Nonlocal Nonlinear Diffusion

Variational Principles, Optimal Control, and Approximation

Master's Thesis

Marcus Johan Schytt

Supervised by: Anton Evgrafov

AALBORG UNIVERSITET
Department of Mathematical Sciences
Skjernvej 4A, 9220 Aalborg Øst, Denmark

June 2023

Abstract

This thesis explores a nonlocal nonlinear thermal diffusion law inspired by peridynamics, a nonlocal reformulation of classical mechanics. The primary objective is to establish an existence theory for nonlocal equilibrium states based on the Dirichlet principle of minimum energy. By rigorously defining and analyzing nonlocal analogues of gradient and divergence operators, we prove the existence and uniqueness of nonlocal equilibrium states, characterizing them as weak solutions to a nonlocal p-Laplacian law. Dual formulations are derived using Kelvin's principle of minimum complementary energy, and their well-posedness is established through the Ladyzhenskaya-Babuška-Brezzi theory and Fenchel-Rockafellar duality. Furthermore, we demonstrate the convergence of nonlocal equilibrium states to their local counterparts as the nonlocal interaction horizon vanishes, drawing on relevant results from Bourgain-Brezis-Mironescu and Ponce.

The scope narrows down to linear diffusion, specifically focusing on the formulation and analysis of nonlocal optimal control and obstacle problems. Interestingly, the analysis of linear-quadratic optimal distributed control problems closely mirrors the corresponding local analysis. However, for locally ill-posed nonlinear control in conductivity, nonlocal analogues yield practical solutions without additional regularization. Similarly, we analyze nonlocal obstacle problems with minimal assumptions on the obstacles. We argue that nonlocal modeling proves advantageous in considering discontinuous conductivities and obstacles, which pose challenges in the local theory.

Lastly, the thesis investigates the numerical approximation of equilibrium states. We demonstrate the extension of the finite element method to the nonlocal case, albeit with increased computational costs. Rigorous analysis confirms the convergence of finite element approximations to the nonlocal equilibrium state, and this is supported by a series of numerical experiments. Additionally, we apply the nonlocal finite element approximations to solve nonlocal optimal control and obstacle problems.

Preface

This thesis concludes my third and fourth semester in the Master's program in Mathematics at the Department of Mathematical Sciences at Aalborg University. The thesis was written between September 2022 and June 2023, and merits 50 ECTS if successfully defended.

During the thesis, I was supervised by my advisor Anton Evgrafov, to whom I am profoundly grateful. It is through his guidance that I discovered nonlocal modelling, and was able to prove the results and conduct the experiments that have shaped this thesis. I am deeply thankful for his support and his collaborative mentorship. Without his assistance, this thesis would not have been possible.

I am also thankful to the Department of Mathematical Sciences for their support and guidance throughout my academic journey. Their provision of a development server has been instrumental in facilitating the execution of experiments and analysis for this thesis.

The thesis assumes that the reader is comfortable with results from graduate-level real and functional analysis, and is familiar with variational principles, convex analysis, and optimization. In addition, the thesis requires knowledge of elliptic partial differential equations and their optimal control. For their approximation, the reader is expected to have some practical and theoretical knowledge of finite element methods.

Contents

Chapter 1

Introduction

In recent decades, there has been a surge in the formulation and application of models characterized by a notion of nonlocality. Nonlocality arises in the modeling of long-range interactions, which are present in a wide variety of contexts. We mention their application in image processing [42] and stochastic processes [52, 53], as well as physical phenomena such as phase transition [9, 19], material fracture [60, 65], and anomalous diffusion arising in flows in porous media, subsurface transport, emulsions, and more [13, 31, 53, 69]. The presence of anomalous diffusion invalidates classical *local* modeling via Fick's law in the language of a partial differential equation (PDE). In contrast, nonlocal models are of integral form. Relaxed regularity requirements allow for improved modeling capabilities, especially for discontinuities.

One of the most exciting applications is the attempt to reformulate classical continuum mechanics from a nonlocal theory [34, 35, 49, 50]. Recently, much attention has been given to the theory of peridynamics [65, 66], which is a nonlocal reformulation of continuum mechanics that does not require spatial derivatives in its formulation. Here, nonlocality appears in a reformulated equation of motion in the form of long-range material bonds, characterized by a force function that includes all material properties. This allows peridynamics to describe a wide range of material behavior. A characteristic of peridynamics is that a material point can only form bonds within the *interaction horizon*. This property makes peridynamics a generalizing theory, since classical elasticity theory can be recovered in the limit when the interaction horizon vanishes. We refer to [51] for an overview of the theory and applications of peridynamics.

Of particular interest in this thesis is the extension of peridynamic theory to thermal diffusion, specifically its equilibrium states. As such, this thesis constructs a mathematical framework for a general nonlocal diffusion law. The generality of the model lies in the description of the diffusion by a nonlocal p-Laplacian law. We consider nonlocal material conductivity distributions, and an extended notion of Dirichlet boundary conditions for nonlocal boundary value problems. We approach the construction by introducing nonlocal analogues of the gradient and divergence operators, as well as a notion of nonlocal heat flux as in [33, 39, 45]. Through these, equilibrium states are defined by variational principles of minimum energy. In addition, we obtain equivalent variational formulations that mimic the weak formulations of classical PDE theory. In the spirit of peridynamics, we will similarly study the limiting behavior of nonlocal diffusion as the interaction horizon vanishes.

The mathematical validity of the variational principles is paramount. Therefore, we aim to prove results on existence and uniqueness. We will see that, in the eyes of the classical calculus of variations, its nonlocal counterpart has surprising qualities [57]. The direct method of Tonelli will be our main tool. Therefore, the notions of continuity and compactness will recur throughout the thesis. First and foremost, we establish the well-posedness of the variational formulation of our nonlocal diffusion law and its dual formulation presented in [39]. Second, we restrict ourselves to the case of linear diffusion and study its application in the field of optimal control. We treat the prototypical problem of linear-quadratic optimal source control and the nonlinear problem of conductivity identification. In addition, we will study how the dual formulation in term of thermal fluxes aids in the analysis of the saddle-point problem of Céa and Malanowski [22]. Finally, we study how the relaxed regularity of the nonlocal framework applies to obstacle problems.

Of practical interest, we will also explore the numerical simulation of the nonlocal diffusion model. Its variational formulations are tractable for Galerkin-type approximations. In particular, we will use a finite element method (FEM) to approximate the nonlocal equilibrium states. Computationally, the nonlocal FEM is more complicated than its local counterpart due to the nonlocal interactions [30, 7]. Implementation methods to overcome these challenges are under active research, and only in the last few years have nonlocal FEM codes become publicly available. We include numerical experiments to settle implementation details and to explore the convergence rate of our Galerkin scheme. Finally, we will use our nonlocal FEM implementation to approximate solutions to optimal distributed control problems and obstacle problems.

1.1 Outline

We realize that nonlocal modeling may be a new concept to the reader, and therefore Chapter 2 serves as a gentle introduction. The scope is limited to one dimension, with the goal of defining a nonlocal derivative analog and establishing an existence theory for its variational problems. Chapter 3 defines the nonlocal gradient and formulates the thermal diffusion law in terms of the Dirichlet principle. In addition, the notion of nonlocal fluxes is introduced together with the nonlocal divergence operator, allowing a dual formulation of the diffusion law. The existence and uniqueness of equilibrium states is established, and the convergence to the limiting local problem is studied. Chapter 4 deals with the formulation and analysis of nonlocal optimal control problems and obstacle problems. Special emphasis is given to the nonlinear problems of parameter identification. In Chapter 5, we discuss the implementation of the finite element method for simulation of nonlocal diffusion. We perform numerical experiments to show convergence and to approximate solutions to both optimal control and obstacle problems.

Chapter 2

An introduction to nonlocality

The following chapter is a brief introduction to the notion of nonlocality. The goal is to define a one-dimensional nonlocal framework which will serve to illustrate key differences between the classical calculus of variations and its nonlocal counterpart. We first introduce the notion of a nonlocal derivative operator which we use to define a class of nonlocal boundary value problems. We proceed by developing an existence theory for their solutions. This involves identifying suitable state spaces in which the direct method in the calculus of variations can be applied. Rather surprisingly, we discover that nonlocal versions of classically unsolvable problems are solvable. The chapter follows the discussion in [57].

2.1 Nonlocal derivatives

Throughout this chapter we will consider a fixed domain, the interval $\Omega = (0,1)$. Classically, we say that a function $u : \Omega \to \mathbb{R}$ is differentiable at a point $x \in \Omega$ if the limit

$$u'(x) = \lim_{x' \to x} \frac{u(x) - u(x')}{x - x'}$$

exists and is finite. Hence, the derivative $u'(x)$ is dependant on the pointwise regularity of u near x for its existence. Modelling via the derivative is therefore limited in its application, as points of discontinuity or cusps do not admit derivatives. An extended definition adopted in the analysis of PDEs is the notion of the weak derivative. We say that u admits a weak derivative if there exists a locally integrable function $u' : \Omega \to \mathbb{R}$, which satisfies an integration by parts formula

$$\int_0^1 u(x)\, \varphi'(x)\, \mathrm{d}x = -\int_0^1 u'(x)\, \varphi(x)\, \mathrm{d}x; \quad \forall \varphi \in C_c^\infty(\Omega).$$

Here, $C_c^\infty(\Omega)$ represents the space of infinitely differentiable functions with compact support in Ω. It is worth noting that if a function u is continuously differentiable, its weak derivative coincides with its usual derivative. However, the concept of weak derivative extends to nondifferentiable functions as well. However, it is important to highlight that functions with jump discontinuities do not possess weak derivatives. Note, both of these derivative notions depend on behavior in infinitesimally small neighborhoods. Therefore, we refer to modeling with these notions as *local*. Our goal is to redefine the notion of derivative to incorporate nonlocal interactions. Nonlocality takes a variety of forms in the literature, but our preliminary presentation considers the following simple idea.

Definition 2.1. Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then we define its *nonlocal derivative* at a distinct pair of points $(x, x') \in \Omega \times \Omega$ as

$$Gu(x, x') = \frac{u(x) - u(x')}{x - x'}.$$

Remark that the nonlocal derivative shares properties with the usual local derivative. First, it is clear from its definition that it may be seen as a linear operator on functions. Second, if the nonlocal derivative Gu vanishes everywhere on the domain Ω, then u is constant. Unlike the local derivative, the nonlocal derivative depends on two points in the domain. Therefore, we cannot just simply substitute it for the usual derivative in classical local problems. Consider the following example.

Example 2.2. The Dirichlet principle is a classical problem in variational calculus, and its study led to the development of the direct method of calculus of variations. We will formulate it through the elliptic Laplace equation

$$\begin{aligned} -u'' &= 0, \quad \text{in } \Omega, \\ u(0) = 0, &\quad u(1) = 1. \end{aligned} \tag{2.1}$$

The Dirichlet principle states that the unique solution to (2.1) is also the unique equilibrium state satisfying a minimum energy principle. That is, we find it as a solution to

$$\begin{aligned} \min_{u \in W^{1,2}(\Omega)} \; & I(u) = \frac{1}{2}\int_0^1 |u'(x)|^2 \,\mathrm{d}x, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 1. \end{aligned} \tag{2.2}$$

The solution is sought in the Sobolev space $W^{1,2}(\Omega)$, and the integral I is referred to as the *Dirichlet energy*. Since we have a nonlocal notion of the derivative, we may adapt the variational formulation of (2.2) to the nonlocal setting. However, the nonlocal derivative is defined with an additional argument, and therefore its analogous problem requires an additional integral to be defined.

The simple nonlocal extension of (2.2) is

$$\begin{aligned} \min_{u \in U} \; & J(u) = \frac{1}{2}\int_0^1 \int_0^1 \left|Gu(x, x')\right|^2 \,\mathrm{d}x' \,\mathrm{d}x, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 1, \end{aligned} \tag{2.3}$$

and we will refer to the integral J as the *nonlocal Dirichlet energy*. At the moment it is unclear in which space U the solution u is sought. We note that the formulation in (2.3) explicitly requires two forms of regularity. First, we want to enforce pointwise boundary conditions. Obviously, this disqualifies candidates like the Lebesgue spaces $L^p(\Omega)$ for $p \in [1, \infty]$. Second, the nonlocal Dirichlet energy has to be a proper functional, that is $J(u) < \infty$ for some $u \in U$. The last requirement is settled by the following definition inspired by the Sobolev spaces.

Definition 2.3. Let $p \in (1, \infty)$ and define the space

$$NW^{1,p}(\Omega) = \{u \in L^p(\Omega) \,|\, Gu \in L^p(\Omega \times \Omega)\},$$

with norm

$$\|u\|_{NW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Gu\|_{L^p(\Omega\times\Omega)}.$$

The choice $U = NW^{1,2}(\Omega)$ makes J proper, but again we are left with the question of how boundary conditions can be enforced. It is well-known that the Sobolev trace theory allows the association of boundary values to equivalence classes of functions, as long as they are regular. Similarly we will see for $NW^{1,p}(\Omega)$, as long as $p \in (1, \infty)$ is sufficiently large. For this, we recall the fractional Sobolev spaces $W^{s,p}(\Omega)$ with exponent $p \in (1, \infty)$ and fractional exponent $s \in (0, 1)$. They are reflexive Banach spaces defined by their norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)}\right)^{1/p},$$

where the *Gagliardo seminorm* is defined

$$|u|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(x')|^p}{|x - x'|^{1+sp}} \,\mathrm{d}x' \,\mathrm{d}x\right)^{1/p}.$$

We remark the connection between the Gagliardo seminorm and the Lebesgue norm of the nonlocal derivative. Indeed, we have the following correspondence

$$\|Gu\|_{L^p(\Omega\times\Omega)} = |u|_{W^{1/p^*,p}(\Omega)},$$

where $p^* \in (1, \infty)$ is the conjugate exponent of p defined by

$$1 = \frac{1}{p} + \frac{1}{p^*}.$$

Thus $NW^{1,p}(\Omega)$ coincides with the fractional Sobolev space $W^{1/p^*,p}(\Omega)$ due to the equivalence of their norms. Since $W^{1/p^*,p}(\Omega)$ is a reflexive Banach space, so is $NW^{1,p}(\Omega)$. Unfortunately, the space we are interested in, $NW^{1,2}(\Omega) = W^{1/2,2}(\Omega)$, is notoriously problematic from a regularity viewpoint. Indeed, for all $s \le 1/2$, the space $W^{s,2}(\Omega)$ contains discontinuous functions, which invalidates the enforcement of pointwise boundary conditions. However, there are other combinations of exponents that are useful in this case. The fractional Sobolev spaces enjoy the following embedding result.

Theorem 2.4. *Let $s \in (0, 1)$ and $p \in (1, \infty)$. Then we have the following embedding results.*

(i) *Assume $q \in [1, p]$. Then the fractional Sobolev space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, i.e. every bounded set in $W^{s,p}(\Omega)$ is relatively compact in $L^q(\Omega)$.*

(ii) *Assume $sp > 1$. Then the fractional Sobolev space $W^{s,p}(\Omega)$ is continuously embedded in $C^{0,\alpha}(\bar\Omega)$ with $\alpha = (sp - 1)/p$, i.e. there exists $C > 0$ such that*

$$\|u\|_{C^{0,\alpha}(\bar\Omega)} \le C \|u\|_{W^{s,p}(\Omega)}, \quad \forall u \in W^{s,p}(\Omega).$$

Proof. The proofs may be found in the excellent introductory monograph on fractional Sobolev spaces [32]. □

Here, we note that $C^{0,\alpha}(\Omega)$ with $\alpha \le 1$ denotes the space of α-Hölder continuous functions. By juggling the exponents, we find a straightforward corollary that applies to our current situation.

Corollary 2.5. *Let $p > 2$. Then $NW^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, and continuously embedded in $C^{0,\alpha}(\Omega)$ with $\alpha = (p-2)/p$.*

Proof. Recall that $NW^{1,p}(\Omega) = W^{s,p}(\Omega)$ with $s = 1/p' = 1 - 1/p$. Theorem 2.4(i) then gives the compact embedding. Since $p > 2$, we have $sp = p - 1 > 1$, which by Theorem 2.4(ii) guarantees the continuous embedding of $W^{s,p}(\Omega)$ in $C^{0,\alpha}(\Omega)$ with $\alpha = (sp-1)/p = (p-2)/p$. □

As a consequence of Corollary 2.5 we see that functions in $NW^{1,p}(\Omega)$ admit Hölder continuous representatives whenever the exponent $p > 2$. Their boundary values are uniquely determined by continuity, and therefore we can impose boundary conditions. Unfortunately, we had to sacrifice the possibility of irregularity in the process. In subsequent chapters we adopt another framework for boundary conditions, so that we can impose boundary conditions, even in the absence of continuity.

With this in mind, we can now study the nonlocal Dirichlet principle (2.3) with a slight modification. For exponents $p > 2$, the problem becomes

$$\begin{aligned} \min_{u \in NW^{1,p}(\Omega)} \quad & J(u) = \frac{1}{p}\int_0^1 \int_0^1 \left|Gu(x,x')\right|^p \, dx' \, dx, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 1, \end{aligned} \tag{2.4}$$

This modified problem is well defined since the functional J is proper and the boundary conditions can be imposed. However, it is important to investigate whether the problem admits any solutions.

2.2 Nonlocal variational calculus

We now proceed to develop an initial calculus of variations for problems governing the nonlocal derivative. We fix the exponent $p > 2$ and consider end-point boundary conditions $u(0) = a$ and $u(1) = b$ for $a, b \in \mathbb{R}$. We study the general problem

$$\begin{aligned} \min_{u \in NW^{1,p}(\Omega)} \quad & J(u) = \int_0^1 \int_0^1 F(x, u(x), Gu(x,x')) \, dx' \, dx, \\ \text{s.t.} \quad & u(0) = a, \quad u(1) = b. \end{aligned} \tag{2.5}$$

Here, we assume that the integrand $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory, that is measurable in its first variable and continuous in the remaining. Additionally, we assume that F satisfies the coercivity condition: There exists some $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$F(x,u,U) \ge \alpha |U|^p + \beta, \quad \forall (x,u,U) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{2.6}$$

To demonstrate that (2.5) admits solutions, we will utilize the direct method of the calculus of variations. Therefore, we need some notion of continuity of the functional J. Surprisingly, it can be obtained under very limited assumptions. The reason, presented in the following lemma, lies in the nature of the nonlocal derivative.

Lemma 2.6. *Let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $NW^{1,p}(\Omega)$. Then there exists a subsequence $\{u_{k'}\}_{k'\in\mathbb{N}}$, and some $u \in NW^{1,p}$, such that as $k' \to \infty$:*

(i) *The subsequence $u_{k'}$ converges to u in $L^p(\Omega)$.*

(ii) *The subsequence $\{Gu_{k'}\}_{k'\in\mathbb{N}}$ converges pointwise to Gu for a.e. $(x, x') \in \Omega\times\Omega$.*

(iii) *The subsequence $\{Gu_{k'}\}_{k'\in\mathbb{N}}$ converges weakly to Gu in $L^p(\Omega\times\Omega)$.*

Proof. By assumption and Corollary 2.5 we know that $\{u_k\}_{k\in\mathbb{N}}$ is relatively compact in $L^p(\Omega)$. Therefore, for (i), we see that there exists some $u \in L^p(\Omega)$ which is the strong and a.e. pointwise limit of some extracted subsequence $\{u_{k'}\}_{k'\in\mathbb{N}}$. The pointwise convergence of $\{u_{k'}\}_{k'\in\mathbb{N}}$ immediately gives us the pointwise convergence of nonlocal derivatives. Indeed, for almost all $x, x' \in \Omega\times\Omega$ the convergence $Gu_{k'}(x, x') \to Gu(x, x')$ follows from the linearity of the limit, and therefore (ii) holds. Lastly, for (iii) we note $\{Gu_{k'}\}_{k'\in\mathbb{N}}$ is bounded in $L^p(\Omega\times\Omega)$, which due to its pointwise convergence and Egorov's theorem also implies the weak convergence $Gu_{k'} \rightharpoonup Gu$ in $L^p(\Omega\times\Omega)$, see e.g. [46, Theorem 13.44]. Consequently, $Gu \in L^p(\Omega\times\Omega)$ which proves the claim $u \in NW^{1,p}$. □

This lemma marks a watershed between the nonlocal and local theory. Notably, we obtain the pointwise convergence of nonlocal derivatives, purely by pointwise convergence of the functions. A similar situation in the Sobolev space $W^{1,p}(\Omega)$ yields only points (i) and (iii), but not (ii). With pointwise convergence of nonlocal derivatives, lower semicontinuity can be established for the functional J by Fatou's lemma.

Proposition 2.7. *Let J be given as in (2.5), and assume that the integrand F is bounded from below. Then we have the following lower semicontinuity properties.*

(i) *J is weakly lower semicontinuous in $NW^{1,p}(\Omega)$.*

(ii) *J is lower semicontinuous in $L^p(\Omega)$.*

Proof. For (i) let $u \in NW^{1,p}(\Omega)$ and assume $u_k \rightharpoonup u$ in $NW^{1,p}(\Omega)$ for some sequence $\{u_k\}_{k\in\mathbb{N}}$. By weak convergence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $NW^{1,p}(\Omega)$ and by Lemma 2.6 we know that we may extract a subsequence $\{u_{k'}\}_{k'\in\mathbb{N}}$ for which $Gu_{k'}$ converges weakly and pointwise to Gv for some $v \in NW^{1,p}(\Omega)$. However, the strong convergence of Lemma 2.6(i) implies that $u_{k'}$ converges weakly to v in $L^p(\Omega)$, which implies $u = v$ due to the continuous embedding of $NW^{1,p}(\Omega)$ in $L^p(\Omega)$. In summary, we have all of the convergence results of Lemma 2.6 with the original limiting point $u \in NW^{1,p}(\Omega)$. If $J(u_k) \to \infty$ then lower semicontinuity is immediate. Therefore we assume WLOG that $J(u_k) \le M$ uniformly for all $k \in \mathbb{N}$ for some $M \in \mathbb{R}$. The pointwise convergence of the sequence and its nonlocal derivatives, together with the continuity of F, gives us the pointwise convergence of the integrand

$$\lim_{k'\to\infty} F(x, u_{k'}(x), Gu_{k'}(x, x')) = F(x, u(x), Gu(x, x')), \tag{2.7}$$

for almost all $(x, x') \in \Omega \times \Omega$. Since F is bounded below, we can apply Fatou's lemma and the pointwise limit in (2.7) to obtain

$$\begin{aligned} J(u) &= \int_0^1 \int_0^1 F(x, u(x), Gu(x, x'))\, \mathrm{d}x'\, \mathrm{d}x \\ &\leq \liminf_{k' \to \infty} \int_0^1 \int_0^1 F(x, u_{k'}(x), Gu_{k'}(x, x'))\, \mathrm{d}x'\, \mathrm{d}x \\ &= \liminf_{k' \to \infty} J(u_{k'}). \end{aligned}$$

Since $J(u_k)$ is bounded below, we could initially choose the subsequence $\{u_{k'}\}_{k' \in \mathbb{N}}$ from a minimizing sequence converging towards $\liminf_{k \to \infty} J(u_k)$. The above arguments would still hold, and we would additionally have

$$\liminf_{k' \to \infty} J(u_{k'}) = \liminf_{k \to \infty} J(u_k),$$

which completes the proof of (i). For (ii) we realize that $u_k \to u$ in $L^p(\Omega)$ implies $u_{k'}(x) \to u(x)$ a.e. in Ω for some subsequence, which in turn implies the pointwise convergence $Gu_{k'} \to Gu$. The proof now follows exactly as for (i). □

Remark that in contrast to the local theory, we obtain weak lower semicontinuity without a convexity assumption. This is striking, since some notion of convexity is necessary for semicontinuity in the local theory [25].

