E}(t)$$

Multiplying this by the integration factor $e^{\omega t}$ gives

$$\omega e^{\omega t} c - \omega e^{\omega t} \int_0^t \tilde{E}(s)\, ds - e^{\omega t} \tilde{E}(t) \leq 0$$

$$\omega e^{\omega t} c - \frac{d}{dt}\Big(e^{\omega t} \int_0^t \tilde{E}(s)\, ds \Big) \leq 0$$

Then integration from 0 to t results in

$$e^{\omega t} c - c - e^{\omega t} \int_0^t \tilde{E}(s)\, ds \leq 0$$

$$e^{\omega t} \Big(c - \int_0^t \tilde{E}(s)\, ds \Big) \leq c$$

$$c - \int_0^t \tilde{E}(s)\, ds \leq e^{-\omega t} c$$

$$\int_t^\infty \tilde{E}(s)\, ds \leq e^{-\omega t} \int_0^\infty \tilde{E}(s)\, ds$$

where c has been reintroduced from (11.54).

From (11.23) $\tilde{E}$ is decreasing, so for every $\theta > 0$

$$\theta \tilde{E}(t+\theta) \leq \int_t^{t+\theta} \tilde{E}(s)\, ds \leq e^{-\omega t} \int_0^\infty \tilde{E}(s)\, ds \leq \frac{e^{-\omega t}}{\omega} \tilde{E}(0)$$

where (11.53) is used in the last inequality. From this

$$\tilde{E}(t+\theta) \leq \frac{e^{\omega\theta}}{\omega\theta} e^{-\omega(t+\theta)} \tilde{E}(0) \quad \theta > 0, \quad t \geq 0$$

The fraction on the right hand side has it's minimum at $\theta = \frac{1}{\omega}$, so inserting this value gives that

$$\tilde{E}(t + \frac{1}{\omega}) \leq e \cdot e^{-\omega(t+\frac{1}{\omega})} \tilde{E}(0) \qquad t \geq 0$$

which shows (11.47). □

Appendix A

Prerequisites

A.1 Homeomorphisms

The following exposition is taken from lecture notes used in the course on operators in Hilbert spaces, held at Aalborg University in 2001. It has been included because a reference to the results given here is needed.

Throughout this section the scalar field will be $\mathbb{C}$.

Definition A.1
Let V be a Banach space. The anti-dual space, V', denotes the set of all conjugate linear functionals on V.

Lemma A.2
Let H be a Hilbert space. Then

- *H' endowed with the norm $\|\phi\| = \sup\{|\langle x, \phi\rangle| \, | x \in H \, , \|x\| \le 1\}$ is a Banach space isometrically, but anti-linearly isomorphic to H^*.*
- *There exists a linear, surjective map $\Phi : H \to H'$ fulfilling*

$$\langle \Phi(x), y\rangle = (x|y)_H \quad \forall \, x, y \in H$$

- *Since $(\xi|\eta)_{H'} := (\Phi^{-1}(\xi)|\Phi^{-1}(\eta))_H$ is an inner product inducing the norm, H' is a Hilbert space, and hereby H and H' become unitarily equivalent.*

Proof:
First it is noted that the operator $\psi : H^* \to H'$, given by complex conjugation, is an anti-linear isometric isomorphism, and thus the first statement holds.

By Riesz' representation theorem there exists, for a Hilbert space H, a conjugate linear, surjective isometry, $\tilde{\Phi} : H \to H^*$. Now let $\Phi : H \to H'$ be defined by $\Phi(x) = (\psi \circ \tilde{\Phi})(x)$, which then becomes a linear surjective isometry, fulfilling

$$\langle \Phi(x), y \rangle = (x|y)_H \quad \forall\, x, y \in H$$

For the last statement, it is noted that $(\xi|\eta)_{H'} := (\Phi^{-1}(\xi)|\Phi^{-1}(\eta))_H$ is an inner product. Now since Φ is a surjective isometry, it follows for all $\phi \in H'$ that

$$\|\phi\| = \sup_{x \in H \setminus \{0\}} \frac{|(\Phi(x)|\phi)|}{\|x\|} \leq \sup_{x \in H \setminus \{0\}} \frac{\|\Phi(x)\|' \|\phi\|'}{\|x\|} = \|\phi\|'$$

where $\|\phi\|'$ denotes the norm on H', induced by the inner product.

Moreover

$$(\|\phi\|')^2 = (\phi|\phi) = |\langle \phi, \Phi^{-1}(\phi) \rangle| \leq \|\phi\| \, \|\Phi^{-1}\phi\| = \|\phi\|^2$$

Thus it follows that

$$\|\phi\| \leq \|\phi\|' \leq \|\phi\|$$

and hence H' is a Hilbert space, unitarily equivalent to H. □

Lemma A.3
Let H and H_1 be Hilbert spaces, and $T \in \mathbb{B}(H, H_1)$. Then there exists a unique $T' \in \mathbb{B}(H_1, H')$ such that

$$\langle T'x, y \rangle = (x|Ty)$$

for all $x \in H_1$ and $y \in H$.

Proof:
Let $T^* \in \mathbb{B}(H_1, H)$ be the usual Hilbert space adjoint, and let $\Phi : H \to H'$ be the surjective isometry of Lemma A.2. Then $T' = \Phi \circ T^*$ is linear, and bounded, so $T' \in \mathbb{B}(H_1, H')$.

Now by Lemma A.2

$$\langle T'x, y \rangle = \langle \Phi T^* x, y \rangle = (T^*x|y) = (x|Ty)$$

for all $x \in H_1$ and $y \in H$.

Now assume that T_1' and T_2' both satisfy the requirements of the lemma. Then

$$\langle (T_1' - T_2')x, y \rangle = (x|Ty - Ty) = 0$$

for all $x \in H_1$ and $y \in H$, so $T_1' = T_2'$. □

Definition A.4
Let V and H be Hilbert spaces. Then we write $V \overset{d}{\hookrightarrow} H$, if $V \subset H$ is dense, and if there exists a $C > 0$ fulfilling

$$\|v\|_V \geq C\|v\|_H \quad \forall\, v \in V$$

Lemma A.5
Let V and H be Hilbert spaces. If $V \overset{d}{\hookrightarrow} H$, then $H' \overset{d}{\hookrightarrow} V'$.

