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On Weak Solutions of von Karmans Equations

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Synopsis

This report gives an account of the proof of existence of weak solutions of the von Karman equations presented by Lions, and the proof of uniqueness presented by Boutet de Monvel and Chueshov. The equations consists of a hyperbolic and an elliptic partial differential equation.
The existence and uniqueness of weak solutions are shown in the vector distribution sense.
The foundation for showing the main results is established in the first three chapters through a number of statements and an introduction to vector distributions and vector valued functions.

Preface

The present report is the result of a project in applied mathematical analysis on the MAT6-term, 2001 at Aalborg University.

The theme of the project is partial differential equations and distribution theory. Vector valued functions and vector distributions are introduced as an aid to verify the proofs of existence and uniqueness of weak solutions of the Karman equations presented in [7] and [8].

The notation used in this report is described in Chapter 1, which also contains a presentation of the von Karman equations, and a definition of weak solutions.

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Contents

Chapter 1

Introduction

The aim of this report is to give an account of the proofs of existence and uniqueness of weak solutions of von Karmans equations, given in [7] and [8] respectively. This chapter contains a short summary of the report, a description of the notation used in the report and presents the von Karman equations and the definition of a weak solution of von Karmans equations.

Chapter 2 contains within reasonable limits the definitions, lemmas and theorems used in Chapter 3,4 and 5.

Chapter 3 presents a proof of continuity of a weak solution to the von Karman equations under the assumption that the other conditions of being a weak solution to the von Karman equations are satisfied.

In Chapter 4 the existence of weak solutions to the von Karman equations is shown, by use of the model presented in Chapter 1, Section 4 in [7] by J.L. Lions.

Chapter 5 contains a proof of uniqueness of the weak solutions to the von Karman equations. The proof follows the model presented in the article [8] by Anne Boutet de Monvel and Igor Chueshov.

1.1 Notation

The notation used in this report is primarily the same as the notation used in [3] and [4]. One exception is the L^p-spaces which is written with an upper index instead of a lower index.

When nothing else is assumed, then $\Omega \in \mathbb{R}^n$ is open, bounded and has a smooth boundary $\partial\Omega$.

The Sobolev space $H^s(\Omega)$ and its norm is for $s \in \mathbb{R}$ defined by

$$\begin{aligned} H^s(\Omega) &= \{u \in \mathscr{D}'(\Omega) \mid u = r_\Omega U \text{ for some } U \in H^s(\mathbb{R}^n)\} \\ \|u\|_{H^s(\Omega)} &= \inf\{\|U\|_{H^s(\mathbb{R}^n)} \mid u = r_\Omega U\} \end{aligned} \tag{1.1}$$

The norm on $H^s(\Omega) \cap H^t(\Omega)$ for $t, s \in \mathbb{R}$ is defined by

$$\|\cdot\|_{H^s(\Omega)\cap H^t(\Omega)} = \|\cdot\|_{H^s(\Omega)} + \|\cdot\|_{H^t(\Omega)} \tag{1.2}$$

Let u be a function on $[0, T] \times \Omega$, then $r_0 u(t, \cdot) = u(0, \cdot)$, i.e. the restriction to $t = 0$, and $r_1 u(t, \cdot) = u'(0, \cdot)$, i.e. the restriction to $t = 0$ of the derivative of u with respect to t.

It is assumed that all Hilbert spaces are separable.

In the report a lot of positive constants $C_i > 0$ for $i \in \mathbb{N}$ are used. The index symbolises that C_i and C_j might not be equal for $i \neq j$. The numbering is started from $i = 1$ within each theorem, lemma or proof.

The von Karman bracket defined below plays a central role in the von Karman equations.

Definition 1.1
Let $u, v \in H^2(\Omega)$, then the von Karman brackets are defined by

$$[u, v] = D_1^2 u D_2^2 v + D_2^2 u D_1^2 v - 2D_{12}^2 u D_{12}^2 v. \tag{1.3}$$

as an element of $\mathscr{D}'(\Omega)$.

The space of vector distributions considered is $\mathscr{D}'(0, T; H^{-2}(\Omega))$, which consists of all bounded linear operators from $C_0^\infty(]0, T[)$ into $H^{-2}(\Omega)$.

Any other notation used in this report is either commonly used, or it is explained in the text.

1.2 The von Karman Equations

The equations considered in this report, which are shown below, are simplifications of the original von Karman equations.

$$u''(t, x) + \Delta^2 u(t, x) - [u(t, x), v(t, x)] = f(t, x) \qquad \text{on} \quad]0, T[\times \Omega \tag{1.4}$$

$$\Delta^2 v(t, x) + [u(t, x), u(t, x)] = 0 \qquad \text{on} \quad]0, T[\times \Omega. \tag{1.5}$$

These equations are considered together with the following boundary and initial conditions for $t \in]0,T[$ and for $x \in \Omega$

$$\gamma_0 u(t,x) = \gamma_0 v(t,x) = 0 \tag{1.6}$$
$$\gamma_1 u(t,x) = \gamma_1 v(t,x) = 0 \tag{1.7}$$
$$r_0 u(t,x) = u_{01}(x) \tag{1.8}$$
$$r_1 u'(t,x) = u_{11}(x). \tag{1.9}$$

The existence and uniqueness of a so called weak solution to the von Karman equations are investigated with the following conditions for the initial data. For $Q =]0,T[\times \Omega$

$$\begin{aligned} f(t,x) &\in L^2(Q) \\ u_{01}(x) &\in H_0^2(\Omega) \\ u_{11}(x) &\in L^2(\Omega). \end{aligned} \tag{1.10}$$

1.3 The Concept of Weak Solutions

The definition of a weak solution to the von Karman equations is the one used in [7].

Definition 1.2
A weak solution to the problem (1.4)-(1.9) *on* $]0,T[\times \Omega$ *are functions* $u(t,x)$ *and* $v(t,x)$ *satisfying*

$$u(t,x) \in L^\infty(0,T; H_0^2(\Omega)) \tag{1.11}$$
$$u'(t,x) \in L^\infty(0,T; L^2(\Omega)) \tag{1.12}$$
$$v(t,x) \in L^\infty(0,T; H_0^2(\Omega)) \tag{1.13}$$

and the following conditions

1. *The equations* (1.4) *and* (1.5) *are satisfied in the vector distribution sense, i.e. they are satisfied in* $\mathscr{D}'(]0,T[; H^{-2}(\Omega))$.
2. *The conditions* (1.6)-(1.9) *are satisfied.*
3. *The functions* $u(t,x)$ *and* $u'(t,x)$ *depends continuously on* t *in the norm topology on* $H_0^2(\Omega)$ *and* $L^2(\Omega)$ *respectively.*

Chapter 2

Preliminaries

This chapter presents a number of definitions, theorems and lemmas, which are necessary to obtain the results in the rest of the report.

2.1 Properties of some Hilbert Spaces

Lemma 2.1
Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function in $\mathbb{R}^n$, that for some $\rho > 0$ satisfies

$$(P(x)|x) \geq 0 \quad \text{for all } x \text{ with } |x| = \rho. \tag{2.1}$$

Then there exists an $x \in \overline{B}(0,\rho)$ for which $P(x) = 0$.

Proof:
Let $P(x)$ be a continuous function in $\mathbb{R}^n$ satisfying (2.1) for some $\rho > 0$. Assume that $P(x) \neq 0$ on $\overline{B}(0,\rho)$. Consider the continuous function

$$x \to -\frac{\rho}{|P(x)|}P(x) \tag{2.2}$$

which is well defined as a map from $\overline{B}(0,\rho)$ into $\overline{B}(0,\rho)$. The ball $\overline{B}(0,\rho)$ is compact and convex, so Brouwer's Fix Point Theorem assures the existence of an x for which

$$x = -\frac{\rho}{|P(x)|}P(x). \tag{2.3}$$

This x satisfies $|x| = \rho$. Moreover

$$(P(x)|x) = -\frac{\rho}{|P(x)|}(P(x)|P(x)) = -\rho|P(x)| < 0 \tag{2.4}$$

which contradicts that $(P(x)|x) \geq 0$ for all x with $|x| = \rho$. Hence the assumption $P(x) \neq 0$ on $\overline{B}(0,\rho)$ is wrong. Hereby the lemma is proved. □

Lemma 2.2
Let H be a Hilbert space, and let $v_n \rightharpoonup v$ on H. Then the sequence (v_n) is bounded in H.

Proof:
The sequence $(\cdot|v_n)_H : H \to \mathbb{C}$ is a set of bounded linear operators. Fix $y \in H$. Since $(y|v_n)_H \to (y|v)_H$ in $\mathbb{C}$, then $\{(y|v_n)_H\}$ is a bounded set in $\mathbb{C}$. By the Principle of Uniform Boundedness [10, p. 31]

$$\sup_{n\in\mathbb{N}} \|v_n\|_H = \sup_{n\in\mathbb{N}} \|v_n\|_{H^*} = \sup_{n\in\mathbb{N}} \|(\cdot \mid v_n)_H\|_{\mathbb{C}} < \infty. \tag{2.5}$$

Hence (v_n) is bounded in H. □

Lemma 2.3
Let H be a Hilbert space. If the sequence $v_n \to v$ in H and the sequence $u_n \rightharpoonup u$ in H, then $(u_n|v_n)_H \to (u|v)_H$.

Proof:
By Cauchy-Schwarz inequality

$$\begin{aligned}|(u_n|v_n)_H - (u|v)_H| &= |(u_n - u|v_n - v)_H + (u_n - u|v)_H + (u|v_n - v)_H| \\ &\leq \|u_n - u\|_H \|v_n - v\|_H + |(u_n - u|v)_H| + |\overline{(v_n - v|u)_H}|\end{aligned}$$

where the right hand side tends to 0. Indeed, the first term tends to 0 since $\|u_n - u\|_H \leq \|u_n\|_H + \|u\|_H$ and (u_n) is bounded (Lemma 2.2), and the last two terms because $(u_n - u)$ and $(v_n - v)$ both tend weakly to 0. □

2.2 The Dirichlet Realisation of Δ^2

In the next theorem the existence and boundedness of the inverse of the Dirichlet realisation Δ_D^2 of Δ^2 is shown using the known result, that the inverse of the Dirichlet realisation Δ_D of Δ is bounded, which is stated without proof. The domain of Δ_D is $H_0^1(\Omega) \cap H^2(\Omega)$ [2, p. 317].

Theorem 2.4
Let $\Omega \subseteq \mathbb{R}^n$ be open. Then

1. *The Dirichlet realisation Δ_D of the operator Δ obtained from the triple $(L^2(\Omega)$, $H_0^1(\Omega), s(u,v))$ with*

$$s(u,v) = \sum_{j=1}^{n} (D_j u|D_j v)_{L^2(\Omega)} \tag{2.6}$$

has a bounded inverse. Indeed, the domain of the realisation Δ_D is $H_0^1(\Omega) \cap H^2(\Omega)$. Then for $u \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists a $C > 0$, so

$$\|u\|_{H_0^1(\Omega)\cap H^2(\Omega)} \leq C\|\Delta_D u\|_{L^2(\Omega)}. \tag{2.7}$$

2. *The Dirichlet realisation Δ_D^2 of the operator Δ^2 obtained from the triple $(L^2(\Omega),\, H_0^2(\Omega), a(u,v))$ with*

$$a(u,v) = (\Delta_D u|\Delta_D v)_{L^2(\Omega)} \tag{2.8}$$

has a bounded inverse. Indeed, the domain of the realisation Δ_D^2 is $H_0^2(\Omega) \cap H^4(\Omega)$.

Proof of part 2:
It is assumed that the first part of the theorem is shown.
Let $v \in D(\Delta_D)$, then

$$\begin{aligned} \text{Re } a(v,v) &= \int_\Omega |\Delta_D v|^2 dx \\ &= \|\Delta_D v\|_{L^2(\Omega)}^2 \\ &\geq C_1\|v\|_{H_0^1(\Omega)\cap H^2(\Omega)} \\ &\geq C_1\|v\|_{H_0^2(\Omega)}. \end{aligned} \tag{2.9}$$

According to Lax-Milgram's Lemma Δ_D^2 has a bounded inverse G_2 [3, p. 2.16].
If $u \in H_0^2(\Omega) \cap H^4(\Omega)$ and $v \in H_0^2(\Omega)$ then by partial integration

$$\left(\Delta^2 u|v\right)_{L^2(\Omega)} = (\Delta u|\Delta v)_{L^2(\Omega)} = a(u,v). \tag{2.10}$$

Hence $u \in D(\Delta_D^2)$. On the other hand $D(\Delta_D^2) \subset H_0^2(\Omega)$, according to the definition of the realisation [3, Section 2.5], also $D(\Delta_D^2) \subseteq H^4(\Omega)$, so $D(\Delta_D^2) = H_0^2(\Omega) \cap H^4(\Omega)$.
□

The anti-dual space V' of a vector space V is the set of all anti-linear functionals on V. The anti-dual space is isometric isomorphic to the dual space, V^* of V.

Theorem 2.5
Let H and V be Hilbert spaces, let V be densely injected in H, and let s be a sesqui-linear form on V, that is V-elliptic. Then the associated operator is a linear homeomorphism $S : V \to V'$.

When s is symmetric Theorem 2.5 is shown by giving V a new Hilbert space structure using the norm $\sqrt{s(v,v)}$, because then S is a linear isometry that identifies V and V'.

In the non-symmetric case, it is used that S and $\tilde{S}$ are injective and has closed range, where $\tilde{S}$ is the operator related to the adjoint sesqui-linear form. Now the identity

$$\langle Su, v\rangle = \overline{s^*(u,v)} = \overline{\langle \tilde{S}v, u\rangle} \quad \text{for} \quad u, v \in V \tag{2.11}$$

gives that $R(S)^{\perp} = \{0\}$, hence $R(S) = V$, and S^{-1} is continuous.

Theorem 2.5 is used on the triple $(L^2(\Omega),\ H_0^2(\Omega), a(u,v))$ defined in Theorem 2.4. Therefore Δ_D^2 is a homeomorphism from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$ since V' is isometric isomorph to V^*. A more general statement which contains this result is presented below.

Theorem 2.6
The operator Δ_D^2 is a homeomorphism from $H^{t+4}(\Omega) \cap H_0^2(\Omega)$ onto $H^t(\Omega)$ for $t \geq -2$, i.e. the inverse is a continuous operator

$$G_2 : H^t(\Omega) \to H^{t+4}(\Omega) \cap H_0^2(\Omega). \tag{2.12}$$

2.3 Some Properties of $H_0^2(\Omega)$

Theorem 2.7
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. The injection J of $H_0^2(\Omega)$ into $L^2(\Omega)$ is compact.

Since the injection of $H_0^2(\Omega)$ into $H^1(\Omega)$ is continuous, Theorem 2.7 is shown if $H^1(\Omega)$ is compactly injected into $L^2(\Omega)$. This can be shown for a set $Q = [0, 2\pi]^n$, since $u \in H^1(Q)$ is equivalent to $(\langle k\rangle c_k)_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$, where c_k is the Fourier coefficients of u, and $\langle k\rangle = (1 + |k|^2)^{\frac{1}{2}}$. Now the operator $K : u \to (\langle k\rangle^{-1} c_k)_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$ is compact, and the inverse operator T is closed. Let $(u_j)_{j\in\mathbb{Z}^n} \subseteq H^1(Q)$ be a bounded sequence, then $(Tu_j)_{j\in\mathbb{Z}^n}$ is bounded in $l^2(\mathbb{Z}^n)$, and hence $(KTu_j)_{j\in\mathbb{Z}^n} = (u_j)_{j\in\mathbb{Z}^n}$ has a convergent subsequence in $L^2(Q)$. Since $C_1\Omega \subseteq Q$ for some $C_1 > 0$ when Ω is bounded, the injection of $H_0^2(\Omega)$ into $L^2(\Omega)$ is compact.

Lemma 2.8
Let Ω be a bounded set in $\mathbb{R}^n$. If

$$u_i \rightharpoonup u \quad \text{in} \quad H_0^2(\Omega) \tag{2.13}$$

then the sequence $(u_i)_{i\in\mathbb{N}}$ is bounded and there exists a subsequence $(u_{i\mu})_{i\mu\in\mathbb{N}}$ of (u_i) with

$$u_{i\mu} \to u \quad \text{in} \quad L^2(\Omega). \tag{2.14}$$

Proof:
Weak convergence of u_i to u in $H_0^2(\Omega)$ implies weak convergence of u_i to u in $L^2(\Omega)$, since for $v \in L^2(\Omega)$, the injection J of $H_0^2(\Omega)$ into $L^2(\Omega)$ is weak-weak continuous. The injection J is compact and (u_i) is bounded (Theorem 2.7 and Lemma 2.2). So $\overline{\{J(u_i)\}}$ is compact, hence $J(u_i)$ has a subsequence $(u_{i\mu})$ that is convergent in the strong topology on $L^2(\Omega)$. Since strong convergence implies weak convergence the limit of $(u_{i\mu})$ is u. $\square$

Lemma 2.9
Let $\Omega \subseteq \mathbb{R}^n$ be open. Then there exists a countable basis for $H_0^2(\Omega)$ consisting of functions in $C_0^\infty(\Omega)$, i.e. there exists a countable set $U \in C_0^\infty(\Omega)$, so $v \in \overline{\operatorname{span}\{U\}}$ for all $v \in H_0^2(\Omega)$.

Proof:
The realisation Δ_D^2 of Δ^2 has a bounded inverse (Theorem 2.4)

$$G_2\colon L^2(\Omega) \overset{\text{bounded}}{\longrightarrow} H_0^2(\Omega) \cap H^4(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega) \tag{2.15}$$

so G_2 is a compact operator in $L^2(\Omega)$. The sesquilinear form $a(u,v)$ is symmetric, so Δ_D^2 is selfadjoint and closed according to Lax-Milgram's Lemma [3, p. 2.16], hence G_2 is selfadjoint [3, Theorem 2.7]. Therefore $L^2(\Omega)$ has an onb. U consisting of a sequence of eigenvectors for G_2 according to the Spectral Theorem of Compact Selfadjoint Operators.

Since $\operatorname{span}((u_k)_{k\in\mathbb{N}})$ is dense in $L^2(\Omega)$ and G_2 is a homeomorphism from $L^2(\Omega)$ to $H_0^2(\Omega) \cap H^4(\Omega)$, then $\operatorname{span}((G_2 u_k)_{k\in\mathbb{N}})$ is dense in $H_0^2(\Omega)\cap H^4(\Omega)$ and hence in $H_0^2(\Omega)$. Now $\operatorname{span}((u_k)_{k\in\mathbb{N}})$ is dense in $H_0^2(\Omega)$, since $\operatorname{span}((G_2 u_k)_{k\in\mathbb{N}}) \subseteq \operatorname{span}((u_k)_{k\in\mathbb{N}})$.