The next step towards applying the direct method is to ensure appropriate compactness of minimizing sequences. Here, a coercivity property of the functional J is crucial. The polynomial coercivity of F given by (2.6) may at first seem insufficient. Indeed, with it we obtain

$$J(u) \geq \alpha \|Gu\|^p_{L^p(\Omega \times \Omega)} + \beta, \quad \forall u \in NW^{1,p}(\Omega). \tag{2.8}$$

However, for coercivity we want $J(u) \to \infty$ as the norm $\|u\|_{NW^{1,p}(\Omega)} \to \infty$. But the norm on $NW^{1,p}(\Omega)$ also relies on $\|u\|_{L^p(\Omega)}$, which is not considered in (2.8). Fortunately, we have nonlocal analogues of the Poincaré inequalities that resolve this debacle.

Proposition 2.8. *Let M be a closed subspace of $NW^{1,p}(\Omega)$ for which the only constant function in M is identically zero. Then there exists some $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|Gu\|_{L^p(\Omega \times \Omega)}, \quad \forall u \in M.$$

Proof. We employ the usual proof by contradiction for Poincaré-type inequalities. Hence for contradiction, we assume that there exists some sequence $\{u_k\}_{k \in \mathbb{N}}$ in M all with $\|u_k\|_{L^p(\Omega)} = 1$, such that

$$\|Gu_k\|_{L^p(\Omega \times \Omega)} \leq \frac{1}{k}, \quad \forall k \in \mathbb{N}.$$

This sequence is bounded in $NW^{1,p}(\Omega)$, hence we apply Lemma 2.6(i) to obtain a subsequence for which $u_{k'} \to u \in L^p(\Omega)$. Both $\{u_{k'}\}_{k' \in \mathbb{N}}$ and $\{Gu_{k'}\}_{k' \in \mathbb{N}}$ are convergent in $L^p(\Omega)$ and $L^p(\Omega \times \Omega)$, respectively, as $Gu_{k'} \to 0$ in $L^p(\Omega \times \Omega)$. Since

$NW^{1,p}(\Omega)$ is Banach, so is the closed subspace M, and as a result $u_{k'}$ converges to some $u \in M$ with $Gu \equiv 0$ and $\|u\|_{L^p(\Omega)} = 1$. But $Gu \equiv 0$ implies that the continuous representative of u in $C^{0,\alpha}(\Omega)$ is constant. Since the only constant in M vanishes everywhere, we obtain a contradiction with $\|u\|_{L^p(\Omega)} = 1$. □

Consider the subspace of functions with vanishing boundary conditions

$$NW_0^{1,p}(\Omega) = \{u \in NW^{1,p}(\Omega) \mid u(0) = u(1) = 0\}.$$

We remark that $NW_0^{1,p}$ is closed. This follows from Theorem 2.4(ii) since convergence in $NW^{1,p}(\Omega)$ implies uniform convergence in $C^{0,\alpha}(\Omega)$. Therefore, vanishing boundary conditions are preserved in the limit. Additionally, we note that the only constant function in $NW_0^{1,p}$ is identically zero. Hence one can choose $M = NW_0^{1,p}$ in Proposition 2.8. This yields a nonlocal version of the classical Poincaré inequality.

Corollary 2.9 (Nonlocal Poincaré inequality). *There exists some $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|Gu\|_{L^p(\Omega\times\Omega)}, \quad \forall u \in NW_0^{1,p}(\Omega).$$

Unsurprisingly, the Poincaré inequality from Corollary 2.9 allows us to define an equivalent norm on $NW_0^{1,p}$. In fact, we see that

$$\|Gu\|_{L^p(\Omega\times\Omega)} \leq \|u\|_{NW^{1,p}(\Omega)} \leq (1 + C)\|Gu\|_{L^p(\Omega\times\Omega)}, \quad \forall u \in NW_0^{1,p}(\Omega), \tag{2.9}$$

where $C > 0$ is given by Corollary 2.9. Hence $\|Gu\|_{L^p(\Omega\times\Omega)}$ is equivalent to the norm $\|u\|_{NW^{1,p}(\Omega)}$ for all $u \in NW_0^{1,p}(\Omega)$. Returning to the coercivity inequality (2.8), we realize that $\|u\|_{NW^{1,p}(\Omega)} \to \infty$ also implies $\|Gu\|_{L^p(\Omega\times\Omega)} \to \infty$.

2.2.1 Existence theorems

It is now straightforward to prove existence under vanishing boundary conditions.

Theorem 2.10. *Assume F is given as in* (2.5) *and satisfies the coercivity condition* (2.6)*. Then the problem*

$$\min_{u \in NW_0^{1,p}(\Omega)} J(u) = \int_0^1\int_0^1 F(x, u(x), Gu(x, x'))\,\mathrm{d}x'\,\mathrm{d}x$$

admits an optimal solution if J is proper.

Proof. To apply the direct method, we begin by noting that J is bounded below by the coercivity condition (2.6)

$$J(u) \geq \alpha\int_0^1\int_0^1 |Gu(x, x')|^p\,\mathrm{d}x'\,\mathrm{d}x + \beta, \quad \forall u \in NW^{1,p}(\Omega).$$

Hence, we can choose a minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ in $NW_0^{1,p}(\Omega)$. Since the functional values $\{J(u_k)\}_{k\in\mathbb{N}}$ are bounded, there exists $M \in \mathbb{R}$ such that for all $k \in \mathbb{N}$

$$M \geq J(u_k) \geq \alpha\int_0^1\int_0^1 |Gu_k(x, x')|^p\,\mathrm{d}x'\,\mathrm{d}x \geq \alpha(1 + C)^{-p}\|u_k\|^p_{NW^{1,p}(\Omega)}.$$

Here, the last inequality follows from (2.9). Thus, u_k is bounded in $NW_0^{1,p}(\Omega)$, and by Lemma 2.6, there exists $u \in NW_0^{1,p}(\Omega)$ and a subsequence $u_{k'} \to u$ in $L^p(\Omega)$. Furthermore, since the coercivity condition also implies that F is bounded below, we can conclude the lower semicontinuity of J in $L^p(\Omega)$. Now, according to Proposition 2.7(ii), we have the limiting inequality

$$J(u) \le \liminf_{k' \to \infty} J(u_{k'}) = \inf_{v \in NW_0^{1,p}(\Omega)} J(v),$$

which proves that $u \in NW_0^{1,p}(\Omega)$ minimizes J. □

The proof of Theorem 2.10 is dependent on the coercivity of J in $NW_0^{1,p}(\Omega)$, which was shown utilizing the nonlocal Poincaré inequality. We will now see that similar arguments also apply in the case of arbitrarily fixed boundary conditions. For $a, b \in \mathbb{R}$ we let

$$NW^{1,p}_{a,b}(\Omega) = \{u \in NW^{1,p}(\Omega) \mid u(0) = a, u(1) = b\}, \quad a, b \in \mathbb{R}.$$

Our approach realizes that $NW^{1,p}_{a,b}(\Omega)$ can be seen as an affine subspace of $NW^{1,p}(\Omega)$ through $NW_0^{1,p}(\Omega)$. To this end we define the affine function

$$u_{a,b}(x) = a + (b - a)x, \quad \forall x \in \Omega.$$

Note that the nonlocal derivative of $u_{a,b}$ is constant. Indeed, for $(x, x') \in \Omega^2$ distinct, we have

$$Gu_{a,b}(x, x') = (b - a)\frac{x - x'}{x - x'} = b - a.$$

Hence $u_{a,b} \in NW^{1,p}_{a,b}(\Omega)$. Now for any $u \in NW^{1,p}_{a,b}(\Omega)$ let $u_0 = u - u_{a,b} \in NW_0^{1,p}(\Omega)$. This gives the affine decomposition

$$u = u_0 + u_{a,b}. \tag{2.10}$$

Lemma 2.11. *Assume F is given as in* (2.5) *and satisfies the coercivity condition* (2.6)*. Then the functional*

$$J(u) = \int_0^1 \int_0^1 F(x, u(x), Gu(x, x'))\,\mathrm{d}x'\,\mathrm{d}x$$

is coercive on $NW^{1,p}_{a,b}(\Omega)$.

Proof. We prove that $J(u) \to \infty$ as $\|u\|_{NW^{1,p}(\Omega)} \to \infty$ for $u \in NW^{1,p}_{a,b}(\Omega)$. For any $u \in NW^{1,p}_{a,b}(\Omega)$ we use the decomposition (2.10). By the coercivity condition (2.6) we have

$$\begin{aligned} J(u) &\ge \|Gu\|^p_{L^p(\Omega\times\Omega)} + \beta \\ &= \|Gu_0 + Gu_{a,b}\|^p_{L^p(\Omega\times\Omega)} + \beta \\ &\ge \left| \|Gu_0\|_{L^p(\Omega\times\Omega)} - \|Gu_{a,b}\|_{L^p(\Omega\times\Omega)} \right|^p + \beta, \end{aligned}$$

due to the reverse triangle inequality. Rearranging, we get

$$\begin{aligned}(\varepsilon^{-1}J(u) - \varepsilon^{-1})^{1/p} &\geq \left|\|Gu_0\|_{L^p(\Omega)} - \|Gu_{a;b}\|_{L^p(\Omega)}\right| \\ &\geq \|Gu_0\|_{L^p(\Omega)} - \|Gu_{a;b}\|_{L^p(\Omega)} \\ &\geq (1+C)^{-1}\|u_0\|_{NW^{1;p}(\Omega)} - \|Gu_{a;b}\|_{L^p(\Omega)};\end{aligned}$$

using the nonlocal Poincaré inequality. Therefore, we have

$$\|u_0\|_{NW^{1;p}(\Omega)} \leq (1+C)\left[(\varepsilon^{-1}J(u) - \varepsilon^{-1})^{1/p} + \|Gu_{a;b}\|_{L^p(\Omega)}\right]. \tag{2.11}$$

We now note that $\|u\|_{NW^{1;p}(\Omega)} \to \infty$ implies $\|u_0\|_{NW^{1;p}(\Omega)} \to \infty$ due to the triangle inequality

$$\|u\|_{NW^{1;p}(\Omega)} \leq \|u_0\|_{NW^{1;p}(\Omega)} + \|u_{a;b}\|_{NW^{1;p}(\Omega)};$$

and since $\|u_{a;b}\|_{NW^{1;p}(\Omega)}$ is constant. From (2.11) we deduce that $J(u) \to \infty$ as $\varepsilon > 0$ and $\|u\|_{NW^{1;p}(\Omega)} \to \infty$. □

Using the direct method as in the proof of Theorem 2.10, we can now establish the existence of solutions under nonhomogeneous boundary conditions.

Theorem 2.12. *Assume F is given as in* (2.5) *and satisfies the coercivity condition* (2.6). *Given boundary conditions $a; b \in \mathbb{R}$, the problem*

$$\min_{u \in NW^{1;p}_{a;b}(\Omega)} J(u) = \int_0^1 \int_0^1 F(x; u(x); Gu(x; x')) \, dx' \, dx;$$

admits an optimal solution if J is proper.

Proof. The proof follows that given in Theorem 2.10, but with a slight modification. In order to guarantee that a minimizing sequence $\{u_k\}_{k \in \mathbb{N}} \subset NW^{1;p}_{a;b}(\Omega)$ is bounded in $NW^{1;p}(\Omega)$, we apply Lemma 2.11. Due to the coercivity condition (2.6), the sequence of functional values $\{J(u_k)\}_{k \in \mathbb{N}}$ is bounded. But by coercivity of J on $NW^{1;p}_{a;b}(\Omega)$ this implies that the sequence of norms $\{\|u_k\|_{NW^{1;p}(\Omega)}\}_{k \in \mathbb{N}}$ is bounded, proving the minimizing sequence is bounded. □

2.3 Examples

We conclude this chapter by providing two examples of local variational problems and their corresponding nonlocal variants. The existence of solutions to the nonlocal problems will be established using the nonlocal variational calculus developed in the previous section.

Example 2.13 (The p-Laplace equation). We first present the p-Laplace equation, which is a generalization of the linear Laplace equation (2.1) for varying exponents $p \in (1; \infty)$.

$$\begin{aligned} -(|u'|^{p-2}u')' &= 0; \quad \text{in } \Omega; \\ u(0) = 0; \quad u(1) &= 1. \end{aligned} \tag{2.12}$$

The p-Laplace equation arises in the description of a variety of nonlinear physical phenomenons, such as some types of anomalous diffusion and fluid flow. Despite its nonlinearity, its solution is still characterized by Dirichlets principle of minimum energy

$$\begin{aligned} \min_{u \in W^{1,p}(\Omega)} \quad & I(u) = \frac{1}{p}\int_0^1 |u'(x)|^p \,\mathrm{d}x, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 1. \end{aligned} \tag{2.13}$$

For $p > 2$ we realize that the the nonlocal version of (2.13) was the problem introduced in (2.4)

$$\begin{aligned} \min_{u \in NW^{1,p}(\Omega)} \quad & J(u) = \frac{1}{p}\int_0^1 \int_0^1 \left|Gu(x,x')\right|^p \,\mathrm{d}x' \,\mathrm{d}x, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 1. \end{aligned} \tag{2.14}$$

Let us now verify that (2.14) admits a unique solution. We have already argued that J is a proper functional on $NW^{1,p}(\Omega)$. In addition, we see that its integrand is of the form $F(x,u,U) = |U|^p$, and hence automatically satisfies the coercivity condition (2.6). As such, Theorem 2.12 promises the existence of a solution. Since the integrand is convex, the functional is also convex and the solution is unique. In Figure 2.1 we provide numerical estimates of the nonlocal solutions obtained by discretizing (2.14).

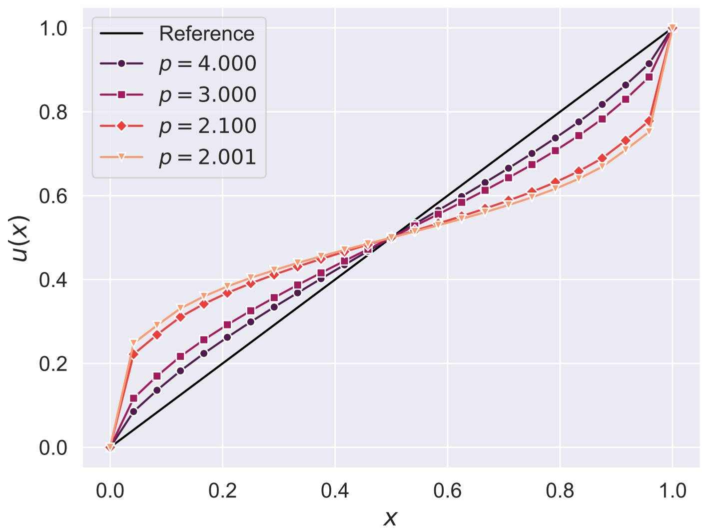


Figure 2.1: Solutions to the nonlocal variational problem (2.14) are shown for different choices of $p > 2$. The classical local solution $u(x) = x$ to (2.12) is plotted as the reference line.

Example 2.14 (The double-well potential). Lastly, we consider a common example of a degenerate local problem.

$$\begin{aligned} \min_{u \in W^{1,4}(\Omega)} \quad & I(u) = \int_0^1 \left||u'(x)|^2 - 1\right|^2 \,\mathrm{d}x + \int_0^1 |u(x)|^2 \,\mathrm{d}x, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{2.15}$$

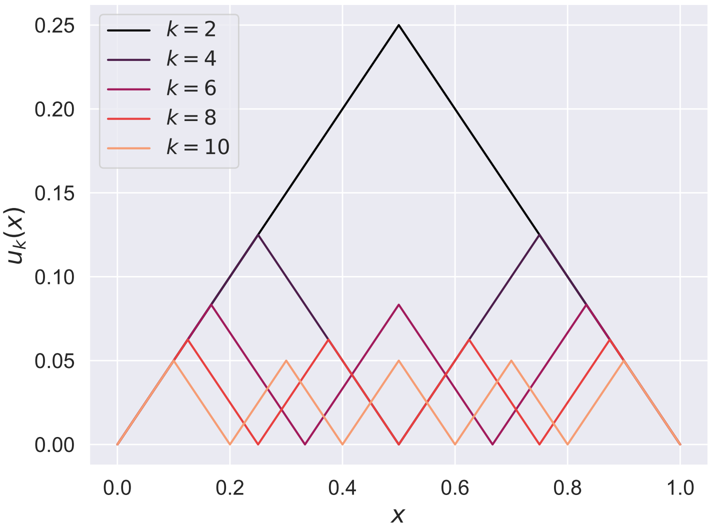


Figure 2.2: The first few sawtooth functions belonging to a minimizing sequence of (2.15). As k increases, the functions become more and more oscillatory.

Despite its simple formulation, it holds that (2.15) admits no solutions. Indeed, a standard exercise shows that one can construct a minimizing sequence, whose limiting functional value is unattainable. For instance, consider the sequence of piecewise affine sawtooth-like functions $\{u_k\}_{k\in\mathbb{N}}$ illustrated in Figure 2.2. Their derivatives are given by the formula

$$u_k'(x) = \begin{cases} +1, & \text{if } \sin(2\pi kx) \geq 0, \\ -1, & \text{else}, \end{cases} \qquad \forall x \in \Omega. \tag{2.16}$$

Note that $|u_k'(x)| = 1$ and $|u_k(x)| \leq 1/(2k)$ for all $x \in \Omega$ and thus

$$I(u_k) = \int_0^1 |u(x)|^2 \, dx \leq \frac{1}{4k^2} \to 0.$$

However, no element $u \in W^{1,4}(\Omega)$ can attain the value $I(u) = 0$. That would imply both of the contradictory properties $u \equiv 0$ and $u' \equiv \pm 1$.

Problem (2.15) poses a challenge due to the lack of convexity in the integrand. Specifically, the double-well potential $\mathbb{R} \ni U \mapsto (U^2 - 1)^2$ is not a convex function. This nonconvexity leads to a lack of semicontinuity of the functional I. However, it is important to note that convexity is not a requirement of the nonlocal variational calculus. To demonstrate this, we consider the nonlocal version of (2.15).

$$\begin{aligned} \min_{u \in \mathcal{N}W^{1,4}(\Omega)} J(u) &= \int_0^1 \int_0^1 \left| |\mathcal{G}u(x, x')|^2 - 1 \right|^2 dx' \, dx + \int_0^1 |u(x)|^2 \, dx, \\ \text{s.t.} \quad & u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{2.17}$$

It is straightforward to see that $u \equiv 0$ implies $J(u) = 1$, hence the functional is proper. Additionally, the coercivity condition (2.6) is satisfied for $p = 4$ for some constants $\alpha > 0$ and $\beta \in \mathbb{R}$. To see this, note that the polynomial

$$p(x) = (1 - \alpha)x^2 - 2x^2 + (1 - \beta), \quad \forall x \in \mathbb{R},$$

is nonnegative on R whenever $1 = (1 \quad)(1 \quad)$ and $\quad < 1$. Inserting $x = U^2$ and rearranging $p(x) \quad 0$ we obtain $(U^2 \quad 1)^2 \quad U^4 + \quad$, which is sufficient since

$$F(u;U) = (U^2 \quad 1)^2 + u^2 \quad (U^2 \quad 1)^2; \quad 8(u;U)\ 2\ \mathrm{R} \quad \mathrm{R}:$$

Thus, F satisfies the coercivity condition, and as a surprising consequence of Theorem 2.10, there must exist a solution to (2.17).

This highlights a fascinating aspect of nonlocality. Namely, nonlocal problems sometimes yield solutions even when their local counterparts do not. This phenomenon makes nonlocal models valuable as approximations of local models, facilitating the study of ill-posed physical behaviour.

Chapter 3

Principles of nonlocal diffusion

The following chapter is devoted to the formulation of a nonlocal diffusion law for equilibrium temperature states. The chapter extends the framework in [39] to the nonlinear case. We begin by recalling the classical local formulation, the principles of which will be replicated by its nonlocal counterpart. The nonlocal theory is constructed through the nonlocal gradient operator. We then introduce the notion of the nonlocal divergence Similar to the approach in [33, 45]. We then proceed with the formulation of the nonlocal diffusion law, which is formulated by a Dirichlet principle. Thereafter, we establish the existence and uniqueness of equilibrium states under nonlocal boundary conditions. This is followed by a section studying the dual formulation of the nonlocal Dirichlet principle, which yields a nonlocal Kelvin principle. Finally, we analyze the localizing property of the nonlocal model as the nonlocality vanishes.