Proof:
This lemma is a special case of Lemma 7.2, and hence the proof of that lemma applies. □

Lemma A.6
Let V and H be Hilbert spaces, with $V \overset{d}{\hookrightarrow} H$. Then there exists a linear isometry $A : V \to V'$ such that

$$\langle Av, w \rangle = (v|w)_V \quad \forall\, v, w \in V$$

Proof:
Follows, as in the proof of Lemma A.2. □

Lemma A.7
Let V be a Hilbert space. To every sesqui-linear form $s : V \times V \to \mathbb{F}$ which is bounded, i.e. there exists a $c > 0$ such that

$$|s(u, v)| \leq c\|u\|_V \|v\|_V \quad \forall\, u, v \in V$$

there corresponds a uniquely determined $S \in \mathbb{B}(V, V')$ such that for all $u, v \in V$

$$s(u, v) = \langle Su, v \rangle \tag{A.1}$$

Proof:
Uniqueness: Assume that there exists $S_1 \in \mathbb{B}(V, V')$ and $S_2 \in \mathbb{B}(V, V')$, such that for all $u, v \in V$

$$s(u, v) = \langle S_1 u, v \rangle$$
$$s(u, v) = \langle S_2 u, v \rangle$$

Then $0 = \langle (S_1 - S_2)u, v \rangle$ for all $u, v \in V$. Thus $(S_1 - S_2)u = 0$ for all $u \in V$, so $S_1 = S_2$.

Existence: Let $\tilde{S}$ be defined as in Definition 2.15 in [Grubb, 1996b, p 2.15]. Since $s(u, v)$ is bounded, so is $\tilde{S}$. Now define $S = \Phi\tilde{S}$. Then, by Lemma A.2

$$\langle Su, v \rangle = \langle \Phi\tilde{S}u, v \rangle = (\tilde{S}u|v) = s(u, v)$$

so $S \in \mathbb{B}(V, V')$ with $\langle Su, v \rangle = s(u, v)$ for all $u, v \in V$. □

Lemma A.8
Let V and H be Hilbert spaces, with $V \overset{d}{\hookrightarrow} H$. Then

$$V \subset H \subset V'$$

Proof:
The inclusions follow by Lemma A.2 and A.5. □

Lemma A.9
Let V and H be Hilbert spaces, with $V \overset{d}{\hookrightarrow} H$. For any bounded sesquilinear form on V and its associated operator (as defined in Lemma A.7), there exists an operator T in H given by restriction:

$$D(T) = S^{-1}(H)$$
$$T = S|_{D(T)}$$

T coincides with the operator $\tilde{T}$ in H given by

$$D(\tilde{T}) = \{u \in V \mid \exists\, x \in H \ \forall v \in V : s(u, v) = (x|v)_H\}$$
$$\tilde{T}u = x$$

$\tilde{T}$ is called the operator associated with the triplet (H, V, s), or the Lax-Milgram operator adjoined to (H, V, s).

Proof:
Assume that $u \in D(T) \subseteq V$. Then there exists an $x \in V$, such that $Su = I'x$. But then it follows, for $v \in V$, that

$$s(u, v) = \langle Su, v \rangle = \langle I'x, v \rangle = (x|v)_H \tag{A.2}$$

so $u \in D(\tilde{T})$.

Now assume that $u \in D(\tilde{T})$. Then there exists an $x \in H$, such that $s(u, v) = (x|v)_H$ for all $v \in V$. But then by (A.2) $Su = I'x$, so that $u \in D(T)$. □

Definition A.10
Let V be a Hilbert space, and $s : V \times V \to \mathbb{F}$ be a bounded sesqui-linear form on V. The adjoint sesqui-linear form is given by

$$s^*(u,v) := \overline{s(v,u)}$$

s is called symmetric, if $s \equiv s^*$.

Lemma A.11
Let V be a Hilbert space, and let $s : V \times V \to \mathbb{F}$ be a bounded sesqui-linear form on V. If $\mathbb{F} = \mathbb{C}$, s is symmetric if and only if $s(u,u)$ is real for all $u \in V$.

Proof:
The lemma follows by the polarization identity. □

Lemma A.12
Let V be a Hilbert space, and let $s : V \times V \to \mathbb{F}$ be a bounded sesqui-linear form on V. Then there exists a unique $\tilde{S} \in \mathbb{B}(V, V')$, such that for all $u, v \in V$

$$s^*(u,v) = \langle \tilde{S}u, v \rangle$$

Proof:
Follows by Lemma A.7 □

Definition A.13
Let V and H be a Hilbert spaces, V a linear subspace of H, with $V \overset{d}{\hookrightarrow} H$, and let $s : V \times V \to \mathbb{F}$ be a sesqui-linear form on H, with $D(s) = V$, such that s is bounded on V. s is said to be V-elliptic, if there exists a $c_0 > 0$ such that

$$\mathrm{Re}\, s(v,v) \geq c_0 \|v\|_V^2$$

and s is called V-coercive, if there exists a $c_0 > 0$ and $k \in \mathbb{R}$ such that

$$\mathrm{Re}\, s(v,v) \geq c_0 \|v\|_V^2 - k\|v\|_H^2$$

Lemma A.14
Let (H, V, s) be a triplet, with $V \overset{d}{\hookrightarrow} H$, s bounded on V, and s V-elliptic. Then the operator associated with the triplet is a linear homeomorphism $S : V \to V'$.

Proof:
First assume, that s is V-elliptic and symmetric. Then $s(\cdot,\cdot)$ defines an

inner product on V. Since s is V-elliptic and bounded, $\sqrt{s(v,v)}$ is a norm on V, equivalent to $\|\cdot\|_V$, and it follows that V with the new structure is complete.

By definition S is then the linear isomorphism identifying V and V'.

If s is V-elliptic, but not symmetric, it follows, by (A.1), that

$$\|Sv\|_{V'} \geq c_0 \|v\|_V \quad \forall\, v \in V$$

and thereby, S is injective and has closed range. By a similar argument, it follows that S' associated with (H, V, s^*) is injective and has closed range. But then

$$\langle Su, v\rangle = s(u,v) = \overline{s^*(v,u)} = \overline{\langle S'v, u\rangle} \quad \forall\, u, v \in V \tag{A.3}$$

gives that $R(S)^\perp = \{0\}$, and thus $R(S) = V'$.

This is seen in the following way: The annihilator of $R(S)$ is

$$\begin{aligned} R(S)^\perp &= \{u \in V \,|\, \langle \phi, u\rangle = 0,\ \forall \phi \in R(S)\} \\ &= \{u \in V \,|\, \langle Sv, u\rangle = 0,\ \forall v \in V\} \end{aligned}$$

Assume $u \in R(S)^\perp$ and $u \neq 0$ then

$$0 = \langle Sv, u\rangle \quad \forall v \in V$$

Further more $\langle Sv, \tilde{u}\rangle = 0$ for all $\tilde{u} = \alpha u$ with $\alpha \in \mathbb{F}$. Then by (A.3)

$$0 = \langle Sv, \tilde{u}\rangle = \langle S'\tilde{u}, v\rangle \quad \forall v \in V$$

so $S'\tilde{u} = 0, \forall \tilde{u} = \alpha u$. The injectivity of S' implies $\tilde{u} = 0$. But then $u = 0$, which contradicts the assumption that $u \neq 0$.