For $N \in \mathbb{N}$ there exists a sequence $w_k^{(N)} \in C_0^\infty(\Omega)$ satisfying

$$\|u_k - w_k^{(N)}\|_{H_0^2(\Omega)} < 2^{-Nk}. \tag{2.16}$$

Let $v \in H_0^2(\Omega)$, and let $\alpha_1, \dots, \alpha_n$ satisfy for $\varepsilon > 0$, that

$$\|v - (\alpha_1 u_1 + \cdots + \alpha_n u_n)\|_{H_0^2(\Omega)} < \frac{\varepsilon}{2} \tag{2.17}$$

and choose $N(k)$ so

$$\|u_k - w_k^{N(k)}\|_{H_0^2(\Omega)} < \frac{\varepsilon}{2n}\frac{1}{1+|\alpha_k|} \quad \text{for } k = 1, \dots, n. \tag{2.18}$$

Then

$$\begin{aligned}
\|v - \sum_{k=1}^n \alpha_k w_k^{N(k)}\|_{H_0^2(\Omega)} &\le \|v - \sum_{k=1}^n \alpha_k u_k\|_{H_0^2(\Omega)} + \sum_{k=1}^n |\alpha_k| \|u_k - w^{N(k)}\|_{H_0^2(\Omega)} \\
&\le \frac{\varepsilon}{2} + \sum_{k=1}^n |\alpha_k| \frac{\varepsilon}{2n}\frac{1}{1+|\alpha_k|} \\
&< \varepsilon.
\end{aligned}$$

Hence $(w_k^{(N)})_{N,k\in\mathbb{N}}$ is a countable set in $C_0^\infty(\Omega)$ and $\text{span}((w_k^{(N)})_{N,k\in\mathbb{N}})$ is dense in $H_0^2(\Omega)$, since ε is arbitrary. □

2.4 Fractional Powers of Δ_D^2

In the proof of Lemma 2.9 it was shown that the inverse of Δ_D^2 is a compact selfadjoint operator and that there exists an orthonormal basis (e_n) of $L^2(\Omega)$ consisting of eigenvectors of $G_2 \equiv (\Delta_D^2)^{-1}$ with corresponding eigenvalues (λ_n). The eigenvalues is a bounded set of positive numbers $\neq 0$ since G_2 is compact and has an inverse. It is assumed that the eigenvalues are arranged in numerical order with the largest first.
The spectrum $\sigma((\Delta_D^2)^{-1}) = \{\lambda | \lambda \text{ is an eigenvalue of } (\Delta_D^2)^{-1}\} \cup 0$. The function $f(t) = t^\alpha$ is then a bounded function on $\sigma((\Delta_D^2)^{-1})$. Now the functional calculus for compact operators can be used to define positive powers of $((\Delta_D^2)^{-1})$. Let $u \in L^2(\Omega)$, then

$$((\Delta_D^2)^{-1})^\alpha u = \sum_{n=1}^{\infty} \lambda_n^\alpha \, (u|e_n)_{L^2(\Omega)}\, e_n \tag{2.19}$$

giving the negative powers $(\Delta_D^2)^{-\alpha}) = ((\Delta_D^2)^{-1})^\alpha$. These operators are injective, because $(\Delta_D^2)^{-\alpha}u = 0$, means that $u = 0$, since all the eigenvalues are positive, and therefore $(u|e_n)_{L^2(\Omega)} = 0$ for all $n \in \mathbb{N}$, hence $\ker((\Delta_D^2)^{-\alpha}) = \{0\}$. The operator $(\Delta_D^2)^{-\alpha}$ is selfadjoint, since $f(t)$ is real valued. Every Hilbert space H can for a densely defined operator T be written as

$$H = \overline{R(T)} \oplus \ker(T^*) \tag{2.20}$$

so $\ker(((\Delta_D^2)^{-\alpha})^*) = \ker((\Delta_D^2)^{-\alpha}) = \{0\}$, and then the range $R((\Delta_D^2)^{-\alpha})$ is dense in $L^2(\Omega)$, hence $(\Delta_D^2)^{-\alpha}$ has a densely defined inverse $(\Delta_D^2)^\alpha$.

By using the operators defined above, it is possible to define norms on the Sobolev spaces that are equivalent with the usual norms, by

$$\|\cdot\|_{(s)} = \|(\Delta_D^2)^{\frac{s}{4}} \cdot \|_{L^2(\Omega)}. \tag{2.21}$$

for $-2 \leq s \leq 2$ and for $s \neq \pm\frac{1}{2}, \pm\frac{3}{2}$.

2.5 Properties of The von Karman Bracket

The definition of the von Karman bracket (Definition 1.1) is for $u, v \in H_0^2(\mathbb{R}^n)$ equivalent to

$$\begin{aligned} [u,v] &= D_1^2(uD_2^2v) + D_2^2(uD_1^2v) - 2D_{12}^2(uD_{12}^2v) &\quad (2.22)\\ &= D_1(D_1uD_2^2v - D_2uD_{12}^2v) + D_2(D_2uD_1^2v - D_1uD_{12}^2v) &\quad (2.23)\end{aligned}$$

in $\mathscr{D}'(\Omega)$.

Lemma 2.10
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. The mapping $u, v \to [u, v]$ is bilinear and continuous $H_0^2(\Omega) \times H_0^2(\Omega) \to H^{-2}(\Omega)$.

Proof:
The von Karman bracket can be written as

$$[u, v] = \sum_{|\beta|=|\alpha|=2} a_{\alpha,\beta} D_\alpha^2 u D_\beta^2 v \tag{2.24}$$

Here α and β are multi-indices of length 2, and the $a_{\alpha,\beta}$'s are constants. For $u, v \in H_0^2(\Omega)$ each term in this sum is a product of two $L^2(\Omega)$ functions, hence $[u, v] \in L^1(\Omega)$. Let $w \in C_0^\infty(\Omega)$ then by Sobolev's Theorem

$$|\langle [u, v], w \rangle| \leq \|[u, v]\|_{L^1(\Omega)} \|w\|_{L^\infty} \leq C_1 \|[u, v]\|_{L^1(\Omega)} \|w\|_{H_0^2(\Omega)} \tag{2.25}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, the inequality above is valid for $w \in H_0^2(\Omega)$, hence $[u, v] \in H^{-2}(\Omega)$.
Continuity is shown by evaluation the norm of one of the terms in (2.24)

$$\begin{aligned} \|a_{\alpha,\beta} D_\alpha^2 u D_\beta^2 v\|_{H^{-2}(\Omega)} &\leq C_2 \|D_\alpha^2 u D_\beta^2 v\|_{L_1((\Omega))} \\ &\leq C_2 \|D_\alpha^2 u\|_{L^2(\Omega)} \|D_\beta^2 v\|_{L^2(\Omega)} \\ &\leq C_2 \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)}. \end{aligned} \tag{2.26}$$

Linearity in the first argument is easily shown by rewriting $[u_1 + u_2, v]$ for $u_1, u_2, v \in H_0^2(\Omega)$, and then the symmetry of the von Karman bracket gives the bilinearity. □

Lemma 2.11
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. The form $([u, v]|w)_{L^2(\Omega)}$ is tri-linear and continuous on $H_0^2(\Omega)$.

Proof:
Let $u, v, w \in C_0^\infty(\Omega)$, then by Schwartz' inequality

$$\begin{aligned} ([u, v]|w)_{L^2(\Omega)} &= \langle [u, v], \overline{w} \rangle \\ &\leq \|[u, v]\|_{H^{-2}} \|w\|_{H_0^2(\Omega)} \\ &\leq C_1 \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)} \|w\|_{H_0^2(\Omega)}. \end{aligned} \tag{2.27}$$

Since the von Karman bracket and the inner product both are continuous on the dense subspace $C_0^\infty(\Omega)$ of $H_0^2(\Omega)$, then (2.27) extends to $u, v, w \in H_0^2(\Omega)$ by continuity.

It can easily be shown that the inner product $([u, v]|w)_{L^2(\Omega)}$ is tri-linear when $u, v, w \in C_0^\infty(\Omega)$, hence it extends to $u, v, w \in H_0^2(\Omega)$ by continuity. □

Lemma 2.12
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. The tri-linear form $([u,v]|w)_{L^2(\Omega)}$ is symmetric on $H_0^2(\Omega)$, i.e.

$$([u,v]|w)_{L^2(\Omega)} = ([\overline{w},u]|\overline{v})_{L^2(\Omega)} = ([v,\overline{w}]|\overline{u})_{L^2(\Omega)} \tag{2.28}$$

Proof:
Let $u, v, w \in C_0^\infty(\Omega)$, then the following is true for the inner product on $L^2(\Omega)$

$$\begin{aligned} ([u,v]|w)_{L^2(\Omega)} &= \left\langle D_1^2(vD_2^2u), \overline{w}\right\rangle + \left\langle D_2^2(vD_1^2u), \overline{w}\right\rangle - \left\langle 2D_{12}(vD_{12}u), \overline{w}\right\rangle \\ &= \left\langle vD_2^2u, D_1^2\overline{w}\right\rangle + \left\langle vD_1^2u, D_2^2\overline{w}\right\rangle - 2\left\langle vD_{12}u, D_{12}\overline{w}\right\rangle \\ &= \left\langle D_1^2\overline{w}D_2^2u, v\right\rangle + \left\langle D_2^2\overline{w}D_1^2u, v\right\rangle - 2\left\langle D_{12}\overline{w}D_{12}u, v\right\rangle \\ &= ([\overline{w},u]|\overline{v})_{L^2(\Omega)} \end{aligned} \tag{2.29}$$

Since the equations above are satisfied on $C_0^\infty(\Omega)$ they extend to $H_0^2(\Omega)$ by continuity. The last equation in (2.28) is shown by using the symmetry of the von Karman bracket. □

2.6 Vector Valued Functions

Consider a measure space $\langle M, \mathcal{R}, \mu\rangle$, where M is a measurable space, $\mathcal{R}$ is a σ-algebra defined on M, and μ is a measure. Let $A \in \mathcal{R}$, and let V be a vector space. Then $f(t)$ is a vector valued function on A if $f(t) \in V$ for a.e. $t \in A$. Three kinds of measurability of vector valued functions taking its values in a Banach space is defined below.

Definition 2.13
Let f be defined on a measure space $\langle M, \mathcal{R}, \mu\rangle$, taking its values in a Banach space X.

1. *f is called* **strongly measurable** *if there is a sequence of measurable functions f_n so that $f_n(x) \to f(x)$ in norm for a.e. $x \in M$ and each f_n being a simple function (taking only finitely many values, each value being taken on a set in $\mathcal{R}$).*

2. *f is called* **Borel measurable** *if $f^{-1}(C) \in \mathcal{R}$ for each open set $C \in X$.*

3. *f is called* **weakly measurable** *if $\langle f(x), \phi\rangle$ is a complex-valued measurable function for each $\phi \in X^*$.*

When a vector valued function takes its values in a Hilbert space the three kinds of measurability are the same.

Theorem 2.14
Let H be a Hilbert space, and let f be a function from a measure space $\langle M, \mathcal{R}, \mu \rangle$ to H. Then the following three statements are equivalent

1. *f is strongly measurable.*
2. *f is Borel measurable.*
3. *f is weakly measurable.*

A proof of this theorem can be found in [11, p. 116].

Definition 2.15
Let $1 \leq p < \infty$, let $\langle M, \mathcal{R}, \mu \rangle$ be a measure space, let $A \in \mathcal{R}$ and let X be a Banach space. Then $L^p(A; X)$ is the space of weakly measurable functions $f(t)$ on A with values in X a.e. for which

$$\left(\int_A \|f(t)\|_X^p d\mu(t) \right)^{\frac{1}{p}} < \infty, \tag{2.30}$$

In addition $L^\infty(A; X)$ is the space of weakly measurable essentially bounded functions on A with values in X a.e., hence for $u(t) \in L^\infty(A; X)$,

$$\operatorname*{ess\,sup}_{t \in A} \|u(t)\|_X < \infty. \tag{2.31}$$

The spaces defined above equipped with the norms (2.30) and (2.31) respectively are Banach spaces.

2.7 Integration of Vector Valued Functions

Theorem 2.16
Let $\langle M, \mathcal{R}, \mu \rangle$ be a measure space, let $A \in \mathcal{R}$, let X be a reflexive Banach space, i.e. $X = X^{**}$, and let $f \in L^1(A; X)$. Then there is a unique element $x \in X$ such that for all $\phi \in X^*$

$$\langle x, \phi \rangle = \int \langle f(\cdot), \phi \rangle. \tag{2.32}$$

Proof:
For all $\phi \in X^*$, the function $t \rightarrow \langle f(t), \phi \rangle$ is measurable, and

$$|\langle f(t), \phi \rangle| \leq \|f(t)\|_X \|\phi\|_{X^*} \in L^1(A), \tag{2.33}$$

so the integral $\int_A \langle f(t), \phi\rangle d\mu(t)$ is well defined.
Now

$$\phi \to \int_A \langle f(t), \phi\rangle d\mu(t) \tag{2.34}$$

is a bounded linear functional on X^*, since (2.33) gives

$$|\int_A \langle f(t), \phi\rangle d\mu(t)| \leq \int_A |\langle f(t), \phi\rangle| d\mu(t) \leq \|\phi\|_{X^*} \int_A \|f(t)\|_X d\mu(t). \tag{2.35}$$

Hence there exists a unique $x \in X^{**} = X$ with the properties

$$\langle x, \phi\rangle = \int \langle f(\cdot), \phi\rangle$$

for all $\phi \in X^*$. □

Theorem 2.16 can be used to define the integral of a vector valued function f.

Definition 2.17
Let X and f satisfy the conditions in Theorem 2.16. The integral of f over A is defined by $\int_A f(t)d\mu(t) = x$.

The definition above is called the Bochner identity.

Lemma 2.18
Let $\langle M, \mathcal{R}, \mu\rangle$ be a measure space, let $A \in \mathcal{R}$, let X be a reflexive Banach space and let $f(t) \in L^1(A; X)$ then

$$\Big\| \int_A f(t)d\mu(t)\Big\|_X \leq \int_A \|f(t)\|_X d\mu(t). \tag{2.36}$$

Proof:
If $\int_A f(t)d\mu(t) = 0_X$ then (2.36) is satisfied. Assume that $\int_A f(t)d\mu(t) \neq 0_X$, then there exists a $\phi \in X^*$, with $\|\phi\|_{X^*} = 1$, satisfying

$$\Big\| \int_A f(t)d\mu(t)\Big\|_X = \int_A \langle f(t), \phi\rangle \, d\mu(t). \tag{2.37}$$

Since (2.35) is satisfied for all $\phi \in X^*$ the lemma is proved. □

The following theorem is Lebesgue's Dominated Convergence Theorem extended to functions valued in a Banach space.

Theorem 2.19
Let $\langle M, \mathcal{R}, \mu\rangle$ be a measure space, let $A \in \mathcal{R}$, let X be a reflexive Banach space, let $f(t)$ be a weakly measurable function valued in X for a.e. $t \in A$ and let $f_n(t) \in$

$L^1(A;X)$, with $f_n(t) \to f(t)$ a.e. on A. If there exists a function $K(t) \in L^1(A)$ for which

$$\|f_n(t)\|_X \le K(t) \quad \text{for all } n \in \mathbb{N} \text{ and for a.e. } t \in A \tag{2.38}$$

then $f(t) \in L^1(A;X)$ and

$$\Big\| \int_A f_n(t)d\mu(t) - \int_A f(t)d\mu(t)\Big\|_X \to 0 \quad \text{for } n \to \infty. \tag{2.39}$$

Proof:
Since $\|f_n(t)\|_X \le M(t)$ for a.e. $t \in A$, so that $\|f(t)\|_X \le M(t)$ for a.e. $t \in A$, and since $f(t)$ is weakly measurable, then $f(t) \in L^1(A;X)$. Now the following integrals are well defined

$$\Big\| \int_A f_n(t)d\mu(t) - \int_A f(t)d\mu(t)\Big\|_X \le \int_A \|f_n(t) - f(t)\|_X d\mu(t) \tag{2.40}$$

according to Lemma 2.18. Now $\|f_n(t) - f(t)\|_X \le 2M(t)$ for a.e. $t \in A$, hence Lebesgue's Dominated Convergence Theorem for complex valued functions gives that the right hand side and hence the left hand side in (2.40) tends to 0 as n tends to infinity. □

Theorem 2.20
Let $\langle M, \mathcal{R}, \mu\rangle$ be a measure space, let $A \in \mathcal{R}$, let X be a reflexive Banach space, and let $f(t,\cdot) \in L^1(A;\mu,X)$ for $t \in \mathbb{R}$ with $f(\cdot,s)$ norm-differentiable on $\mathbb{R}$ for a.e $s \in A$. Assume that $\frac{\partial}{\partial t}f(t,s)$ is weakly measurable. If there exists a function $K(s) \in L^1(A,\mu)$, satisfying $|\frac{\partial}{\partial t}f(\cdot,s)| \le K(s)$ for a.e. $s \in A$, then $\int_A f(t,s)d\mu(s)$ is differentiable and

$$\frac{\partial}{\partial t}\int_A f(t,s)d\mu(s) = \int_A \frac{\partial}{\partial t} f(t,s)d\mu(s). \tag{2.41}$$

Proof:
Assume that f is a real valued, vector valued function, since f is differentiable on $\mathbb{R}$ for a.e. $s \in A$ it is also weakly differentiable. Then for a.e. $s \in A$, and for $\phi \in X^*$, with $\|\phi\|_{X^*} = 1$, it follows by the Mean Value Theorem

$$\begin{aligned}
\Big| \Big\langle \frac{1}{h}(f(t+h) - f(t)), \phi \Big\rangle - \Big\langle \frac{\partial}{\partial t} f(t), \phi \Big\rangle \Big| & \\
&\le \|\frac{\partial}{\partial t} f(t)\|_X + |\frac{1}{h}(\langle f,\phi\rangle(t+h) - \langle f,\phi\rangle(t))| \\
&\le K(s) + \Big| \frac{\partial}{\partial t}\langle f,\phi\rangle(t+\theta h)\Big| \\
&\le K(s) + \Big| \Big\langle \frac{\partial}{\partial t} f(t+\theta h), \phi \Big\rangle \Big| \\
&\le 2K(s)
\end{aligned} \tag{2.42}$$

Now Theorem 2.19 gives that $\|\frac{1}{h}(f(t+h)-f(t))-\frac{\partial}{\partial t}f(t)\|_X \to 0$ for $h \to 0$.
If f is a complex valued function the theorem is shown by splitting f into the two real valued functions $\operatorname{Re} f$ and $\operatorname{Im} f$. □

2.8 Ordinary Differential Equations

In this section three theorems are presented, which are used in Chapter 4 to show the existence of an approximated solution of von Karmans equations.