3.1 Local reference model

Throughout the chapter we let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, that is a nonempty, connected, and open set, with Lipschitz boundary Γ. We assume that the conductivity distribution of Ω is heterogeneous and isotropic, and is described by $\kappa \in L^\infty(\Omega)$ which satisfies $\kappa \in [\underline{\kappa}, \overline{\kappa}] \subset (0, \infty)$ almost everywhere in Ω. Subject to a volumetric heat source $f \in L^2(\Omega)$ under homogeneous Dirichlet boundary conditions, the equilibrium temperature state is found as the solution of

$$\begin{aligned} -\operatorname{div}(\kappa \nabla u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma. \end{aligned} \tag{3.1}$$

The problem (3.1) is known as the generalized Poisson equation. Here $\nabla v = [\partial_i v]_{i=1}^d$ denotes the gradient of a scalar function v and $\operatorname{div} \psi = \sum_{i=1}^d \partial_i \psi_i$ denotes the divergence of a vector function ψ. Note that the partial derivative ∂_i with respect to the ith variable is understood in the weak sense. The corresponding heat flux is found as $q = -\kappa \nabla u$ through Fick's first law. It is important to mention that a variety of heat transport phenomena can be modelled using Neumann, Robin or mixed boundary conditions. However, for simplicity we limit ourselves to Dirichlet conditions. The weak formulation of (3.1) can be posed as a linear variational problem

$$\text{Find } u \in U: \quad a(u; v) = \ell(v), \quad \forall v \in U, \tag{3.2}$$

defined by the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall u,v \in U,$$

and linear functional

$$\ell(v) = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in U.$$

For clarity we omit the variables in the integrals whenever they are clear from the context. The linear variational problem (3.2) is analyzed by fixing the appropriate Sobolev space $U = H_0^1(\Omega)$, and unique solvability is deduced by invoking the Lax-Milgram theorem. To this end, it is necessary to obtain notions of coercivity and boundedness of the bilinear form and functional. For details, see [17, 18]. Alternatively, the properties of the bilinear form gives an equivalent formulation of (3.2) through a minimum energy principle. Indeed, the equilibrium temperature state satisfies Dirichlet's Principle, and is found as the unique minimizer of

$$\min_{v \in U} I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x. \tag{3.3}$$

Here $|\cdot|$ denotes the Euclidean norm and I is the Dirichlet energy functional. A dual interpretation may be formulated by considering the equilibrium heat flux. By Kelvin's principle, the equilibrium heat flux of the equilibrium temperature state is found as the minimizer of the complementary energy functional

$$\min_{\sigma \in Q} I(\sigma) = \frac{1}{2}\int_\Omega \kappa^{-1}|\sigma|^2 \, \mathrm{d}x, \tag{3.4}$$

among all suitable heat fluxes σ with $\operatorname{div} \sigma = f$. Mathematically, Kelvin's principle can be derived as the dual problem of Dirichlet's principle (3.3) or by rewriting (3.2) as a mixed variational problem in the Sobolev space $Q = H(\operatorname{div};\Omega)$ [14]. We will explore the details in the nonlocal case. This alternative description of heat diffusion is useful in applications where the heat flux is the variable of importance.

The above principles can be generalized to the case of nonlinear diffusion. If we modify the exponent in the Dirichlet energy functional to be $p \in (1,\infty)$, we may formulate a generalized Dirichlet principle. Subject to nonlinear diffusion and a volumetric heat source $f \in L^{p'}(\Omega)$, p' the conjugate exponent of p, the equilibrium temperature state is found as the minimizer of

$$\min_{v \in U} I(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x. \tag{3.5}$$

In this case, the appropriate Sobolev space is $U = W_0^{1,p}(\Omega)$. Similar to before, the minimum energy principle can be equivalently characterized by a variational problem that takes the same form as (3.2). However, it is now defined by the bivariate functional

$$a(u,v) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall u,v \in U. \tag{3.6}$$

This results in a nonlinear variational problem, which is the weak formulation of the generalized p-Laplace equation

$$\begin{aligned} -\operatorname{div}\left(\kappa|\nabla u|^{p-2}\nabla u\right) &= f; \quad \text{in } \Omega; \\ u &= 0; \quad \text{on } \partial\Omega. \end{aligned} \tag{3.7}$$

Here, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is known as the *p-Laplacian* of a function $u : \Omega \to \mathbb{R}$. It is possible to verify the existence of a unique solution to (3.5), and equivalently prove it to be a weak solution to (3.7), by applying the direct method of calculus of variations and utilizing convexity arguments [6, Section 6.11]. Once again, we can formulate a dual Kelvin principle in the nonlinear setting, and such has been done in the context of developing mixed finite element methods for the p-Laplace equation, see e.g. [24, 40, 41]. In this case, Kelvins minimum complementary energy principle is of the form

$$\min_{\sigma\in Q} I(\sigma) = \frac{1}{p}\int_\Omega \kappa^{1-p}|\sigma|^p \, dx; \tag{3.8}$$

where the equilibrium heat flux is sought among heat fluxes σ with $\operatorname{div}\sigma = f$. The nonlinear heat flux is defined by the relation $q = -\kappa|\nabla u|^{p-2}\nabla u$, and it is sought in the Sobolev space

$$Q = \{q \in L^{p'}(\Omega;\mathbb{R}^d) \,|\, \operatorname{div} q \in L^{p'}(\Omega)\}.$$

In the ensuing chapter, all the presented variational problems and minimum energy principles are reobtained in the nonlocal setting.

3.2 Nonlocal operators

Similar to the previous chapter, we first introduce a nonlocal notion of the gradient. Since we treat the extension of the bond-based peridynamic model to thermal diffusion, we first fix the *nonlocal interaction horizon* $\delta > 0$, which limits the range at which material bonds can form. With caution, we remark that this implies that points near the boundary of the domain Ω may interact with parts of its exterior. As such, we consider the *interaction domain*

$$\Omega_\delta = \cup_{x\in\Omega} B_\delta(x);$$

where $B_\delta(x)$ denotes the open ball in $\mathbb{R}^d$ with radius δ and center in $x \in \mathbb{R}^d$. In particular, we let $B_\delta = B_\delta(0)$. The set $\Gamma_\delta = \Omega_\delta \setminus \Omega$ will be referred to as the *nonlocal boundary* of Ω. Figure 3.1 displays some examples.

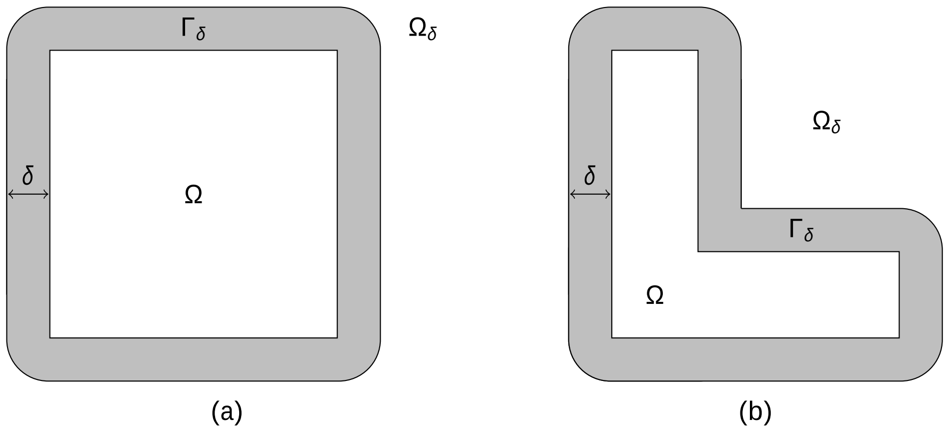


Figure 3.1: Two examples of Lipschitz domains Ω and their corresponding interactions domains Ω_δ, (a) is a square domain, (b) is an L-shaped domain. Their respective nonlocal boundaries Γ_δ are shaded grey, and the interaction horizon δ is marked.

3.2.1 The nonlocal gradient

We first define the nonlocal gradient. We consider an extension of the definition given in the previous chapter. In this case, the interaction horizon is taken into account. Encoding the strength of the nonlocal interactions, we fix a *nonlocal kernel* $\omega_\delta : \mathbb{R}^d \to [0, \infty)$ with support in B_δ. We will assume it to be radial, i.e.

$$\omega_\delta(x) = \omega_\delta(|x|), \quad \forall x \in B_\delta.$$

We elaborate on its remaining properties after the following definition.

Definition 3.1. Let $u : \mathbb{R}^d \to \mathbb{R}$. Then the *nonlocal gradient* of u is defined

$$\mathcal{G}_\delta u(x, x') = (u(x) - u(x'))\,\omega_\delta(x - x'), \quad \forall (x, x') \in \mathbb{R}^d \times \mathbb{R}^d.$$

Remark that the nonlocal gradient is defined on the entirety of $\mathbb{R}^d \times \mathbb{R}^d$. However, due to the interaction horizon, its support resides entirely within $\Omega_\delta \times \Omega_\delta$. Throughout this chapter we fix $p \in (1, \infty)$, and we view the nonlocal gradient as an unbounded linear operator

$$\mathcal{G}_\delta : D(\mathcal{G}_\delta) \subset L^p(\Omega) \to L^p(\Omega_\delta \times \Omega_\delta).$$

Here, we note that $D(\mathcal{G}_\delta)$ denotes the domain of $\mathcal{G}_\delta$. In order to consider the nonlocal gradient of functions defined on Ω, we implicitly extend the functions by zero outside of their defined domains. This allows us to apply the nonlocal gradient operator to functions defined on Ω even if they are not defined everywhere in $\mathbb{R}^d$. The goal of this section is to explore the properties of the nonlocal gradient. Naturally, the properties of the kernel are defining for the nonlocality, and therefore it has to be chosen suitably. The simplest kernel is perhaps the constant kernel

$$\omega_\delta(x) = c_\delta \chi_{B_\delta}(x), \quad \forall x \in \mathbb{R}^d.$$

Here χ_M is understood as the indicator function on the set $M \subset \mathbb{R}^d$. Similarly, a continuous cutoff function such as

$$\omega_\varepsilon(x) = c_\varepsilon(\varepsilon - |x|)^\alpha \chi_{B_\varepsilon}(x), \quad \forall x \in \mathbb{R}^d, \quad \forall \varepsilon, \alpha > 0,$$

is also admissible. Note that these are both examples of bounded integrable kernels. In contrast, one may also choose the singular kernel

$$\omega_\varepsilon(x) = \frac{c_\varepsilon}{|x|^\alpha} \chi_{B_\varepsilon}(x), \quad \forall x \in \mathbb{R}^d \setminus \{0\},$$

with an appropriate choice of $\alpha > 0$. For $s \in (0, 1)$, one such choice is $\alpha = d/p + s$ in spirit of the fractional Sobolev spaces. Regardless of the choice of kernel, the constant $c_\varepsilon > 0$ is chosen to normalize the kernel in such a way that

$$\int_{\mathbb{R}^d} |x|^p \omega_\varepsilon(x)^p \, \mathrm{d}x = K_{p,d}^{-1}. \tag{3.9}$$

Here $K_{p,d} > 0$ is defined by

$$K_{p,d} = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} |s \cdot e|^p \mathrm{d}s,$$

where S^{d-1} is the $(d-1)$-dimensional unit sphere, $|S^{d-1}|$ its Lebesgue measure, and $e \in S^{d-1}$ is chosen arbitrarily. This choice of normalization is intrinsically related to the approximation of Sobolev spaces by their nonlocal counterparts, as first seen in [15] and further developed in [58, 59]. In the same vein, we will use (3.9) in Section 3.5 to mathematically describe the localization of our nonlocal model as $\varepsilon \searrow 0$. For practical reasons, we give an example of the constant $K_{p,d}$ and a normalization constant c_ε.

Example 3.2. Assume $d = p = 2$ and consider the continuous cutoff kernel with $\varepsilon > 0$ and $\alpha = 2$

$$\omega_\varepsilon(x) = c_\varepsilon(\varepsilon^2 - |x|^2)^\alpha \chi_{B_\varepsilon}(x), \quad \forall x \in \mathbb{R}^d.$$

Let us determine $c_\varepsilon > 0$ such that (3.9) is satisfied. The constant $K_{2,2}$ is readily found

$$K_{2,2} = \frac{1}{|S^1|} \int_{S^1} |s \cdot e|^2 \mathrm{d}s = \frac{1}{2\pi} \int_0^{2\pi} \cos(t)^2 \mathrm{d}t = \frac{1}{2}.$$

Meanwhile, the right-hand side of (3.9) can be computed by switching to polar coordinates

$$K_{2,2}^{-1} = c_\varepsilon^2 \int_{B_\varepsilon} |x|^2(\varepsilon^2 - |x|^2)^{2\alpha} \, \mathrm{d}x = 2\pi c_\varepsilon^2 \int_0^\varepsilon r^3(\varepsilon^2 - r^2)^{2\alpha} \, \mathrm{d}r = \frac{c_\varepsilon^2 \pi \varepsilon^{4\alpha+4}}{2(2\alpha+1)(\alpha+1)}.$$

Rearranging the previous two equations, we get

$$c_\varepsilon^2 = \frac{4(2\alpha+1)(\alpha+1)}{\pi \varepsilon^{4\alpha+4}}.$$

We note that as $\varepsilon \searrow 0$, the normalization constant $c_\varepsilon \nearrow \infty$.

With an appropriate kernel at hand, we begin our analysis of the nonlocal gradient G_δ as an unbounded linear operator. We start with the following density result.

Proposition 3.3. *Let the domain of G_δ be denoted U_δ. Then U_δ is dense in $L^p(\Omega)$.*

Proof. We prove that U_δ belongs to a dense subset of $L^p(\Omega)$. Let $\varphi \in C_c^{0,1}(\Omega)$ and note that its extension by zero resides in $C_c^{0,1}(\mathbb{R}^d)$. Then

$$\begin{aligned}\int_\Omega \int_\Omega |G_\delta \varphi(x,x')|^p \,\mathrm{d}x'\,\mathrm{d}x &= \int_\Omega \int_\Omega |\varphi(x) - \varphi(x')|^p \omega_\delta(x-x')^p \,\mathrm{d}x'\,\mathrm{d}x \\ &= \int_\Omega \int_\Omega \frac{|\varphi(x)-\varphi(x')|^p}{|x-x'|^p} |x-x'|^p \omega_\delta(x-x')^p \,\mathrm{d}x'\,\mathrm{d}x \\ &\leq \|\varphi\|^p_{C^{0,1}(\Omega)} \int_\Omega \int_\Omega |x-x'|^p \omega_\delta(x-x')^p \,\mathrm{d}x'\,\mathrm{d}x \\ &\leq \|\varphi\|^p_{C^{0,1}(\Omega)} \int_\Omega \int_{\mathbb{R}^d} |z|^p \omega_\delta(z)^p \,\mathrm{d}z\,\mathrm{d}x \\ &\leq |\Omega| \|\varphi\|^p_{C^{0,1}(\Omega)} K_{p,d}^{-1},\end{aligned}$$

where the third inequality follows from the normalization assumption (3.9). This shows that $G_\delta \varphi \in L^p(\Omega \times \Omega)$, which implies that $C_c^{0,1}(\Omega) \subset U_\delta$. Since $C_c^{0,1}(\Omega)$ is dense in $L^p(\Omega)$, so is U_δ. □

The domain U_δ will serve as the nonlocal analog of the equilibrium temperature state space $U = H_0^1(\Omega)$ discussed in Section 3.1. It is important to note that the boundary conditions imposed in $H_0^1(\Omega)$ (in the trace sense) are not enforced in U_δ. Instead, the practice of extending functions by zero outside their defined domains leads to a different notion of boundary conditions in the nonlocal setting. We will discuss this in more detail later.

In order to study variational principles on U_δ, we need to study its compactness properties. In particular, we will show that U_δ is a reflexive Banach space. To do this, we will rely on a series of results that shed light on the properties of the nonlocal gradient. Initially, we will equip U_δ with the graph norm

$$\|u\|_{G_\delta} = \left(\|u\|^p_{L^p(\Omega)} + \|G_\delta u\|^p_{L^p(\Omega\times\Omega)}\right)^{1/p}.$$

Here, the notation is inspired by the graph of G_δ, which we denote $\mathcal{G}(G_\delta)$. Subject to this norm, the nonlocal gradient trivially becomes a bounded linear operator on U_δ. Indeed for all $u \in U_\delta$, we see that

$$\|G_\delta u\|^p_{L^p(\Omega\times\Omega)} \leq \|u\|^p_{L^p(\Omega)} + \|G_\delta u\|^p_{L^p(\Omega\times\Omega)} = \|u\|^p_{G_\delta},$$

and hence it has operator norm $\|G_\delta\|_{U_\delta^*} \leq 1$. Here U_δ^* denotes the dual space of U_δ. In connection we have the following result.

Proposition 3.4. *The graph of G_δ is closed in $L^p(\Omega) \times L^p(\Omega \times \Omega)$.*

Proof. In order to prove the result, we assume that $\{u_k, q_k\}_{k\in\mathbb{N}} \subset \mathcal{G}(G_\delta)$ is a sequence which converges in $L^p(\Omega) \times L^p(\Omega\times\Omega)$ to the pair (u, q). By extracting a subsequence indexed by $k' \in \mathbb{N}$ we may obtain pointwise convergence. That is, for almost all $(x, x') \in \Omega \times \Omega$ we have

$$\mathcal{G}_\delta u(x, x') = \lim_{k' \to \infty} \mathcal{G}_\delta u_{k'}(x, x') = \lim_{k' \to \infty} q_{k'}(x, x') = q(x, x').$$

Here, the first equality follows from the pointwise convergence of $u_{k'}$, while the last inequality follows from the pointwise convergence of $q_{k'}$. This suggests $\mathcal{G}_\delta u = q$ and $(u, q) \in G(\mathcal{G}_\delta)$. Thus, we find that the graph of $\mathcal{G}_\delta$ is closed. □

Similar to the previous chapter, we find it handy that the pointwise convergence of a function automatically implies the same for its nonlocal gradient. We can now draw the desired conclusion by using Proposition 3.4.

Proposition 3.5. *The space $(U_\delta, \|\cdot\|_{\mathcal{G}_\delta})$ is a reflexive Banach space.*

Proof. Consider the canonical linear isometry $T : U_\delta \to L^p(\Omega) \times L^p(\Omega \times \Omega)$ defined

$$T(u) = (u, \mathcal{G}_\delta u), \quad \forall u \in U_\delta.$$

It is clear that the range $R(T)$ coincides with $G(\mathcal{G}_\delta)$. By Proposition 3.4, $R(T)$ is a closed subspace of the reflexive Banach space $L^p(\Omega) \times L^p(\Omega \times \Omega)$, and so it inherits the properties of reflexivity and completeness. Finally, since T is an isometry between U_δ and the reflexive Banach space $R(T)$, U_δ also inherits those properties. □

In future proofs it will come in handy to have a nonlocal Poincaré inequality, which will provide us knowledge about a function given that we have information about its nonlocal gradient. In addition, it will provide an equivalent norm for U_δ, which as an added bonus will simplify computations. In order to prove such an inequality, we need more information on the behaviour of the nonlocal gradient. Following the steps of [38] we give the following compactness result from [58, Theorem 1.2], here stated in the language of our framework.

Lemma 3.6 (Ponce inequality). *Let $\delta_0 > 0$ and let $\{\delta_k\}_{k \in \mathbb{N}} \subset (0, \delta_0)$ be a sequence of interaction horizons converging to zero. Assume $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega)$ is a bounded sequence and that there exists some $M > 0$ for which*

$$\int_{\Omega_{\delta_k}} \int_{\Omega_{\delta_k}} |\mathcal{G}_{\delta_k} u_k(x, x')|^p \, dx' \, dx \le M, \quad \forall k \in \mathbb{N}.$$

Then $\{u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^p(\Omega_0)$. Any limit point $u \in L^p(\Omega_0)$ is also in $W^{1,p}(\Omega_0)$ and satisfies

$$\int_{\Omega_0} |\nabla u|^p \, dx \le \limsup_{k \to \infty} \int_{\Omega_{\delta_k}} \int_{\Omega_{\delta_k}} |\mathcal{G}_{\delta_k} u_k(x, x')|^p \, dx' \, dx. \tag{3.10}$$

In contrast to Lemma 2.6, which we used to prove the Poincaré inequality given in Proposition 2.8, Lemma 3.6 considers a sequence of different gradient operators, defined by a corresponding vanishing sequence of interaction horizons. This result is our first glimpse into the localization property of the nonlocal gradient. The obtained inequality for the local limit points (3.10) is known as the Ponce inequality. With it, we can prove the nonlocal Poincaré inequality.

Proposition 3.7 (Nonlocal Poincaré inequality). *There exists some $\delta_0 > 0$ and a constant $C(\delta_0) > 0$, such that for all $\delta \in (0, \delta_0)$*

$$\|u\|_{L^p(\Omega)} \leq C(\delta_0) \|\mathcal{G}_\delta u\|_{L^p(\Omega_\delta \times \Omega_\delta)}, \quad \forall u \in U_\delta. \tag{3.11}$$

Proof. We take the usual approach of proof by contradiction. Therefore, we assume there exists a sequence $\{\delta_k\}_{k \in \mathbb{N}}$ in $(0, \delta_0)$, for some $\delta_0 > 0$, which converges to zero, and that for all $k \in \mathbb{N}$ we find corresponding $u_k \in U_{\delta_k}$ satisfying

$$\|u_k\|_{L^p(\Omega)} > k \|\mathcal{G}_{\delta_k} u_k\|_{L^p(\Omega_{\delta_k} \times \Omega_{\delta_k})},$$

and $\|u_k\|_{L^p(\Omega)} = 1$ for all $k \in \mathbb{N}$. Hence

$$\|\mathcal{G}_{\delta_k} u_k\|_{L^p(\Omega_{\delta_k} \times \Omega_{\delta_k})} \to 0 \text{ as } k \to \infty.$$

For all $k \in \mathbb{N}$ we define $M_k = \sup_{j \geq k} \left\|\mathcal{G}_{\delta_j} u_j\right\|^p_{L^p(\Omega_{\delta_j} \times \Omega_{\delta_j})} < \infty$, and we get the uniform bound

$$\int_{\Omega_{\delta_k}} \int_{\Omega_{\delta_k}} |\mathcal{G}_{\delta_k} u_k(x, x')|^p \, \mathrm{d}x' \, \mathrm{d}x \leq M_k \leq M_0. \tag{3.12}$$

As $\delta_k \searrow 0$ we can use Lemma 3.6 to see that $\{u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^p(\Omega_0)$. If we consider such a limit point $u \in W^{1,p}(\Omega_0)$, we can use the Ponce inequality (3.10) to get

$$\int_{\Omega_0} |\nabla u|^p \, \mathrm{d}x \leq \lim_{k \to \infty} \int_{\Omega_{\delta_k}} \int_{\Omega_{\delta_k}} |\mathcal{G}_{\delta_k} u_k(x, x')|^p \, \mathrm{d}x' \, \mathrm{d}x = 0.$$

This implies that $\nabla u \equiv 0$, hence u is constant almost everywhere. We can prove $u \equiv 0$ on $\Omega_0 \setminus \overline{\Omega}$ by pointwise convergence of the sequence $\{u_k\}_{k \in \mathbb{N}}$. Indeed, since we extend functions by zero outside their domains $\lim_{k \to \infty} u_k(x) = 0$ for all $x \in \Omega_0 \setminus \overline{\Omega}$, which implies the same almost everywhere for the limit u. Since u is constant, it must vanish everywhere in Ω_0. But, due to the convergence in $L^p(\Omega_0)$, we must have that $\|u\|_{L^p(\Omega_0)} = 1$ which is a contradiction. □

It is important for us to note that the above nonlocal Poincaré inequality (3.11) only holds true if $\delta > 0$ is chosen sufficiently small, that is $\delta \in (0, \delta_0)$. Throughout the rest of this thesis, we assume that δ is chosen as such, and we define the corresponding constant $C_P = C(\delta_0) > 0$. For practical reasons, we will also assume that all of the numerically considered interaction horizons are permissible.