It now follows, by the inverse mapping theorem [Reed and Simon, 1980, p. 83] that S^{-1} is continuous. □

A.2 Sobolev spaces

In this section, a few lemmas and definitions concerning Sobolev spaces will be stated.

Throughout this section the scalar field will be $\mathbb{C}$. Except the last two, Lemma A.29 and A.30, where the scalar field is $\mathbb{R}$.

First the different Sobolev spaces is introduced:

Definition A.15
For all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, is defined by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega)\, \forall |\alpha| \leq k\}$$

with the norm

$$\|u\|_{k,p} = \begin{cases} (\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max\limits_{|\alpha| \leq k} \|D^\alpha u\|_\infty & \text{if } p = \infty \end{cases}$$

When $p = 2$, $W^{k,p}(\Omega)$ is written $H^k(\Omega)$.

For all $s, t \in \mathbb{R}$ the anisotropic Sobolev space $H^{(s,t)}(\mathbb{R}^n)$ is defined as the space of $u \in \mathscr{S}'(\mathbb{R}^n)$, such that

$$\langle \xi \rangle^s \langle \xi' \rangle^t \hat{u}(\xi) \in L^2(\mathbb{R}^n)$$

for $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, with norm

$$\|u\|_{(s,t)} = (2\pi)^{-\frac{n}{2}} \|\langle \xi \rangle^s \langle \xi' \rangle^t \hat{u}(\xi)\|_0$$

When $t = 0$, $H^{(s,t)}(\mathbb{R}^n)$ is written $H^s(\mathbb{R}^n)$.

If Ω is an open and bounded subset of $\mathbb{R}^n$, then the anisotropic Sobolev space $H^{(s,t)}(\bar{\Omega})$ is defined by

$$H^{(s,t)}(\bar{\Omega}) = \{u \mid \exists U \in H^{(s,t)}(\mathbb{R}^n) \, : \, u = r_\Omega U\}$$

with norm

$$\|u\|_{(s,t),\Omega} = \inf\{\|U\|_{(s,t)} | u = r_\Omega U\}$$

When $t = 0$, $H^{s,t}(\bar{\Omega})$ is written $H^s(\bar{\Omega})$.

For all $s \in \mathbb{R}$ the Sobolev space $H_0^s(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, is defined by

$$H_0^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$$

with the norm on $H^s(\mathbb{R}^n)$.

Remark: It should be noted, that in [Lions and Magenes, 1972], $H_0^s(\overline{\Omega})$ is defined as the closure of $\mathscr{D}(\Omega)$ in $H^s(\bar{\Omega})$. Thus elements of $H_0^s(\overline{\Omega})$ are only defined on Ω. It can be proved that if $s \neq n + \frac{1}{2}$, $n \in \mathbb{Z}$, then the two definitions coincide, if the elements defined on Ω is identified with their extension by zero outside Ω. However, in lack of time, this lemma is left out.

Lemma A.16
For all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is a Banach space.

For all $s, t \in \mathbb{R}$, $H^{(s,t)}(\mathbb{R}^n)$ is a Hilbert space.

For all $s, t \in \mathbb{R}$, $H^{(s,t)}(\bar{\Omega})$ is a Hilbert space.

Proof:
The proof of this lemma is an easy excercise in use of the definitions, and thus it is left out. However, proofs can be found in [Adams, 1978, p. 45] and [Grubb, 1996a, p. 483f]. □

Definition A.17
Let X be a normed space and Y a subspace of X. Then the annihilator $Y^\perp$ is defined by

$$Y^\perp = \{\phi \in X^* \mid \langle \phi, y \rangle = 0\,,\, \forall y \in Y\}$$

Definition A.18
Let X be a Banach space and Y be a closed subspace of X. The quotient space X/Y denotes the set of equivalence classes under the equivalence relation $\sim$, where $x \sim y$ iff $x - y \in Y$.

Lemma A.19
Let $\Omega \subset \mathbb{R}^n$ be smooth, open and bounded and let $s \in \mathbb{R}$, then $H^s(\bar{\Omega}) \simeq H^s(\mathbb{R}^n)/H_0^s(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$.

Remark: By definition of the norm on $H^s(\bar{\Omega})$ it follows that

$$\begin{aligned}\|u\|_s &= \inf\{\|U\|_s \,|\, u = r_\Omega U\} \\ &= \inf\{\|U\|_s \,|\, U - u \in H_0^s(\overline{\mathbb{R}^n \setminus \overline{\Omega}})\} \\ &= \inf\{\|u - V\|_s \,|\, V \in H_0^s(\overline{\mathbb{R}^n \setminus \overline{\Omega}})\}\end{aligned}$$

so the norm on $H^s(\bar{\Omega})$ is exactly the quotient norm.

Proof:
Assume that $u \in H^s(\bar{\Omega})$. Then there exists a $U \in H^s(\mathbb{R}^n)$ such that $u = r_\Omega U$. But then $u - U \in H_0^s(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$, so consequently

$$u \in H^s(\mathbb{R}^n)/H_0^s(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$$

□

Lemma A.20
Let $\Omega \subset \mathbb{R}^n$ *be open, bounded and smooth, and let* $s \in \mathbb{R}$. *Then* $H^{-s}(\bar{\Omega})^* = H_0^s(\overline{\Omega})$

Remark: Special care should be taken when $s = 0$. The space $H^0(\bar{\Omega})$ is given by

$$H^0(\bar{\Omega}) = \{u \mid \exists U \in H^0(\mathbb{R}^n) : u = r_\Omega U\}$$

and the space $H_0^0(\overline{\Omega})$ is given by

$$H_0^0(\overline{\Omega}) = \{u \in H^0(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$$

If one identifies $u \in H_0^0(\overline{\Omega})$ with $r_\Omega u$, it follows that $u \in H^0(\bar{\Omega})$ and that

$$H_0^0(\overline{\Omega}) = L^2(\Omega) = L^2(\Omega)^* = H^0(\bar{\Omega})^*$$

Proof:
By [Pedersen, 1995, p. 67], the dual space of a quotient space, X/Y, can be identified with the annihilator, $Y^\perp$, of Y in X^*.