Theorem 2.21
Let $\Omega \in \mathbb{R}\times\mathbb{C}$ be open, let $F(t,g):\Omega\to\mathbb{C}$ with $F(\cdot,g)\in L^1(t_0-\varepsilon,t_0+\varepsilon)$ for some ε and $t_0\in\mathbb{R}$, and let $F(t,\cdot)$ be continuous. Let $g_0\in\mathbb{C}$, then

$$\begin{aligned}\frac{dg}{dt} &= F(t,g)\\ g(t_0) &= g_0\end{aligned} \tag{2.43}$$

has a continuous solution $g(t)$ on some interval $]t_0-t_\varepsilon,t_0+t_\varepsilon[$ if and only if there exists some continuous $g(t)$ which satisfies

$$g(t)=g_0+\int_{t_0}^{t}F(\sigma,g(\sigma))d\sigma \tag{2.44}$$

for all $t\in]t_0-t_\varepsilon,t_0+t_\varepsilon[$.

Theorem 2.22
Let $\Omega \in \mathbb{R}\times\mathbb{C}$ be open, let $F(t,g):\Omega\to\mathbb{C}$ with $F(\cdot,g)\in L^1(t_0-\varepsilon,t_0+\varepsilon)$ for some ε and $t_0\in\mathbb{R}$, and let $F(t,\cdot)$ be continuous. Let F satisfy

$$|F(t,g)-F(t,h)|\le C_1|g-h| \tag{2.45}$$

for a.e. $(t,g),(t,h)\in\Omega$. Then there exists a $t_\varepsilon>0$ so (2.44) has a unique solution on $]t_0-t_\varepsilon,t_0+t_\varepsilon[$.

These two theorems can be proved using the same proof ideas as in [6], since the only difference is that $F(t,\cdot)\in L^1(t_0-\varepsilon,t_0+\varepsilon)$ instead of being continuous, and that it takes its values in $\mathbb{C}$ instead of $\mathbb{R}$. Indeed, continuity is used to ensure integrability, a quality which L^1 functions also possess on a measurable set, and to ensure a supremum of F, but an essential supremum is enough. The analysis with a complex valued function F can be done by doing the analysing for the two real valued functions $\operatorname{Re} F$ and $\operatorname{Im} F$ separately.

Theorem 2.23
Let $U\subset\mathbb{C}$ be open and bounded, and let $[a,b]\subset\mathbb{R}$ be bounded. Let $\Omega\subseteq[a,b]\times U$, let $F(t,g):\Omega\to\mathbb{C}$ with $F(\cdot,g)\in L^1([a,b])$, and let $F(t,\cdot)$ be continuous on U, and satisfy (2.45). Let $g(t)$ be a solution to (2.43) defined on a maximal subinterval $]a_0,b_0[$ of $[a,b]$. Assume

- *There exists an* $\varepsilon > 0$, *so* $\overline{g(]b_0-\varepsilon, b_0[)} \subset U$.
- *There exists a* $B > 0$, *so* $\operatorname{ess\,sup} |F(t, g(t))| \leq B$ *for all* $t \in (b_0 - \varepsilon, b_0)$.

Then $b_0 = b$

Proof:
The function $g(t)$ solves (2.43), so

$$g(t) = g_0 + \int_{t_0}^{t} F(u, g(u))du. \tag{2.46}$$

Let $t_1, t_2 \in (b_0 - \varepsilon, b_0)$, then

$$|g(t_1) - g(t_2)| \leq \int_{t_2}^{t_1} |F(u, g(u))|du \leq B|t_1 - t_2|. \tag{2.47}$$

A Cauchy sequence (t_i) of numbers is formed by letting $t_i \in]b_0 - \varepsilon_i, b_0[$, with $\varepsilon_i \to 0$ for $i \to \infty$, hence $(g(t_i))$ is a Cauchy sequence, and the limit $g_{b_0} \in U$ of $g(t)$ exists for $t \to b_0$.
Assume that $b_0 \neq b$, then Theorem 2.21 assure the existence of a h satisfying

$$\begin{aligned} \frac{dh}{dt} &= F(t, h) \\ h(b_0) &= g_{b_0} \end{aligned} \tag{2.48}$$

on some interval $]b_0 - t_h, b_0 + t_h[$, for $t_h > 0$.
Now $g'(t) = h'(t)$ on some open interval $]t_{gh}, b_0[$, hence on this interval $g(t) - h(t) = C$, where C is a constant. Since the limits of $g(t)$ and $h(t)$ on (t_{gh}, b_0) are equal for $t \to b_0$ the constant C must be 0. Therefore the function

$$\tilde{g}(t) = \begin{cases} g(t) & \text{on }]a_0, b_0[\\ h(t) & \text{on } [b_0, t_h + b_0[\end{cases} \tag{2.49}$$

is a solution to (2.43) on the interval $]a_0, t_h + b_0[$, contradicting that the interval $]a_0, b_0[$ is maximal, hence the assumption $b_0 \neq b$ must be wrong. □

2.9 Norms on Sobolev Spaces

In the rest of the report the norm on the Sobolev space $H^s(\Omega)$ is written as $\|\cdot\|_s$, which is easier to read when a lot of norms are involved.

In this section $(e_n)_{n\in\mathbb{N}}$ is a basis of $L^2(\Omega)$ consisting of the eigenvectors of Δ_D^2, with the corresponding eigenvalues (λ_n), where $0 < \lambda_1 \leq \lambda_2 \cdots$. Consider

$$\Delta_D^2 e_n = \lambda_n e_n \tag{2.50}$$

The left hand side $\lambda_n e_n \in L^2(\Omega)$, so the right hand side $\Delta_D^2 e_n \in L^2(\Omega)$ and by Theorem 2.6 $e_n \in H^4(\Omega)$, but then $\lambda_n e_n \in H^4(\Omega)$, so $\Delta_D^2 e_n \in H^4(\Omega)$ and therefore $e_n \in H^8(\Omega)$. Continuing this way it can be seen by using Sobolevs Theorem that

$$e_n \in \bigcap_{s\in\mathbb{N}} H^{4s}(\Omega) \hookrightarrow C^\infty(\overline{\Omega}) \tag{2.51}$$

For $u \in L^2(\Omega)$ let $P_N u$ be the projection of u on $\operatorname{span}(e_1, \dots, e_N)$, i.e.

$$P_N u = \sum_{n=1}^{N} (u|e_n)_{L^2(\Omega)}\, e_n \tag{2.52}$$

Then $P_N u \in C^\infty(\overline{\Omega})$, a fact which will be used in the rest of this chapter without reference.

Note that the λ's are not the same, but the reciprocal of the λ's used in Section 2.4. With the new convention

$$(\Delta^2)^\alpha e_n = \lambda_n^\alpha e_n. \tag{2.53}$$

Lemma 2.24
The von Karman bracket satisfies for $j = 1, 2$ and $0 < \psi < 1$ that

$$\|[u, v]\|_{-j} \leq C_1 \|u\|_{2-\psi} \|v\|_{3-j+\psi}, \tag{2.54}$$

and for $j = 0, 1$ and $0 < \psi \leq \theta < 1$

$$\|[u, v]\|_{-j-\theta} \leq C_2 \|u\|_{2-\theta+\psi} \|v\|_{3-j-\psi}. \tag{2.55}$$

Lemma 2.25
Let $f(x) \in H_0^1(\Omega)$ then there exists $N_0 > 0$ so that

$$\max_{x\in\Omega} |(P_N f)(x)| \leq C_1 (\log(1 + \lambda_N))^{1/2} \|f\|_1 \tag{2.56}$$

for $N \geq N_0$.

Proof:
Let $\phi \in C_0^\infty(\Omega)$, then it follows from Cauchy-Schwarz inequality for $\sigma > 0$, that

$$\begin{aligned}
\max_{x\in\Omega} |\phi(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |\hat{\phi}(x)| dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \langle x \rangle^{\sigma+1} \langle x \rangle^{-\sigma-1} |\hat{\phi}(x)| dx \\
&= \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \langle x \rangle^{2(\sigma+1)} |\hat{\phi}(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^2} \langle x \rangle^{-2(\sigma+1)} dx \right)^{1/2} \\
&= \|\phi\|_{1+\sigma} \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.57}$$

Let $g \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ and $\tilde{g} \in H^{1+\sigma}(\mathbb{R}^2)$ with $\operatorname{supp} \tilde{g} \subset \Omega$ and $r_\Omega \tilde{g} = g$. It is possible to find a sequence $(\tilde{\phi}_k) \in C_0^\infty(\mathbb{R}^2)$ that converges to $\tilde{g}$ in $H^{1+\sigma}(\mathbb{R}^2)$, i.e. for $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, so for $k \geq N_0$

$$\|\tilde{\phi}_k - \tilde{g}\|_{1+\sigma} < \varepsilon. \tag{2.58}$$

Sobolevs Theorem gives $H^{1+\sigma}(\mathbb{R}^2) \subset C(\mathbb{R}^2)$, hence the sequence $\tilde{\phi}_k$ also converges to $\tilde{g}$ in $C(\mathbb{R}^2)$, i.e. there exists a $M_0 \in \mathbb{N}$, satisfying for $k \geq M_0$, that

$$\max_{x\in\mathbb{R}^2} |\tilde{\phi}_k(x) - \tilde{g}(x)| < \epsilon. \tag{2.59}$$

Therefore for $k \geq \max\{N_0, M_0\}$, then by (2.57)

$$\begin{aligned}\max_{x\in\mathbb{R}^2}|\tilde{g}(x)| &\leq \max_{x\in\mathbb{R}^2}|\tilde{g}(x) - \tilde{\phi}_k(x)| + \max_{x\in\omega}|\tilde{\phi}_k(x)| \\ &\leq \varepsilon + C_1\sigma^{-1/2}\|\tilde{\phi}_k(x)\|_{1+\sigma} \\ &\leq \varepsilon + C_1\sigma^{-1/2}(\varepsilon + \|\tilde{g}\|_{1+\sigma}) \\ &\leq C_2\varepsilon + C_1\sigma^{-1/2}\|\tilde{g}\|_{1+\sigma}.\end{aligned} \tag{2.60}$$

Hence for $g \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ and for $\sigma > 0$

$$\max_{x\in\Omega} |g(x)| \leq C_1\sigma^{-1/2}\|g\|_{1+\sigma} \tag{2.61}$$

since ε was arbitrary.
The projection $P_N f \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$, so by using (2.21) and (2.52) for $0 < \sigma < 1$, then

$$\max_{x\in\Omega} |(P_N f)(x)| \leq C_1\sigma^{-1/2}\lambda_N^{\sigma/4}\|(\Delta_D^2)^{1/4} f\|_{L^2(\Omega)} \leq C_3\sigma^{-1/2}\|f\|_1. \tag{2.62}$$

Since $0 < (\log(1 + \lambda_N))^{-1} < 1/2$ for N large enough, then (2.56) is shown for $\sigma = (\log(1 + \lambda_N))^{-1}$. □

Lemma 2.26
Let $f(x) \in H^\sigma(\Omega)$ for $0 < \sigma \leq 1$. Then for some $C > 0$, and $1 < p < (1 - \sigma)^{-1}$

$$\|f\|_{L^{2p}(\Omega)} \leq C\left(\pi\frac{p-1}{\sigma p - p + 1}\right)^{\frac{p-1}{2p}}\|f\|_\sigma. \tag{2.63}$$

Proof:
Let $g(x) \in H^\sigma(\mathbb{R}^2)$ be an extension of $f(x)$. Let $\frac{1}{2p} + \frac{1}{\tilde{p}} = 1$. Evaluating the $L^{\tilde{p}}$-norm of $\hat{g}$ using Hölders inequality gives for $\tilde{\sigma} = \tilde{p}\sigma(2 - \tilde{p})^{-1}$

$$\|\hat{g}\|_{L^{\tilde{p}}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \langle x\rangle^{-\frac{\tilde{p}\sigma}{2}}\langle x\rangle^{\frac{\tilde{p}\sigma}{2}}|\hat{g}(x)|^{\tilde{p}}dx\right)^{1/\tilde{p}} \tag{2.64}$$

$$\leq \left(\int_{\mathbb{R}^2}\langle x\rangle^{2\sigma}|\hat{g}(x)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^2}\langle x\rangle^{-2\tilde{\sigma}}dx\right)^{\frac{2-\tilde{p}}{2\tilde{p}}} \tag{2.65}$$

which is finite since the first integral equals $\|g\|_\sigma$, and since $2\tilde{\sigma} > n = 2$, making the second integral finite. Therefore $\hat{g} \in L^{\tilde{p}}(\mathbb{R}^2)$ and Theorem 7.1.13 in [5] gives that $\hat{g}(x)$ is mapped continuously into L^{2p} by the Fourier transformation.
Theorem 8.4 in [3] gives for $h(x) \in \mathscr{S}(\mathbb{R}^2)$, that $h(x) = (2\pi)^{-2}\overline{\mathscr{F}}\hat{h}(x)$, the co-Fourier transform of the Fourier transform. Hence for $h(x) \in \mathscr{S}(\mathbb{R}^2)$

$$\begin{aligned} h(x) &= (2\pi)^{-2}\int_{\mathbb{R}^2} e^{ix\cdot\psi}\hat{h}(\psi)d\psi \\ &= (2\pi)^{-2}\int_{\mathbb{R}^2} e^{-ix\cdot\psi}\hat{h}(-\psi)d\psi \\ &= C_1\hat{\hat{h}}(-x). \end{aligned} \tag{2.66}$$

Since the Fourier transformation is continuous on $\mathscr{S}(\mathbb{R}^2)$, the consideration above extends to $\mathscr{S}'(\mathbb{R}^2)$ by continuity, and therefore to $H^\sigma(\mathbb{R}^2)$. Hence

$$\|f\|_{L^{2p}(\Omega)} \leq \|g\|_{L^{2p}(\mathbb{R}^2)} = C_1\|\hat{\hat{g}}\|_{L^{2p}(\mathbb{R}^2)} \leq C_2\|\hat{g}\|_{L^{\tilde{p}}(\mathbb{R}^2)}. \tag{2.67}$$

The second integral in (2.65) can be calculated and an evaluation based on the size of p and σ gives an upper limit $\left(\pi\frac{p-1}{\sigma p-p+1}\right)^{\frac{p-1}{2p}}$ to the integral. Now

$$\|f\|_{L^{2p}(\Omega)} \leq C_2\left(\pi\frac{p-1}{\sigma p-p+1}\right)^{\frac{p-1}{2p}}\|g\|_\sigma \tag{2.68}$$

Which is true for all $g \in H^\sigma(\mathbb{R}^2)$, with $g(x)|_\Omega = f(x)$, hence it is also true for infimum of $\|g\|_\sigma$ and thereby (2.63) is shown. □

Lemma 2.27
Let $f(x) \in L^2(\Omega)$ and $g(x) \in H^1_0(\Omega)$. Then there exists $N_0 > 0$ so that

$$\|(P_Nf)g\|_{L^2(\Omega)} \leq C(\log(1+\lambda_N))^{1/2}\|f\|_{L^2(\Omega)}\|g\|_1 \tag{2.69}$$

for all $N \geq N_0$. The constant C does not depend on N.

Proof:
Hölders inequality gives for $0 < \theta < 1$,

$$\begin{aligned} \|(P_Nf)g\|^2_{L^2(\Omega)} &\leq \|\,|(P_Nf)|^2|g|^2\|_{L^1(\Omega)} \\ &\leq \|\,|(P_Nf)|^2\|_{L^{\frac{1}{1-\theta}}(\Omega)}\|\,|g|^2\|_{L^{\frac{1}{\theta}}(\Omega)} \\ &= \|P_Nf\|^2_{L^{\frac{2}{1-\theta}}(\Omega)}\|g\|^2_{L^{\frac{2}{\theta}}(\Omega)}. \end{aligned} \tag{2.70}$$

Lemma 2.26 is used with $p = (1-\theta)^{-1}$ and $\sigma = 2\theta$, and then (2.21) is used giving

$$\|P_Nf\|_{L^{\frac{2}{1-\theta}}} \leq C_1\|P_Nf\|_{2\theta} \leq C_1\|(\Delta^2_D)^{\theta/2}P_Nf\|_{L^2(\Omega)} \leq C_1\lambda_N^{\theta/2}\|f\|_{L^2(\Omega)} \tag{2.71}$$

for $0 < \sigma < \theta$.
Let $p = \theta^{-1}$ and $\sigma = 1$, then Lemma 2.26 gives

$$\|g\|_{L^{\frac{2}{\theta}}} \leq \left(\pi \frac{1-\theta}{\theta}\right)^{(1-\theta)/2} \|g\|_1. \tag{2.72}$$

Together (2.70), (2.71) and (2.72) gives

$$\|(P_N f)g\|_{L^2(\Omega)} \leq C_2 \theta^{-1/2} \lambda_N^{\theta/2} \|f\|_{L^2(\Omega)} \|g\|_1 \tag{2.73}$$

If $\theta = (\log(1+\lambda_N))^{-1}$, then (2.69) is satisfied. □

Lemma 2.28
Let $u \in H^\beta(\Omega)$ and let $v \in H^{1-\beta}(\Omega)$ for $0 < \beta < 1$. Then the following statements are true

$$\|uv\|_{L^2(\Omega)} \leq C_1 \|u\|_\beta \|v\|_{1-\beta} \tag{2.74}$$

$$\|uv\|_{\beta-1} \leq C_2 \|u\|_\beta \|v\|_{L^2(\Omega)}. \tag{2.75}$$

Proof:
According to Hölders inequality

$$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^{\frac{2}{1-\beta}}} \|v\|_{L^{\frac{2}{\beta}}} \tag{2.76}$$

Since $H^{1-\beta}(\Omega)$ is continuously embedded into $L^{\frac{2}{\beta}}$, and hence H^β into $L^{\frac{2}{1-\beta}}$ for $0 < \beta \leq 1$ [1, Theorem 5.4], then (2.76) leads to (2.74).
Furthermore $L^{\frac{2}{2-\beta}}(\Omega)$ is continuously embedded into $H^{1-\beta}(\Omega)$. Therefore by Hölders inequality

$$\|uv\|_{\beta-1} \leq C_1 \|uv\|_{L^{\frac{2}{2-\beta}}(\Omega)} \leq C_1 \|u\|_{L^{\frac{2p}{2-\beta}}(\Omega)} \|v\|_{L^{\frac{2q}{2-\beta}}(\Omega)} \tag{2.77}$$

for $p^{-1} + q^{-1} = 1$. Let $q = 2 - \beta$ and $p = \frac{2-\beta}{1-\beta}$, then

$$\|uv\|_{\beta-1} \leq C_1 \|u\|_{\frac{2}{1-\beta}} \|v\|_{L^2(\Omega)} \tag{2.78}$$

Now continuity of the embedding of $H^\beta(\Omega) \subset L^{\frac{2}{1-\beta}}(\Omega)$ gives (2.75). □

Chapter 3

Continuity of Weak Solutions

A Weak solution u of von Karmans equations has to be continuously depending on the time, or else it makes no sense to give an initial condition on u, likewise u' has to be continuously depending on t. In order to show that these functions are in fact continuous if the other conditions of being a weak solution are satisfied, a number of statements has to be shown. First a sequence of functions are chosen.