As promised, we can now define an equivalent norm on U_δ.

Corollary 3.8. *Define the norm*

$$\|u\|_{U_\delta} = \|\mathcal{G}_\delta u\|_{L^p(\Omega_\delta \times \Omega_\delta)}, \quad \forall u \in U_\delta.$$

Then the space $(U_\delta, \|\cdot\|_{U_\delta})$ is a reflexive Banach space. In particular, U_δ becomes a Hilbert space, if $p = 2$, with inner product

$$\langle u, v \rangle_{U_\delta} = \int_{\Omega_\delta} \int_{\Omega_\delta} \mathcal{G}_\delta u \mathcal{G}_\delta v \, \mathrm{d}x' \, \mathrm{d}x, \quad \forall u, v \in U_\delta. \tag{3.13}$$

Proof. Given the constant $C_P > 0$ from the nonlocal Poincaré inequality, we have the following equivalence of norms

$$\|u\|_U^p \le \|u\|_{\mathcal{G}}^p = \|u\|_{L^p(\Omega)}^p + \|\mathcal{G}u\|_{L^p(\Omega\times\Omega)}^p \le (1 + C_P^p)\,\|u\|_U^p\,.$$

This proves the first claim. It is immediate to see that the norm is induced by the inner product $\|u\|_U = (\langle u,u\rangle_U)^{1/2}$, when $p = 2$. □

With this choice of norm, it is again immediate to see that the nonlocal gradient is continuous on U. In fact, this time as an isometry. In particular, this implies that the nonlocal gradient is an injective operator. Before our next step, we prove a final property of the nonlocal gradient.

Proposition 3.9. *The range $R(\mathcal{G})$ is closed in $L^p(\Omega\times\Omega)$.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in U for which $\mathcal{G}u_k \to q$ in $L^2(\Omega\times\Omega)$. This implies that $\{\mathcal{G}u_k\}_{k\in\mathbb{N}}$ is Cauchy in $L^2(\Omega\times\Omega)$, which in turn implies $\{u_k\}_{k\in\mathbb{N}}$ is Cauchy in U, since $\mathcal{G}$ is an isometry. Corollary 3.8 implies that the sequence u_k has a limit $u \in U$ due to the fact that U is a Banach space. Moreover, since $\mathcal{G}$ is a closed operator (as stated in Proposition 3.4), we can conclude that $\mathcal{G}u = q$, which proves that the range of $\mathcal{G}$ is closed. □

3.2.2 The nonlocal divergence

Now, let's focus on constructing a nonlocal divergence operator. Like the nonlocal gradient, we will treat and analyze the nonlocal divergence as an unbounded linear operator. In contrast to the local divergence, which acts on vector fields, the nonlocal divergence consider the nonlocal interactions. As such, it acts on scalar functions defined on $\Omega\times\Omega$. To ease the use of notation, we introduce the integral notation

$$\langle u,v\rangle_\Omega = \int_\Omega u(x)v(x)\,\mathrm{d}x,$$
$$\langle q,\psi\rangle_{\Omega\times\Omega} = \int_\Omega\int_\Omega q(x,x')\,\psi(x,x')\,\mathrm{d}x'\,\mathrm{d}x,$$

for measurable scalar functions $u,v : \Omega \to \mathbb{R}$, and $q,\psi : \Omega\times\Omega \to \mathbb{R}$. We define the nonlocal divergence as follows.

Definition 3.10. Let $\mathcal{D} : D(\mathcal{D}) \subseteq L^{p'}(\Omega\times\Omega) \to L^{p'}(\Omega)$ be the negative adjoint of $\mathcal{G} : U \subseteq L^p(\Omega) \to L^p(\Omega\times\Omega)$. Then $\mathcal{D}$ denotes the *nonlocal divergence*. In particular, it is the unique linear operator satisfying

$$\langle \mathcal{D}q, u\rangle_\Omega = -\langle q, \mathcal{G}u\rangle_{\Omega\times\Omega}, \quad \forall q \in D(\mathcal{D}), u \in U\,. \tag{3.14}$$

It is immediate to see that the nonlocal divergence is well-defined since the nonlocal gradient is densely defined, see Proposition 3.3. It should be noted that our characterization, as expressed by equation (3.14), relies on identifying the dual Lebesgue spaces $L^p(\Omega)^*$ and $L^p(\Omega\times\Omega)^*$ with their corresponding Riesz representations, namely $L^{p'}(\Omega)$ and $L^{p'}(\Omega\times\Omega)$. Remark that the characterization defines the nonlocal gradient to satisfy an integration by parts formula in conjunction with the nonlocal gradient, similar to the definitions of local operators. Further justification of this definition is presented in [33] by relating the local physical balance laws to the nonlocal vector calculus of [45]. In the latter paper, the authors develop a nonlocal vector calculus using the nonlocal operators, which generalizes Gauss' theorem and Green's identities from the classical vector calculus.

We analyze the properties of the nonlocal divergence in a similar manner as we did for the nonlocal gradient.

Proposition 3.11. *Let $Q = D(D)$. Then Q is dense in $L^p(\quad)$.*

Proof. We will show that $C^{0,1}(\quad) \quad Q$. To this end we let $\; \in C^{0,1}(\quad)$, and we will show that we may find $D \quad \in L^p(\;)$ satisfying (3.14). Let $u \in U$, and for $\varepsilon > 0$ let

$$O_\varepsilon = \{(x, x') \in \quad \mid |x - x'| > \varepsilon\}.$$

Utilizing the Lebesgue dominated convergence theorem (DCT) we get

$$\begin{aligned} \langle \; , G u \rangle &= \int \int (x, x')(u(x) - u(x'))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x \\ &= \lim_{\varepsilon \downarrow 0} \iint_{O_\varepsilon} (x, x')(u(x') - u(x))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

Under this limit we claim that one can apply Fubini's theorem to find

$$\begin{aligned} &\iint_{O_\varepsilon} (x, x')(u(x') - u(x))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x \\ &\quad = \iint_{O_\varepsilon} (x, x')u(x')\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x - \iint_{O_\varepsilon} (x, x')u(x)\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x \\ &\quad = \iint_{O_\varepsilon} u(x)(\; (x', x) - \; (x, x'))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x. \end{aligned}$$

Here we utilize that ω is radial and exchange the variables in the first term. To verify the assumptions of Fubini's theorem, we first consider $\; (x, x') = \; (x, x')/|x - x'|$. Since $\;$ is Lipschitz continuous, we may consider the Lipschitz extension of $\;$ to the closure $\overline{O_\varepsilon}$. Now $\;$ is continuous on $\overline{O_\varepsilon}$, and therefore $C = \sup_{O_\varepsilon} \;$ is finite. Consequently, we may apply Hölder's inequality twice to see that

$$\begin{aligned} &\iint_{O_\varepsilon} \frac{| \; (x, x')|}{|x - x'|}|u(x')||x - x'|\,\omega(x - x')\,\mathrm{d}x\,\mathrm{d}x' \\ &\qquad \le C \int |u(x')| \int |x - x'|\,\omega(x - x')\,\mathrm{d}x\,\mathrm{d}x' \\ &\qquad \le C \int |u(x')|\,\mathrm{d}x' \int_B |z|\,\omega(z)\mathrm{d}z \\ &\qquad \le C| \; |^{1/p}|B|^{1/p} K_{p,d}^{p} \|u\|_{L^p(\;)} < \infty. \end{aligned}$$

Returning to (3.15) we can utilize the DCT again to obtain

$$\begin{aligned} \langle q, G u \rangle &= \lim_{\varepsilon \downarrow 0} \iint_{O_\varepsilon} u(x)(\; (x', x) - \; (x, x'))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x \\ &= \int u(x) \int (\; (x', x) - \; (x, x'))\,\omega(x - x')\,\mathrm{d}x'\,\mathrm{d}x. \end{aligned}$$

By definition, we recognize that

$$D\ (x) = \int\ (\ (x';x) \quad (x;x'))\,\omega\ (x \quad x')\,\mathrm{d}x'; \quad \forall x \in\ ;$$

if $D\ \in L^p\ (\ )$. In particular, we find that the nonlocal divergence of is bounded. For $x \in$ and by another application of Hölder's inequality we get

$$\begin{aligned} |D\ (x)| &\quad \int\ |\ (x';x) \quad (x;x')|\,\omega\ (x \quad x')\,\mathrm{d}x' \\ &= \int\ \frac{|\ (x';x) \quad (x;x')|}{|x \quad x'|}|x \quad x'|\,\omega\ (x \quad x')\,\mathrm{d}x' \\ &\quad 2\|\ \|_{C^{0,1}(\ )}|\ |^{1/p}\ K_{p,d}^{p}. \end{aligned}$$

So $\|D\ \|_{L^p\ (\ )} < \infty$ and as a result $C^{0,1}(\ ) \quad Q$. Since $C^{0,1}(\ )$ is dense in $L^p\ (\ )$, density of Q is immediate. □

The proof of Proposition 3.11 gives a formula for the nonlocal divergence

Corollary 3.12. *Let $q \in C^{0,1}(\ )$. Then the nonlocal divergence of q can be represented by the formula*

$$D\ q(x) = \int\ (q(x';x) \quad q(x;x'))\,\omega\ (x \quad x')\,\mathrm{d}x'; \quad \forall x \in\ . \tag{3.16}$$

As a straightforward consequence of D being an adjoint operator, we automatically get the following property of the domain Q .

Proposition 3.13. *Define the norm*

$$\|q\|_{Q} = \left(\|q\|_{L^p\ (\ )} + \|D\ q\|_{L^p\ (\ )}\right)^{1/p}.$$

Then the space $(Q\ ;\|\ \|_{Q}\ )$ is a reflexive Banach space.

Proof. Note that the nonlocal divergence D is a closed operator by virtue of being an adjoint operator [18, Proposition 2.17]. Since the norm $\|\ \|_{Q}$ is the graph norm arising from D , the arguments follow the proof of Proposition 3.5. □

Using the fact that the nonlocal divergence is a closed operator by virtue of being an adjoint operator, we conclude the proof. The following proposition summarizes the remaining properties inherited by D .

Proposition 3.14. *The nonlocal divergence $D\ : Q\ \to L^p\ (\ )$ enjoys the following properties:*

(i) *The range $R(D\ )$ is closed in $L^p\ (\ )$.*

(ii) *D is surjective onto $L^p\ (\ )$.*

(iii) *D is a bounded linear operator.*

Proof. Since G is a closed operator, $R(D)$ is closed if and only if $R(G)$ is closed, see e.g. [18, Theorem 2.19]. Hence (i) follows by Proposition 3.9. Since the nonlocal gradient is injective, and the range of D is closed, we get the following equalities

$$R(D) = N(G)^{\perp} = L^p(\Omega).$$

Here $N(G)$ denotes the kernel of G, and $M^{\perp}$ denotes the orthogonal subspace of a subspace M. This shows (ii). Lastly, (iii) follows from the closed graph theorem, since the nonlocal divergence D is closed and Q is a Banach space. □

In future discussions we will consider the following notion. For $f \in L^p(\Omega)$ we define the closed affine subspace

$$Q(f) = \{q \in Q \mid D q = f\}.$$

Due to the surjectivity of the nonlocal divergence D, we know that the affine space $Q(f)$ is nonempty. By carefully choosing an element of $Q(f)$ for all $f \in L^p(\Omega)$, we will now show that D admits a bounded right inverse.

Proposition 3.15. *There exists $C > 0$ such that for all $f \in L^p(\Omega)$ there exists some $q \in Q(f)$ satisfying*

$$\|q\|_Q \leq C\|f\|_{L^p(\Omega)}.$$

Proof. From Proposition 3.14 we know that $D : Q \to L^p(\Omega)$ is a bounded surjective operator. Let us now denote $N = N(D)$ and consider the quotient space Q/N. Since N is closed Q/N is a Banach space with quotient norm

$$\|\tilde{q}\|_{Q/N} = \inf_{n \in N} \|q + n\|_Q, \quad \forall \tilde{q} \in Q/N, \tag{3.17}$$

where $q \in Q$ is some fixed representative of $\tilde{q}$. Note that the infimum in (3.17) is attained since Q is a reflexive Banach space. Working in the quotient space, bounded surjectivity of D induces a bijective bounded linear quotient operator $\tilde{D}$: $Q/N \to L^p(\Omega)$ defined by

$$\tilde{D}\tilde{q} = D q; \quad \forall \tilde{q} \in Q/N.$$

By the open mapping theorem $\tilde{D}$ is an open map, and hence its inverse operator $(\tilde{D})^{-1} : L^p(\Omega) \to Q/N$ is a bounded linear bijection. Therefore, for each $f \in L^p(\Omega)$ there exists $\tilde{q} \in Q/N$, such that $(\tilde{D})^{-1} f = \tilde{q}$, which implies that

$$\|\tilde{q}\|_{Q/N} \leq C\|f\|_{L^p(\Omega)},$$

where $C > 0$ is given by the boundedness of $(\tilde{D})^{-1}$. Since the implicit infimum on the left-hand side is attained, there exists some $q \in Q(f)$ which satisfies the inequality

$$\|q\|_Q \leq C\|f\|_{L^p(\Omega)},$$

as we wanted. □

3.3 Nonlocal diffusion

Using the previously defined nonlocal operators, we now proceed to formulate a nonlocal diffusion law. Similar to the local formulation, we address the question of domain conductivity. To do so, we introduce a nonlocal conductivity distribution $\in L^\infty()$. In contrast to the local case, the nonlocal conductivity distribution takes values on . For physical reasons, we assume that is symmetric, meaning that $(x;x') = (x';x)$ almost everywhere in . Additionally, similar to its local counterpart, we assume that $\in [\underline{\ };\overline{\ }]$ almost everywhere in . Just like the nonlocal kernel, the nonlocal conductivity distribution allows for further characterization of the nonlocal interactions.

We move on to consider the nonlocal analog of the generalized p-Laplace equation introduced in (3.7), which characterized the local equilibrium states. Given a volumetric heat source $f \in L^{p'}()$, we study the generalized nonlocal p-Laplace equation

$$\begin{aligned} \mathcal{D}(|\mathcal{G}u|^{p-2}\mathcal{G}u) &= f; \quad \text{in } ;\\ u &= 0; \quad \text{in } : \end{aligned} \tag{3.18}$$

We let $\mathcal{L}u = \mathcal{D}(|\mathcal{G}u|^{p-2}\mathcal{G}u)$ denote the *nonlocal p-Laplacian* of $u : \to \mathbb{R}$. Utilizing the divergence formula in (3.16), we formally write the nonlocal p-Laplacian with the formula

$$\mathcal{L}u(x) = 2\int |u(x) - u(x')|^{p-2}(u(x) - u(x'))\,\omega(x;x')^p\,\mathrm{d}x'; \quad \forall x \in : \tag{3.19}$$

Depending on the choice of nonlocal kernel, it may be necessary to consider the formula (3.19) in the principal value sense. For the case of $p = 2$, the nonlocal 2-Laplacian arises in the formulation of the peridynamic equilibrium equation. The general nonlocal p-Laplacian has been studied extensively in the literature, with applications to anomalous diffusion and as fractional p-Laplacian operators. A comprehensive citation of the related works would be impractical, but notable studies include [2, 3]. The nonlocal p-Laplacian shares various properties with its local counterpart, and for further discussion, we refer to [47] and the references therein.

Unique to the nonlocal approach, we impose a *volume constraint* on the nonlocal boundary , as opposed to a boundary condition on in the local case. Unlike the local case, where a boundary condition on is typically imposed, requiring regularity of the temperature state, in the nonlocal case, it severely restricts the admissible nonlocal kernels. Recall our findings in Chapter 2. In our present case, we consider a homogeneous Dirichlet volume constraint, which enforces $u \equiv 0$ on . It is important to note that this constraint is automatically satisfied for temperature states $u \in \mathcal{U}$ since they are implicitly extended to vanish outside . It is also possible to impose inhomogeneous Dirichlet volume constraints by extending the states with a given function $g \in L^p()$. Additionally, nonlocal Neumann and Robin-type volume constraints can also be prescribed. Further details can be found in [33].

Let us clarify that we consider solutions of (3.18) in the weak sense. In the present case, we define them by the following procedure. Assuming that $u \in \mathcal{U}$ solves the generalized nonlocal p-Laplace equation, we test the first equation of (3.18) with arbitrary $v \in U$ and integrate over Ω. Formally invoking the integration by parts formula (3.14), we obtain the weak formulation

$$\int_{\Omega}\int_{\Omega} \gamma |\mathcal{G} u|^{p-2}\mathcal{G} u\,\mathcal{G} v \,\mathrm{d}x' \,\mathrm{d}x = \int_{\Omega} f v \,\mathrm{d}x, \quad \forall v \in U. \tag{3.20}$$

This inspires the following definition.

Definition 3.16. We say $u \in U$ is a solution to (3.18) if it satisfies (3.20).

Observe that (3.20) is the direct nonlocal analog of the weak formulation characterizing the generalized p-Laplacian equation.

3.3.1 Nonlocal Dirichlet principle

On par with the local analysis, we will study the variational problem in (3.20) through an equivalent minimum energy principle. Accordingly, we formulate the *nonlocal Dirichlet principle* as the nonlocal analogue of (3.5). It states that the nonlocal equilibrium temperature state $u \in U$, which solves (3.18), is equivalently found as a minimizer of

$$\min_{v \in U} J(v) = \frac{1}{p}\int_{\Omega}\int_{\Omega} \gamma |\mathcal{G} v|^{p} \,\mathrm{d}x' \,\mathrm{d}x - \int_{\Omega} f v \,\mathrm{d}x. \tag{3.21}$$

Here the functional $J : U \to \mathbb{R}$ denotes the nonlocal Dirichlet energy. Following the exact steps of the local theory, we argue the existence of a solution to (3.21) by application of the direct method, and we will derive (3.20) as the characterizing optimality condition. The following series of results provide the necessary tools.

Lemma 3.17. *The functional $J : U \to \mathbb{R}$ is strictly convex.*

Proof. We first define the function $\varphi(r) = r^p$ and note that it is strictly convex on $[0, \infty)$ since $p > 1$. With it, the first integrand of $J : U \to \mathbb{R}$ may be written as

$$\gamma(x, x')\,\varphi(|\mathcal{G} v(x, x')|), \quad \forall (x, x') \in \Omega \times \Omega, \ \forall v \in U. \tag{3.22}$$

For arbitrary $(x, x') \in \Omega \times \Omega$, $v, w \in U$, and $\lambda \in [0, 1]$, we see that

$$\begin{aligned}
\gamma(x, x')\,\varphi(|\mathcal{G}(\lambda v + (1-\lambda) w)(x, x')|) &\\
&\le \gamma(x, x')\,\varphi(\lambda|\mathcal{G} v(x, x')| + (1-\lambda)|\mathcal{G} w(x, x')|)\\
&\le \gamma(x, x')\,\lambda\varphi(|\mathcal{G} v(x, x')|) + (1-\lambda)\varphi(|\mathcal{G} w(x, x')|).
\end{aligned}$$

Here the first inequality follows from the triangle inequality and the linearity of the nonlocal gradient, while the second inequality follows from the convexity of φ. We remark that the inequalities may be strict. Indeed, if $v \neq w$, then $\mathcal{G} v \neq \mathcal{G} w$ due to injectivity of the nonlocal gradient. If $|\mathcal{G} v| \neq |\mathcal{G} w|$, then the second inequality is strict for almost every $(x, x') \in \Omega \times \Omega$ due to the strict convexity of φ. If not, then $|\mathcal{G} v| = |\mathcal{G} w|$, which in turn implies $\mathcal{G} v = -\mathcal{G} w$. In this case the first inequality becomes strict for almost every $(x, x') \in \Omega \times \Omega$, since $\lambda\mathcal{G} v$ and $(1-\lambda)\mathcal{G} w$ have different signs. Integrating over $\Omega \times \Omega$ shows the strict convexity of the functional

$$U \ni v \mapsto \frac{1}{p}\int_{\Omega}\int_{\Omega} \gamma |\mathcal{G} v|^{p} \,\mathrm{d}x' \,\mathrm{d}x. \tag{3.23}$$

Strict convexity of the nonlocal Dirichlet energy follows since it is the sum of the linear functional

$$U \ni v \mapsto \int_{\Omega} f v \,\mathrm{d}x; \tag{3.24}$$

and the strictly convex functional (3.23). □

Proposition 3.18. *The nonlocal Dirichlet energy $J : U \to \mathbb{R}$ satisfies:*

(i) *J is continuous in U.*

(ii) *J is weakly lower semicontinuous in U.*

Proof. For (i) we consider the strictly convex term (3.23) and the linear term (3.24) separately. To this end, we first notice that (3.23) is bounded for all $v \in U$ in the sense that

$$\int_{\Omega} \int_{\Omega \cup \Omega_I} \gamma |\mathcal{G} v|^p \,\mathrm{d}x' \,\mathrm{d}x \leq \int_{\Omega \cup \Omega_I} \int_{\Omega \cup \Omega_I} \gamma |\mathcal{G} v|^p \,\mathrm{d}x' \,\mathrm{d}x = \|v\|_U^p . \tag{3.25}$$

Note that the reverse triangle inequality together with (3.25) yields

$$\begin{aligned} \left\| \gamma^{1/p}|\mathcal{G} v| - \gamma^{1/p}|\mathcal{G} w| \right\|^p_{L^p(\Omega \times (\Omega\cup\Omega_I))} &= \int_{\Omega} \int_{\Omega \cup \Omega_I} \gamma \big| |\mathcal{G} v| - |\mathcal{G} w| \big|^p \,\mathrm{d}x' \,\mathrm{d}x \\ &\leq \int_{\Omega} \int_{\Omega \cup \Omega_I} \gamma |\mathcal{G} (v - w)|^p \,\mathrm{d}x' \,\mathrm{d}x \\ &\leq \|v - w\|_U^p . \end{aligned} \tag{3.26}$$

By applying another reverse triangle inequality to the left-hand side of (3.26) we get

$$\left| \left\| \gamma^{1/p}\mathcal{G} v \right\|_{L^p(\Omega \times (\Omega\cup\Omega_I))} - \left\| \gamma^{1/p}\mathcal{G} w \right\|_{L^p(\Omega \times (\Omega\cup\Omega_I))} \right|^p \leq \|v - w\|_U^p .$$

This implies that $v \mapsto \left\| \gamma^{1/p}\mathcal{G} v \right\|_{L^p(\Omega \times (\Omega\cup\Omega_I))}$ is continuous from $U \to \mathbb{R}$. The function $\varphi(r) = r^p$ from the proof of Lemma 3.17 is continuous from $[0, \infty) \to \mathbb{R}$, and there we have continuity $U \to \mathbb{R}$ of the scaled composition

$$v \mapsto \frac{1}{p} \varphi\left(\left\| \gamma^{1/p}\mathcal{G} v \right\|_{L^p(\Omega \times (\Omega\cup\Omega_I))} \right) = \frac{1}{p} \int_{\Omega} \int_{\Omega \cup \Omega_I} \gamma |\mathcal{G} v|^p \,\mathrm{d}x' \,\mathrm{d}x$$

This proves continuity of the strictly convex term (3.23). Continuity of the linear term (3.24) follows immediately by Hölder's inequality and the nonlocal Poincaré inequality. Indeed, for all $v \in U$ we have

$$\left| \int_{\Omega} f v \,\mathrm{d}x \right| \leq \|f\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} \leq C_P \|f\|_{L^{p'}(\Omega)} \|v\|_U .$$

Hence J is a continuous functional as the sum of two other continuous functionals, which concludes the proof of (i). We realize that (ii) follows immediately. Indeed, from (i) we know J is continuous, hence also lower semicontinuous. From Lemma 3.17 we know J is convex, which means that it is also weakly lower semicontinuous. □

Lemma 3.19. *The functional $J : U \to \mathbb{R}$ is coercive.*

Proof. Instead of showing that $J(v) \to \infty$ as $\|v\|_U \to \infty$, we equivalently prove that the lower level sets of J are bounded. To this end, let $\alpha \in \mathbb{R}$ and assume $v \in U$ satisfies $J(v) \leq \alpha$, i.e.

$$\frac{1}{p}\int_\Omega \int_\Omega |Gv|^p \, dx' \, dx - \int_\Omega f v \, dx \leq \alpha .$$

Rearranging and applying the boundedness of the linear term (3.24), we obtain the following string of inequalities

$$\underline{\lambda}\|v\|_U^p \leq \int_\Omega \int_\Omega |Gv|^p \, dx' \, dx \leq pC_P \|f\|_{L^p(\Omega)} \|v\|_U + p\alpha .$$

Dividing through by $\underline{\lambda}$ and $\|v\|_U$, we obtain

$$\|v\|_U^{p-1} \leq \frac{p}{\underline{\lambda}} \left(C_P \|f\|_{L^p(\Omega)} + \frac{\alpha}{\|v\|_U} \right).$$

We now present two cases. If $\alpha \geq 0$ and $\|v\|_U \geq 1$, then

$$\|v\|_U \leq \left[\frac{p}{\underline{\lambda}} \left(C_P \|f\|_{L^p(\Omega)} + \alpha \right) \right]^{p'-1};$$

since $(p-1)(p'-1) = 1$. However, if $\alpha < 0$ and $\|v\|_U \geq 1$, then

$$\|v\|_U \leq \left[\frac{p}{\underline{\lambda}} \left(C_P \|f\|_{L^p(\Omega)} \right) \right]^{p'-1}.$$

In summary, we see that

$$\|v\|_U \leq \max\left\{ 1; \left[\frac{p}{\underline{\lambda}} \left(C_P \|f\|_{L^p(\Omega)} + |\alpha| \right) \right]^{p'-1} \right\};$$

which proves that the lower level set for α is bounded in U. □

We are now in a good position to prove that there exists a unique state satisfying the nonlocal Dirichlet principle.