It is noted that $H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}}) \hookrightarrow H^{-s}(\mathbb{R}^n)$.

By Lemma A.19, $H^{-s}(\bar{\Omega}) \simeq H^{-s}(\mathbb{R}^n)/H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$.

By [Grubb, 1996b, p. 9.10] the dual space of $H^{-s}(\mathbb{R}^n)$ is $H^s(\mathbb{R}^n)$. Thus the dual space of $H^{-s}(\bar{\Omega})$ is given by the annihilator of $H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$ in $H^s(\mathbb{R}^n)$, i.e.

$$H^{-s}(\bar{\Omega})^* = \{\phi \in H^s(\mathbb{R}^n) \mid \langle \phi, y \rangle = 0\,,\, \forall y \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})\}$$

It should be noted, that if $y \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$ then $\operatorname{supp} y \subseteq \overline{\mathbb{R}^n \setminus \overline{\Omega}}$, so $\langle y, \phi \rangle = 0$ for all $\phi \in e_\Omega \mathscr{D}(\Omega)$ by [Grubb, 1996b, p. 6.10].

Assume that $\psi \in \{\phi \in H^s(\mathbb{R}^n) \mid \langle \phi, y \rangle = 0\,,\, \forall y \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})\}$, and that $\operatorname{supp} \psi \cap \mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset$. Then there exists an open set $\omega \subseteq \operatorname{supp} \psi \cap \mathbb{R}^n \setminus \overline{\Omega}$ such that ψ is positive on ω, and a w such that $w \equiv 1$ on ω and $\operatorname{supp} w \subset \mathbb{R}^n \setminus \overline{\Omega}$. Then $e_\omega w \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$, and

$$\langle \psi, e_\omega w \rangle = \int_{\mathbb{R}^n} \psi e_\omega w \; dx \geq \int_\omega \psi e_\omega w \; dx > 0$$

which is a contradiction, and hence $\operatorname{supp} \psi \subset \overline{\Omega}$, so $\psi \in H_0^s(\overline{\Omega})$.

Lastly assume that $\psi \in H_0^s(\overline{\Omega})$. By Lemma A.28 below, $e_\Omega \mathscr{D}(\Omega)$ is dense in $H_0^s(\overline{\Omega})$, so there exists $\psi_m \in e_\Omega \mathscr{D}(\Omega)$ such that $\psi_m \to \psi$ in $H_0^s(\overline{\Omega})$ as $m \to \infty$. Thus, for all $y \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})$

$$\langle \psi, y \rangle = \lim_{m \to \infty} \langle \psi_m, y \rangle = 0$$

and consequently $\psi \in \{\phi \in H^s(\mathbb{R}^n) \mid \langle \phi, y \rangle = 0\,, \forall y \in H_0^{-s}(\overline{\mathbb{R}^n \setminus \overline{\Omega}})\}$.

Thus $H^{-s}(\bar{\Omega})^* = H_0^s(\overline{\Omega})$. □

Lemma A.21
Let $\Omega \subset \mathbb{R}^n$ be smooth, open and bounded. Then $H_0^s(\overline{\Omega})^* = H^{-s}(\bar{\Omega})$.

Proof:
By Lemma A.16, $H^{-s}(\bar{\Omega})$ is a Hilbert space, and hence reflexive. Then by Lemma A.20, $H^{-s}(\bar{\Omega})^* = H_0^s(\overline{\Omega})$, so $H_0^s(\overline{\Omega})^* = H^{-s}(\bar{\Omega})^{**} = H^{-s}(\bar{\Omega})$. □

Lemma A.22
Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth. Then there exists a countable basis, $\{w_j\}$, for $H_0^2(\overline{\Omega})$ consisting of $\mathscr{D}(\Omega)$-functions.

Proof:
From Lemma 4.6 and its proof, the eigenvectors $\{e_k\}$ of G_2 form an orthonormal basis for $L^2(\Omega)$. Then since $L^2(\Omega)$ is dense in $H^{-2}(\bar{\Omega})$, $\{e_k\}$ is an orthonormal basis for $H^{-2}(\bar{\Omega})$ by [Pedersen, 2000, p. 40]. By Theorem 4.1 G_2 is a homeomorphism from $H^{-2}(\bar{\Omega})$ to $H_0^2(\overline{\Omega})$, and hence $\text{span}\{G_2 e_k\} = \text{span}\{e_k\}$ is dense in $H_0^2(\overline{\Omega})$.

For each $\epsilon > 0$ and each $v \in H_0^2(\overline{\Omega})$ there exists $N \in \mathbb{N}$ and $a_n \in \mathbb{R}$, $n = 1, \ldots, N$ such that

$$\|v - \sum_{n=1}^{N} a_n e_n\|_2 < \epsilon$$

since $\text{span}\{e_k\}$ is dense in $H_0^2(\overline{\Omega})$.

Moreover, for each $j \in \mathbb{N}$ and each e_k there exists a $w_k^j \in \mathscr{D}(\Omega)$ such that

$$\|e_k - w_k^j\| < 2^{-jk} \tag{A.4}$$

Now the set $\{w_k^j\}$ is considered, and it is claimed that $\{w_k^j\}$ has dense span in $H_0^2(\overline{\Omega})$.

Thus let $\epsilon > 0$ be given and let $a_1, \ldots, a_n$ be given such that

$$\|v - \sum_{n=1}^{N} a_n e_n\| < \frac{\epsilon}{2}$$

By (A.4) there exists for each $k = 1, \ldots, N$ a $j(k) \in \mathbb{N}$ such that

$$\|e_k - w_k^{j(k)}\| < \frac{\epsilon}{2N} \frac{1}{1 + |a_k|}$$

Then

$$\begin{aligned}
\|v - \sum_{n=1}^{N} a_n w_n^{j(n)}\| &\leq \|v - \sum_{n=1}^{N} a_n e_n\| + \sum_{n=1}^{N} |a_n| \|e_n - w_n^{j(n)}\| \\
&< \frac{\epsilon}{2} + \sum_{n=1}^{N} \frac{\epsilon}{2N} \frac{|a_n|}{1 + |a_n|} \\
&< \epsilon
\end{aligned}$$

so $\{w_k^j\}$ has dense span in $H_0^2(\overline{\Omega})$.

Now $(w_k^j)_{k,j \in \mathbb{N}}$ is countable, and will hereforth be denoted $(\tilde{w}_m)$, $m \in \mathbb{N}$.