It is possible to find a function $h \in C_0^\infty(\mathbb{R}^n)$, which satisfies

$$h \geq 0, \int_{\mathbb{R}^n} h(x)dx = 1, \text{ supp } h \ \in \overline{B}(0,1)$$

and hence a sequence of functions $h_j(x) = j^n h(jx)$ satisfying for $j \in \mathbb{N}$

$$h_j \geq 0, \int_{\mathbb{R}^n} h_j(x)dx = 1, \text{ supp } h_j \ \in \overline{B}(0,\frac{1}{j}). \tag{3.1}$$

Lemma 3.1
Let $V,\ H,\ V^*$ be three Hilbert spaces, with $V \subset H \subset V^*$ each dense in the following, and with continuous injections. If $u \in L^2\left(0,T;V\right)$ and $u' \in L^2\left(0,T;V^*\right)$ then there exists a sequence of functions (u_j), that are infinitely differentiable from $[0,T]$ to V, with the properties

$$u_j \to u \qquad \textit{in} \ \ L^2_{\text{loc}}\left(]0,T[;V\right) \tag{3.2}$$

$$u'_j \to u' \qquad \textit{in} \ \ L^2_{\text{loc}}\left(]0,T[;V^*\right). \tag{3.3}$$

Proof:
Let h_j be given as in (3.1) for $n=1$, and let

$$\tilde{u}(t) = \begin{cases} u(t) & \text{on } [0,T] \\ 0 & \text{on } \mathbb{R}\backslash[0,T] \end{cases} \tag{3.4}$$

then the following sequence of functions will have the desired properties when restricted to $[0, T]$

$$u_j(t) = (h_j * \tilde{u})(t) = \int_{\mathbb{R}} h_j(t-s)\tilde{u}(s)ds. \tag{3.5}$$

Each u_j is well defined, since the integrand is in $L^1(\mathbb{R}; V)$ by Hölders inequality. In order to show that u_j is infinitely differentiable the difference between two function values is rewritten as

$$\begin{aligned} u_j(t+\tau) - u_j(t) &= \int_{\mathbb{R}} \big(h_j(t+\tau-s) - h_j(t-s)\big)\tilde{u}(s)ds \\ &= \int_{\mathbb{R}} \big(h_j'(t-s)\tau + |\tau|\varepsilon(\tau, t-s)\big)\tilde{u}(s)ds \\ &= \tau \int_{\mathbb{R}} h_j'(t-s)\tilde{u}(s)ds + |\tau| \int_{\mathbb{R}} \varepsilon(\tau, t-s)\tilde{u}(s)ds \\ &= \tau(h_j' * \tilde{u})(t) + |\tau| \int_{\mathbb{R}} \varepsilon(\tau, t-s)\tilde{u}(s)ds. \end{aligned} \tag{3.6}$$

It can be assumed that $|\tau| \leq 1$. First Taylor expansion is used to evaluate the term $|\tau|\varepsilon(\tau, t-s)$ [3, p. 1.6]

$$\begin{aligned} |\tau|\varepsilon(\tau, t-s) &= h_j(t+\tau-s) - h_j(t-s) - h_j'(t-s)\tau \\ &= \int_0^1 (1-\theta)h_j''(t-s+\tau\theta)\tau^2 d\theta, \end{aligned} \tag{3.7}$$

therefore

$$\begin{aligned} |\varepsilon(\tau, t-s)| &\leq |\tau| \int_0^1 (1-\theta)|h_j''(t-s+\tau\theta)|d\theta \\ &\leq |\tau| \sup_{t\in\mathbb{R}} |h_j''(t)| \leq |\tau| C_1. \end{aligned} \tag{3.8}$$

Which leads to the evaluation

$$\Big\| \int_{\mathbb{R}} \varepsilon(\tau, t-s)\tilde{u}(s)ds \Big\|_V \leq |\tau| C_1 \Big\| \int_{\mathbb{R}} \tilde{u}(s)ds \Big\|_V \leq |\tau| C_1 \int_{\mathbb{R}} \|\tilde{u}(s)\|_V ds \tag{3.9}$$

where the right hand side and hence the left tends to 0 as τ tends to 0, since $\tilde{u}(s) \in L^2_{\text{comp}}(\mathbb{R}; V) \subseteq L^1(\mathbb{R}; V)$.
So for $\tau \to \infty$, the norm

$$\Big\| \frac{1}{\tau}(u_j(t+\tau) - u_j(t)) - (h_j' * \tilde{u}(t)) \Big\|_V \to 0 \tag{3.10}$$

Therefore u_j is differentiable w.r.t t on $[0, T]$ for all $j \in \mathbb{N}$ with $\frac{\partial u_j}{\partial t} = (h_j' * \tilde{u})(t)$. Continuing the same way, it can be shown that u_j is infinitely differentiable w.r.t t.

To prove (3.2) it is enough for an arbitrary compact interval $[a,b] \subseteq]0,T[$ to show that $\|u_j - u\|_{L^2(a,b;V)}$ tends to 0 as j tends to infinity, but since $\|u_j - u\|_{L^2(a,b;V)} \leq \|u_j - \tilde{u}\|_{L^2(\mathbb{R},V)}$, it can be done by evaluating the following norm

$$\begin{aligned}
\|u_j - \tilde{u}\|^2_{L^2(\mathbb{R},V)} &= \int_{\mathbb{R}} \|u_j(t) - u(t)\|^2_V dt \\
&= \int_{\mathbb{R}} \Big\| \int_{\mathbb{R}} h_j(s)(\tilde{u}(t-s) - \tilde{u}(t))ds \Big\|^2_V dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|h_j(s)(\tilde{u}(t-s) - \tilde{u}(t))\|_V ds \right)^2 dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h_j(s)\|\tilde{u}(t-s) - \tilde{u}(t)\|^2_V ds \right) \left(\int_{\mathbb{R}} h_j(s) ds \right) dt.
\end{aligned} \tag{3.11}$$

In the second step Lemma 2.18 is used, and in the last step $h_j(s)$ is placed outside the norm since $h_j(s) \geq 0$, and then Hölder's inequality is applied.
Now, by the translation invariance of the Lebesgue measure

$$\left| \int_{\mathbb{R}} h_j(s) \int_{\mathbb{R}} \|\tilde{u}(t-s) - \tilde{u}(t)\|^2 dt \; ds \right| \quad \leq \int_{\mathbb{R}} h_j(s) 4\|u\|^2_{L^2(\mathbb{R},V)} ds = 4\|u\|^2_{L^2(\mathbb{R},V)}.$$

Therefore Fubini's Theorem can be used on (3.11), leading to

$$\|u_j - \tilde{u}\|^2_{L^2(\mathbb{R},V)} \leq \int_{\mathbb{R}} h_j(s) \int_{\mathbb{R}} \|\tilde{u}(t-s) - \tilde{u}(t)\|^2_V dt \; ds. \tag{3.12}$$

Because $\tilde{u} \in L^2(\mathbb{R};V)$ it is weakly measurable, hence strongly measurable (Theorem 2.14), then for $\varepsilon > 0$ there exists a simple measurable function $v(t)$ with the property $\|\tilde{u} - v\|_{L^2(\mathbb{R},V)} < \frac{\varepsilon^{1/2}}{3}$. Let $s \in [-\frac{1}{j}, \frac{1}{j}]$, then

$$\begin{aligned}
&(\int_{\mathbb{R}} \|\tilde{u}(t-s) - \tilde{u}(t)\|^2_V dt)^{1/2} \\
&= \|\tilde{u}(t-s) - \tilde{u}(t)\|_{L^2(\mathbb{R},V)} \\
&\leq \|\tilde{u}(t-s) - v(t-s)\|_{L^2(\mathbb{R},V)} + \|v(t-s) - v(t)\|_{L^2(\mathbb{R},V)} + \|v(t) - \tilde{u}(t)\|_{L^2(\mathbb{R},V)} \\
&= 2\|u - v\|_{L^2(\mathbb{R},V)} + \|v(t-s) - v(t)\|_{L^2(\mathbb{R},V)} \\
&< \frac{2\varepsilon^{1/2}}{3} + \|v(t-s) - v(t)\|_{L^2(\mathbb{R},V)}.
\end{aligned} \tag{3.13}$$

The function v is written as

$$v(t) = \sum_{k=1}^{N} a_k 1_{A_k}(t) \tag{3.14}$$

Where $a_k \in V$ and 1_{A_k} is the characteristic function of the measurable set A_k. It is assumed that $\operatorname{supp} v \in [-\frac{1}{2}, T + \frac{1}{2}]$ and $s < \frac{1}{2}$. By using Theorem 9.5 in [12], since

$\sum_{k=1}^{N} 1_{A_k} \in L^2(\Omega)$, then

$$\begin{aligned}\|v(t-s)-v(t)\|^2_{L^2(\mathbb{R},V)} &= \int_{\mathbb{R}} \|v(t-s)-v(t)\|^2_V dt \\ &= \int_{[-1,T+1]} \|\sum_{k=1}^{N} a_k(1_{A_k}(t)-1_{A_k}(t-s))\|^2_V dt \\ &\le \sup_{l\in\{1,\dots,N\}} \|a_l\|^2_V \int_{[-1,T+1]} |\sum_{k=1}^{N}(1_{A_k}(t)-1_{A_k}(t-s))|dt \\ &\le C_2 \int_{[-1,T+1]} sdt \\ &= C_2(T+2)s\end{aligned} \tag{3.15}$$

which tends to 0 as s tends to 0, so for j large enough $\|v(t-s)-v(t)\|_{L^2(\mathbb{R},V)} < \frac{\varepsilon^{1/2}}{3}$. Therefore

$$\begin{aligned}\|u_j-\tilde{u}\|^2_{L^2(\mathbb{R},V)} &\le \int_{\mathbb{R}} h_j(s) \int_{\mathbb{R}} \|\tilde{u}(t-s)-\tilde{u}(t)\|^2_V ds \\ &< \int_{\mathbb{R}} h_j(s)\varepsilon ds = \varepsilon\end{aligned} \tag{3.16}$$

when j is sufficiently large, and (3.2) is hereby shown.

If it is shown that $\|u_j'-u'\|_{L^2(a,b;V^*)} \to 0$ for an arbitrary compact interval $[a,b] \subseteq]0,T[$ then (3.3) is proven. Consider

$$\|(h_j' * \tilde{u})(t)-\tilde{u}'(t)\|_{V^*} = \| \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j'(s)\tilde{u}(t-s)ds - \tilde{u}'(t)\|_{V^*}. \tag{3.17}$$

Let $\phi(s) \in C_0^\infty(\mathbb{R})$, then

$$\begin{aligned}\langle \partial_s(h_j(s)\tilde{u}(t-s)), \phi(s)\rangle &= \langle \tilde{u}(t-s), -h_j(s)\partial_s\phi(s)\rangle \\ &= \langle \tilde{u}(t-s), (\partial_s h_j(s))\phi(s) - \partial_s(h_j(s)\phi(s))\rangle \\ &= \left\langle h_j'(s)\tilde{u}(t-s), \phi(s)\right\rangle - \langle h_j(s)\tilde{u}'(t-s), \phi(s)\rangle\end{aligned}$$

Now the following is well defined for j large enough by Hölders inequality

$$\begin{aligned}&\int_{-\frac{1}{j}}^{\frac{1}{j}} \partial_s(h_j(s)\tilde{u}(t-s))ds \\ &\qquad = \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j'(s)\tilde{u}(t-s)ds - \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j(s)\tilde{u}'(t-s)ds.\end{aligned} \tag{3.18}$$

The integral on the right hand side equals 0_V, so

$$\int_{-\frac{1}{j}}^{\frac{1}{j}} h_j'(s)\tilde{u}(t-s)ds = \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j(s)\tilde{u}'(t-s)ds. \tag{3.19}$$

and therefore

$$\begin{aligned}\int_a^b \|(h_j' * \tilde{u})(t) - \tilde{u}'(t)\|_{V^*} dt &= \int_a^b \| \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j(s)\tilde{u}'(t-s)ds - \tilde{u}'(t)\|_{V^*} dt \\ &= \int_a^b \| \int_{-\frac{1}{j}}^{\frac{1}{j}} h_j(s)(\tilde{u}'(t-s) - \tilde{u}'(t))ds\|_{V^*} dt \\ &\underset{j\to\infty}{\to} 0\end{aligned} \tag{3.20}$$

Which is shown in the same way as for u above, since $u' \in L^2(0,T;V^*)$, and V^* is a Hilbert space like V. □

Lemma 3.2
Let V, H, V^* be three Hilbert spaces, where $V \subset H \subset V^*$, each dense in the following, and with continuous injections. If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V^*)$ the following equation holds in the distribution sense on $]0,T[$

$$\frac{\partial}{\partial t}\|u(t)\|_H^2 = 2\left\langle u'(t), \overline{u(t)}\right\rangle. \tag{3.21}$$

Proof:
Let $u_j(t)$ be given as the restriction of the u_j's defined in (3.5) to $]0,T[$. Since $u_j :]0,T[\to H$ is differentiable

$$\frac{\partial}{\partial t}\|u_j(t)\|_H^2 = \frac{\partial}{\partial t}\left(u_j(t)|u_j(t)\right)_H = 2\left(u_j'(t)|u_j(t)\right)_H = 2\left\langle u_j'(t), \overline{u_j(t)}\right\rangle \tag{3.22}$$

Since $u_j(t)$ and $u_j'(t)$ converges to $u(t)$ and $u'(t)$ in $L^2_{\text{loc}}(]0,T[;V)$ and $L^2_{\text{loc}}(]0,T[;V^*)$ respectively, then $u_j(t)$ and $u_j'(t)$ converges to $u(t)$ and $u'(t)$ on V and V^* respectively a.e. for $t \in]0,T[$. Therefore the duality $\left\langle u_j'(t), \overline{u_j(t)}\right\rangle$ converges to $\left\langle u'(t), \overline{u(t)}\right\rangle$ for $j \to \infty$, which is shown in the same way as Lemma 2.3, where the inner product is replaced by a duality.
Now by (3.5), Lemma 3.1 and Hölder's inequalities $\|u'(t)\|_{V^*}\|u\|_V$ is a integrable majorant to $\left\langle u_j'(t), \overline{u_j(t)}\right\rangle$. Hence Theorem 2.19 gives that $\left\langle u_j'(t), \overline{u_j(t)}\right\rangle$ converges to $\left\langle u'(t), \overline{u(t)}\right\rangle$ in $L^1([a,b])$ for an arbitrary compact interval $[a,b] \subset]0,T[$, hence in $L^1_{\text{loc}}(]0,T[)$.
Likewise $\|u_j(t)\|_H^2 = (u_j(t)|u_j(t))_H \to (u(t)|u(t))_H = \|u(t)\|_H^2$ for $j \to \infty$, and since ∂_t is continuous in $\mathscr{D}'(]0,T[)$, then $\partial_t\|u_j(t)\|_H^2 \to \partial_t\|u(t)\|_H^2$ in the distribution sense, and hence (3.21) is shown in the distribution sense. □

Lemma 3.3
Let X and Y be two Banach spaces with $X \subset Y$, and the injection dense and continuous. Then the injection of Y^* into X^* is dense and continuous.

Proof:
Let I be the injection $X \hookrightarrow Y$, then the adjoint operator $I^*\colon Y^* \to X^*$. Let $x \in X$ and $y^* \in Y^*$, then

$$\langle Ix, y^* \rangle = \langle x, I^* y^* \rangle \tag{3.23}$$

Now I^* is injective, since for $I^* y^* = 0_{X^*}$, then $(I \cdot |y^*)$ is the zero-functional on the dense subspace X of Y, hence $y^* = 0_{Y^*}$.
Deneseness of the injection I^* is shown by using that every norm closed convex subset of a normed space is weakly closed [9, p. 66], so that $\overline{R(I^*)}$ equals the w^* closure of $R(I^*)$. Assume that $\overline{R(I^*)} \neq X^*$, then Proposition 2.4.10 in [9] for every $x^* \in X^* \backslash \overline{R(I^*)}$ gives the existence of an $x' \in (\overline{R(I^*)})^\perp \subset X$ such that

$$\langle x', x^* \rangle \neq 0. \tag{3.24}$$

But the identity (3.23) is true for all $x \in X$ and $y^* \in Y^*$, hence for $y^* \in Y^*$

$$\langle Ix', y^* \rangle = \langle x', I^* y^* \rangle = 0 \tag{3.25}$$

since $x' \in (\overline{R(I^*)})^\perp$, but I is injective, so $x' = 0_X$, which contradicts (3.24), hence the assumption $\overline{R(I^*)} \neq X^*$ must be wrong. □

Lemma 3.4
Let X and Y be two Banach spaces with $X \subset Y$, and the injection dense and continuous. Let ϕ be weakly continuous on $[0,T]$ with values in Y and let $\phi \in L^\infty(0,T;X)$, then ϕ is weakly continuous with values in X.