Theorem 3.20. *There exists a unique solution $u \in U$ to* (3.21)*.*

Proof. We can establish the existence of a minimizer using the direct method. First, we show that J is a proper functional by verifying that $J(v) < \infty$ for all $v \in U$. Next, consider a minimizing sequence $\{u_k\}_{k \in \mathbb{N}} \subset U$. Since J is proper, the sequence of values $J(u_k)_{k \in \mathbb{N}}$ is bounded above. From Lemma 3.19, we know that J is coercive. Therefore, the minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$ is bounded in U. Since U is a reflexive Banach space (Corollary 3.8), the minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$ is weakly relatively compact. Thus, we can extract a convergent subsequence $\{u_{k'}\}_{k' \in \mathbb{N}}$ that converges weakly to some $u \in U$. By Proposition 3.18, the functional J is weakly lower semicontinuous. Hence, we have $J(u) \leq \liminf_{k' \to \infty} J(u_{k'}) = \inf_U J$. This shows that u is a minimizer. To prove the uniqueness of the minimizer, we use Lemma 3.17, which states the strict convexity of J. Therefore, there can be no other minimizer. In conclusion, the direct method ensures the existence and uniqueness of a state $u \in U$ satisfying the nonlocal Dirichlet principle. □

We now want to prove that the state $u \in U$ satisfying the nonlocal Dirichlet principle (3.21) is indeed the solution to the generalized nonlocal p-Laplace equation (3.18). To this end, we derive (3.20) as the necessary and sufficient optimality conditions for (3.21).

Theorem 3.21. *The unique solution $u \in U$ of* (3.21) *is equivalently the unique $u \in U$ satisfying* (3.20). *In particular, we find u as the nonlocal equilibrium temperature state subject to nonlocal nonlinear diffusion.*

Proof. Let us study the first variation of J along the lines of [6, Section 6.11]. Note that $u \in U$ is a solution to (3.21) if and only if $J(u) \leq J(v)$ for all $v \in U$. By rearranging this and inserting $v = u + tv$ for $t > 0$, we may obtain the inequality

$$\frac{1}{p}\int \int \frac{1}{t}\left(|G u + tG v|^p - |G u|^p\right) \mathrm{d}x' \,\mathrm{d}x - \int f v \,\mathrm{d}x \geq 0. \tag{3.27}$$

Utilizing the DCT, we consider (3.27) in the limit for $t \searrow 0$. Towards this, we need to verify the assumptions of the DCT. Therefore, we fix $(x, x') \in$ and consider the first integrand of (3.27). We now define $\psi(t) = |G u + tG v|^p$ for $t \in \mathbb{R}$, and note that

$$\psi'(t) = \begin{cases} p|G u + tG v|^{p-2}(G u + tG v)G v, & \text{if } t \neq -G u/G v, \\ 0, & \text{else}, \end{cases} \quad \forall t \in \mathbb{R},$$

is continuous since $p > 1$. Using the fundamental theorem of calculus, we see that the first integrand of (3.27) can be expressed as

$$\begin{aligned} \frac{1}{t}\left(|G u + tG v|^p - |G u|^p\right) &= \frac{1}{t}\left(\psi(t) - \psi(0)\right) \\ &= \frac{1}{t}\int_0^t \psi'(s)\,\mathrm{d}s \\ &= \frac{p}{t}\int_0^t |G u + tG v|^{p-2}(G u + tG v)G v \,\mathrm{d}s. \end{aligned}$$

With the goal of dominating this expression, we consider its absolute value for $t \leq 1$

$$\begin{aligned} \frac{1}{t}\left|\int_0^t \psi'(s)\,\mathrm{d}s\right| &\leq \frac{p}{t}\int_0^t |G u + sG v|^{p-1}|G v| \,\mathrm{d}s \\ &\leq \frac{p}{t}\int_0^t (|G u| + s|G v|)^{p-1}|G v| \,\mathrm{d}s \\ &\leq p\,(|G u| + |G v|)^{p-1}|G v|. \end{aligned} \tag{3.28}$$

Here, the first inequality follows from the triangle inequality for integrals, and the second inequality uses the standard Euclidean triangle inequality. The third inequality follows because $t \leq 1$ implies that $|G u| + s|G v| \leq |G u| + |G v|$ for all $s \leq t$. As a consequence, we may use Hölder's inequality to deduce that the term in (3.28) is integrable on . Indeed, since $(p-1)p' = p$, we initially note that

$$\begin{aligned} \left\|(|G u| + |G v|)^{p-1}\right\|_{L^{p'}()}^{p'} &= \||G u| + |G v|\|_{L^p()}^p \\ &\leq \left(\|G u\|_{L^p()} + \|G v\|_{L^p()}\right)^p \\ &< \infty, \end{aligned}$$

where the first inequality follows due to the triangle inequality, and the strict inequality follows since $u, v \in U$. We can now apply Hölder's inequality

$$
\begin{aligned}
p \int \int & (|G u| + |G v|)^{p-1} |G v| \, dx' \, dx \\
&\leq p \bar{\gamma} \int \int (|G u| + |G v|)^{p-1} |G v| \, dx' \, dx \\
&\leq p \bar{\gamma} \left\| (|G u| + |G v|)^{p-1} \right\|_{L^{p'}(\;)} \|G v\|_{L^{p}(\;)} \\
&< \infty ,
\end{aligned}
$$

which proves that the majorant (3.28) is integrable. As such, it is valid to apply the DCT in (3.27). Since

$$
{}'(0) = p |G u|^{p-2} G u G v,
$$

we take the limit $t \searrow 0$ and obtain the inequality

$$
\int \int |G u|^{p-2} G u G v \, dx' \, dx - \int f v \, dx \geq 0, \quad \forall v \in U,
$$

which yields the equality in (3.20) after inserting $v = -v$. The fact that (3.20) is sufficient for optimality in (3.21) follows from standard convexity arguments. Let us follow the steps taken in [47, Theorem 3.9]. We assume $u \in U$ satisfies (3.20) for arbitrary $v \in U$. Inserting $v = u - v$ yields the equality

$$
\int \int |G u|^{p} \, dx' \, dx - \int f u \, dx = \int \int |G u|^{p-2} G u G v \, dx' \, dx - \int f v \, dx. \tag{3.29}
$$

The right-hand side can be bounded as follows using Young's inequality for products

$$
\begin{aligned}
\int \int & |G u|^{p-2} G u G v \, dx' \, dx - \int f v \, dx \\
&\leq \int \int |G u|^{p-1} |G v| \, dx' \, dx - \int f v \, dx \\
&\leq \frac{1}{p'} \int \int |G u|^{p} \, dx' \, dx + \frac{1}{p} \int \int |G v|^{p} \, dx' \, dx - \int f v \, dx.
\end{aligned}
$$

Once again, we use the fact that $(p-1)p' = p$. Rearranging this with (3.29) yields

$$
\begin{aligned}
\frac{1}{p} \int \int |G u|^{p} \, dx' \, dx - \int f u \, dx &= \left(1 - \frac{1}{p'}\right) \int \int |G u|^{p} \, dx' \, dx - \int f u \, dx \\
&\leq \frac{1}{p} \int \int |G v|^{p} \, dx' \, dx - \int f v \, dx,
\end{aligned}
$$

which shows that u is a minimizer of the nonlocal Dirichlet energy. Hence $u \in U$ minimizes (3.21) if and only if it solves (3.20). □

Before we move on to consider the dual formulation of nonlocal diffusion, we establish the following a priori stability bound for equilibrium temperature states.

Corollary 3.22. *Assume $u \in U$ solves the nonlocal equation* (3.18). *Then there exists some $C > 0$ independent of both $f \in L^{p'}(\Omega)$ and $\lambda > 0$ such that*

$$\|u\|_U \le C\|f\|_{L^{p'}(\Omega)}^{p'-1}.$$

Proof. By setting $v = u$ in equation (3.20) and applying the Hölder inequality, as well as the nonlocal Poincaré inequality, we obtain

$$\lambda\|u\|_U^p \le \int_\Omega\int_\Omega |\mathcal{G}u|^p \, dx' \, dx = \int_\Omega f u \, dx \le C_p\|f\|_{L^{p'}(\Omega)}\|u\|_U.$$

Dividing through by λ and $\|u\|_U$, we get

$$\|u\|_U^{p-1} \le \lambda^{-1}C_p\|f\|_{L^{p'}(\Omega)}.$$

The result thus follows by realizing $(p-1)(p'-1) = 1$. □

3.4 Dual formulations

We turn our attention towards the dual formulations of the nonlocal diffusion law presented in the previous section. We consider two different approaches. In the first approach, we aim to rewrite the weak variational problem (3.20) by introducing the nonlocal heat flux as an auxiliary variable. This leads to a mixed variational formulation. However, in order to obtain a more convenient reformulation, we need to introduce an additional nonlocal operator that is closely connected to the nonlocal gradient. Since the nonlocal divergence is densely defined, we may introduce its negative adjoint $\mathcal{D}^*: D(\mathcal{D}^*) \subset L^p(\Omega) \to Q^*$. Let us briefly summarize it properties.

Proposition 3.23. *Let $\mathcal{D}^*: D(\mathcal{D}^*) \subset L^p(\Omega) \to Q^*$ be the adjoint of the nonlocal divergence $\mathcal{D}: Q \subset L^{p'}(\Omega \times \Omega) \to L^{p'}(\Omega)$. Then we have:*

(i) $D(\mathcal{D}^*) = L^p(\Omega)$.

(ii) $R(\mathcal{D}^*) = N(\mathcal{D})^\perp$.

(iii) $\mathcal{D}^*$ *is closed, injective, and there exists some $C > 0$ such that*

$$\|v\|_{L^p(\Omega)} \le C\|\mathcal{D}^* v\|_{Q^*}, \quad \forall v \in L^p(\Omega).$$

(iv) $\mathcal{D}^* u$ *is represented by $\mathcal{G}u$ for all $u \in U$.*

Proof. For (i) let $u \in L^p(\Omega)$ and note that for $q \in Q$ it holds that

$$|\mathcal{D}^* u(q)| = |\langle u, \mathcal{D}q\rangle_{L^2(\Omega)}| \le \|u\|_{L^p(\Omega)}\|\mathcal{D}q\|_{L^{p'}(\Omega)} \le \|u\|_{L^p(\Omega)}\|q\|_Q.$$

Therefore, we see that $L^p(\Omega) \subseteq D(\mathcal{D}^*)$, which implies that they coincide. By Proposition 3.14 we know $\mathcal{D}$ has closed range, and is surjective. Hence, the closed range theorem implies (ii), and a characterization of operator surjectivity given in [18, Theorem 2.20]) implies (iii). For (iv) let $(u, q) \in U \times Q$ and note that

$$\mathcal{D}^* u(q) = -\langle u, \mathcal{D}q\rangle_{L^2(\Omega)} = \langle \mathcal{G}u, q\rangle_{L^2(\Omega\times\Omega)}.$$

Thus $\mathcal{D}^* u = \mathcal{G}u$ as operators in Q^*. □

In the second approach, we explore the convex dual problem of the nonlocal Dirichlet principle (3.21) using the Fenchel-Rockafellar duality theory. By applying this theory, we establish a strong duality between the primal and dual problems. To enable this approach, we introduce another nonlocal operator, now associated with the nonlocal divergence. Consider the Banach space adjoint $\mathcal{G}^*: L^{p'}(\Omega\times\Omega) \to \mathcal{U}^*$ of the nonlocal gradient $\mathcal{G}: \mathcal{U} \to L^p(\Omega\times\Omega)$, which we view as a bounded linear operator. Similar to before, we summarize the necessary properties.

Proposition 3.24. *Let $\mathcal{G}^*: L^{p'}(\Omega\times\Omega) \to \mathcal{U}^*$ be the Banach space adjoint of the nonlocal gradient $\mathcal{G}: \mathcal{U} \to L^p(\Omega\times\Omega)$. Then we have:*

(i) *$\mathcal{G}^*$ is surjective. In particular, for any $\varphi \in \mathcal{U}^*$ there exists $\psi \in L^{p'}(\Omega\times\Omega)$ such that*

$$\varphi(v) = \mathcal{G}^*(\psi)(v), \quad \forall v \in \mathcal{U},$$

with $\|\varphi\|_{\mathcal{U}^*} = \|\psi\|_{L^{p'}(\Omega\times\Omega)}$.

(ii) *$\mathcal{G}^* q$ is represented by $-\mathcal{D} q$ for all $q \in \mathcal{Q}$.*

Proof. We note that the adjoint $\mathcal{G}^*$ is defined by the formula

$$\mathcal{G}^*(\psi)(v) = \int_\Omega \int_\Omega \psi\, \mathcal{G} v \, dx' \, dx, \quad \forall v \in \mathcal{U}, \forall \psi \in L^{p'}(\Omega\times\Omega). \tag{3.30}$$

For (i) Let us recall that the nonlocal gradient $\mathcal{G}: \mathcal{U} \to L^p(\Omega\times\Omega)$ is an isometry, by the definition of $\|\cdot\|_{\mathcal{U}}$. Therefore, $\mathcal{G}: \mathcal{U} \to R(\mathcal{G})$ is a bounded linear bijection. Since the range $R(\mathcal{G})$ is closed, see Proposition 3.9, $\mathcal{G}: \mathcal{U} \to R(\mathcal{G})$ is in fact an isometric isomorphism due to the classical bounded inverse theorem. Let us now consider $\varphi \in \mathcal{U}^*$ and define the functional $T: R(\mathcal{G}) \to \mathbb{R}$ by the formula

$$T\psi = \varphi(\mathcal{G}^{-1}\psi), \quad \forall \psi \in R(\mathcal{G}).$$

It is immediate to see that T is linear, and in addition it is actually bounded. In fact $\|T\|_{R(\mathcal{G})^*} = \|\varphi\|_{\mathcal{U}^*}$ since

$$\|T\|_{R(\mathcal{G})^*} = \sup_{\psi \in R(\mathcal{G})} \frac{\varphi(\mathcal{G}^{-1}\psi)}{\|\psi\|_{L^p(\Omega\times\Omega)}} = \sup_{\psi \in R(\mathcal{G})} \frac{\varphi(\mathcal{G}^{-1}\psi)}{\left\|\mathcal{G}^{-1}\psi\right\|_{\mathcal{U}}} = \sup_{v \in \mathcal{U}} \frac{\varphi(v)}{\|v\|_{\mathcal{U}}} = \|\varphi\|_{\mathcal{U}^*}.$$

Consequently, we can apply the Hahn-Banach theorem to obtain the continuous extension $T: L^p(\Omega\times\Omega) \to \mathbb{R}$ with $\|T\|_{L^p(\Omega\times\Omega)^*} = \|\varphi\|_{\mathcal{U}^*}$. It remains to consider the Riesz representation $L^p(\Omega\times\Omega)^* = L^{p'}(\Omega\times\Omega)$, and hence there exists some $\psi \in L^{p'}(\Omega\times\Omega)$ with

$$Tq = \int_\Omega \int_\Omega \psi\, q \, dx' \, dx, \quad \forall q \in L^p(\Omega\times\Omega),$$

and $\|T\|_{L^p(\Omega\times\Omega)^*} = \|\psi\|_{L^{p'}(\Omega\times\Omega)}$. Now take $q = \mathcal{G} v$ for $v \in \mathcal{U}$ to see that

$$\varphi(v) = T(\mathcal{G} v) = \int_\Omega \int_\Omega \psi\, \mathcal{G} v \, dx' \, dx, \quad \forall v \in \mathcal{U}.$$

This proves (i). Now for (ii) recall the nonlocal integration by parts formula defining the nonlocal divergence $\mathcal{D}$. It is the restriction of (3.30) on $\mathcal{Q}$. Hence, for $q \in \mathcal{Q}$, the nonlocal divergence $\mathcal{D} q$ is a representation of the functional $-\mathcal{G}^* q$. Consequently, we may view $-\mathcal{G}^*|_{\mathcal{Q}} = \mathcal{D}$. □

3.4.1 Mixed variational formulation

Inspired by the local case, we define the nonlocal heat flux as

$$q = \quad |G\ u|^{p\ \ 2} G\ u, \tag{3.31}$$

where $u \in U$ is the equilibrium temperature state. This definition resembles the local Fourier's law, where the heat flux is proportional to the temperature gradient. However, in the nonlocal setting, the heat flux is determined by the nonlocal gradient $G\ u$. The term $|G\ u|^{p\ \ 2}$ serves as a nonlocal power-law weighting factor, akin to the one present in local nonlinear diffusion. By including the nonlocal heat flux in the formulation, we introduce a variable that captures the characteristics of the heat transfer in the nonlocal diffusion process.

The mixed variational formulation is a system of two equations. The first equation is composed of the following inversion formula.

Lemma 3.25. *Let $u \in U$ be the equilibrium temperature state and let q be the nonlocal flux defined in* (3.31). *Then $q \in L^p(\quad)$ and we have*

$$G\ u = \quad ^{1\ \ p} |q|^{p\ \ 2} q. \tag{3.32}$$

Proof. First let us prove that $q \in L^p(\quad)$. This is once again a consequence of $(p \quad 1)p \quad = p$, as

$$\|q\|^p_{L^p(\quad)} = \int \int \quad ^p |G\ u|^p \, dx' \, dx \quad ^{-p} \|u\|^p_U < \infty.$$

Meanwhile, the inversion formula is obtained by inserting (3.31) into the right-hand side of (3.32)

$$\begin{aligned}
^{1\ \ p} |q|^{p\ \ 2} q &= \quad ^{1\ \ p} | \quad |G\ u|^{p\ \ 2} G\ u|^{p\ \ 2} \quad |G\ u|^{p\ \ 2} G\ u \\
&= \quad ^{1\ \ p} \quad ^{p\ \ 1} \quad |G\ u|^{(p\ \ 2)(p\ \ 2)+(p\ \ 2)+(p\ \ 2)} G\ u \\
&= |G\ u|^{(p\ \ 1)(p\ \ 2)+(p\ \ 2)} G\ u \\
&= |G\ u|^{(p\ \ 1)(p\ \ 1)\ \ 1} G\ u \\
&= G\ u,
\end{aligned}$$

since $(p \quad 1)(p \quad 1) = 1$. $\square$

The flux inversion formula (3.32) is also found in the local literature [8, 40, 41]. The second equation of the mixed formulation is the following unsurprising observation.