$(\tilde{w}_m)$ can be assumed to be a linear independent sequence, so by an application of the Gram-Schmidt proces, there exists an orthonormal sequence (w_m) such that $\text{span}\{w_m\} = \text{span}\{\tilde{w}_m\}$. Then by [Pedersen, 1995, p. 83] $\{w_m\}$ is an orthonormal basis for $H_0^2(\overline{\Omega})$. □

Lemma A.23
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then

$$W^{k,p}(\Omega) \overset{d}{\hookrightarrow} W^{l,q}(\Omega)$$

whenever $k, l \geq 0$, $p, q < \infty$, and

$$k - \frac{n}{p} \geq l - \frac{n}{q}$$

Moreover the embedding is compact whenever the inequality is strict.

Proof:
See [Adams, 1978, pp. 95, 144] □

Definition A.24
Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $a < b$. Let X and Y be Hilbert spaces and $X \overset{d}{\hookrightarrow} Y$ and $m \in \mathbb{N}$. Then we define

$$W(a, b; m; X, Y) = \{u | u \in L^2(a, b; X), \frac{\partial^m}{\partial t^m} u \in L^2(a, b; Y)\}$$

where the derivatives $\partial_t^m u$ are in the sense of distributions.

Theorem A.25
Let B, B_1, B_2 be Banach spaces, with B_1, B_2 reflexive. If $B_1 \subset B \subset B_2$ with continuous injections, $B_1 \overset{comp}{\hookrightarrow} B$, then $W(a, b; 1; B_1, B_2) \overset{comp}{\hookrightarrow} L^2(a, b; B)$.

Proof:
See [Lions, 1969, p. 58ff]. □

Corollary A.26
$W(0, T; 1; H_0^2(\overline{\Omega}), L^2(\Omega)) \overset{d}{\hookrightarrow} L^2(0, T; L^2(\Omega))$ compactly.

Proof:
Let $B_1 = H_0^2(\overline{\Omega})$, $B = B_2 = L^2(\Omega)$. Then, since $H_0^2(\overline{\Omega}) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega)$ the corollary follows immediately from Theorem A.25. □

Lemma A.27
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then $C^\infty(\overline{\Omega})$ is dense in $H^s(\bar{\Omega})$.

Proof:
Let $u \in H^s(\bar{\Omega})$. Then there exists a $U \in H^s(\mathbb{R}^n)$ such that $u = r_\Omega U$. By [Grubb, 1996b, p. 9.7] $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, so there exists a sequence (U_m) in $\mathscr{S}(\mathbb{R}^n)$ such that $U_m \to U$ in $H^s(\mathbb{R}^n)$ as $m \to \infty$. Define $u_m = r_\Omega U_m$. Then $u_m \in C^\infty(\overline{\Omega})$ and

$$\|u_m - u\|_s \leq \|U_m - U\|_s \to 0$$

as $m \to \infty$. □

Lemma A.28
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. Then $e_\Omega \mathscr{D}(\Omega)$ is dense in $H_0^s(\overline{\Omega})$.

Proof:
By lack of time, the proof of this lemma is not written in full detail. However, an outline of the proof is:

Let a partion of unity, (ψ_j), on $\overline{\Omega}$ be given. Then for some j, $\operatorname{supp}\psi_j$ will be contained in the interior of Ω with a positive distance to Γ. Using a regularizing sequence, $\psi_j u$, can then be approximated with $e_\Omega\mathscr{D}(\Omega)$-functions. For the other ψ_k's, a diffeomorfism is used to transform the problem into one given on the half space $\mathbb{R}^n_+$. Then by a translation and regularization, $\psi_k u$ can be approximated by $\mathscr{D}(\mathbb{R}^n_+)$-functions for each k, and hence by the diffeomorfism, this also holds for the functions on $\overline{\Omega}$. □

Lemma A.29
Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth. Then $\|\cdot\|_\Delta = \|\Delta\cdot\|_0$ is a norm on $H^2_0(\overline{\Omega})$ equivalent to $\|\cdot\|_2$.

Proof:
Define

$$|u|_2 = \Big(\sum_{i,j=1,2} \|\partial^2_{ij} u\|_0^2\Big)^{\frac{1}{2}} \tag{A.5}$$

which is a seminorm on $H^2_0(\overline{\Omega})$.

Obviously

$$\frac{1}{2}|u|_2^2 \leq \|u\|_2^2$$

Because Ω is bounded, Poincaré's inequality can be applied repeatedly for $\phi \in \mathscr{D}(\Omega)$ to give

$$\|\phi\|_2^2 = \sum_{|\alpha|\leq 2} \|\partial^\alpha\phi\|_0^2 \leq c \sum_{|\alpha|=2} \|\partial^\alpha\phi\|_0^2 \leq c_1|\phi|_2^2 \tag{A.6}$$

The denseness of $\mathscr{D}(\Omega)$ in $H^2_0(\overline{\Omega})$ gives that (A.6) holds for $\phi \in H^2_0(\overline{\Omega})$, since also $|\cdot|_2$ is continuous on $H^2_0(\overline{\Omega})$, and thus there exists constants $c_2, c_3 > 0$ such that

$$c_2|u|_2 \leq \|u\|_2 \leq c_3|u|_2$$

for all $u \in H^2_0(\overline{\Omega})$.

Now Green's formula gives for all $\phi, \psi \in \mathscr{D}(\Omega)$ that

$$\begin{aligned}
\sum_{i,j=1,2} \int_\Omega \partial_{ij}^2 \phi \partial_{ij}^2 \psi \, dx &= -\int_\Omega \sum_{i,j=1,2} \partial_{iij}^3 \phi \partial_j \psi \, dx \\
&= \int_\Omega \sum_{i,j=1,2} \partial_{ii}^2 \phi \partial_{jj}^2 \psi \, dx \\
&= \int_\Omega \sum_{i=1,2} \partial_{ii}^2 \phi \sum_{j=1,2} \partial_{jj}^2 \psi \, dx \\
&= \int_\Omega \Delta\phi \Delta\psi \, dx
\end{aligned}$$

so since $\mathscr{D}(\Omega)$ is dense in $H_0^2(\overline{\Omega})$ these relations holds for all $\phi, \psi \in H_0^2(\overline{\Omega})$.

Hence $\|\cdot\|_\Delta$ is a norm, equivalent to $\|\cdot\|_2$ on $H_0^2(\overline{\Omega})$. □

Lemma A.30
Let $\Omega \subset \mathbb{R}^2$ be open, bounded and smooth. Then the map $(\cdot|\cdot)_\Delta : H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega}) \to \mathbb{R}$ given by

$$(u|v)_\Delta = \int_\Omega \Delta u \Delta v \, dx$$

is an inner product on $H_0^2(\overline{\Omega})$.