Proof:
Since the injection of X in Y is dense and continuous, the dual space Y^* is dense and continuously embedded in X^* (Lemma 3.3).
Let $\eta \in Y^*$, then

$$\eta(\phi(t)) = \langle \phi(t), \eta \rangle \to \langle \phi(t_0), \eta \rangle \quad \text{for } t \to t_0,\ \forall\ t_0 \in [0,T]. \tag{3.26}$$

Let

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{on } [0,T] \\ 0 & \text{on } \mathbb{R}\backslash[0,T], \end{cases} \tag{3.27}$$

let h_j be given as in (3.1) for $n = 1$, and let

$$\phi_j(t) = (h_j * \tilde{\phi})(t) = \int_{\mathbb{R}} h_j(t-s)\tilde{\phi}(s)ds. \tag{3.28}$$

Then $\phi_j(t)$ satisfies for $j \in \mathbb{N}$ (Lemma 2.18)

$$
\begin{aligned}
\|\phi_j(t)\|_X &= \Big\| \int_{B(0,\frac{1}{j})} h_j(t-s)\tilde{\phi}(s)ds \Big\|_X \\
&\leq \int_{B(0,\frac{1}{j})} h_j(t-s)\|\tilde{\phi}(s)\|_X ds \\
&\leq \|\phi\|_{L^\infty(0,T;X)} \int_{\mathbb{R}} h_j(t-s)ds \\
&= \|\phi\|_{L^\infty(0,T;X)}.
\end{aligned}
\tag{3.29}
$$

Since $\|\tilde{\phi}(s)\|_X < \infty$ a.e. Definition 2.17 and (3.26) gives for all $\eta \in Y^*$

$$
\begin{aligned}
\langle \phi_j(t) - \phi(t), \eta \rangle &= \Big\langle \int_{\mathbb{R}} h_j(s) \left(\tilde{\phi}(t-s) - \tilde{\phi}(t) \right) ds, \eta \Big\rangle \\
&= \int_{\mathbb{R}} h_j(s) \langle \tilde{\phi}(t-s) - \tilde{\phi}(t), \eta \rangle ds \\
&\underset{j\to\infty}{\to} 0.
\end{aligned}
\tag{3.30}
$$

This is seen by using Theorem 2.19, since $\tilde{\phi}$ is weakly continuous, and therefore for $j \to \infty$, $h_j(s)\langle \tilde{\phi}(t), \eta \rangle \to h_j(s)\langle \tilde{\phi}(t-s), \eta \rangle$, since $s \in B(0, \frac{1}{j})$.
Because $\phi_j(t) \in X$ by (3.29) for all $j \in \mathbb{N}$ and for all $t \in [0,T]$,

$$
|\langle \phi_j(t), \eta \rangle| \leq \|\phi_j(t)\|_X \|\eta\|_{X^*} \leq \|\phi\|_{L^\infty(0,T;X)} \|\eta\|_{X^*}, \tag{3.31}
$$

so in the limit $j \to \infty$

$$
|\langle \phi(t), \eta \rangle| \leq \|\phi\|_{L^\infty(0,T;X)} \|\eta\|_{X^*} \quad \forall \eta \in Y^*, \ t \in [0,T]. \tag{3.32}
$$

The inequality is also true for $\eta \in X^*$ since $Y^* \subset X^*$ densely, so $\phi(t) \in X^{**}$ for all $t \in [0,T]$. Now for all $t \in [0,T]$ and for all $\eta \in X^*$

$$
\langle \phi_j(t) - \phi(t), \eta \rangle \underset{j\to\infty}{\to} 0. \tag{3.33}
$$

Therefore $\phi(t)$ is in the w^*-closure of X^{**}, which equals the w^*-closure of X, which again equals the norm closure of X, hence $\phi(t) \in X$ for all $t \in [0,T]$ and

$$
\|\phi(t)\|_X \leq \|\phi\|_{L^\infty(0,T;X)} \quad \forall \ t \in [0,T]. \tag{3.34}
$$

Let $\eta \in X^*$, and let for $\varepsilon > 0$, $\eta_\varepsilon \in Y^*$ satisfy

$$
\|\eta - \eta_\varepsilon\|_{X^*} \leq \frac{\varepsilon}{3\|\phi\|_{L^\infty(0,T;X)}} \tag{3.35}
$$

Now

$$
\begin{aligned}
|\langle \phi(t) - \phi(t_0), \eta \rangle| &\leq |\langle \phi(t) - \phi(t_0), \eta - \eta_\varepsilon \rangle| + |\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle| \\
&\leq \frac{2}{3}\varepsilon + |\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle|
\end{aligned}
$$

The last term tends to 0 as t tends to t_0 since ϕ is weakly continuous is Y, so there exists a $\delta > 0$ satisfying $|t - t_0| < \delta$

$$|\langle \phi(t) - \phi(t_0), \eta \rangle| \leq \varepsilon \tag{3.36}$$

Since ε was arbitrary the lemma is proved. □

Theorem 3.5
Let V, H, V^* be three Hilbert spaces, where $V \subset H \subset V^*$, each dense in the following, and with continuous injections. If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V^*)$ then u is almost everywhere equal to a continuous function from $[0,T]$ into H.

Proof:
The function $t \to \langle u'(t), u(t) \rangle$ can be considered as the composition of two Borel measurable function as follows

$$t \;\to\; \begin{pmatrix} u'(t) \\ u(t) \end{pmatrix} \;\to\; \langle u'(t), u(t) \rangle$$

Hence $\langle u'(t), u(t) \rangle$ is Borel measurable, and Hölder's inequality shows that the integral of $\langle u'(t), u(t) \rangle$ on $[0,T]$ is finite. Therefore (3.21) gives

$$u \in L^\infty(0,T;H) \tag{3.37}$$

Now for $\phi \in C_0^\infty([0,T])$ integration by parts of $\int_0^T u(t)\phi'(t)dt$ shows that condition 2 in Lemma 1.1 in [13, p. 250] is satisfied, hence u is a.e. equal to a continuous function from $[0,T]$ into V^*.

Therefore by Lemma 3.4 u is weakly continuous on $[0,T]$ with values in H, and then for $t \to t_0 \in [0,T]$

$$(u(t)|u(t_0))_H \to (u(t_0)|u(t_0))_H = \|u(t_0)\|_H^2. \tag{3.38}$$

The theorem is shown if the following is satisfied for all $t_0 \in [0,T]$ for $t \to t_0$

$$\|u(t) - u(t_0)\|_H^2 \to 0. \tag{3.39}$$

Rewriting the norm gives

$$\|u(t) - u(t_0)\|_H^2 = \|u(t)\|_H^2 + \|u(t_0)\|_H^2 - 2\,(u(t)|u(t_0))_H \tag{3.40}$$

Integrating (3.21) from t to t_0, and using Hölders inequality gives for $t \to t_0$

$$\begin{aligned} \|u(t)\|_H^2 - \|u(t_0)\|_H^2 &\leq 2\left|\int_{t_0}^t \langle u'(s), u(s) \rangle_{(V^*,V)}\, ds\right| \\ &\leq 2\|u'\|_{L^2(0,T;V^*)}\|u\|_{L^2(0,T;V)}|t - t_0| \\ &\to 0. \end{aligned} \tag{3.41}$$

Hence (3.39) follows from (3.38), (3.40) and (3.41). □

Theorem 3.6
Let Ω be an open and bounded set, let $u(t,x)$ and $v(t,x)$ be defined on $[0,T]\times\Omega$, with

$$u(t,x) \in L^{\infty}(0,T;H_0^2(\Omega)) \tag{3.42}$$
$$u'(t,x) \in L^{\infty}(0,T;L^2(\Omega)) \tag{3.43}$$

satisfying

- *Equation* (1.4) *and* (1.5) *are satisfied in distribution sense, i.e. they are satisfied in* $\mathscr{D}'(]0,T[,H^{-2}(\Omega))$.
- *The conditions* (1.6)-(1.9) *are satisfied.*

Then the following is also satisfied

1. *The function* $v(t,x) \in L^{\infty}(0,T;H_0^2(\Omega))$.
2. *The functions* $u(t,x)$ *and* $u'(t,x)$ *depends continuously on* t *in the norm topology on* $H_0^2(\Omega)$ *and* $L^2(\Omega)$ *respectively.*

Proof:
Since $[u(t),u(t)]$ is a sum of products of $L^2(\Omega)$ functions, for all $t\in[0,T]$, then $[u(t),u(t)]\in L^{\infty}(0,T;L^1(\Omega))$, because for each term in the sum the norm of the product in $L^1(\Omega)$ is less than or equal to the product of the norms in $L^2(\Omega)$, hence the $L^1(\Omega)$-norm will still be essentially bounded on $[0,T]$.
Let $\varepsilon>0$, let $g\in L^1(\Omega)$ and let $\phi\in H_0^{1+\varepsilon}(\Omega)$, then Sobolevs Theorem gives

$$|\,(g|\phi)\,| \le \|g\|_{L^1(\Omega)}\|\phi\|_{L^{\infty}(\Omega)} \le C\|g\|_{L^1(\Omega)}\|\phi\|_{H_0^{1+\varepsilon}(\Omega)}. \tag{3.44}$$

So $L^1(\Omega)\subseteq H^{-1-\varepsilon}(\Omega)$, and therefore

$$[u(t),u(t)] \in L^{\infty}(0,T;H^{-1-\varepsilon}(\Omega)) \tag{3.45}$$

Now G_2 is a bounded operator from $H^{-1-\varepsilon}(\Omega)$ to $H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)$ (Theorem 2.6) for $\varepsilon>0$, so

$$v(t) = -G_2([u(t),u(t)]) \in L^{\infty}(0,T;H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)) \tag{3.46}$$

Since $\|v(t)\|_{H_0^2(\Omega)} \le \|v(t)\|_{H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)}$ the first part of the theorem is shown.

The von Karman bracket satisfies $[\overline{u}(t),v(t)]\in L^{\infty}(0,T;L^1(\Omega))$, since the norm $\|v(t)\|_{L^2(\Omega)}$ is essentially bounded on $[0,T]$. Therefore the von Karman bracket $[u(t),v(t)]\in L^{\infty}(0,T;H^{-2}(\Omega))$. Now $u''\in L^{\infty}(0,T;H^{-2}(\Omega))$, since (1.4) is satisfied in distribution sense, and all the other terms are in $L^{\infty}(0,T;H^{-2}(\Omega))$. Theorem 3.5 is used on both u and u' giving that $u,u'\in C([0,T];L^2(\Omega))$. □

Chapter 4

Existence of Weak Solutions

In this chapter the existence of weak solutions to the von Karman equations in the stationary case and in the time dependent case will be treated.

It is assumed in the rest of the report, that Δ and Δ^2 are the Dirichlet Realisations Δ_D and Δ_D^2 respectively.
It is also assumed that $\Omega \in \mathbb{R}^2$ is open and bounded and has a smooth boundary $\partial\Omega$. Let $Q = [0, T] \times \Omega$.

4.1 The Stationary Case

In this section the existence of weak solutions of stationary von Karman equations will be shown. It is assumed in this section that the functions are reel valued in order to be able to use Lemma 2.1. The problem reduces to

$$\Delta^2 u(x) - [u(x), v(x)] = f(x) \tag{4.1}$$
$$\Delta^2 v(x) + [u(x), u(x)] = 0 \tag{4.2}$$

In the rest of the report writing the x dependents is omitted, it should be clear from the text whether a function (or distribution) depends on x.

In the stationary case a weak solution of von Karmans equations is defined as:

Definition 4.1
A weak solution of von Karmans equations consists of two functions $u, v \in H_0^2(\Omega)$ which solves (4.1)-(4.2) in distribution sense, i.e. in $\mathscr{D}'(\Omega)$.

It is possible to show the existence of a weak solution to the von Karman equations under the assumption that $f(x) \in H^{-2}(\Omega)$, and not necessarily in $L^2(\Omega)$, as it is assumed in the time dependent case.

Theorem 4.2
Let $f(x) \in H^{-2}(\Omega)$. Then the von Karman equations (4.1) - (4.2) *has a weak solution.*

Proof:
Let $w_1, \dots, w_m, \dots$ be a basis for $H_0^2(\Omega)$ consisting of functions in $C_0^\infty(\Omega)$, and assume that the w_is are an onb. for $H_0^2(\Omega)$ (Lemma 2.9). For $1 \leq i \leq m \in \mathbb{N}$ a function $u_m \in \text{span}\{w_1, \dots, w_m\}$ that solves the following duality between functions in $\mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$

$$\left\langle \Delta^2 u_m + [u_m, G_2([u_m, u_m])], w_i \right\rangle = \langle f, w_i \rangle \tag{4.3}$$

is desired, where G_2 is the inverse to Δ_D^2, which is well defined by Theorem 2.4.

Let $v_m \in H_0^2(\Omega)$ be defined by

$$v_m = -G_2([u_m, u_m]) \tag{4.4}$$

for $u_m \in \text{span}\{w_1, \dots, w_m\}$.
Then u_m solves (4.3) if it solves the following for $1 \leq i \leq m$

$$\begin{aligned} \left\langle \Delta^2 u_m + [u_m, v_m], w_i \right\rangle &= \langle f, w_i \rangle & (4.5)\\ \Delta^2 v_m + [u_m, u_m] &= 0. & (4.6) \end{aligned}$$

A function $u_m \in \text{span}\{w_1, \dots, w_m\}$ can for $\xi \in \mathbb{R}^m$ be written as $u_m(\xi) = \sum_{i=1}^m \xi_i w_i$. Let $P\colon \mathbb{R}^m \to \mathbb{R}^m$ be given by

$$P(\xi) = (\eta_1, \dots, \eta_m) \tag{4.7}$$

where

$$\eta_i = \left\langle \Delta^2 u_m(\xi) + [u_m(\xi), v_m(\xi)], w_i \right\rangle - \langle f, w_i \rangle. \tag{4.8}$$

For each $i \in \{1, \dots, m\}$ the first term in (4.8) depends continuously on ξ according to Lemma 2.11, since the first term equals $(\Delta^2 u_m(\xi) + [u_m(\xi), v_m(\xi)] | w_i)_{L^2(\Omega)}$ and G_2 is a continuous operator, so v_m depends continuously on ξ. The last term in (4.8) is just a constant for each i. Therefore $P(\xi)$ is a continuous function on $\mathbb{R}^m$.
The aim is to use Lemma 2.1, so consider

$$\begin{aligned} (P(\xi)|\xi) &= \sum_{i=1}^m \eta_i \xi_i \\ &= \sum_{i=1}^m \left(\left\langle \Delta^2 u_m(\xi) - [u_m(\xi), v_m(\xi)], \xi_i w_i \right\rangle - \langle f, \xi_i w_i \rangle \right) \\ &= \left\langle \Delta^2 u_m(\xi), u_m(\xi) \right\rangle - \left\langle [u_m(\xi), v_m(\xi)], u_m(\xi) \right\rangle - \langle f, u_m(\xi) \rangle \\ &= \|\Delta u_m\|_{L^2(\Omega)}^2 - \left\langle [u_m(\xi), u_m(\xi)], v_m(\xi) \right\rangle - \langle f \big| u_m(\xi) \rangle \\ &= \|\Delta u_m(\xi)\|_{L^2(\Omega)}^2 + \|\Delta v_m(\xi)\|_{L^2(\Omega)}^2 - \langle f, u_m(\xi) \rangle \end{aligned}$$

According to Schwartz' inequality [3, eq. (9.31)] and Theorem 2.4

$$|\langle f, u_m(\xi)\rangle| \leq \|f\|_{H^{-2}(\Omega)}\|u_m(\xi)\|_{H_0^2(\Omega)} \leq C_1\|\Delta u_m(\xi)\|_{L^2(\Omega)}, \tag{4.9}$$

then

$$(P(\xi),\xi) \geq \|\Delta u_m(\xi)\|^2_{L^2(\Omega)} + \|\Delta v_m(\xi)\|^2_{L^2(\Omega)} - C_1\|\Delta u_m(\xi)\|_{L^2(\Omega)}. \tag{4.10}$$

Hence $(P(\xi),\xi) \geq 0$ if

$$\|\Delta u_m(\xi)\|_{L^2(\Omega)} \geq C_1. \tag{4.11}$$

This should be satisfied for all ξ, with $|\xi| = \rho$, for some $\rho > 0$. Let $|\xi| = 1$, and let $\xi' = s\xi$ for some $s > 0$. Since the w_i's are linearly independent

$$\|\Delta u_m(\xi')\|_{L^2(\Omega)} = \int_\Omega \Big|\sum_{i=1}^m s\Delta(\xi_i w_i(x))\Big|^2 dx = s^2\|\Delta u_m(\xi)\|_{L^2(\Omega)}. \tag{4.12}$$

Hence it is possible to satisfy (4.11) for t large enough if $\|\Delta u_m(\xi)\|_{L^2(\Omega)} \neq 0$ for all ξ.
Assume that $\|\Delta u_m(\xi)\|_{L^2(\Omega)} = 0$ for some ξ, then

$$\|u_m(\xi)\|_{H_0^2(\Omega)} \leq C_2\|\Delta u_m(\xi)\|_{L^2(\Omega)} = 0. \tag{4.13}$$

Hence $\|u_m(\xi)\|_{H_0^2(\Omega)} = 0$, so $u_m(\xi) = 0_{H_0^2(\Omega)}$ which contradicts that $|\xi| = 1$ and the w_i's are independent. Therefore the assumption that $\|\Delta u_m(\xi)\|_{L^2(\Omega)} = 0$ for some ξ must be wrong. So it is possible to satisfy $(P(\xi'),\xi') \geq 0$ for all ξ' with $|\xi'| = \rho$ for some $\rho > 0$.

Now according to Lemma 2.1 there exists a $u_m \in \operatorname{span}\{w_1,\ldots,w_m\}$ that solves (4.5) and hence (4.3).