Proposition 3.26. *Let $u \in U$ be the equilibrium temperature state. Then its nonlocal flux q satisfies $D\ q = f$. In particular, $q \in Q\ (f)$.*

Proof. Let $v \in U$. By (3.20), we see that

$$|\langle q, G\ v\rangle \quad | = \left| \int \int \quad |G\ u|^{p\ \ 2} G\ u G\ v \, dx' \, dx \right| = \left| \int f v \, dx \right|$$
$$C_p \|f\|_{L^p(\ )} \|v\|_U \ ;$$

where the last inequality follows the continuity of the linear form. This shows $q \in Q^*$. The defining integration by parts formula (3.14) gives the result. Indeed,

$$\langle \mathcal{D} q, v \rangle = \int_\Omega \int_\Omega |\mathcal{G} u|^{p-2} \mathcal{G} u \, \mathcal{G} v \, dx' \, dx = \int_\Omega f v \, dx;$$

and therefore we see that

$$\int_\Omega (\mathcal{D} q - f) v \, dx = 0; \quad \forall v \in U. \tag{3.33}$$

Since U is dense in $L^p(\Omega)$, (3.33) can be extended by continuity to hold for all $v \in L^p(\Omega)$. Consequently, we must have $\mathcal{D} q = f$ in $L^{p'}(\Omega)$, which is what we wanted. □

Testing the equations in Lemma 3.25 and Proposition 3.26, we are lead to conclude that the nonlocal equilibrium state-flux pair $(u; q) \in U \times Q(f)$ satisfies the mixed variational problem

$$\text{Find } (u; q) \in L^p(\Omega) \times Q : \quad \begin{aligned} (Aq)(\tau) + (\mathcal{D}^* u)(\tau) &= 0; \quad \forall \tau \in Q; \\ \langle \mathcal{D} q; v \rangle &= \langle f; v \rangle; \; \forall v \in L^p(\Omega); \end{aligned} \tag{3.34}$$

where we define the nonlinear operator $A : Q \to Q^*$ by

$$(Aq)(\tau) = \langle \lambda^{1-p'} |q|^{p'-2} q; \tau \rangle_{\Omega \times \Omega}; \quad \forall q; \tau \in Q; \tag{3.35}$$

and we identify $\mathcal{G}^* = \mathcal{D}$ by Proposition 3.23(iv). Considering further auxiliary variables, one may propose inversion formulas different from (3.32) that provide variational systems equivalent to (3.34), see e.g. [24]. We will now demonstrate the well-posedness of the mixed problem (3.34). The existence and uniqueness of a solution will be established using a variational approach inspired by the classical theory of Ladyzhenskaya-Babuška-Brezzi (LBB) [14]. We will present a proof following the framework outlined in [8].

First, we establish a preparatory lemma.

Lemma 3.27. *Let $A : Q \to Q^*$ be the operator defined in* (3.35)*. Then A is well-defined and there exist some $C > 0$ for which*

$$\|Aq\|_{Q^*} \leq C \|q\|^{p'-1}_{L^{p'}(\Omega \times \Omega)}; \quad \forall q \in Q.$$

Proof. Given $q \in Q$ it is clear from its definition that Aq is a linear functional. To prove its boundedness, we apply Hölder's inequality to see that

$$\begin{aligned} \|Aq\|_{Q^*} &= \sup_{\tau \in Q} \frac{\left| \langle \lambda^{1-p'} |q|^{p'-2} q; \tau \rangle_{\Omega \times \Omega} \right|}{\|\tau\|_Q} \\ &\leq \sup_{\tau \in Q} \underline{\lambda}^{1-p'} \frac{\left\| q^{p'-1} \right\|_{L^p(\Omega \times \Omega)} \|\tau\|_{L^{p'}(\Omega \times \Omega)}}{\|\tau\|_Q} \\ &\leq \underline{\lambda}^{1-p'} \left\| q^{p'-1} \right\|_{L^p(\Omega \times \Omega)} \\ &= \underline{\lambda}^{1-p'} \|q\|^{p'-1}_{L^{p'}(\Omega \times \Omega)}. \end{aligned}$$

Here the second inequality follows since $\|\tau\|_{L^{p'}(\Omega \times \Omega)} \leq \|\tau\|_Q$, and our by now so beloved identity $(p'-1)p = p'$ gives us the final equality. □

We can now state and prove the following result.

Theorem 3.28. *There exists a unique pair* $(u;q) \in L^p(\Omega) \times Q_\delta$ *solving* (3.34)*. In addition, there exists two constants* $C_1;C_2 > 0$ *such that*

$$\begin{aligned} \|q\|_{Q_\delta} &\le C_1 \|f\|_{L^{p'}(\Omega)}; \\ \|u\|_{L^p(\Omega)} &\le C_2 \|f\|_{L^{p'}(\Omega)}^{p'-1}. \end{aligned} \tag{3.36}$$

Proof. We proceed by recasting (3.34) with the classical LBB procedure. Since the nonlocal divergence is surjective, we know that $Q_\delta(f)$ is nonempty, and in particular by Proposition 3.15 we know that there exists some $q_f \in Q_\delta(f)$ and a constant $C_1 > 0$ such that

$$\|q_f\|_{Q_\delta} \le C_1 \|f\|_{L^{p'}(\Omega)}. \tag{3.37}$$

Applying the superposition $q = q_f + q_0$, with $q_0 \in Q_\delta(0)$ we are left with

$$\text{Find } (u;q_0) \in L^p(\Omega) \times Q_\delta(0) : \; A(q_0 + q_f)(\psi) + (D_\delta^* u)(\psi) = 0; \quad \forall \psi \in Q_\delta. \tag{3.38}$$

To solve this formulation, it is useful to consider the following auxiliary problem

$$\text{Find } q_0 \in Q_\delta(0) : \quad A(q_f + q_0)(\psi) = 0; \quad \forall \psi \in Q_\delta(0). \tag{3.39}$$

Let us write out the the equation in (3.39). Assuming $q_0 \in Q_\delta(0)$ is a solution, the superposition $q = q_f + q_0$ satisfies

$$\int_\Omega \int_\Omega \delta^{1-p} |q|^{p-2} q \psi \, dx' dx = 0; \quad \forall \psi \in Q_\delta(0). \tag{3.40}$$

Using similar arguments as in Theorem 3.21, we realize that (3.40) is the necessary and sufficient optimality condition for q_0 to be the minimizer of

$$\min_{q_0 \in Q_\delta(0)} J(q_f + q_0) = \frac{1}{p} \int_\Omega \int_\Omega \delta^{1-p} |q_f + q_0|^p \, dx' dx. \tag{3.41}$$

Luckily, we have already studied the solvability of (3.41). In fact, the arguments are analogous to those given in the proof of Theorem 3.20. Consequently, there exists a unique $q_0 \in Q_\delta(0)$ satisfying (3.39). The equality in (3.39) implies that $A(q_f + q_0) = Aq \in N(D_\delta)^\perp$. Due to Proposition 3.23(ii), we know that

$$N(D_\delta)^\perp = R(D_\delta^*);$$

and therefore there must exist some $u \in L^p(\Omega)$ for which

$$Aq = D_\delta^* u; \quad \text{in } Q_\delta^*. \tag{3.42}$$

Rearranging, we realize that the pair $(u;q) \in L^p(\Omega) \times Q_\delta$ satisfies the original mixed formulation (3.34). We now argue its uniqueness. Given $q_f \in Q_\delta(f)$ we found $q_0 \in Q_\delta(0)$ as the unique minimizer of (3.41). If instead $\tilde{q}_f \in Q_\delta(f)$ was chosen distinct from q_f, then its $\tilde{q}_0$ would be found as the unique minimizer of

$$\min_{\tilde{q}_0 \in Q_\delta(0)} J(\tilde{q}_f + \tilde{q}_0).$$

But $q = q_f + q_0 = \tilde{q}_f + \tilde{q}_0$ since uniqueness of q_0 forces $\tilde{q}_0 = q_f - \tilde{q}_f + q_0$. Hence the superposition $q = q_f + q_0$ is unique. Consequently, u is also unique due to the injectivity of D^* established in Proposition 3.23(iii). Finally, we prove the stability estimates (3.36). Since $q = q_f + q_0$ is the unique minimizer of J in Q, we see that

$$\|q\|^p_{L^p(\Omega\times\Omega)} \le p\,\lambda^{-1} J(q_f + q_0) \le p\,\lambda^{-1} J(q_f) \le \left(\frac{\Lambda}{\lambda}\right)^{p-1} \|q_f\|^p_{L^p(\Omega\times\Omega)}.$$

Inserting (3.37), we see that

$$\|q\|_{L^p(\Omega\times\Omega)} \le \left(\frac{\Lambda}{\lambda}\right)^{1/p} \|q_f\|_Q \le \left(\frac{\Lambda}{\lambda}\right)^{1/p} C_1 \|f\|_{L^p(\Omega)},$$

which proves the first estimate of (3.36). Towards the second estimate, we apply the stability estimate from Lemma 3.27 to see that there exists some $C_2 > 0$ such that

$$\|Aq\|_{Q^*} \le C_2 \|q\|^{p-1}_{L^p(\Omega\times\Omega)}.$$

We now invoke the equality in (3.42) and the constant $C_3 > 0$ found in Proposition 3.23(iii) to see that

$$\|u\|_{L^p(\Omega)} \le C_3 \|Aq\|_{Q^*} \le C_2 C_3 \|q\|^{p-1}_{L^p(\Omega\times\Omega)}.$$

Inserting the first estimate of (3.36), we get some $C > 0$ satisfying

$$\|u\|_{L^p(\Omega)} \le C \|f\|^{p-1}_{L^p(\Omega)}.$$

□

Our proof relies heavily on the surjectivity of the nonlocal divergence, and the injectivity of its negative adjoint. Both of these properties are fundamental pieces in the study of mixed variational problems. In fact, they are intimately related to the famous ***LBB condition***, also known as the ***inf-sup condition***. The LBB condition for our mixed problem (3.34) is the following:

$$\text{There exists } \beta > 0 \text{ s.t. } \inf_{v \in L^p(\Omega)} \sup_{\psi \in Q} \frac{\langle D^*\psi, v\rangle}{\|v\|_{L^p(\Omega)} \|\psi\|_Q} \ge \beta. \tag{3.43}$$

In general, the LBB condition is a sufficient condition to guarantee the results of Theorem 3.28. However, when the heat flux space Q is finite dimensional, the LBB condition becomes necessary to preserve the uniqueness of the heat flux. This fact makes the numerical approximation of mixed problems difficult, as violation of the LBB condition can lead to nonphysical behavior of numerical solutions. Well-known examples include the checkerboard oscillations, spurious modes, and locking phenomena observed by unstable finite element approximations [14, 16, 36]. The LBB condition holds in the present case.

Corollary 3.29. *The LBB condition* (3.43) *holds.*

Proof. The result can follow from either Proposition 3.15 or Proposition 3.23(iii). In fact, the three statements are equivalent, and we refer to [62, Lemma A.1] for a proof. Here we present the proof using Proposition 3.23(iii). To this end, we first note that

$$\inf_{v\in L^p(\Omega)} \sup_{\tau\in Q} \frac{\langle D^*\tau, v\rangle}{\|v\|_{L^p(\Omega)}\|\tau\|_Q} = \inf_{v\in L^p(\Omega)} \sup_{\tau\in Q} \frac{Dv(\tau)}{\|v\|_{L^p(\Omega)}\|\tau\|_Q} = \inf_{v\in L^p(\Omega)} \frac{\|Dv\|_{Q^*}}{\|v\|_{L^p(\Omega)}}.$$

Now we use that Proposition 3.23(iii) gives us some $C > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq C\|Dv\|_{Q^*}.$$

Consequently, we get the result

$$\inf_{v\in L^p(\Omega)} \sup_{\tau\in Q} \frac{\langle D^*\tau, v\rangle}{\|v\|_{L^p(\Omega)}\|\tau\|_Q} \geq C^{-1} > 0.$$

□

3.4.2 Nonlocal Kelvin principle

We have now shown that the equilibrium state-flux pair $(u, q) \in U \times Q$ is found as the unique solution to the mixed variational problem (3.34). Reading from the proof of Theorem 3.28, we see that the equilibrium heat flux $q \in Q$ satisfies a minimum energy principle. Indeed, it is found as the minimizer of

$$\min_{\tau\in Q(f)} J(\tau) = \frac{1}{p}\int_\Omega \int_\Omega \gamma^{1-p} |\tau|^p \, dx' dx. \tag{3.44}$$

This may be seen as a nonlocal analog of the Kelvin principle introduced in (3.8). Another way to obtain (3.44) is through Fenchel-Rockafellar duality. Indeed (3.44) arises as the dual problem of the nonlocal Dirichlet principle (3.21). We conclude this section by detailing the proof. To this end, we will briefly introduce some notions from convex analysis. For further details we refer to [6, 18]. We begin with recalling the definition of the convex conjugate.

Definition 3.30. Let U be a Banach space, and let $F : U \to (-\infty, \infty]$ be a proper functional. Then the *convex conjugate* of F is the functional $F^* : U^* \to (-\infty, \infty]$ defined by

$$F^*(\varphi) = \sup_{v\in U} \varphi(v) - F(v), \quad \forall \varphi \in U^*.$$

It is important to note that for F to have a well-defined convex conjugate F^*, F must satisfy certain properties. Specifically, F should be proper, convex, and lower semicontinuous. When these conditions are satisfied, F^* is also proper and convex. The convex conjugate plays a fundamental role in convex analysis and optimization. Let us briefly consider its geometric interpretation. The supremum defining F^* determines supporting hyperplanes of the epigraph of F, each associated with a $\varphi \in U^*$. The points $v \in U$ that attain the supremum value $F^*(\varphi)$ are exactly the support points of the supporting hyperplane associated to φ. In particular, the points $v \in U$ attaining $F^*(0)$ are support points of the $\varphi = 0$ hyperplane and are in fact minimizers of F. With this notion, we formulate the classical Fenchel-Rockafellar duality theorem [6, Theorem 9.8.1].

Theorem 3.31. *Let U, Q be two Banach spaces, and assume $F : Q \to (-\infty, \infty]$ and $G : U \to (-\infty, \infty]$ are proper, convex, and lower semicontinuous functionals. Given a linear continuous operator $K : U \to Q$, consider the primal problem*

$$\min_{v \in U} L(v) = F(Kv) + G(v), \tag{3.45}$$

and its corresponding dual problem

$$\max_{\sigma \in Q^*} L^*(\sigma) = -F^*(\sigma) - G^*(-K^*\sigma). \tag{3.46}$$

If there exists $v_0 \in U$ for which $L(v_0) < \infty$ and F is continuous at Kv_0, then the dual problem (3.46) *admits a solution $q \in Q^*$. Additionally, if the primal problem* (3.45) *admits a solution $u \in U$, then strong duality holds, and there is no duality gap*

$$L(u) = L^*(q). \tag{3.47}$$

Let us apply Theorem 3.31 to the nonlocal Dirichlet principle. First, we identify the spaces $U = \mathcal{U}$ and $Q = L^p(\Omega \times \Omega)$, and we consider the nonlocal gradient $K = \mathcal{G} : \mathcal{U} \to L^p(\Omega \times \Omega)$. By defining

$$F(\sigma) = \frac{1}{p}\int_\Omega \int_\Omega |\sigma|^p \, dx' \, dx, \quad \forall \sigma \in L^p(\Omega \times \Omega), \tag{3.48}$$

$$G(v) = -\int_\Omega f v \, dx, \qquad \forall v \in \mathcal{U}, \tag{3.49}$$

we see that the primal problem (3.45) becomes the nonlocal Dirichlet principle from (3.21), i.e. $L = J$. Note that we have previously argued in Lemma 3.17 and Proposition 3.18 that both functionals are proper, convex and continuous. In addition, we know that the nonlocal gradient $\mathcal{G} : \mathcal{U} \to L^p(\Omega \times \Omega)$ is continuous. Hence, the assumptions of Theorem 3.31 are satisfied. We move on to determine the corresponding dual problem (3.46).

To this end, let us first study the dual space $\mathcal{U}^*$. Since $\mathcal{U} \subset L^p(\Omega)$, we may view $L^{p'}(\Omega) \subset \mathcal{U}^*$. Indeed, consider $g \in L^{p'}(\Omega)$ and define $\varphi_g : \mathcal{U} \to \mathbb{R}$ by the formula

$$\varphi_g(v) = \int_\Omega g v \, dx, \quad \forall v \in \mathcal{U}.$$

It is immediate to see that φ_g is a bounded linear operator on $\mathcal{U}$ since

$$|\varphi_g(v)| \le \|g\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} \le C_P \|g\|_{L^{p'}(\Omega)} \|v\|_{\mathcal{U}}, \quad \forall v \in \mathcal{U}. \tag{3.50}$$

Here we utilize the Hölder inequality, and the nonlocal Poincaré inequality gives the constant $C_P > 0$ as usual. Therefore, $\varphi_g \in \mathcal{U}^*$ is represented by $g \in L^{p'}(\Omega)$. However, this is not the only way in which elements of $\mathcal{U}^*$ can be represented. From Proposition 3.24(i) we quickly observe that there exists some $\sigma_g \in L^{p'}(\Omega \times \Omega)$ for which

$$\varphi_g(v) = \int_\Omega \int_\Omega \sigma_g \mathcal{G} v \, dx' \, dx = \int_\Omega g v \, dx \quad \forall v \in \mathcal{U}.$$

As a consequence, we may apply (3.50) to obtain

$$\left|\int_\Omega\int_\Omega \psi_g G v \,\mathrm{d}x' \mathrm{d}x\right| \le \|g\|_{L^{p'}(\Omega)}\|v\|_{L^p(\Omega)}, \quad \forall v \in U,$$

hence we have $\psi_g \in Q$ with $D\psi_g = g$. More generally, if $\varphi \in U^*$ admits $\psi_\varphi \in Q$, then we may write

$$\varphi(v) = \int_\Omega\int_\Omega \psi_\varphi G v \,\mathrm{d}x' \mathrm{d}x = \int_\Omega D\psi_\varphi v \,\mathrm{d}x, \quad \forall v \in U,$$

and we have $\varphi = \varphi_{D\psi_\varphi}$. Hence φ has a representative $D\psi_\varphi \in L^{p'}(\Omega)$ with

$$\left|\int_\Omega\int_\Omega \psi_\varphi G v\right| \le \|D\psi_\varphi\|_{L^{p'}(\Omega)}\|v\|_{L^p(\Omega)}, \quad \forall v \in U. \tag{3.51}$$

Therefore, we view $\varphi \in L^{p'}(\Omega) \subset U^*$. Recall that (3.51) is defining for Q being the domain of the unbounded linear operator $D : Q \subset L^{p'}(\Omega\times\Omega) \to L^{p'}(\Omega)$. Consequently, if $\varphi \in U^*$ admits $\psi_\varphi \notin Q$ then for all $c \ge 0$ there exists some $\tilde{v} \in U$ for which

$$|\varphi(\tilde{v})| = \left|\int_\Omega\int_\Omega \psi_\varphi G \tilde{v}\right| > c\|\tilde{v}\|_{L^p(\Omega)}. \tag{3.52}$$

Consequently φ admits no $L^{p'}(\Omega)$ representative, and we write $\varphi \in U^* \setminus L^{p'}(\Omega)$. This observation allows us to compute the convex conjugate of G.

Lemma 3.32. *Let the functional $G : U \to \mathbb{R}$ be defined as in (3.49). Then*

$$G^*(\varphi) = \begin{cases} 0, & \text{if } \varphi \in L^{p'}(\Omega) \text{ with } \varphi = -f, \\ \infty, & \text{else,} \end{cases} \quad \forall \varphi \in U^*.$$

Proof. For $\varphi \in L^{p'}(\Omega)$ we have

$$G^*(\varphi) = \sup_{v\in U} \langle \varphi, v\rangle + \langle f, v\rangle = \sup_{v\in U} \langle \varphi + f, v\rangle. \tag{3.53}$$

We realize that if $\varphi = -f$, then $G^*(u) = 0$. If not, then $\langle \varphi + f, v\rangle \ne 0$ for at least one $v \in U$, since U is dense in $L^p(\Omega)$. Taking scalar multiples of this v proves that the supremum is unbounded. Consider now $\varphi \in U^* \setminus L^{p'}(\Omega)$. For such φ we see from (3.52) that for all $c \ge 0$ there exists some $\tilde{v} \in U$ for which

$$G^*(\varphi) = \sup_{v\in U} \varphi(v) + \langle f, v\rangle \ge \varphi(\tilde{v}) + \langle f, \tilde{v}\rangle > c\|\tilde{v}\|_{L^p(\Omega)} + \langle f, \tilde{v}\rangle.$$

By the Hölder inequality, we see that

$$G^*(\varphi) > \left(c - \|f\|_{L^{p'}(\Omega)}\right)\|\tilde{v}\|_{L^p(\Omega)}.$$

Choosing $c > \|f\|_{L^{p'}(\Omega)}$ and taking scalar multiples of the corresponding $\tilde{v} \in U$ proves that $G^*(\varphi) = \infty$ since the supremum is scale invariant. □

Recall from Proposition 3.24(ii) that $G\ (\ ) = \ D$ for $\in Q$. Consequently, the last term in the dual problem may be computed for $\in L^p(\ )$ as

$$G(\ G\ (\ )) = \begin{cases} 0; & \text{if } G\ (\ ) \in L^p(\ ) \text{ with } D\ = f; \\ \infty; & \text{else;} \end{cases}$$
$$= \begin{cases} 0; & \text{if } \in Q(\ f); \\ \infty; & \text{else.} \end{cases}$$

It remains to find the convex conjugate of F.

Lemma 3.33. *Let the functional* $F : L^p(\ ) \to \mathbb{R}$ *be defined as in* (3.48). *Then*

$$F(q) = \frac{1}{p}\int\int \ ^{1-p}|q|^p \, dx' \, dx; \quad \forall q \in L^p(\ ).$$

Proof. We first define the scalar function $j : \ \mathbb{R} \to \mathbb{R}$ by

$$j(x; x'; r) = \frac{1}{p}\ (x; x')|r|^p; \quad \forall (x; x') \in \ ; \quad \forall r \in \mathbb{R}.$$

With it, it is clear that

$$F(\ ) = \int\int j(x; x';\ (x; x'))\, dx' \, dx; \quad \forall \ \in L^p(\ ).$$

Invoking [6, Theorem 9.3.3], we see that the convex conjugate of such a functional is given by

$$F(q) = \int\int j\ (x; x'; q(x; x'))\, dx' \, dx; \quad \forall q \in L^p(\ ); \tag{3.54}$$

where j is the convex conjugate of j with respect to the last variable. To ease the use of notation we will write $= \ (x; x')$ and $j\ (\ ) = j\ (x; x';\ )$ for fixed $(x; x') \in$. Now let $s \in \mathbb{R}$ and consider the definition of the convex conjugate

$$j\ (s) = \sup_{r \in \mathbb{R}} sr - \frac{1}{p}\ |r|^p. \tag{3.55}$$

The concave function $(r) = sr - \frac{1}{p}\ |r|^p$ attains it maximum on $\mathbb{R}$ at the point $\bar{r} \in \mathbb{R}$ with $'(\bar{r}) = 0$. This leads us to the equation

$$0 = \ '(\bar{r}) = s - \ |\bar{r}|^{p-2}\bar{r} \implies s = \ |\bar{r}|^{p-2}\bar{r}. \tag{3.56}$$

We now recall the inversion formula (3.32) from Lemma 3.25, and realize that it is applicable to (3.56). Hence, we obtain

$$\bar{r} = \ ^{1-p}|s|^{p-2}s.$$

Inserting this optimum in (3.55) yields us

$$j\ (s) = \ ^{1-p}|s|^p - \frac{1}{p}\ \left|\ ^{1-p}|s|^{p-2}s\right|^p$$
$$= \ ^{1-p}|s|^p - \frac{1}{p}\ ^{1-(p-1)p}|s|^{(p-1)p}$$
$$= \frac{1}{q}\ ^{1-p}|s|^p;$$

by using the identities $(p-1)p = p$ and $1/q = 1 - 1/p$. Inserting this back into (3.54) gives us the result. □

Applying the above results we see that the dual problem (3.46) takes the form

$$\max_{\in L^{p}(\;)} L(\;) = \frac{1}{p}\int\int {}^{1\;p} |\;|^{p}\, dx' dx \quad \begin{cases} 0; & \text{if } \in Q(\; f); \\ \infty; & \text{else.} \end{cases}$$

Equivalently, we may write the dual problem as

$$\max_{\in Q(\; f)} L(\;) = \frac{1}{p}\int\int {}^{1\;p} |\;|^{p}\, dx' dx;$$

where the dual variables are sought among $\in Q$ with $D\; = f$. Consequently, the optimal dual variable represents the negative thermal flux of the nonlocal equilibrium temperature state. Since the optimal value is invariant whether we consider or , we can equivalently seek dual variables satisfying $D\; = f$. Rewriting the dual problem as a minimization problem

$$\min_{\in Q(f)} \frac{1}{p}\int\int {}^{1\;p} |\;|^{p}\, dx' dx;$$

we regain the nonlocal Kelvin principle.