Proof:
That $(\cdot|\cdot)_\Delta$ is symmetric and linear follows by definition.

Let $u \in H_0^2(\overline{\Omega})$. Then

$$(u|u)_\Delta = \|u\|_\Delta^2 \geq 0$$

Moreover, since Ω is bounded, it follows from Lemma A.29 that $(u|u)_\Delta = 0$ if and only if $u = 0$. □

A.3 Vector valued distributions and -functions

Definition A.31
Let $I \subset \mathbb{R}$ be an open interval, and let X be a Banach space. Then the space of vector valued distributions $\mathscr{D}'(I; X)$ is the space of continuous linear maps $\mathscr{D}(I) \to X$ on $\mathscr{D}(I)$, equipped with the topology of uniform convergence on the bounded sets of $\mathscr{D}(I)$.

Definition A.32
Let $1 \leq p \leq \infty$, $I \subset \mathbb{R}$ *and let* X *be a Banach space. Then* $L^p(I, X)$ *denotes the space of weakly Lebesgue measurable functions on* I *with values in* X, *such that when* $1 \leq p < \infty$

$$\Big(\int_I ||f(t)||_X^p \, dt\Big)^{1/p} < \infty \tag{A.7}$$

and

$$\operatorname*{ess\,sup}_{t \in I} ||f(t)||_X < \infty \tag{A.8}$$

when $p = \infty$.

The following two lemmas are easy extensions from the case of real- and complex valued L^p-spaces.

Lemma A.33
Let X *be a Banach space and let* $I \subset \mathbb{R}$. *Then* $L^p(I; X)$ *is a Banach space, when* $1 \leq p \leq \infty$, *with the norm given by*

$$\|u\|_{L^p(I;X)} = \begin{cases} \big(\int_I ||f(t)||_X^p \, dt\big)^{1/p} & \text{if } p < \infty \\ \operatorname*{ess\,sup}_{t \in I} ||f(t)||_X & \text{if } p = \infty \end{cases}$$

Moreover if X *is a Hilbert space, then* $L^2(I; X)$ *is a Hilbert space, the inner product being given by*

$$(u|v)_{L^2(I;X)} = \int_I (u(t)|v(t))_X \, dt \tag{A.9}$$

Lemma A.34
Let X *be a Banach space, and let* $I \subset \mathbb{R}$. *Then* $L^p(I; X) \subset L^q(I; X)$ *whenever* $p > q$ *and* I *has finite measure. Consequently* $L^p_{loc}(I; X) \subset L^q_{loc}(I; X)$ *whenever* $p > q$.

To give an identification of vector valued functions as vector valued distribution, as done for real- or complex valued functions, the following lemma is included:

Lemma A.35
Let X *be a Banach space, and let* $I \subset \mathbb{R}$ *be open. Then* $L^1_{loc}(I; X) \subset \mathscr{D}'(I; X)$, *the identification being given by the identity (in X)*

$$\langle f, \phi \rangle = \int_I f(t)\phi(t) \, dt$$

for $f \in L^1_{loc}(I;X)$ *and for all* $\phi \in \mathscr{D}(I)$.

Proof:
Let a sequence of sets, (K_j), be given, such that

$$K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \dots \subset \overset{\circ}{K}_j \subset K_j \subset \dots$$

and

$$\bigcup_{j \in \mathbb{N}} \overset{\circ}{K}_j = I$$

and define the functional

$$\Lambda_f : \phi \mapsto \int_I f\phi \, dx$$

Then by [Grubb, 1996b, p. 5.3], Λ_f is continuous if and only if

$$\Lambda_f : C^\infty_{K_j}(I) \to X$$

is continuous for all $j \in \mathbb{N}$.

But since $f \in L^1_{\text{loc}}(I;X)$,

$$\|\Lambda_f(\phi)\|_X = \Big\| \int_{K_j} f(x)\phi(x) \, dx \Big\|_X \leq \sup_{K_j} |\phi| \|f\|_{L^1(I;X)}$$

□

Definition A.36
Let X *be a Banach space, and let* $u \in \mathscr{D}'(I;X)$. *Then* $\partial_t : \mathscr{D}'(I;X) \to \mathscr{D}'(I;X)$ *is defined by*

$$\langle \partial_t f, \phi \rangle = -\langle f, \partial_t \phi \rangle$$

Lemma A.37
$\partial_t : f \to \partial_t f$ *is continuous on* $\mathscr{D}'(I;X)$.

Proof:
The proof follows exactly as for $\mathscr{D}'(I)$. □

Next a theorem follows, which allows one to make sense of the integral of a vector valued function:

Theorem A.38
Let X be a reflexive Banach space, and let $f \in L^1(I;X)$. Then there exists a unique element $x \in X$ such that

$$\langle x, \phi \rangle = \int_I \langle f(t), \phi \rangle \, dt \tag{A.10}$$

for all $\phi \in X^*$.

Proof:
Let $\phi \in X^*$ be arbitrary. Then it is noted, since $f(t)$ is t-measurable and ϕ is continuous, that

$$\langle f(t), \phi \rangle$$

is t-measurable.

Moreover

$$|\langle f(t), \phi \rangle| \leq \|f(t)\|_X \|\phi\|_{X^*}$$

so, since $f \in L^1(I;X)$,

$$\Big| \int_I \langle f(t), \phi \rangle \, dt \Big| \leq \|f\|_{L^1(I;X)} \|\phi\|_{X^*}$$

and hence

$$\phi \mapsto \int_I \langle f(t), \phi \rangle \, dt$$

is a continuous linear functional on X^*, and hence there exists an $x \in X^{**} = X$, such that

$$\langle x, \phi \rangle = \int_I \langle f(t), \phi \rangle \, dt$$

Now assume that x and y both fulfil (A.10). Then for all $\phi \in X^*$,

$$\langle x - y, \phi \rangle = \int_I \langle f(t) - f(t), \phi \rangle \, dt = 0$$

so $x = y$, and hence x is uniquely determined. □

This theorem gives a way of defining the integral of a vector valued function:

Definition A.39
Let X be a reflexive Banach space, and let $f \in L^1(I;X)$. Then

$$x = \int_I f(t)\,dt$$

is defined as the unique element of X, such that (A.10) *holds.*

Remark: By definition of the integral of f, when $f \in L^1(I;X)$, it follows that

$$\langle \int_I f\,dt, \eta \rangle = \int_I \langle f(t), \eta \rangle\,dt \tag{A.11}$$

for all $\eta \in X^*$. This identity is known as Bochner's identity.