Choose a sequence of functions $(u_m)_{m\in\mathbb{N}}$ each satisfying (4.3) for $i \leq m$. Then it follows from (2.7) that

$$\begin{aligned} \|u_m\|^2_{H_0^2(\Omega)} + \|v_m\|^2_{H_0^2(\Omega)} &\leq C_3(\|\Delta u_m\|^2_{L^2(\Omega)} + \|\Delta v_m\|^2_{L^2(\Omega)}) \\ &= C_3\langle f, u_m\rangle \\ &\leq C_4\|\Delta u_m\|_{L^2(\Omega)} \\ &\leq C_4\|u_m\|_{H_0^2(\Omega)}. \end{aligned} \tag{4.14}$$

So the sequence $(u_m)_{m\in\mathbb{N}}$ and the corresponding sequence $(v_m)_{m\in\mathbb{N}}$ are contained in a bounded set in $H_0^2(\Omega)$. Since $H_0^2(\Omega)$ is a closed subspace of a Hilbert space, $H_0^2(\Omega)$ is a Hilbert space, hence it can be identified with its own dual space. According to Alaoglu's Theorem [9, p. 70] the unit ball and hence every bounded set in $H_0^2(\Omega)$ is w^*-compact, so there exists subsequences $(u_\mu)_{\mu\in\mathbb{N}}$ and $(v_\mu)_{\mu\in\mathbb{N}}$ satisfying

$$\begin{aligned} u_\mu &\to u \quad \text{weakly on } H_0^2(\Omega) \\ v_\mu &\to v \quad \text{weakly on } H_0^2(\Omega). \end{aligned} \tag{4.15}$$

According to Theorem 2.8 there exists subsequences $(u_\gamma)_{\gamma\in\mathbb{N}}$ and $(v_\gamma)_{\gamma\in\mathbb{N}}$ of $(u_\mu)_{\mu\in\mathbb{N}}$ and $(v_\mu)_{\mu\in\mathbb{N}}$ respectively satisfying

$$\begin{aligned} u_\gamma \to u &\quad \text{strongly on } L^2(\Omega) \\ v_\gamma \to v &\quad \text{strongly on } L^2(\Omega). \end{aligned} \tag{4.16}$$

Let i be fixed, with $\gamma \geq i$, then

$$\left(\Delta^2 u_\gamma | w_i\right)_{L^2(\Omega)} - ([u_\gamma, v_\gamma]|w_i)_{L^2(\Omega)} = \langle f, w_i\rangle. \tag{4.17}$$

Now it follows from Lemma 2.3, that

$$\left(\Delta^2 u_\gamma | w_i\right)_{L^2(\Omega)} = \left(u_\gamma | \Delta^2 w_i\right)_{L^2(\Omega)} \to \left(u | \Delta^2 w_i\right)_{L^2(\Omega)} = \left(\Delta^2 u | w_i\right)_{L^2(\Omega)}. \tag{4.18}$$

Furthermore Lemma 2.12 and Lemma 2.11 gives

$$([u_\gamma, v_\gamma]|w_i)_{L^2(\Omega)} = ([w_i, u_\gamma]|v_\gamma)_{L^2(\Omega)} \to ([w_i, u]|v)_{L^2(\Omega)} = ([u, v]|w_i)_{L^2(\Omega)} \tag{4.19}$$

Therefore for all $i \in \mathbb{N}$

$$\left\langle \Delta^2 u, w_i \right\rangle - \left\langle [u, v], w_i \right\rangle = \langle f, w_i\rangle \tag{4.20}$$

Now (4.20) is true for any finite linear combination of the w_i's, hence for all $w \in H_0^2(\Omega)$. Because $C_0^\infty(\Omega) \in H_0^2(\Omega)$, then $u,\ v$ solves (4.1) in the distribution sense.

By the definition of v_γ, it follows by Lemma 2.12 and Lemma 2.11 that

$$\begin{aligned} 0 &= \left(\Delta^2 v_\gamma | w_i\right)_{L^2(\Omega)} + ([u_\gamma, u_\gamma]|w_i)_{L^2(\Omega)} \\ &= \left(v_\gamma | \Delta^2 w_i\right)_{L^2(\Omega)} + ([w_i, u_\gamma]|u_\gamma)_{L^2(\Omega)} \\ &\to \left(v | \Delta^2 w_i\right)_{L^2(\Omega)} + ([w_i, u]|u)_{L^2(\Omega)} \\ &= \left(\Delta^2 v | w_i\right)_{L^2(\Omega)} + ([u, u]|w_i)_{L^2(\Omega)} \end{aligned}$$

hence u, v solves (4.2) in $\mathscr{D}'(\Omega)$. □

4.2 The Time Dependent Case

In this section existence of weak solutions of the time dependent von Karman equations is shown. The time dependent von Karman equations are given by

$$u''(t,x) + \Delta^2 u(t,x) - [\overline{u(t,x)}, v(t,x)] = f(t,x) \qquad \text{on} \quad]0,T[\times\,\Omega \tag{4.21}$$

$$\Delta^2 v(t,x) + [u(t,x), u(t,x)] = 0 \qquad \text{on} \quad]0,T[\times\,\Omega. \tag{4.22}$$

These von Karman equations are evaluated with the following boundary and initial conditions for $t \in [0,T]$ and for $x \in \Omega$

$$\gamma_0 u(t,x) = \gamma_0 v(t,x) = 0 \tag{4.23}$$
$$\gamma_1 u(t,x) = \gamma_1 v(t,x) = 0 \tag{4.24}$$
$$r_0 u(t,x) = u_{01}(x) \tag{4.25}$$
$$r_1 u'(t,x) = u_{11}(x) \tag{4.26}$$

The problem is investigated with the following properties of the initial data for $Q =]0,T[\times\Omega$

$$\begin{aligned} f(t,x) &\in L^2(Q) \\ u_0(x) &\in H_0^2(\Omega) \\ u_1(x) &\in L^2(\Omega). \end{aligned} \tag{4.27}$$

The inverse G_2 of Δ^2 (Theorem 2.4) is used to eliminate v from (4.21), when $u(t) \in H_0^2(\Omega)$, then

$$v(t) = -G_2([u(t), u(t)]) \tag{4.28}$$

Therefore if $u(t) \in H_0^2(\Omega)$ a.e. on $[0,T]$, and then (4.21) is equivalent to

$$u''(t) + \Delta^2 u(t) + [\overline{u(t)}, G_2([u(t), u(t)])] = f(t) \tag{4.29}$$

which does not depend on $v(t)$.

Theorem 4.3
Let (4.27) be satisfied. Then the problem (4.21)-(4.26) has a weak solution (Definition 1.2).

Proof:
Let $w_1, \ldots, w_m, \ldots$ be a basis for $H_0^2(\Omega)$ consisting of functions in $C_0^\infty(\Omega)$ for all $t \in [0,T]$ and assume that the w_is is an onb. for $H_0^2(\Omega)$. Let $u_m(t) \in \operatorname{span}\{w_1, \ldots, w_m\}$ for all $t \in [0,T]$, then $u_m(t)$ can be written as

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i \tag{4.30}$$

for some coefficients $g_{im}(t)$.

A solution is desired to the following system of m equations for $1 \le j \le m$

$$\begin{aligned} \left(u_m''(t), w_j\right)_{L^2(\Omega)} + (\Delta u_m(t), \Delta w_j)_{L^2(\Omega)} + \left([\overline{u_m(t)}, G_2([u_m(t), u_m(t)])], w_j\right)_{L^2(\Omega)} \\ = (f(t), w_j)_{L^2(\Omega)} \end{aligned} \tag{4.31}$$

with initial conditions

$$u_m(0) = u_{01m} \in \text{span}[w_1, \dots, w_m], \qquad u_{01m} \to u_{01} \quad \text{in } H_0^2(\Omega) \tag{4.32}$$
$$u'_m(0) = u_{11m} \in \text{span}[w_1, \dots, w_m], \qquad u_{11m} \to u_{11} \quad \text{in } L^2(\Omega). \tag{4.33}$$

For $t \in [0,T]$ and for $u_m(t)$ defined by (4.30) equation (4.31) can be written as a matrix equation

$$\begin{bmatrix} (w_1|w_1)_{L^2(\Omega)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (w_m|w_m)_{L^2(\Omega)} \end{bmatrix} \begin{bmatrix} g''_{1m}(t) \\ \vdots \\ g''_{mm}(t) \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(t, g_{1m}(t)) \\ \vdots \\ \tilde{F}_m(t, g_{mm}(t)) \end{bmatrix} \tag{4.34}$$

Since the matrix $[(w_i|w_j)]$ is invertible the matrix equation has a solution, giving

$$g''_{im}(t) = F_i(t, g_{im}(t)) \tag{4.35}$$

Where F_i depends continuously on g_{im}, and $F_i(\cdot, g_{im}) \in L^1([0,T])$. The second order ordinary differential equation (4.35) is a composition of two equations of the type described in Theorem 2.21 - Theorem 2.23, with the initial conditions given by (4.32) and (4.33). Therefore (4.35) and thereby (4.31) has solutions satisfying

$$\begin{aligned} u_m(t) &\in C^1([0,T], C_0^\infty(\Omega)) \\ u'_m(t) &\in C([0,T], C_0^\infty(\Omega)) \end{aligned} \tag{4.36}$$

These solutions also satisfy that $u''_{im}(t) \in L^2([0,T]; C_0^\infty(\Omega))$, because they solve (4.31), with $f(t) \in L^2(Q)$.

Define for $t \in [0,T]$ the sequence $(v_m(t))_{m\in\mathbb{N}} \in H_0^2(\Omega)$ by

$$v_m(t) = -G_2([u_m(t), u_m(t)]) \tag{4.37}$$

Now v_m can be inserted in (4.31) giving

$$\begin{aligned} \big(u''_m(t), w_j\big)_{L^2(\Omega)} + (\Delta u_m(t), \Delta w_j)_{L^2(\Omega)} - \big([\overline{u_m(t)}, v_m(t)], w_j\big)_{L^2(\Omega)} & \\ = (f(t), w_j)_{L^2(\Omega)} & \end{aligned} \tag{4.38}$$

for $1 \le j \le m$.

Multiplying the first term in (4.38) by $g'_{jm}(t)$ and adding the equations for $j = 1, \dots, m$ gives by using Leibniz' Formula and Theorem 2.20

$$\begin{aligned} \sum_{j=1}^m g'_{jm}(t)\,(u''_m(t)|w_j)_{L^2(\Omega)} &= \sum_{j=1}^m \int_\Omega u''_m(t)\overline{g'_{jm}(t)w_j}dx \\ &= \int_\Omega u''_m(t)\overline{u'_m(t)}dx \\ &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}u'_m(t)\overline{u'_m(t)}dx \\ &= \frac{1}{2}\frac{\partial}{\partial t}\|u'_m(t)\|^2_{L^2(\Omega)} \end{aligned}$$

Making the same evaluation on the second term in (4.38) gives that $(u_m(t))_{m\in\mathbb{N}}$ and $(v_m(t))_{m\in\mathbb{N}}$ solves

$$\frac{1}{2}\frac{\partial}{\partial t}\big(\|u_m'(t)\|_{L^2} + \|\Delta u_m(t)\|_{L^2}^2\big) - \big([\overline{u_m(t)}, v_m(t)]|u_m'(t)\big)_{L^2(\Omega)} = (f(t)|u_m'(t))_{L^2(\Omega)} \quad (4.39)$$

It is possible to rewrite the third term using Lemma 2.12 and Theorem (2.20)

$$\begin{aligned}
-\big([\overline{u_m(t)}, v_m(t)]\big|u_m'(t)\big)_{L^2(\Omega)}
&= -\,\big([\overline{u_m(t)}, \overline{u_m'(t)}]\big|\overline{v_m(t)}\big)_{L^2(\Omega)} \\
&= -\,\big(D_1^2\overline{u_m}D_2^2\overline{u_m'} + D_2^2\overline{u_m}D_1^2\overline{u_m'} - 2D_1D_2\overline{u_m}D_1D_2\overline{u_m'}\big|\overline{v_m}\big)_{L^2(\Omega)} \\
&= -\,\big(\frac{\partial}{\partial t}(D_1^2\overline{u_m}D_2^2\overline{u_m} - D_1D_2\overline{u_m}D_1D_2\overline{u_m})\big|\overline{v_m}\big)_{L^2(\Omega)} \\
&= -\,\frac{1}{2}\big(\frac{\partial}{\partial t}[\overline{u_m(t)}, \overline{u_m(t)}]\big|\overline{v_m(t)}\big)_{L^2(\Omega)} \\
&= \frac{1}{2}\big(\Delta^2\overline{v_m'(t)}, \overline{v_m(t)}\big)_{L^2(\Omega)} \\
&= \frac{1}{4}\frac{\partial}{\partial t}\|\Delta v_m(t)\|_{L^2(\Omega)}^2
\end{aligned}$$

In the last step $\frac{\partial}{\partial t}$ and Δ are interchanged since $v_m(t)$ is continuously differentiable with respect to t, and infinitely differentiable with respect to x.

Hence (4.39) is equivalent to

$$\frac{1}{2}\frac{\partial}{\partial t}\big(\|u_m'(t)\|_{L^2(\Omega)}^2 + \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta v_m(t)\|_{L^2(\Omega)}^2\big) = (f(t)|u_m'(t))_{L^2(\Omega)} \quad (4.40)$$

Integrating this with respect to t gives

$$\begin{aligned}
\frac{1}{2}(\|u_m'(t)\|_{L^2(\Omega)}^2 + \|\Delta u_m(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta v_m(t)\|_{L^2(\Omega)}^2) \\
= \frac{1}{2}(\|u_m'(0)\|_{L^2(\Omega)}^2 + \|\Delta u_m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta v_m(0)\|_{L^2(\Omega)}) \\
+ \int_0^t (f(\sigma)|u_m'(\sigma))_{L^2(\Omega)}d\sigma \quad (4.41)
\end{aligned}$$

The sequences $\{u_{01m}\}$ and $\{u_{11m}\}$ are both bounded (Lemma 2.2), so $\{\Delta u_{01m}\}$ is bounded since $\|u_{01m}\|_{H_0^2(\Omega)} \geq \|\Delta u_{01m}\|_{L^2(\Omega)}$, therefore

$$\|u_{11m}\|_{L^2(\Omega)}^2 + \|\Delta u_{01m}\|_{L^2(\Omega)}^2 \leq C_1 \quad (4.42)$$

Since the initial conditions (4.32) and (4.33) are satisfied by u_m, then

$$v_m(0) = -G_2([u_{01m}, u_{01m}]) \quad (4.43)$$

The last term is a composition of continuous operators on u_{01m}, so the sequence $(v_m(0))_{m\in\mathbb{N}}$ is bounded.

The integral in (4.41) is also bounded, which is shown by Cauchy-Schwarz' inequality since

$$\begin{aligned}|\int_0^t (f(\sigma)|u_m'(\sigma))_{L^2(\Omega)}d\sigma| &\le \int_0^T \|f(\sigma)\|_{L^2(\Omega)}\|u_m'(\sigma)\|_{L^2(\Omega)}d\sigma \\ &\le \left(\int_0^T \|f(\sigma)\|^2_{L^2(\Omega)}d\sigma\right)^{1/2}\left(\int_0^T \|u_m'(\sigma)\|^2_{L^2(\Omega)}d\sigma\right)^{1/2} \\ &\le C_2.\end{aligned}$$

Therefore there exists a constant $C_3 > 0$, satisfying

$$\|u_m'(t)\|^2_{L^2(\Omega)} + \|\Delta u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\Delta v_m(t)\|^2_{L^2(\Omega)} \le C_3 \tag{4.44}$$

For $t \in [0,T]$ it follows by (2.7) that

$$\begin{aligned}\{u_m(t)\},\{v_m(t)\} &\quad \text{is bounded in } L^\infty(0,T;H_0^2(\Omega)) \\ \{u_m'(t)\} &\quad \text{is bounded in } L^\infty(0,T;L^2(\Omega))\end{aligned}$$

The Banach space $L^\infty(0,T;X)$ is the dual space to the Banach space $L^1(0,T;X)$, so it is possible according to Alauglus Theorem to extract subsequences $(u_\mu)_{\mu\in\mathbb{N}}$ and $(v_\mu)_{\mu\in\mathbb{N}}$ satisfying

$$\begin{aligned} u_\mu \to u \quad w^* \text{ on} &\quad L^\infty(0,T;H_0^2(\Omega)) \\ v_\mu \to v \quad w^* \text{ on} &\quad L^\infty(0,T;H_0^2(\Omega)) \\ u_\mu' \to u' \quad w^* \text{ on} &\quad L^\infty(0,T;L^2(\Omega))\end{aligned} \tag{4.45}$$

Since $(u_\mu)_{\mu\in\mathbb{N}}$ converges to u on $L^\infty(0,T;H_0^2(\Omega))$ considered with the w^*-topology, then it converges weakly to u on $L^\infty(0,T;H_0^2(\Omega))$ (considered with the norm-topology). Likewise $(u_\mu')_{\mu\in\mathbb{N}}$ converges weakly to u' on $L^\infty(0,T;L^2(\Omega))$.
Let $W = \{w|w \in L^2(0,T;H_0^2(\Omega)), w' \in L^2(0,T;L^2(\Omega))$. Then Theorem 5.1 in [7, p. 58] gives that W is compactly injected into $L^2(0,T;L^2(\Omega))$.
By evaluating the norms it is seen that $L^\infty(0,T;H_0^2(\Omega))$ is continuously injected into $L^2(0,T;H_0^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ is continuously injected into $L^2(0,T;L^2(\Omega))$. Hence the sequence $(u_\mu)_{\mu\in\mathbb{N}}$ is compactly injected into $L^2(0,T;L^2(\Omega))$. Therefore $(u_\mu)_{\mu\in\mathbb{N}}$ has a subsequence $(u_\gamma)_{\gamma\in\mathbb{N}}$ that converges strongly in $L^2(0,T;L^2(\Omega))$. Since $L^2(0,T;L^2(\Omega))$ is continuously injected into $L^2(Q)$, then the subsequence satisfies

$$u_\gamma \to u \text{ on } L^2(Q). \tag{4.46}$$

The corresponding sequence $(v_\gamma)_{\gamma\in\mathbb{N}}$ converges according to (4.45) to v in the w^* topology on $L^\infty(0,T;H_0^2(\Omega))$.

Let for $1 \leq j \leq j_0$, $\phi_j \in C^1([0,T])$, let $\phi_j(T) = 0$ and let

$$\psi(t,x) = \sum_{j=1}^{j_0} \phi_j(t) w_j(x) \tag{4.47}$$

The following integral can for $\gamma \geq j_0$ be written as

$$\begin{aligned}
&\int_0^T \Big(\sum_{j=1}^{j_0} \{ \overline{\phi_j(t)} \left(u_\gamma''(t,x) | w_j(t,x) \right)_{L^2(\Omega)} \} \Big) dt \\
&= \int_0^T \left(u_\gamma''(t,x) | \psi(t,x) \right)_{L^2(\Omega)} dt \\
&= \int_0^T \int_\Omega u_\gamma''(t,x) \overline{\psi(t,x)} dx dt \\
&= \int_\Omega \int_0^T u_\gamma''(t,x) \overline{\psi(t,x)} dt dx \\
&= \int_\Omega \Big([u_\gamma'(t,x)\overline{\psi(t,x)}]_0^T - \int_0^T u_\gamma'(t,x) \overline{\psi'(t,x)} dt \Big) dx \\
&= - \left(u_{11\gamma} | \psi(0,x) \right)_{L^2(\Omega)} - \int_0^T \left(u_\gamma'(t,x) | \psi'(t,x) \right)_{L^2(\Omega)} dt
\end{aligned} \tag{4.48}$$

Fubinis Theorem is used twice, because in both cases the integrand is absolutely integrable by Cauchy-Schwarz' inequality.