Theorem 3.34. *Let $(u; q) \in U \times Q$ be the equilibrium state-flux pair solving the nonlocal equation* (3.18). *Assume that $J : U \to \mathbb{R}$ and $J : Q \to \mathbb{R}$ denote to the nonlocal Dirichlet and complementary energy functionals, respectively. Then we have the equality*

$$J(u) + J(q) = 0. \tag{3.57}$$

Proof. This is the result of Theorem 3.31 since all the assumptions are satisfied. Indeed, our derivations have shown that $L = J$ on U and that $L = J$ on $Q(f)$. As such, the equality (3.57) is in fact (3.47). □

3.5 Local convergence

We will conclude this chapter by studying the localization of the presented nonlocal model. In particular, we will show that the nonlocal equilibrium temperature states converge to the local equilibrium state as the interaction horizon vanishes. To do this, we first recall their defining diffusion laws. The equilibrium temperature state under local nonlinear diffusion satisfies the Dirichlet principle

$$\min_{v \in W_0^{1,p}(\;)} I(v) = \frac{1}{p}\int |\nabla v|^{p}\, dx \quad \int f v\, dx. \tag{3.58}$$

Our goal is to show that the local state, which we denote by $u \in W_0^{1,p}(\;)$, can be approximated by a sequence of nonlocal equilibrium temperature states all satisfying the nonlocal Dirichlet principle

$$\min_{v \in U} J(v) = \frac{1}{p}\int\int |G\; v|^{p}\, dx' dx \quad \int f v\, dx. \tag{3.59}$$

We denote the sequence of nonlocal states as $\{u_\delta\}_{\delta\searrow 0}$, and we write the Dirichlet energy J_δ with subscript δ to signify the dependence. Throughout this discussion $\lim_{\delta\searrow 0}$ will mean the limit for a discrete vanishing sequence of interaction horizons $\delta \in (0, \delta_0)$. Both problems are specified by a volumetric heat source $f \in L^{p'}(\Omega)$, and the local conductivity distribution $\kappa \in L^{\infty}(\Omega)$. Indeed, by extending the local conductivity by $\underline{\kappa}$ outside of Ω, we define the nonlocal conductivity distribution as the mean of the two point conductivities

$$\kappa_\delta(x,x') = \frac{\kappa(x) + \kappa(x')}{2}, \quad \forall (x,x') \in \Omega_\delta \times \Omega_\delta.$$

Note here that we propose that the nonlocal conductivity distribution arises as the arithmetic mean of the local conductivity. It will be clear from the following survey that this choice is not exclusive. Indeed, both the geometric and harmonic means are also common admissible choices.

The analysis in this section is motivated by the important paper [15] which catalyzed the research on nonlocal approximations of Sobolev spaces. We present the main results of this paper. First we have [15, Theorem 1], which presents the following inequality.

Proposition 3.35. *Let $u \in W_0^{1,p}(\Omega)$. Then there exists some $C > 0$ such that*

$$\int_{\Omega_\delta} \int_{\Omega_\delta} |G_\delta u|^p \, dx' \, dx \leq C K_{p,d}^{-1} \|u\|^p_{W^{1,p}(\Omega)}. \tag{3.60}$$

In particular, we have $u \in U_\delta$.

Remark that Proposition 3.35 implies that our nonlocal state space U_δ acts as an intermediate space between $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$. Indeed, we have

$$W_0^{1,p}(\Omega) \subset U_\delta \subset L^p(\Omega).$$

The constant $K_{p,d}$ in (3.60) appears due to the normalization of the nonlocal kernel given in (3.9). This choice is important since it provides the correct scaling for the following convergence result of [15, Theorem 2 & Corollary 1].

Proposition 3.36. *Let $u \in W_0^{1,p}(\Omega)$. Then we have the following convergence result*

$$\lim_{\delta\searrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} |G_\delta u|^p \, dx' \, dx = \|u\|^p_{W^{1,p}(\Omega)}.$$

In particular, we have

$$\lim_{\delta\searrow 0} \int_{\Omega_\delta} |G_\delta u(\cdot, x')|^p \, dx' \to |\nabla u(\cdot)|^p, \text{ in } L^1(\Omega_0).$$

A simple consequence of Proposition 3.36 shows that that the nonlocal Dirichlet energies of the local equilibrium state converges towards the local Dirichlet energy.

Proposition 3.37. *Let $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ and $J_\delta : U_\delta \to \mathbb{R}$ denote the local and nonlocal Dirichlet energies from* (3.58)-(3.59)*, respectively. If $u \in W_0^{1,p}(\Omega)$ denotes the local equilibrium state, then*

$$I(u) = \lim_{\delta\searrow 0} J_\delta(u).$$

Proof. Utilizing the $L^1(\Omega_0)$-convergence from Proposition 3.36, we know that

$$\lim_{\delta\searrow 0}\int_{\Omega}\int_{\Omega}\sigma_\delta(x)|G_\delta u(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x=\int_{\Omega}\sigma(x)|\nabla u(x)|^p\,\mathrm{d}x;$$

since $\sigma\in L^1(\Omega_0)$. Here we note that the right-hand side integral is taken over Ω since $\nabla u\equiv 0$ outside of Ω. Applying Fubini's theorem and renaming the variables on the left-hand side, we similarly obtain

$$\lim_{\delta\searrow 0}\int_{\Omega}\int_{\Omega}\sigma_\delta(x')|G_\delta u(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x=\int_{\Omega}\sigma(x)|\nabla u(x)|^p\,\mathrm{d}x;$$

since the nonlocal gradient is anti-symmetric. As the nonlocal conductivity distribution $\sigma_\delta(x;x')$ arises from the arithmetic mean, we then have

$$\begin{aligned}
\lim_{\delta\searrow 0}J_\delta(u)&=\lim_{\delta\searrow 0}\frac{1}{p}\int_{\Omega}\int_{\Omega}\sigma_\delta(x;x')|G_\delta u(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x-\int_{\Omega}f(x)u(x)\,\mathrm{d}x\\
&=\lim_{\delta\searrow 0}\left(\frac{1}{2p}\int_{\Omega}\int_{\Omega}\sigma_\delta(x)|G_\delta u(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x-\int_{\Omega}f(x)u(x)\,\mathrm{d}x\right.\\
&\qquad\left.+\frac{1}{2p}\int_{\Omega}\int_{\Omega}\sigma_\delta(x')|G_\delta u(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x-\int_{\Omega}f(x)u(x)\,\mathrm{d}x\right)\\
&=\frac{1}{p}\int_{\Omega}\sigma(x)|\nabla u(x)|^p\,\mathrm{d}x-\int_{\Omega}f(x)u(x)\,\mathrm{d}x\\
&=I(u);
\end{aligned}$$

which proves the result. □

In order to study the limiting behaviour of the nonlocal equilibrium states, we need to establish a notion of compactness for vanishing interaction horizons. We previously introduced Lemma 3.6, which provided the Ponce inequality

$$\int_{\Omega_0}|\nabla\tilde{u}|^p\,\mathrm{d}x\le\limsup_{k\to\infty}\int_{\Omega_k}\int_{\Omega_k}|G_{\delta_k}u_k|^p\,\mathrm{d}x'\,\mathrm{d}x;$$

for limit points $\tilde{u}\in L^p(\Omega_0)$ of certain uniformly bounded sequences of $\{u_{\delta_k}\}_{k\in\mathbb{N}}$. However, the present case requires a similar result which also accounts for the convergence of nonlocal conductivities. To this end, we introduce the following generalized Ponce inequality [54, 55].

Lemma 3.38 (Generalized Ponce inequality). *Let $\{\delta_k\}_{k\in\mathbb{N}}\subset(0;\delta_0)$ be a sequence of interaction horizons converging to zero. Assume $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega)$ is a bounded sequence and that there exists some $M>0$ for which*

$$\int_{\Omega_k}\int_{\Omega_k}|G_{\delta_k}u_k(x;x')|^p\,\mathrm{d}x'\,\mathrm{d}x\le M;\quad\forall k\in\mathbb{N}:$$

Then $\{u_k\}_{k\in\mathbb{N}}$ is relatively compact in $L^p(\Omega_0)$. Any limit point $\tilde{u}\in L^p(\Omega_0)$ is also in $W^{1;p}(\Omega_0)$ and satisfies

$$\int_{\Omega_0}|\nabla\tilde{u}|^p\,\mathrm{d}x\le\liminf_{k\to\infty}\int_{\Omega_k}\int_{\Omega_k}|G_{\delta_k}u_k|^p\,\mathrm{d}x'\,\mathrm{d}x:\tag{3.61}$$

The question now is whether the result obtained in Proposition 3.37 also holds for the sequence of nonlocal equilibrium states. The following theorem establishes the localizing property.

Theorem 3.39. *Following the previous assumptions we have*

$$\min_{W_0^{1,p}(\Omega)} I = \lim_{\delta \to 0} \min_{U_\delta} J_\delta .$$

In particular, assume $\{u_\delta\}_{\delta \to 0}$ is a sequence of nonlocal equilibrium states, then $\|u_\delta - u\|_{L^p(\Omega)} \to 0$ and

$$I(u) = \lim_{\delta \to 0} J_\delta(u_\delta).$$

Proof. We first realize that the sequence of nonlocal equilibrium states $\{u_\delta\}_{\delta \to 0}$ satisfies the assumptions of Lemma 3.38. Indeed, the a priori stability estimates obtained in the proof of Corollary 3.22 provides us a constant $C > 0$ independent of $\delta \in (0, \delta_0)$ for which we can uniformly bound the nonlocal gradients

$$\int_{\Omega} \int_{\Omega_\delta} |\mathcal{G}_\delta u_\delta|^p \, \mathrm{d}x' \, \mathrm{d}x \le C \|f\|^{p'}_{L^{p'}(\Omega)}, \quad \forall \delta \in (0, \delta_0);$$

and due to the nonlocal Poincaré inequality of Proposition 3.7, there exists another constant $C > 0$ independent of $\delta \in (0, \delta_0)$ for which

$$\|u_\delta\|_{L^p(\Omega)} \le C \|f\|^{\frac{1}{p-1}}_{L^{p'}(\Omega)}, \quad \forall \delta \in (0, \delta_0).$$

Hence, the sequence of states is uniformly bounded. This allows us extract a subsequence, which converges to $\bar{u} \in W^{1,p}(\Omega_0)$ strongly in $L^p(\Omega_0)$. For this discussion, we will not relabel subsequences. Since each $u_\delta \in U_\delta$ is extended by zero outside of the domain Ω, so is $\bar{u}$. Indeed, by the strong convergence in $L^p(\Omega_0)$ we can extract a further subsequence which converges pointwise almost everywhere in Ω_0. Hence we have the pointwise limit

$$\lim_{\delta \to 0} u_\delta(x) = \bar{u}(x) = 0, \quad \text{for a.e. } x \in \Omega_0 \setminus \Omega.$$

Appealing to the characterization of [18, Proposition 9.18] again implies that $\bar{u} \in W_0^{1,p}(\Omega)$. The generalized Ponce inequality now yields

$$\frac{1}{p} \int_{\Omega} |\nabla \bar{u}|^p \, \mathrm{d}x - \int_{\Omega} f \bar{u} \le \liminf_{\delta \to 0} \frac{1}{p} \int_{\Omega} \int_{\Omega_\delta} |\mathcal{G}_\delta u_\delta(x, x')|^p \, \mathrm{d}x' \, \mathrm{d}x - \int_{\Omega} f u_\delta \, \mathrm{d}x,$$

since $u_\delta \to \bar{u}$ strongly in $L^p(\Omega)$. Recalling that the local state $u \in W_0^{1,p}(\Omega)$ satisfies the local Dirichlet principle, we get

$$\frac{1}{p} \int_{\Omega} |\nabla u|^p \, \mathrm{d}x - \int_{\Omega} f u \, \mathrm{d}x \le \frac{1}{p} \int_{\Omega} |\nabla \bar{u}|^p \, \mathrm{d}x - \int_{\Omega} f \bar{u} \, \mathrm{d}x.$$

In summary, we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\int_{\Omega} |\nabla u|^p \,\mathrm{d}x - \int_{\Omega} f u \,\mathrm{d}x \\
&\le \frac{1}{p}\int_{\Omega} |\nabla \bar{u}|^p \,\mathrm{d}x - \int_{\Omega} f \bar{u} \,\mathrm{d}x \\
&\le \liminf_{\varepsilon \searrow 0} \frac{1}{p}\int_{\Omega}\int_{\Omega} |G_\varepsilon u_\varepsilon(x;x')|^p \,\mathrm{d}x' \,\mathrm{d}x - \int_{\Omega} f u_\varepsilon \,\mathrm{d}x \\
&= \liminf_{\varepsilon \searrow 0} J_\varepsilon(u_\varepsilon).
\end{aligned}
$$

The reverse inequality follows from Proposition 3.37. Indeed, we see that

$$
\begin{aligned}
I(u) &= \frac{1}{p}\int_{\Omega} |\nabla u|^p \,\mathrm{d}x - \int_{\Omega} f u \,\mathrm{d}x \\
&= \lim_{\varepsilon \searrow 0} \frac{1}{p}\int_{\Omega}\int_{\Omega} |G_\varepsilon u|^p \,\mathrm{d}x' \,\mathrm{d}x - \int_{\Omega} f u \,\mathrm{d}x \\
&\ge \limsup_{\varepsilon \searrow 0} \frac{1}{p}\int_{\Omega}\int_{\Omega} |G_\varepsilon u_\varepsilon|^p \,\mathrm{d}x' \,\mathrm{d}x - \int_{\Omega} f u_\varepsilon \,\mathrm{d}x \\
&= \limsup_{\varepsilon \searrow 0} J_\varepsilon(u_\varepsilon).
\end{aligned}
$$

Here, the inequality follows since the nonlocal states satisfy the nonlocal Dirichlet principle. Gathering both of these estimates, we obtain the equality

$$
I(u) = \lim_{\varepsilon \searrow 0} J_\varepsilon(u_\varepsilon).
$$

We now claim that $\bar{u} = u$. This is however immediate from the previous arguments showing

$$
\lim_{k \to \infty} J_{\varepsilon_k}(u_k) \le I(u) \le I(\bar{u}) \le \lim_{k \to \infty} J_{\varepsilon_k}(u_k);
$$

implying that $I(u) = I(\bar{u})$. This demonstrates that $\bar{u}$ satisfies the local Dirichlet principle, and therefore it has to be the unique local equilibrium state u. This proves that the limiting point is unique and that the entire sequence of nonlocal states $\{u_\varepsilon\}_{\varepsilon \searrow 0}$ converges to the local equilibrium state u in $L^p(\Omega)$. □

Chapter 4

Nonlocal optimal control & obstacle problems

The goal of the following chapter is to further illustrate the applicability of our nonlocal diffusion law. We will show that it can substitute the local diffusion law in prototypical model problems in optimal control, as well as in the famous obstacle problem. Of course, the nonlocal formulations differ from their local counterparts, however, they are derived analogously. In this chapter we will consider only the linear case with $p = 2$ in order to simplify the exposition. First, we consider the linear-quadratic problem of optimal control in the volumetric heat source. Nonlocal optimal control in the source was first studied in [28] for $p = 2$, and subsequently extended in [21, 56] for $p \neq 2$. We then turn our attention towards optimal control in the conductivity distribution. Its study is made difficult by the nonlinear relation between conductivities and optimal equilibrium states. We first consider the problem of parameter identification as in [29], and then the nonlocal saddle-point problem studied in [4, 38, 39] extending the local case proposed in [22]. In particular, we demonstrate that the analysis of the latter problem benefits from a dual formulation of nonlocal diffusion. Lastly, we consider the classical obstacle problems. We discuss their nonlocal variants as presented in [20, 44]. Specifically for the nonlocal case, we will see that the analysis of nonlocal obstacle problems is not necessarily complicated by irregular obstacles.

4.1 Distributed control

The first problem we consider seeks to identify the volumetric heat source which provides the best reconstruction of a target equilibrium temperature state. This amounts to a linear-quadratic optimal control problem known as distributed control. In this section, we will assume that the material body Ω has a homogeneous conductivity distribution. The case with inhomogeneous conductivity distributions extends similarly. Let us recall the local problem.

Assume a *target equilibrium state* $u \in L^2(\Omega)$ arises from the linear diffusion law

$$
\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
$$

whose volumetric heat source $f \in L^2(\Omega)$ is unknown. Our goal is to estimate it. To this end, we formulate a minimization problem whose solution minimizes the difference between the realized and target equilibrium states. Specifically, we seek a solution to

$$\begin{aligned} \min_{u;f} \; & I(u;f) = \frac{1}{2}\|u - \hat{u}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|f\|^2_{L^2(\Omega)}; \\ \text{s.t.} \quad & -\Delta u = f; \quad \text{in } \Omega; \\ & u = 0; \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Remark that we specify the problem with an additional quadratic term parameterized by a parameter $\beta \geq 0$. As we will soon see, its role is to regularize the problem.

It is straightforward to see that the nonlocal analogue of (4.1) is

$$\begin{aligned} \min_{u;f} \; & J_\delta(u;f) = \frac{1}{2}\|u - \hat{u}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|f\|^2_{L^2(\Omega)}; \\ \text{s.t.} \quad & -\mathcal{L}_\delta u = f; \quad \text{in } \Omega; \\ & u = 0; \quad \text{in } \Gamma_\delta; \end{aligned} \tag{4.2}$$

where $\mathcal{L}_\delta$ denotes the nonlocal 2-Laplacian for some fixed $\delta > 0$. Note that the only difference between (4.1) and (4.2) is the physical enforcement of a local or nonlocal diffusion law. As a standard approach of the classical control theory, we will consider the reduced form of (4.2). It is formulated using the control-to-state operator.

Definition 4.1. The *control-to-state operator* $S_\delta : L^2(\Omega) \to U_\delta$ of (4.2) is defined as the operator which maps a heat source $f \in L^2(\Omega)$ to its corresponding equilibrium state $S_\delta f \in U_\delta$.

The control-to-state operator is clearly well-defined, since for each heat source $f \in L^2(\Omega)$ there exists a unique corresponding equilibrium state $u \in U_\delta$, which satisfies the nonlocal diffusion law imposed in (4.2). Hence $S_\delta f = u$. We remark the following properties of the control-to-state operator.

Lemma 4.2. *The control-to-state operator $S_\delta : L^2(\Omega) \to U_\delta$ of* (4.2) *is a bounded linear operator.*

Proof. Bounded linearity follows immediately since $p = 2$. Indeed, let $f; g \in L^2(\Omega)$ be two distinct heat sources, and let $\alpha; \beta \in \mathbb{R}$. The state induced by $\alpha f + \beta g$ satisfies the variational formulation (3.20), and for all $v \in U_\delta$ it holds that

$$\int_{\Omega_\delta}\int_{\Omega_\delta} \mathcal{G}_\delta S_\delta(\alpha f + \beta g)\mathcal{G}_\delta v \,\mathrm{d}x' \,\mathrm{d}x = \int_\Omega (\alpha f + \beta g) v \,\mathrm{d}x.$$

Expanding the right-hand side and writing the variational formulations for f and g separately yields

$$\begin{aligned} \int_\Omega (\alpha f + \beta g) v \,\mathrm{d}x &= \int_{\Omega_\delta}\int_{\Omega_\delta} \mathcal{G}_\delta(\alpha S_\delta f)\mathcal{G}_\delta v \,\mathrm{d}x' \,\mathrm{d}x + \int_{\Omega_\delta}\int_{\Omega_\delta} \mathcal{G}_\delta(\beta S_\delta g)\mathcal{G}_\delta v \,\mathrm{d}x' \,\mathrm{d}x \\ &= \int_{\Omega_\delta}\int_{\Omega_\delta} \mathcal{G}_\delta(\alpha S_\delta f + \beta S_\delta g)\mathcal{G}_\delta v \,\mathrm{d}x' \,\mathrm{d}x; \end{aligned}$$

which proves that $S(\alpha f + \beta g) = \alpha S f + \beta S g$. Finally, boundedness is obtained by the stability estimate given by Corollary 3.22. It gives some constant $C > 0$ such that

$$\|Sf\|_U \leq C\|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega),$$

which is what we desired. □

We introduce the *reduced functional* $J_\gamma(f) = J_\gamma(Sf, f)$ and simply reformulate the problem (4.2) as

$$\min_{f \in A_{ad}} J_\gamma(f). \tag{4.3}$$

Here $A_{ad} \subseteq L^2(\Omega)$ denotes the set of *admissible controls*. Solvability of (4.3) is the subject of the following theorem.

Theorem 4.3. *Let $A_{ad} = L^2(\Omega)$, and assume that $\gamma > 0$. Then* (4.3) *admits a unique optimal control $f^* \in A_{ad}$.*

Proof. We employ the direct method. First we note that the reduced functional

$$J_\gamma(f) = \frac{1}{2}\|Sf - u_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|f\|^2_{L^2(\Omega)}. \tag{4.4}$$

is bounded below by zero and is proper. In addition, it is both convex, and continuous as a sum of norms composed with the continuous affine function $f \mapsto Sf - u_d$, see Lemma 4.2. Consequently, it is weakly lower semicontinuous. Due to the last quadratic term in (4.4), the reduced functional J_γ is in fact strictly convex, since $\gamma > 0$. The same term also gives the coercivity of J_γ. Indeed, we see that

$$J_\gamma(f) \geq \frac{\gamma}{2}\|f\|^2_{L^2(\Omega)},$$

which implies that $J_\gamma(f) \to \infty$ as $\|f\|_{L^2(\Omega)} \to \infty$. Since coercivity and weak lower semicontinuity is established, the existence of an optimal control $f \in L^2(\Omega)$ follows from the direct method. Uniqueness of f is immediate due to the strict convexity of the reduced functional J_γ. □

Note that the proof relies on the coercivity of the reduced functional J_γ, which in turn holds since $\gamma > 0$. When $\gamma = 0$ further assumptions need to be made. For practical purposes it may be necessary to constrain the set of admissible controls A_{ad}. A common practical choice is the set of box constraints. Fixing two bounding heat sources $\underline{f}, \overline{f} \in L^2(\Omega)$ satisfying $\underline{f} \leq \overline{f}$ a.e. in Ω, their corresponding *box constraints* are defined by

$$A_{ad} = \{f \in L^2(\Omega) \mid \underline{f} \leq f \leq \overline{f}, \text{ a.e. in } \Omega\}. \tag{4.5}$$

Imposing such constraints on the optimal control provides a different approach towards solvability.