The last theorem is a version of Lebesgue's theorem on dominated convergence, adapted to vector valued functions:

Theorem A.40
Let X be a reflexive Banach space, let $I \subset \mathbb{R}$ and let f be weakly Lebesgue measurable function defined on I. Furthermore let $f_n \in L^1(I;X)$ be a sequence of functions such that $f_n(t) \to f(t)$ a.e. on I. If there exists a function $K \in L^1(I)$, such that

$$||f_n(t)||_X \leq K(t) \tag{A.12}$$

a.e. on I for all n, then $f \in L^1(I;X)$ and

$$\lim_{n\to\infty} \int_I f_n(t)\,dt \to \int_I f(t)\,dt$$

in norm on X.

Proof:
The proof follows the lines of Lebesgue's theorem on dominated convergence, [Rudin, 1987, p. 26] □

A.4 Ordinary Differential Equations

Theorem A.41
Let $G \subset \mathbb{R}^m$ be open and let $f : \,]t_0 - \epsilon, t_0 + \epsilon[\, \times G \to \mathbb{R}^m$ be given such that for $i = 1, \ldots m$, $f_i(\cdot, x) \in L^1(]\,t_0 - \epsilon, t_0 + \epsilon\,[)$ for all $x \in G$ and $f_i(t, \cdot)$

is continuous for all $t \in \,]\, t_0 - \delta, t_0 + \delta)$. *Then the initial value problem.*

$$\frac{\partial x}{\partial t} = f(t,x) \tag{A.13}$$

$$x(t_0) = x_0 \tag{A.14}$$

has a solution on $]\, t_0 - \delta, t_0 + \delta\, [$ *if and only if there exists an absolutely continuous function* $x(t)$ *on* $]\, t_0 - \delta, t_0 + \delta\, [$ *that satisfies*

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau \tag{A.15}$$

for all $t \in \,]\, t_0 - \delta, t_0 + \delta\, [$.

This can be proved as in [Jensen, 1993], by using [Hartman and Mikusiński, 1961, p. 99] to give absolute continuity of the solution.

Theorem A.42
Let $G \subset \mathbb{R}^m$ *be open,* $G_1 = \,]\, t_0 - \epsilon, t_0 + \epsilon\, [\, \times G$ *and let* $f : G_1 \to \mathbb{R}^m$ *be such that for all* $i = 1, \ldots, m$, $f_i(\cdot, x) \in L^1(]\, t_0 - \epsilon, t_0 + \epsilon\, [)$ *and* $f_i(t, \cdot)$ *is continuous. Assume that for all* $i = 1, \ldots, m$, f_i *is Lipschitz continuous, i.e.*

$$\|f_i(t,x) - f_i(t,y)\| \leq C\|x - y\|$$

for $(t,x), (t,y) \in G_1$ a.e. *Then there exists a* $\delta > 0$ *such that* (A.15) *has a unique solution on* $]\, t_0 - \delta, t_0 + \delta\, [$.

The proof follows as in [Jensen, 1993] or [Khalil, 1996, p. 68ff], where the integrability is ensured by $f(\cdot, g)$ being integrable and $|\int_{t_0}^{t} g(\tau)\, d\tau| \leq \int_{t_0}^{t} |g(\tau)|\, d\tau = \|g\|_{L^1}$ is used instead of supremum.

Definition A.43
Assume that $x(t)$ *is a solution on* $]\, a, b\, [$ *to* $x' = f(t,x)$ *and that* $y(t)$ *is a solution on* $]\, \alpha, \beta\, [$ *to* $x' = f(t,x)$. *If* $]\, a, b\, [\, \subset \,]\, \alpha, \beta\, [$ *and* $y|_{]a,b[} = x$, *then the solution* $y(t)$ *is called an extension of* $x(t)$.

If furthermore $]\, a, b\, [\, \subset \,]\, \alpha, \beta\, [$ *and* $]\, a, b\, [\, \neq \,]\, \alpha, \beta\, [$, *then* $y(t)$ *is said to be a proper extension of* $x(t)$.

Definition A.44
If $x(t)$ *is a solution to* $x' = f(t,x)$, *and* x *has no proper extension, then* $x(t)$ *is said to be a maximal solution.*

Theorem A.45
Let $G_1 = \,]\,a, b\,[\, \times G$ *and let* $f : G_1 \to \mathbb{R}^m$*. If* $x(t)$ *is a maximal solution to* $x' = f(t, x)$*, defined on* $]\,t_0, t_1\,[$*, then for all compact sets* $K \subset G_1$ *there exists a* $\delta > 0$ *such that* $(t, x(t)) \notin K$*, whenever* $t - t_0 < \delta$ *or* $t_1 - t < \delta$*.*

The proof follows as a variant of the proof given in [Khalil, 1996, p. 77].

Appendix B

Nomenclature

Numbers in brackets refer to the page, on which they are defined.

Symbol		Description
		General Notation
X_w	:	The vector space X equipped with the weak topology.
$C_0^m(\Omega)$	:	The space of m times continuously differential functions with compact support.
$C^\infty(\overline{\Omega})$	:	The space of infinitely differentiable functions, defined on Ω, such that there exists a C^∞-extension to $\overline{\Omega}$.
$\mathscr{D}(\Omega)$	:	The space of $C_0^\infty(\Omega)$ functions with the limes Frechet topology.
$\mathscr{S}(\mathbb{R}^n)$	:	The space of Schwartz functions.
$\mathscr{D}'(\Omega)$	:	The dual space of $\mathscr{D}(\Omega)$.
$\mathscr{E}'(\Omega)$	:	The dual space of $C^\infty(\Omega)$.
$\mathscr{S}'(\mathbb{R}^n)$	:	The dual space of $\mathscr{S}(\mathbb{R}^n)$.
$\mathscr{D}'(a,b;X)$	:	The space of bounded linear operators from $\mathscr{D}(\,]a,b[\,)$ to X.
$L^p(a,b;X)$	:	The spaces of L^p-functions with values in X, (173).
$W^{k,p}(\Omega)$	:	The L^p-based Sobolev space of positive integer order k, (165).
$H^s(\mathbb{R}^n)$	:	The L^2-based Sobolev space of order $s \in \mathbb{R}$, (165).
$H^{(s,t)}(\mathbb{R}^n)$	:	The anisotropic Sobolev spaces of order $(s,t) \in \mathbb{R} \times \mathbb{R}$, (165).

Continued...