Now multiplying (4.38) by $\overline{\phi_j}$, adding the equations for $j = 1, \ldots, j_0$ and integrating with respect to t gives for $\gamma \geq j_0$

$$\begin{aligned}
&- \int_0^T \left(u_\gamma' | \psi' \right)_{L^2(\Omega)} dt + \int_0^T \left(\Delta u_\gamma | \Delta \psi \right)_{L^2(\Omega)} dt - \int_0^T \left([\overline{u_\gamma}, v_\gamma] | \psi \right)_{L^2(\Omega)} dt \\
&\qquad = \int_0^T (f|\psi)_{L^2(\Omega)} dt + \left(u_{11\gamma} | \psi(0) \right)_{L^2(\Omega)}
\end{aligned} \tag{4.49}$$

Integration of a continuous function over the interval $[0,T]$ is a continuous operator, hence Theorem (2.3) and Lemma (2.12) gives

$$\begin{aligned}
&\int_0^T \left([\overline{u_\gamma}, v_\gamma] | \psi \right)_{L^2(\Omega)} dt = \int_0^T \left([\overline{\psi}, \overline{u_\gamma}] | \overline{v_\gamma} \right)_{L^2(\Omega)} dt \\
&\qquad \to \int_0^T \left([\overline{\psi}, \overline{u}] | \overline{v} \right)_{L^2(\Omega)} dt = \int_0^T \left([\overline{u}, v] | \psi \right)_{L^2(\Omega)} dt
\end{aligned} \tag{4.50}$$

For $\gamma \to \infty$ (4.49) becomes

$$\begin{aligned}
&- \int_0^T \left(u' | \psi' \right)_{L^2(\Omega)} dt + \int_0^T \left(\Delta u | \Delta \psi \right)_{L^2(\Omega)} dt - \int_0^T \left([\overline{u}, v] | \psi \right)_{L^2(\Omega)} dt \\
&\qquad = \int_0^T (f|\psi)_{L^2(\Omega)} dt + \left(u_{11} | \psi(0) \right)_{L^2(\Omega)}
\end{aligned} \tag{4.51}$$

As a help to show that $u_{11} = u'(0)$ a positive function $\eta(t) \in C_0^\infty([-T,T])$, satisfying

$$\begin{aligned} \eta(t) &= 1 \quad \text{for } t \in [0, \frac{T}{4}] \\ \eta(t) &= 0 \quad \text{for } t \in [\frac{3T}{4}, T] \\ \eta(t) = \eta(-t) &\leq 1 \quad \text{for all } t \\ \int_{\mathbb{R}} \eta(t)dt &= 1. \end{aligned} \tag{4.52}$$

is chosen. Let $w \in \text{span}\{w_1, \dots, w_{j_0}\}$. Then a sequence of functions η_k is defined by

$$\eta_k(t) = \eta(kt)w \tag{4.53}$$

This sequence fulfils the critirias for being defined by (4.47), hence solves (4.51) for $\eta_k = \psi$. For the first term in (4.51)

$$\begin{aligned} &\Big| \int_0^T (u'(t)|\eta'(t))_{L^2(\Omega)}\, dt - (u'(0)|w)_{L^2(\Omega)} \Big| \\ &\qquad = \Big| \int_0^T k\eta'(kt)\, (u'(t)|w)_{L^2(\Omega)}\, dt - (u'(0)|w)_{L^2(\Omega)} \Big| \\ &\qquad = \Big| \int_0^T k\eta'(kt)\, (u'(t) - u'(0)|w)_{L^2(\Omega)}\, dt \Big| \\ &\qquad \leq \sup_{t \in [0, \frac{T}{k}]} \|u'(t) - u'(0)\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \Big| \int_0^T k\eta'(kt)dt \Big| \\ &\qquad = \sup_{t \in [0, \frac{T}{k}]} \|u'(t) - u'(0)\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \end{aligned} \tag{4.54}$$

Which tends to 0 for $k \to \infty$ by Theorem 3.6, since (4.45) is satisfied.

The second term in (4.51) is evaluated by using Cauchy-Schwarz' inequality twice

$$\begin{aligned} \Big| \int_0^T (\Delta u|\Delta \eta_k)_{L^2(\Omega)}\, dt \Big| &\leq \int_0^T \|\Delta u\|_{L^2(\Omega)} \|\Delta \eta_k\|_{L^2(\Omega)} dt \\ &\leq \int_0^T \|\Delta u\|_{L^2(\Omega)} \|\Delta w\|_{L^2(\Omega)} |\eta_k(t)| dt \\ &\leq \|\Delta u\|_{L^2(Q)} \|\Delta w\|_{L^2(\Omega)} \|\eta_k(t)\|_{L^2([0,T])} \end{aligned} \tag{4.55}$$

On the interval $[0,T]$ the support of η_k is contained in $]0, \frac{3T}{4k}[$, hence

$$\|\eta_k(t)\|^2_{L^2([0,T])} \leq \int_0^T |\eta_k|^2 dt \leq \int_0^{\frac{3T}{4k}} 1 dt \to 0 \tag{4.56}$$

for $k \to \infty$, hence the second term in (4.51) tends to 0.
The third term in (4.51) gives for $\psi = \eta_k$

$$\Big| \int_0^T ([\overline{u}, v]|\eta_k)_{L^2(\Omega)} \Big| \leq C_3 \|u\|_{L^2(Q)} \|v\|_{L^2(Q)} \|w\|_{L^2(Q)} \|\eta_k\|^2_{L^2([0,T])} \tag{4.57}$$

which tends to 0 as $k \to \infty$. The forth term also tends to 0. Therefore (4.51) reduces to

$$(u'(0)|w)_{L^2(\Omega)} = (u_{11}|w) \tag{4.58}$$

Since $\eta_k(0) = w$. The equation above is true for $j_0 \in \mathbb{N}$, i.e. for $w \in \operatorname{span}\{w_j | j \in \mathbb{N}\}$ which is dense in $L^2(\Omega)$, hence $u'(0) = u_{11}$.

The inner products in (4.51) can also be written as dualities, giving for ψ defined as in (4.47)

$$-\int_0^T \left\langle u', \overline{\psi'} \right\rangle dt + \int_0^T \left\langle \Delta u, \Delta\overline{\psi} \right\rangle dt - \int_0^T \left\langle [\overline{u}, v], \overline{\psi} \right\rangle dt = \int_0^T \left\langle f, \overline{\psi} \right\rangle dt + \left\langle u'(0), \overline{\psi(0)} \right\rangle \tag{4.59}$$

The time derivative of $u'(t)$ is considered, it exists as an element of $\mathscr{D}'(0, T; H^{-2}(\Omega))$. Let $\phi(t) \in C_0^\infty(]0, T[)$, then $u''(t)\colon \phi(t) \to H^{-2}(\Omega)$, giving the duality

$$\langle u''(t), \phi(t) \rangle = -\langle u'(t), \phi'(t) \rangle \text{ in } H^{-2}(\Omega) \tag{4.60}$$

This is a functional on $H_0^2(\Omega)$, hence for $j \in \mathbb{N}$

$$\langle \langle u''(t), \phi(t) \rangle, w_j \rangle = -\langle \langle u'(t), \phi'(t) \rangle, w_j \rangle \tag{4.61}$$

Since $u' \in C(0, T; L^2(\Omega))$ the last term in (4.61) is written as an integral and by the Bochner identity (Definition 2.17)

$$\begin{aligned} -\langle \langle u''(t), \phi(t) \rangle, w_j \rangle &= \left\langle \int_0^T u'(t)\phi'(t)dt, w_j \right\rangle \\ &= \int_0^T \langle u'(t)\phi'(t), w_j \rangle \, dt \\ &= \int_0^T \phi'(t) \langle u'(t), w_j \rangle \, dt \\ &= \int_0^T \langle u'(t), \phi'(t) w_j \rangle \, dt \end{aligned} \tag{4.62}$$

The function $\phi(t)w_j$ is defined as described in (4.47), so for $\overline{\psi} = \phi(t)w_j$ equation (4.59) is substituted into (4.62), which leads to

$$\begin{aligned} \langle \langle u''(t), \phi(t) \rangle, w_j \rangle &= \int_0^T \left\langle \Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t), \phi(t) w_j \right\rangle dt \\ &= \int_0^T \left\langle (\Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t))\phi(t), w_j \right\rangle dt \\ &= \left\langle \int_0^T (\Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t))\phi(t) dt, w_j \right\rangle \\ &= \left\langle \left\langle \Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t), \phi(t) \right\rangle, w_j \right\rangle \end{aligned} \tag{4.63}$$

The duality in (4.63) is satisfied when w_j is replaced with any finite linear combinations of the w_i's, and an evaluation of the limit gives that it is satisfied for any $w \in H_0^2(\Omega)$. Hence

$$\langle u''(t), \phi(t)\rangle = \left\langle \Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t), \phi(t) \right\rangle \text{ in } H^{-2}(\Omega). \tag{4.64}$$

The test function $\phi(t) \in C_0^\infty(]0,T[)$ is arbitrary, so

$$u''(t) = \Delta^2 u(t) - [\overline{u(t)}, v(t)] - f(t) \text{ in } \mathscr{D}'(0,T; H^{-2}(\Omega)). \tag{4.65}$$

Hereby (4.21) is shown in the vector distribution sense.

Lemma 2.11 and Lemma 2.12 shows by introduction of a $w \in C_0^\infty(\Omega)$, that

$$\left\langle \Delta^2 v(t), w \right\rangle = \langle [u,u], w\rangle \tag{4.66}$$

which extends to $w \in H_0^2(\Omega)$. Therefore $\Delta^2 v(t) = [u,u]$ in $H^{-2}(\Omega)$, then they are also equal in $\mathscr{D}'(0,T; H^{-2}(\Omega))$, and (4.22) is solved in the vector distribution sense.

It is shown in Chapter 3 that if the other conditions in Definition 1.2 are satisfied, then (1.13) and Condition 3 are automatically satisfied. □

Chapter 5

Uniqueness of Weak Solutions

In this chapter uniqueness of weak solutions to the von Karman equations is shown. The problem treated is again

$$u''(t,x) + \Delta^2(t,x) - [\overline{u(t,x)}, v(t,x)] = f(t,x) \qquad \text{on} \quad]0,T[\times\ \Omega \tag{5.1}$$
$$\Delta^2 v(t,x) + [u(t,x), u(t,x)] = 0 \qquad \text{on} \quad]0,T[\times\ \Omega \tag{5.2}$$

with the following boundary and initial conditions for $t \in]0,T[$ and for $x \in \Omega$

$$\gamma_0 u(t,x) = \gamma_0 v(t,x) = 0 \tag{5.3}$$
$$\gamma_1 u(t,x) = \gamma_1 v(t,x) = 0 \tag{5.4}$$
$$r_0 u(t,x) = u_{01}(x) \tag{5.5}$$
$$r_1 u'(t,x) = u_{11}(x). \tag{5.6}$$

There is a small difference between this problem and the problem treated by Boutet de Monvel and Chueshov in [8]. In the problem they treat f do not depend on time, but since this makes no difference in the proof of uniqueness of weak solutions, which will be seen later, it is assumed that f do depend on time. So the assumptions on the initial data are as before

$$\begin{aligned} u_{01}(x) &\in H_0^2(\Omega) \\ u_{11}(x) &\in L^2(\Omega) \\ f(t,x) &\in L^2(Q). \end{aligned} \tag{5.7}$$

Boutet de Monvel and Choeshov's definition of a weak solution to the problem above also seems to differ a little from the definition used in Chapter 4. Their definition is:

Definition 5.1
The functions $u(t,x)$ and $v(t,x)$ are a weak solution of the problem (5.1) - (5.6) on the interval $[0,T]$ if

$$u(t,x) \in L^\infty(0,T; H_0^2(\Omega)) \quad \text{and} \quad u'(t,x) \in L^\infty(0,T; L^2(\Omega)) \tag{5.8}$$

and if the following conditions are satisfied

1. *The equations* (5.1) *and* (5.2) *are satisfied in the vector distribution sense.*

2. (5.3)-(5.6) *is satisfied.*

3. *The vector-valued function* $t \to (u(t), u'(t)) \in H_0^2(\Omega) \times L^2(\Omega)$ *is weakly continuous.*

Condition 3 in Definition 5.1 does not seem to be satisfied by the weak solution defined in Chapter 1, but a closer inspection will show that it is.

Assume for $t \to t_0$, that

$$\begin{aligned} u(t) &\to u(t_0) \text{ in } H_0^2(\Omega) \\ u'(t) &\to u'(t_0) \text{ in } L^2(\Omega) \end{aligned} \tag{5.9}$$

where the convergences are in the norm topology on $H_0^2(\Omega)$ and $L^2(\Omega)$ respectively. Let $z_1 \in H^{-2}(\Omega)$ and let $z_2 \in L^2(\Omega)$. Then a functional $\Lambda \in (H_0^2(\Omega) \times L^2(\Omega))^*$ is given by

$$\Lambda(u(t), u'(t)) = \langle z_1, u(t) \rangle + \int_\Omega z_2 u'(t) dx \tag{5.10}$$

Weak continuity of $(u(t), u'(t)) \in H_0^2(\Omega) \times L^2(\Omega)$ is shown if the following tends to 0 for $t \to t_0$, since Λ is arbitrary,

$$\begin{aligned} &\Lambda((u(t), u'(t)) - (u(t_0), u'(t_0))) \\ &\qquad = \langle z_1, u(t) \rangle + \int_\Omega z_2 u'(t) dx - \langle z_1, u(t_0) \rangle - \int_\Omega z_2 u'(t_0) dx \\ &\qquad = (z_1 | u(t) - u(t_0))_{L^2(\Omega)} + (z_2 | u'(t) - u'(t_0))_{L^2(\Omega)} . \end{aligned} \tag{5.11}$$

Both terms tends to 0, because of (5.9).

Therefore it is shown in Chapter 4 that weak solutions in the sense described in Definition 5.1 do exist.

Before stating and proving the main result of this chapter some notation is presented, and a couple of lemmas are shown.

Let $u_1, u_2 \in H_0^2(\Omega)$, let $u = u_1 - u_2$, let for $i = 1, 2$

$$v_i = -G_2([u_i, u_i]), \tag{5.12}$$

which is well defined by Theorem 2.6, and finally let $v = v_1 - v_2$.

The operator P_N is defined as in Section 2.9, i.e. as the projection in $L^2(\Omega)$ onto the space spanned by the first N eigenvectors of Δ^2, when the eigenvectors are listed so the corresponding eigenvalues satisfy $0 < \lambda_1 \le \lambda_2 \cdots$. Let $P_N u = u_{(N)}$ etc.

Lemma 5.2
Let $u_1, u_2 \in H_0^2(\Omega)$ and let $\|u_j\|_2 \leq R$ for some $R > 0$. Then there exists a $\beta > 0$, and $N_0 \in \mathbb{N}$, so

$$\|[\overline{u_1}, v]\|_{-1} \leq C_1 \log(1 + \lambda_N)\|u_1 - u_2\|_1 + C_2 \lambda_{N+1}^{-\beta} \tag{5.13}$$

for $N \geq N_0$. The positive constants C_1 and C_2 only depend on R and β.

Proof:
The von Karman bracket can be written as

$$[\overline{u_1}, v] = \sum_{i,j,k,l=1}^{2} \alpha_{ijkl} D_i(D_{jk}^2 \overline{u_1} D_l v) \qquad \text{for } i,j,k,l \in \{-1, 0, 1\}. \tag{5.14}$$

Let $z = D(D^2 \overline{u_1} Dv)$ represent a term in this sum, where D and D^2 are differential operators of first and second order respectively, with constant coefficients.
Lemma 2.24 implies that $[u_i, u_i] \in H^{-1-\theta}(\Omega)$, for $i = 1, 2$ and for $0 < \theta < 1$. Hence Theorem 2.6 implies that $v \in H_0^2(\Omega) \cap H^{2+\delta}(\Omega)$ for $0 < \delta < 1$, so $Dv \in H_0^1(\Omega) \cap H^{1+\delta}(\Omega)$. Now Sobolevs Theorem gives that $Dv \in C_{L^\infty}(\Omega)$. Let $Q_N = I - P_N$. Then

$$\begin{aligned}
\|[\overline{u_1}, v]\|_{-1} &\leq C_1 \|z\|_{-1} \\
&\leq C_2 \|(D^2 \overline{u_1} Dv)\|_0 \\
&\leq C_2 \max_{x \in \Omega} |Dv(x)| \left(\int_\Omega |D^2 \overline{u_1(x)}|^2 dx \right)^{\frac{1}{2}} \\
&\leq C_3 \max_{x \in \Omega} |Dv(x)| \\
&\leq C_4 \left(\max_{x \in \Omega} |(P_N Dv)(x)| + \max_{x \in \Omega} |(Q_N Dv)(x)| \right).
\end{aligned} \tag{5.15}$$

Then Lemma 2.25 gives for $N \geq N_0$

$$\max_{x \in \Omega} |(P_N Dv)(x)| \leq C_5 (\log(1 + \lambda_N))^{\frac{1}{2}} \|Dv\|_1 \leq C_6 (\log(1 + \lambda_N))^{\frac{1}{2}} \|v\|_2 \tag{5.16}$$

Since $Dv \in H_0^1(\Omega) \cap H^{1+\delta}(\Omega)$, the following is obtained by using (2.61) with $4\beta = \sigma$ and $0 < \beta < \frac{1}{4}$, and by using (2.21) and (2.53)

$$\begin{aligned}
\max_{x \in \Omega} |(Q_N Dv)(x)| &\leq C_7 \|(Q_N Dv)(x)\|_{1+4\beta} \\
&= C_8 \|(\Delta^2)^{\frac{1}{4}+\beta} (Q_N Dv)(x)\|_0 \\
&\leq C_9 \lambda_{N+1}^{-\beta} \|(\Delta^2)^{\frac{1}{4}+2\beta} (Q_N Dv)(x)\|_0 \\
&\leq C_{10} \lambda_{N+1}^{-\beta} \|v\|_{2+8\beta}.
\end{aligned} \tag{5.17}$$

Rewriting v gives $v = -G_2([u, u_1 + u_2])$. Now it follows form Theorem 2.6, that

$$\|v\|_{H^{s+4}(\Omega) \cap H_0^2(\Omega)} = \|v\|_{H^{s+4}(\Omega)} + \|v\|_{H_0^2(\Omega)} \leq C_{11} \|[u, u_1 + u_2]\|_s \tag{5.18}$$

so for $s+4 = 2+8\beta$

$$\|v\|_{2+8\beta} \le C_{11}\|[u, u_1+u_2]\|_{8\beta-2} \tag{5.19}$$

and for $j = 0$, $8\beta - 2 = -\theta$ and $\theta = \psi$ it follows from Lemma 2.24, that

$$\|v\|_{2+8\beta} \le C_{12}\|u\|_{2-\theta+\psi}\|u_1+u_2\|_{3-\psi} \le C_{12}\|u\|_2(\|u_1\|_2 + \|u_2\|_2) \le C_{13}. \tag{5.20}$$

Summing up these evaluations

$$\|[\overline{u_1}, v]\|_{-1} \le C_{14}(\log(1+\lambda_N))^{\frac{1}{2}}\|v\|_2 + C_{15}\lambda_{N+1}^{-\beta} \tag{5.21}$$

for $0 < \beta < \frac{1}{8}$ and $N \ge N_0$.
Since the von Karman bracket is linear in both arguments, it follows from Theorem 2.6, that

$$\begin{aligned} \|v\|_2 &\le C_{16}\|[u, u_1+u_2]\|_{-2} \\ &\le C_{17}(\|[u_{(N)}, u_1+u_2]\|_{-2} + \|[Q_N u, u_1+u_2]\|_{-2}). \end{aligned} \tag{5.22}$$

Now Lemma 2.24 gives for $j = 2$, and $\psi = 4\beta$

$$\begin{aligned} \|[Q_N u, u_1+u_2]\|_{-2} &\le C_{18}\|Q_N u\|_{2-4\beta}\|u_1+u_2\|_{1+4\beta} \\ &\le C_{19}\|(\Delta_D^2)^{\frac{1}{2}-\beta}Q_N u\|_0\|u_1+u_2\|_2 \\ &\le C_{20}\lambda_{N+1}^{-\beta}\|Q_N u\|_2 \\ &\le C_{21}\lambda_{N+1}^{-\beta}. \end{aligned} \tag{5.23}$$

By using the rewritten form (2.22) of the von Karman bracket and Lemma 2.25

$$\begin{aligned} \|[u_{(N)}, u_1+u_2]\|_{-2} &\le \|u_{(N)}D^2 v\|_0 \\ &\le C_{22}\max_{x\in\Omega}\|D^2 v\|_0 \\ &\le C_{23}\{\log(1+\lambda_N)\}^{\frac{1}{2}}\|u\|_1. \end{aligned} \tag{5.24}$$

Hereby the Lemma is shown. □

Lemma 5.3
Let $u_1, u_2 \in H_0^2(\Omega)$ and $\|u_j\|_2 \le R$, for some $R > 0$. Then there exists a $\beta > 0$ and $N_0 \in \mathbb{N}$, so

$$\|[\overline{u}, v_2]\|_{-1} \le C_1 \log(1+\lambda_N)\|u\|_1 + C_2\lambda_{N+1}^{-\beta} \tag{5.25}$$

for $N \ge N_0$. The constants C_1 and C_2 only depends on R and β.

Proof:
The von Karman bracket $[\overline{u}, v_2]$ is rewritten by using (2.23). Let

$$z = D(D\overline{u}D^2(v_2)) = D(D\overline{u}D^2G\{D(Du_2D^2u_2)\}) = z(D\overline{u}, Du_2, D^2u_2)$$

represent a term in this sum, where D and D_2 are differential operators with constant coefficients. Because of the linearity of all the operators involved, z can be partitioned in the following way

$$z = z_1(Q_N D\overline{u}, Du_2, D^2u_2) + z_2(P_N D\overline{u}, Q_N Du_2, D^2u_2) + z_3(P_N D\overline{u}, P_N Du_2, D^2u_2) \tag{5.26}$$

The norm of each z_j in $H^{-1}(\Omega)$ is evaluated separately. Lemma 2.28 gives

$$\begin{aligned} \|z_1\|_{-1} &\leq C_1 \|Q_N Du D^2 G_2(D(Du_2 D^2 u_2))\|_0 \\ &\leq C_2 \|Q_N Du\|_{1-\beta} \|D^2 u_2 G_2(D(Du_2 D^2 u_2))\|_\beta \end{aligned} \tag{5.27}$$

If (2.21) and (2.53) is used on the first norm on the right hand side it leads to

$$\|Q_N Du\|_{1-\beta} \leq \lambda_{N+1}^{-\beta/4} \|Du\|_{1-2\beta} \leq C_3 \lambda_{N+1}^{-\beta/4} \|u\|_{2-2\beta} \leq C_4 \lambda_{N+1}^{-\beta/4} \tag{5.28}$$

and the second norm on the right hand side in (5.27) is evaluated by using Lemma 2.28 and Theorem 2.6

$$\begin{aligned} \|D^2 u_2 G_2(D(Du_2 D^2 u_2))\|_\beta &\leq C_5 \|Du_2 D^2 u_2\|_{\beta-1} \\ &\leq C_6 \|Du_2\|_\beta \|D^2 u_2\|_0 \\ &\leq C_7 \|u_2\|_2. \end{aligned} \tag{5.29}$$

Therefore

$$\|z_1\|_{-1} \leq C_8 \lambda_{N+1}^{-\beta/4}. \tag{5.30}$$

The evaluation on z_2 is made using the same methods as above, giving

$$\begin{aligned} \|z_2\|_{-1} &\leq C_9 \|Du\|_{1-\hat{\beta}} \|Q_N Du_2\|_{\hat{\beta}} \|D^2 u_2\|_0 \\ &\leq C_{10} \|u\|_{2-\hat{\beta}} \lambda_{N+1}^{-\beta/4} \|Q_N Du_2\|_{\hat{\beta}-\beta} \|u_2\|_2 \\ &\leq C_{11} \lambda_{N+1}^{-\beta/4} \end{aligned} \tag{5.31}$$

and for z_3

$$\begin{aligned} \|z_3\|_{-1} &\leq C_{12} \|(P_N Du) D^2 G_2(D((P_N Du_2)(D^2 u_2)))\|_0 \\ &\leq C_{13} \{\log(1+\lambda_N)\}^{\frac{1}{2}} \|Du\|_0 \|(P_N Du_2)(D^2 u_2)\|_{-1} \\ &\leq C_{14} \{\log(1+\lambda_N)\}^{\frac{1}{2}} \|u\|_1 \max_{x\in\Omega} |P_N Du_2(x)| \|D^2 u_2\|_0 \\ &\leq C_{15} \log(1+\lambda_N) \|u\|_1. \end{aligned} \tag{5.32}$$

Hereby the Lemma is shown. □

Theorem 5.4
The von Karman equations (5.1)-(5.2) *with the conditions* (5.3)-(5.6) *has a unique weak solution.*

Proof:
The proof of this theorem consists of an analysis of the difference between two solutions $u_1(t)$ and $u_2(t)$ to the von Karman equations. Let $u = u_1(t) - u_2(t)$, and let $P_N u(t) = u_{(N)}(t)$. Since u_1 and u_2 solves (5.1) with (5.2) defining v_1 and v_2, then u satisfies

$$\begin{aligned} &u''(t) + \Delta^2 u(t) = M(t) \\ &\gamma_0 u = \gamma_1 u = u|_{t=0} = u'|_{t=0} = 0 \end{aligned} \tag{5.33}$$

where

$$M(t) = [\overline{u_1(t)}, v_1(t)] - [\overline{u_2(t)}, v_2(t)] \tag{5.34}$$

The projection P_N and the differentiations commute, so $u_{(N)}$ solves the following problem

$$\begin{aligned} &u''_{(N)}(t) + \Delta^2 u_{(N)}(t) = P_N M(t) \\ &\gamma_0 u_{(N)} = \gamma_1 u_{(N)} = u_{(N)}|_{t=0} = u'_{(N)}|_{t=0} = 0 \end{aligned} \tag{5.35}$$

By forming the scalar product with $\partial_t u_{(N)}(t)$ and integrating with respect to t, then (5.35) gives

$$\begin{aligned} \int_0^t \Big(\big(\partial_t^2 u_{(N)} | \partial_t u_{(N)}\big)_{L^2(\Omega)} + \big(\Delta^2 u_{(N)} | \partial_t u_{(N)}\big)_{L^2(\Omega)} \Big) d\tau \\ = \int_0^t \big((P_N M)(t) | \partial_t u_{(N)}\big)_{L^2(\Omega)} \, d\tau. \end{aligned} \tag{5.36}$$

According to Theorem 2.20 the differentiation with respect to t can be moved inside the integration with respect to x in the term $\partial_t \left(\partial_t u_{(N)} | \partial_t u_{(N)}\right)_{L^2(\Omega)}$. Hence Leibniz' formula gives for the first term in the integrand on the left hand side in (5.36)

$$\begin{aligned} \operatorname{Re}\Big(\big(\partial_t^2 u_{(N)}(t) | \partial_t u_{(N)}(t)\big)_{L^2(\Omega)}\Big) &= \frac{1}{2}\partial_t \left(\partial_t u_{(N)}(t) | \partial_t u_{(N)}(t)\right)_{L^2(\Omega)} \\ &= \frac{1}{2}\partial_t \|\partial_t u_{(N)}(t)\|_0 \end{aligned} \tag{5.37}$$

Differentiation with respect to t and $(\Delta^2)^\alpha$ commutes when used on $u_{(N)}$. Indeed for $\alpha \in \mathbb{R}$ it follows by (2.53) and Theorem 2.20, that

$$\begin{aligned} \partial_t (\Delta^2)^\alpha u_{(N)} &= \partial_t (\Delta^2)^\alpha \sum_{n=1}^{N} (u|e_n)_{L^2(\Omega)} \, e_n \\ &= \partial_t \sum_{n=1}^{N} \lambda_n^\alpha \, (u|e_n)_{L^2(\Omega)} \, e_n \\ &= \sum_{n=1}^{N} \lambda_n^\alpha \, (\partial_t u|e_n)_{L^2(\Omega)} \, e_n \\ &= (\Delta^2)^\alpha \partial_t u_{(N)}. \end{aligned} \tag{5.38}$$

The second term in the integrand on the left hand side in (5.36) is therefore evaluated by Leibniz' formula, giving

$$\begin{aligned}\operatorname{Re}\Big(\big(\Delta^2 u_{(N)}(t)|\partial_t u_{(N)}(t)\big)_{L^2(\Omega)}\Big) &= \Big((\Delta^2)^{\frac{1}{2}} u_{(N)}(t)|\partial_t(\Delta^2)^{\frac{1}{2}} u_{(N)}(t)\Big)_{L^2(\Omega)} \\ &= \frac{1}{2}\partial_t\Big((\Delta^2)^{\frac{1}{2}} u_{(N)}(t)|(\Delta^2)^{\frac{1}{2}} u_{(N)}(t)\Big)_{L^2(\Omega)} \\ &= \frac{1}{2}\partial_t\|(\Delta^2)^{\frac{1}{2}} u_{(N)}(t)\|_{L^2(\Omega)}.\end{aligned} \tag{5.39}$$

Hence for the left hand side in (5.36), it follows by (2.21) and (5.35),

$$\begin{aligned}&\Big|\int_0^t \big(\big(\partial_t^2 u_{(N)}|\partial_t u_{(N)}\big)_{L^2(\Omega)} + \big(\Delta^2 u_{(N)}|\partial_t u_{(N)}\big)_{L^2(\Omega)}\big)d\tau\Big| \\ &\qquad \geq \Big|\int_0^t \operatorname{Re}\big(\big(\partial_t^2 u_{(N)}|\partial_t u_{(N)}\big)_{L^2(\Omega)} + \big(\Delta^2 u_{(N)}|\partial_t u_{(N)}\big)_{L^2(\Omega)}\big)d\tau\Big| \\ &\qquad = \frac{1}{2}\int_0^t \partial_t\Big(\|\partial_t u_{(N)}(\tau)\|_0^2 + \|(\Delta^2)^{\frac{1}{2}} u_{(N)}(\tau)\|_0^2\Big)d\tau \\ &\qquad = \frac{1}{2}(\|\partial_t u_{(N)}(t)\|_0^2 + \|(\Delta^2)^{\frac{1}{2}} u_{(N)}(t)\|_0^2) \\ &\qquad \geq C_1(\|\partial_t u_{(N)}(t)\|_{-1}^2 + \|u_{(N)}(t)\|_2^2) \\ &\qquad \geq C_2(\|\partial_t u_{(N)}(t)\|_{-1}^2 + \|u_{(N)}(t)\|_1^2).\end{aligned} \tag{5.40}$$

The right hand side in (5.36) can be evaluated using Schwartz' inequality

$$\begin{aligned}\Big|\int_0^t \big((P_N M)(\tau)|\partial_t u_{(N)}(\tau)\big)_{L^2(\Omega)} d\tau\Big| &\leq \int_0^t \Big|\Big\langle (P_N M)(\tau), \overline{\partial_t u_{(N)}(\tau)}\Big\rangle\Big| d\tau \\ &\leq \int_0^t \|(P_N M)(\tau)\|_{-1}\|\partial_t u_{(N)}(\tau)\|_1 d\tau \\ &= C_3\int_0^t \|(P_N M)(\tau)\|_{-1}\|\partial_t u_{(N)}(\tau)\|_{-1} d\tau.\end{aligned} \tag{5.41}$$

Altogether (5.36) - (5.41) gives

$$\|\partial_t u_{(N)}(t)\|_{-1}^2 + \|u_{(N)}(t)\|_1^2 \leq C_4\int_0^t \|(P_N M)(\tau)\|_{-1}\|\partial_t u_{(N)}(\tau)\|_{-1} d\tau. \tag{5.42}$$

The projection P_N is bounded, and $\|\cdot\|_{-1}$ is a continuous function, so $\|P_N\cdot\|_{-1}$ is bounded, with an operator norm smaller than 1, therefore

$$\int_0^t \|(P_N M)(t)\|_{-1}\|\partial_t u_{(N)}(t)\|_{-1} d\tau \leq \int_0^t \|(M)(t)\|_{-1}\|\partial_t u(t)\|_{-1} d\tau \tag{5.43}$$

Hence

$$\|\partial_t u_{(N)}(t)\|_{-1}^2 + \|u_{(N)}(t)\|_1^2 \leq C_4\int_0^t \|(M)(t)\|_{-1}\|\partial_t u(t)\|_{-1} d\tau \tag{5.44}$$

for all $N \in \mathbb{N}$. Since the operator P_N is bounded the left hand side in (5.44) converges for $N \to \infty$, which leads to

$$\|\partial_t u(t)\|_{-1}^2 + \|u(t)\|_1^2 \leq C_5 \int_0^t \|(M)(\tau)\|_{-1} \|\partial_t u(\tau)\|_{-1} d\tau \tag{5.45}$$

for all $t \in \,]0, T[\,$.

After rewriting $M(t)$, the norm can be evaluated using Lemma 5.2 and Lemma 5.3. For $t \in]0, T[$ there exists an N_0, so

$$\begin{aligned} \|M(t)\|_{-1} &\leq \|[\overline{u}, v_2]\|_{-1} + \|[\overline{u_1}, v]\|_{-1} \\ &\leq C_6 \log(1 + \lambda_N)\|u\|_1 + C_7 \lambda_{N+1}^{-\beta} \end{aligned} \tag{5.46}$$

for $N \geq N_0$ and for some $\beta \in]0, T[$. Let

$$\Psi(t) = \|\partial_t u(t)\|_{-1}^2 + \|u(t)\|_1^2. \tag{5.47}$$

Now is follows from (5.45), that

$$\Psi(t) \leq C_8 \log(1 + \lambda_N) \int_0^t \|u(\tau)\|_1 \|\partial_t u(\tau)\|_{-1} d\tau + C_9 \lambda_{N+1}^{-\beta} \int_0^t \|\partial_t u(\tau)\|_{-1} d\tau \tag{5.48}$$

The product of the norms in the first integrand on the right hand side are replaced by $\Psi(t)$, since $ab \leq \frac{1}{2}(a^2 + b^2)$ for $a, b \in \mathbb{R}$. The second integral is evaluated by using that $\partial_t u \in L^\infty(0, T; L^2(\Omega))$, and $\|\cdot\|_{-1} \leq \|\cdot\|_0$, giving

$$\Psi(t) \leq C_8 \log(1 + \lambda_N) \int_0^t \Psi(\tau) d\tau + C_{10} T \lambda_{N+1}^{-\beta} \tag{5.49}$$

Hence Grönwalls Lemma, gives

$$\Psi(t) \leq C_{10} T \lambda_{N+1}^{-\beta} (1 + \lambda_N)^{C_8 t} \tag{5.50}$$

For N large enough

$$\begin{aligned} \lambda_{N+1}^{-\beta}(1 + \lambda_N)^{C_8 t} &= (1 + \lambda_N)^{C_8 t - \beta} \left(\frac{1 + \lambda_N}{\lambda_{N+1}} \right)^\beta \\ &\leq (1 + \lambda_N)^{C_8 t - \beta} \left(\frac{1}{\lambda_{N+1}} + 1 \right)^\beta \\ &\leq (1 + \lambda_N)^{C_8 t - \beta} 2^\beta. \end{aligned} \tag{5.51}$$

Therefore the right hand side in (5.50) tends to 0 for $N \to \infty$, when $C_8 t - \beta < 0$, i.e. when $0 \leq t < t_0 = \frac{\beta}{C_8}$. Therefore $u_1(t) = u_2(t)$ on the interval $[0, t_0[$, hence the value of $u(\frac{t_0}{2}) = 0$. Now the problem (5.35) can be evaluated starting at $\frac{t_0}{2}$ instead of 0 by a translation of the time variable, giving (5.50), but now for the translated time interval. The constants β and C_8 only depends on the norm of the solutions u_1 and u_2 in $L^\infty(0, TH_0^2(\Omega))$. Hence $u(t) = 0$ on the interval $[0, \frac{3t_0}{2}[$ with the original convention for the time variable. Continuing this way it is shown, that $u(t) = 0$ for $t \in [0, T[\,$, meaning that the solution is unique. □

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