Symbol		Description
$H^s(\bar{\Omega})$	:	The space of distributions on Ω, which are restrictions of elements of $H^s(\mathbb{R}^n)$, (165).
$H^{(s,t)}(\bar{\Omega})$	:	The space of distributions on Ω, which are restrictions of elements of $H^{(s,t)}(\mathbb{R}^n)$, (165).
$H_0^s(\overline{\Omega})$	:	The space of elements of $H^s(\mathbb{R}^n)$, which are supported in $\overline{\Omega}$, (165).
$(\cdot\|\cdot)_\Delta$	:	An inner product on $H_0^2(\overline{\Omega})$, (172).
$(\cdot\|\cdot)_s$	:	Inner product on $H^s(\bar{\Omega})$. If $s = 0$, s may be left out.
$\\|\cdot\\|_\Delta$	:	The equivalent norm on $H_0^2(\overline{\Omega})$ given by $\\|\Delta\cdot\\|_0$, (171).
$\\|\cdot\\|_{s,p}$	:	Norm on $W^{s,p}(\bar{\Omega})$, (165).
$\\|\cdot\\|_s$	:	Norm on $H^s(\bar{\Omega})$, (165).
$\\|\cdot\\|_{(m,s)}$	:	Norm on $H^{(m,s)}(\mathbb{R}^n)$, (165).
$\\|\cdot\\|_{(m,s),\Omega}$	:	Norm on $H^{(m,s)}(\bar{\Omega})$, (165).
Δ	:	The Laplacian.
Δ^2	:	The biharmonic operator.
Δ_D^2	:	The Dirichlet realization of the biharmonic operator.
G_2	:	The inverse of the Dirichlet realization of the biharmonic operator, (58).
$\mathrm{OP}(p(x,\xi))$	:	The pseudo-differential operator with symbol $p(x,\xi)$.
γ_0	:	The first Dirichlet trace.
γ_1	:	The second Dirichlet trace.
B_1	:	A boundary operator, (12).
B_2	:	A boundary operator, (12).
$[\cdot,\cdot]$	:	The Monge-Ampére form, (65).
$M_1(\cdot,\cdot)$	:	A bilinear form related to the Monge-Amére form, (65).
$M_2(\cdot,\cdot)$	:	A bilinear form related to the Monge-Amére form, (65).
ν	:	The unit exterior normal vector field.
τ	:	A tangential vector field.
$U \hookrightarrow V$	:	$U \subset V$ with a continuous embedding.
$U \overset{\mathrm{d}}{\hookrightarrow} V$	:	U is a dense subset of V, and U has a continuous embedding in V, (161).
$U \overset{\mathrm{comp}}{\hookrightarrow} V$	:	$U \overset{\mathrm{d}}{\hookrightarrow} V$ with a compact embedding.
(ρ,ω)	:	Polar coordinates in $\mathbb{R}^n$.
$x_n \rightharpoonup x$	:	x_n converges weakly to x.

Continued...

Symbol		Description
		Elliptic Boundary Value Problems
$\Psi^m(\hat{\Omega})$	:	The class of pseudo-differential operators of order m on $\hat{\Omega}$, (35).
$\Psi^m_{\mathrm{phg}}(\Gamma)$	:	The pseudo-differential operators of order m on $\hat{\Omega}$ with polyhomogeneous symbols, (44).
A	:	An elliptic differential operator.
$\tilde{A}$	:	A parametrix to A.
A_j	:	Differential boundary operators, (38).
A^c	:	A boundary potential, (38).
B	:	A system of elliptic differential boundary operators.
P	:	The Calderón projector, (40).
S	:	A pseudo-differential operator, (44).
S'	:	A pseudo-differential operator, (44).
S''	:	A pseudo-differential operator, (44).
R_1	:	A negligible operator, (37).
R_2	:	A negligible operator, (37).
p	:	The symbol of P.
$\begin{pmatrix} T_1 & T_2 \end{pmatrix}$	:	A parametrix to $\binom{A}{B}$, (46).
$\langle \cdot, \cdot \rangle_s$	:	Duality between a Sobolev space of order s and its dual.
$\mathrm{Ker}(T)$	:	The kernel of T.
$\mathrm{Ran}(T)$	:	The range of T.
$\mathrm{Coker}(T)$	:	The co-kernel of T.
$\mathrm{Ind}(T)$	:	The index of the Fredholm operator T.
$S \equiv T$	:	S is an approximation of T, (36).
$\hat{\Omega}$	:	An open set containing Ω, (36).
Γ	:	The boundary of Ω.
e_Ω	:	An extension operator from Ω to $\hat{\Omega}$, (38).
γ	:	The m first Dirichlet traces.
δ	:	A boundary measure, (37).
μ	:	The maximal transversal order of the boundary operators, (34).
		Dynamical Systems and Stability
$U(t,x)$	:	A dynamical system, (71).
V	:	A Lyapunov function, (72).
W	:	A function associated to a Lyapunov function, (72).
$E(t)$	:	The energi of the dynamical system, (75).

Continued...

Symbol		Description
		Continuity of Weak Solutions
$C(\overline{I}, X_w)$	:	The space of weakly continuous functions with values in X.
$C(\overline{I}, X)$	:	The space of norm continuous functions with values in X.
		Uniqueness of Weak Solutions
$\{\lambda_i\}$	:	The set of eigenvalues for Δ_D^2, (62).
$\{e_i\}$	:	The set of eigenvectors for Δ_D^2, (62).
P_N	:	The projector in $L^2(\Omega)$ on span$\{e_1, \ldots, e_N\}$.
		Stationary von Karman Equations
B	:	An operator on $H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$, (116).
C	:	An operator on $H_0^2(\overline{\Omega})$, (118).
θ_0	:	A function in $H^2(\bar{\Omega})$, (119).
Λ	:	An operator on $H_0^2(\overline{\Omega})$, (119).
j	:	A non-linear functional on $H_0^2(\overline{\Omega})$, (120).
		Boundary Stabilization of von Karman Plates
∂_τ	:	Differential operator in tangential direction, (130).
γ_τ	:	Restriction of ∂_τ to Γ, (130).
μ	:	Poisson's ratio - a constant $0 < \mu < \frac{1}{2}$.
$a(u,v)$	:	Bilinear form on $H^2(\bar{\Omega})$, (131).
$\tilde{a}(u,v)$	:	Bilinear form on $H^2(\bar{\Omega})$, (141).
$E(t)$	:	Energy function for the von Karman equations, (138).
g_1, g_2	:	Controls through which the von Karman equations are stabilized, (140).
$\tilde{E}(t)$	:	Augmented energy of the von Karman equations, (141).

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