Theorem 4.4. *Assume A_{ad} is weakly compact. Then* (4.3) *admits an optimal control $f^* \in A_{ad}$, which is unique if $\gamma > 0$*

Proof. Similar to the proof of Theorem 4.3 we apply the direct method. The reduced functional J is still weakly lower semicontinuous, even if $\lambda = 0$. However, coercivity is not given unless $\lambda > 0$. Since coercivity provides the direct method with weak compactness for minimizing sequences, we can exchange it for the condition that minimizing sequences are assumed to be weakly compact. This is exactly the assumption we have made for A. Thus, the direct method finds an optimal control $f \in L^2(\Omega)$. Similar to before, the optimal control is unique if $\lambda > 0$. □

We remark that the box constraints defined in (4.5) satisfies the assumptions of Theorem 4.4. In general any nonempty, closed, bounded, and convex subset of $L^2(\Omega)$ is also weakly compact.

Optimality system

In the subsequent chapter, we will explore the numerical approximation of the present linear quadratic control problem. To this end, it will prove useful to know the conditions for optimality of controls. We will embed the control-to-state operator $S : L^2(\Omega) \to L^2(\Omega)$ for the purposes of this discussion. In this form, it remains a bounded linear operator due to the continuous embedding of U in $L^2(\Omega)$. Realizing that (4.3) is a convex minimization problem posed in the Hilbert space $L^2(\Omega)$, we find that any optimal source control $f \in A$ has to satisfy the necessary and sufficient optimality condition

$$\langle \nabla J(f), g - f \rangle \geq 0, \quad \forall g \in A. \tag{4.6}$$

Here, the gradient of the reduced functional $\nabla J : L^2(\Omega) \to L^2(\Omega)$ is understood in the Fréchet sense. The gradient may be explicitly found as

$$\nabla J(f) = S^*(Sf - u) + \lambda f, \quad \forall f \in L^2(\Omega), \tag{4.7}$$

where $S^* : L^2(\Omega) \to L^2(\Omega)$ is the Hilbert space adjoint of S. The nature of the adjoint is settled in following lemma.

Lemma 4.5. *The control-to-state operator $S : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint.*

Proof. Let $f, g \in L^2(\Omega)$ and consider the variational formulation of their states

$$\int_\Omega \int_\Omega G(Sf) G v \, dx' \, dx = \int_\Omega f v \, dx, \quad \forall v \in U,$$
$$\int_\Omega \int_\Omega G(Sg) G v \, dx' \, dx = \int_\Omega g v \, dx, \quad \forall v \in U.$$

Testing these with $v = Sg$ and $v = Sf$, respectively, we obtain the equalities

$$\langle f, Sg \rangle = \int_\Omega \int_\Omega G\, Sg\, G\, Sf \, dx' \, dx = \langle Sf, g \rangle,$$

which proves that $S^* = S$, that is S is self-adjoint. □

This inspires the following definition.

Definition 4.6. Let $\bar f \in L^2(\Omega)$ be an optimal control of (4.3). Then we define the *adjoint state* of $\bar f$ as the unique solution $\bar w \in U_\Omega$ to

$$\begin{aligned} -\mathcal{L}_\delta \bar w &= S_\delta \bar f - u_\omega, && \text{in } \Omega, \\ \bar w &= 0, && \text{in } \Omega_I. \end{aligned}$$

Let us now reconsider (4.6).

Proposition 4.7. *Let $\bar f \in L^2(\Omega)$. Then $\bar f$ is an optimal control of (4.3) if and only if there exists some $\bar w \in L^2(\Omega)$ satisfying the optimality system*

$$\begin{aligned} \langle \bar w + \alpha \bar f, g - \bar f\rangle &\geq 0, && \forall g \in A_{ad}, \\ -\mathcal{L}_\delta \bar w &= S_\delta \bar f - u_\omega, && \text{in } \Omega, \\ \bar w &= 0, && \text{in } \Omega_I. \end{aligned} \tag{4.8}$$

Proof. By inserting the formula for the gradient (4.7) into (4.6), we get exactly (4.8). Here the last two equations of (4.8) arise from Lemma 4.5 noting that

$$\bar w = S_\delta^*(S_\delta \bar f - u_\omega) = S_\delta(S_\delta \bar f - u_\omega).$$

Since (4.6) is both a necessary and sufficient optimality condition, the result follows. $\square$

We conclude our discussion on source control by examining a specific case that compares the nonlocal optimal controls to their local counterparts.

Example 4.8. We consider the special case where we allow $A_{ad} = L^2(\Omega)$ and assume that the regularization parameter is present $\alpha > 0$. Now, the inequality in (4.8) takes the form of an equality

$$\langle \bar w + \alpha \bar f, g\rangle = 0, \quad \forall g \in L^2(\Omega),$$

by linearity in $L^2(\Omega)$. In particular, this implies that $\bar w + \alpha \bar f = 0$, and hence the optimal control and its adjoint state $(\bar f, \bar w) \in L^2(\Omega) \times U_\Omega$ has to satisfy the relation

$$\bar f = -\frac{1}{\alpha}\bar w, \tag{4.9}$$

and the optimality system

$$\begin{aligned} -\mathcal{L}_\delta S_\delta \bar f &= -\alpha^{-1}\bar w, && \text{in } \Omega, & \qquad -\mathcal{L}_\delta \bar w &= S_\delta \bar f - u_\omega, && \text{in } \Omega, \\ S_\delta \bar f &= 0, && \text{in } \Omega_I, & \bar w &= 0, && \text{in } \Omega_I. \end{aligned} \tag{4.10}$$

We remark that the corresponding local optimality conditions share the relation (4.9), and the optimality system (4.10) is exactly the nonlocal equivalent of the local optimality system

$$\begin{aligned} -\Delta S\bar f &= -\alpha^{-1}\bar w, && \text{in } \Omega, & \qquad -\Delta \bar w &= S\bar f - u_\omega, && \text{in } \Omega, \\ S\bar f &= 0, && \text{in } \partial\Omega, & \bar w &= 0, && \text{in } \partial\Omega. \end{aligned}$$

Here $S : L^2(\Omega) \to L^2(\Omega)$ denotes the local control-to-state operator. This further illustrates the extension of the local theory to the nonlocal framework.

4.2 Control in the conductivity

Let us now consider a different type of optimal control problem. Specifically, we will now keep the volumetric heat source fixed, and instead assume control over the thermal conductivity distribution. The analysis of optimal control in the source was relatively simple, due to the fact that equilibrium temperature states depend linearly on their heat sources. In the case of conductivity control, the analysis of the considered control problems is complicated by the fact that the equilibrium temperature states are nonlinear with respect to the conductivity. As such, further considerations must be made in order to apply the direct method. Be that as it may, our objective will be twofold. First, we study the optimal control with respect to the functional from the previous section. This problem is known as parameter identification, in which we seek to estimate the material properties of a spatial domain, here the thermal conductivity, from an observed temperature state generated by a known heat source. Second, we consider the saddle-point problem of Ceá and Malankowski [22], which seeks to identify the optimal thermal conductivity distribution, which minimizes the weighted average of the equilibrium temperature state. This problem is a prototype of the famous compliance minimization problem from topology optimization, which seeks to maximize the stiffness of a structure under a specified load. For more details we refer to [1] and the references in [4, 38, 39].

The analysis of both problems requires good continuity properties of their control-to-state operators. Again, we first recall the local state equation. Given a volumetric heat source $f \in L^2(\Omega)$, we assume that the corresponding equilibrium temperature state u arises from the linear diffusion law

$$
\begin{aligned}
-\mathrm{div}(\kappa \nabla u) &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

specified by an unknown conductivity distribution $\kappa \in L^\infty(\Omega)$. Analogously, the nonlocal state equation arises from the nonlocal diffusion law

$$
-\mathcal{D}(\Theta \mathcal{G} u) = f \text{ in } \Omega, \tag{4.11}
$$

$$
u = 0 \text{ in } \Omega_{\mathcal{I}}, \tag{4.12}
$$

where the nonlocal conductivity distribution $\Theta \in L^\infty(\Omega \cup \Omega_{\mathcal{I}} \times \Omega \cup \Omega_{\mathcal{I}})$ is unknown. In order to define the corresponding control-to-state operator, we recall that the nonlocal conductivity distribution was subject to the following assumptions

$$
\mathcal{A}_\Theta = \{\Theta \in L^\infty(\Omega \cup \Omega_{\mathcal{I}} \times \Omega \cup \Omega_{\mathcal{I}}) \mid \Theta \in [\underline{\theta}, \overline{\theta}], \text{ a.e. in } \Omega \cup \Omega_{\mathcal{I}} \times \Omega \cup \Omega_{\mathcal{I}}\}.
$$

In order to differentiate between the previous and present discussions, we define the control-to-state operator as follows.

Definition 4.9. The *conductivity-to-state operator* $S_\Theta : \mathcal{A}_\Theta \to \mathcal{U}_\Theta$ of (4.11) is defined as the operator which maps a nonlocal conductivity distribution $\Theta \in \mathcal{A}_\Theta$ to its corresponding equilibrium state $S_\Theta \Theta \in \mathcal{U}_\Theta$.

Note that we now relate the conductivity-to-state operator S_Θ to the state equation (4.11), since it will not change between the upcoming control problems. Unfortunately, S_Θ is no longer linear due to the nonlinearity of its variational formulation.

As a consequence, it will be more difficult to apply the direct method to obtain optimal controls. Alas, we present two different approaches, each of which requiring additional assumptions and consulting different topologies on $L^\infty(\Omega\times\Omega)$.

The first approach is found in [29]. Their main tool is the strong continuity of the conductivity-to-state operator.

Proposition 4.10. *The conductivity-to-state operator $S : \mathcal{A} \to U$ is Lipschitz continuous, i.e. there exists $C > 0$ such that*

$$\|S\sigma - S\tilde{\sigma}\|_U \leq C\|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega\times\Omega)}, \quad \forall \sigma, \tilde{\sigma} \in \mathcal{A}.$$

Proof. The proof follows along the lines of [29, Lemma 3.1]. Let $\sigma, \tilde{\sigma} \in \mathcal{A}$ be two conductivities, and consider the variational formulations of their corresponding states

$$\begin{aligned} \int_\Omega\int_\Omega \sigma G(S\sigma) G v \,\mathrm{d}x' \,\mathrm{d}x &= \int_\Omega f v \,\mathrm{d}x, \quad \forall v \in U, \\ \int_\Omega\int_\Omega \tilde{\sigma} G(S\tilde{\sigma}) G v \,\mathrm{d}x' \,\mathrm{d}x &= \int_\Omega f v \,\mathrm{d}x, \quad \forall v \in U. \end{aligned} \tag{4.13}$$

Denoting $u = S\sigma$ and $\tilde{u} = S\tilde{\sigma}$, we fix $v \in U$ and realize that comparing equations in (4.13) yields

$$\begin{aligned} \int_\Omega\int_\Omega \sigma G(u - \tilde{u}) G v \,\mathrm{d}x' \,\mathrm{d}x &= \int_\Omega\int_\Omega \sigma G u G v \,\mathrm{d}x' \,\mathrm{d}x - \int_\Omega\int_\Omega \sigma G \tilde{u} G v \,\mathrm{d}x' \,\mathrm{d}x \\ &= \int_\Omega\int_\Omega \tilde{\sigma} G \tilde{u} G v \,\mathrm{d}x' \,\mathrm{d}x - \int_\Omega\int_\Omega \sigma G \tilde{u} G v \,\mathrm{d}x' \,\mathrm{d}x \\ &= \int_\Omega\int_\Omega (\tilde{\sigma} - \sigma) G \tilde{u} G v \,\mathrm{d}x' \,\mathrm{d}x. \end{aligned}$$

We now insert $v = u - \tilde{u}$ and obtain the estimate

$$\begin{aligned} \alpha\|u - \tilde{u}\|_U^2 &\leq \|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega\times\Omega)} \int_\Omega\int_\Omega |G\tilde{u} G(u - \tilde{u})| \,\mathrm{d}x' \,\mathrm{d}x \\ &\leq \|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega\times\Omega)} \|\tilde{u}\|_U \|u - \tilde{u}\|_U, \end{aligned}$$

through the Hölder and Cauchy-Schwarz inequalities. Invoking the stability estimate of Corollary 3.22, there exists some $C > 0$ independent of $\tilde{\sigma}$ (but dependent of α and β) such that $\|\tilde{u}\|_U \leq C\|f\|_{L^2(\Omega)}$. All that remains is to rearrange the previous inequalities to get

$$\|S\sigma - S\tilde{\sigma}\|_U = \|u - \tilde{u}\|_U \leq \alpha^{-1} C\|f\|_{L^2(\Omega)}\|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega\times\Omega)}.$$

That is what we wanted. ☐

Wishing to invoke the usual strong topology on $L^\infty(\Omega\times\Omega)$, a similarly strong compactness property is needed. Therefore we assume that the admissible conductivity distributions admit uniformly bounded weak derivatives. This leads us to study an alternative set of admissible controls

$$\mathcal{A}_\kappa = \{\sigma \in \mathcal{A} \cap W^{1,\infty}(\Omega\times\Omega) \mid \|\sigma\|_{W^{1,\infty}(\Omega\times\Omega)} \leq \kappa\},$$

for some $\kappa \in (0, \infty)$. Regrettably, this set excludes certain discontinuous conductivity distributions which may arise in practical applications. On the other hand, it will simplify the line of reasoning. We present the following result.

Proposition 4.11. *Let $\in (0; \infty)$. Then the set of admissible controls A is compact in $L^\infty($ $)$.*

Proof. The Sobolev embedding theorem given in [Bre10, Theorem 9.16] asserts that $W^{1;\infty}($ $)$ is compactly embedded in $L^\infty($ $)$. Since A, by definition, is bounded in $W^{1;\infty}($ $)$, relative compactness in $L^\infty($ $)$ follows. But A is closed in $L^\infty($ $)$ due to the bounds $\_$ $^-$ for all $\in A$. □

The second approach follows [38] and keeps A as defined. As such, this approach does allow discontinuous conductivity distributions. However, continuity and compactness properties become much harder to obtain. Unlike the first approach, we now look for continuity in the weak -topology on $L^\infty($ $)$. In the local case this approach is naive, since the conductivity-to-state operator is generally not continuous with respect to the weak -topology [1, 67]. In stark contrast, the nonlocal conductivity-to-state operator is, if U is compactly embedded in $L^2($ $)$. We demonstrate this by considering the arguments of [5, 38], in which U is continuously embedded in a fractional Sobolev space. To this end, it proves sufficient to assume that the nonlocal kernel satisfies the additional assumption

$$\omega(x) \quad \frac{c_s}{|x|^{\frac{d}{2}+s}}; \quad \forall x \in B_{=2} \setminus \{0\}; \tag{4.14}$$

for some fractional exponent $s \in (0;1)$ and constant $c_s > 0$.

Lemma 4.12. *Assume* (4.14) *holds with some $s > 0$. Then U is continuously embedded in $W^{s;2}($ $)$. In particular, U is compactly embedded in $L^2($ $)$.*

Proof. Let $u \in U$. By (4.14) there exists $s \in (0;1)$ and $c_s > 0$ for which we can establish the inequality

$$\begin{aligned} \|u\|_U^2 &= \int \int |u(x) \quad u(x')|^2 \omega(x \quad x')^2 \, \mathrm{d}x' \, \mathrm{d}x \\ & \int \int_{\cap B_{=2}(x)} |u(x) \quad u(x')|^2 \omega(x \quad x')^2 \, \mathrm{d}x' \, \mathrm{d}x \\ & c_s \int \int_{\cap B_{=2}(x)} \frac{|u(x) \quad u(x')|^2}{|x \quad x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x. \end{aligned}$$

Invoking [12, Proposition 6.1], there exists some $C_1 > 0$, for which we see that the last expression is bounded below as

$$|u|^2_{W^{s;2}(\ )} \quad C_1 \int \int_{\cap B_{=2}(x)} \frac{|u(x) \quad u(x')|^2}{|x \quad x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x.$$

This gives us an initial inequality

$$|u|^2_{W^{s;2}(\ )} \quad c_s^{\ 1} C_1 \|u\|_U^2 .$$

Invoking a fractional Poincaré inequality, there exists another constant $C_2 > 0$ with which we can establish

$$\|u\|^2_{L^2(\ )} \quad C_2 |u|^2_{W^{s;2}(\ )}$$

Such an inequality can be obtained by considering either Proposition 3.7 with the nonlocal kernel

$$\omega_\delta(x) = \frac{c}{|x|^{\frac{d}{2}+s}}, \quad \forall x \in B_\delta \setminus \{0\};$$

or the arguments presented in [33, Lemma 4.3]. Consequently, we can write

$$\|u\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\Omega_\delta)} \leq C_2 |u|^2_{W^{s,2}(\Omega_\delta)} \leq c_s^{-1} C_1 C_2 \|u\|^2_U;$$

where the equality follows since $u \in U$ is extended by zero outside of Ω. We have

$$\|u\|^2_{W^{s,2}(\Omega)} \leq \|u\|^2_{W^{s,2}(\Omega_\delta)} = \|u\|^2_{L^2(\Omega_\delta)} + |u|^2_{W^{s,2}(\Omega_\delta)} \leq c_s^{-1} C_1 (1 + C_2) \|u\|^2_U;$$

which shows the continuous embedding of U in $W^{s,2}(\Omega)$ The compact embedding in $L^2(\Omega)$ is now a consequence of the embedding theorems for fractional Sobolev spaces, see [32, Theorem 7.1]. □

We can now prove the following continuity result.

Proposition 4.13. *Assume* (4.14) *holds. Then $S : A \to U$ is weakly continuous with respect to the weak*-topology on $L^\infty(\Omega_\delta)$. Specifically, if $\{\sigma_k\}_{k \in \mathbb{N}} \subset A$ is a sequence of conductivity distributions converging in the weak*-topology to $\sigma \in A$, which we denote $\sigma_k \overset{*}{\rightharpoonup} \sigma$, then*

$$\lim_{k \to \infty} \|S\sigma_k - S\sigma\|_U = 0.$$

Proof. We give the proof found in [38, Theorem 2.3]. Assume we have a sequence $\{\sigma_k\}_{k\in\mathbb{N}} \subset A$ for which $\sigma_k \overset{*}{\rightharpoonup} \sigma$. We denote their corresponding states $u = S\sigma$, and $u_k = S\sigma_k$ for all $k \in \mathbb{N}$. The variational formulation of the equilibrium states is used to establish the equalities

$$\begin{aligned} J(u;\sigma) &= \frac{1}{2}\int\int \sigma|\mathcal{G}u|^2 \, dx' \, dx - \int f u \, dx = -\frac{1}{2}\int f u \, dx; \\ J(u_k;\sigma_k) &= \frac{1}{2}\int\int \sigma_k|\mathcal{G}u_k|^2 \, dx' \, dx - \int f u_k \, dx = -\frac{1}{2}\int f u_k \, dx; \end{aligned} \tag{4.15}$$

written in terms of the nonlocal Dirichlet energy $J(\cdot;\cdot)$, which we have supplied with an additional argument denoting the dependence on the conductivity. We now realize that

$$\begin{aligned} 0 \leq \underline{\sigma}\|u_k - u\|^2_U &\leq \int\int \sigma_k|\mathcal{G}(u_k - u)|^2 \, dx' \, dx \\ &= \int\int \sigma_k|\mathcal{G}u_k|^2 \, dx' \, dx + \int\int \sigma_k|\mathcal{G}u|^2 \, dx' \, dx \\ &\quad - 2\int\int \sigma_k \mathcal{G}u \, \mathcal{G}u_k \, dx' \, dx \\ &= 2J(u_k;\sigma_k) + 2J(u;\sigma_k) + 2\int f u_k \, dx + 2\int f u \, dx \\ &\quad - 2\int\int \sigma_k \mathcal{G}u \, \mathcal{G}u_k \, dx' \, dx \\ &= 2J(u;\sigma_k) - 2J(u_k;\sigma_k). \end{aligned} \tag{4.16}$$

Here, left-hand side equalities of (4.15) gives the second equality, and the right-hand side equalities of (4.15), together with the variational formulations, gives the last equality. It is now evident that if the convergence for $k \to \infty$ is well behaved, the terms in (4.16) will cancel in the limit. This would prove our promised result. Let us consider the first term in (4.16). Testing the weak-$*$ convergence $\sigma_k \overset{*}{\rightharpoonup} \sigma$ with $|Gu|^2 \in L^1(\Omega \times \Omega)$, we obtain

$$\begin{aligned} \lim_{k \to \infty} J(u; \sigma_k) &= \lim_{k \to \infty} \frac{1}{2} \int_\Omega \int_\Omega \sigma_k |Gu|^2 \, \mathrm{d}x' \, \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x \\ &= \frac{1}{2} \int_\Omega \int_\Omega \sigma |Gu|^2 \, \mathrm{d}x' \, \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x \\ &= J(u; \sigma). \end{aligned}$$

As for the second term in (4.16), we consider both its $\limsup$ and $\liminf$. Recalling the nonlocal Dirichlet principle, u is the unique minimizer of the Dirichlet energy $J(\cdot; \sigma)$, and similarly u_k is the unique minimizer of $J(\cdot; \sigma_k)$ for all $k \in \mathbb{N}$. Hence we may establish the following inequality

$$\limsup_{k \to \infty} J(u_k; \sigma_k) \le \limsup_{k \to \infty} J(u; \sigma_k) = J(u; \sigma). \tag{4.17}$$

On the other hand, we may extract a subsequence $\{(u_{k'}; \sigma_{k'})\}_{k' \in \mathbb{N}}$ for which

$$\lim_{k' \to \infty} J(u_{k'}; \sigma_{k'}) = \liminf_{k \to \infty} J(u_k; \sigma_k). \tag{4.18}$$

We now recall that U is compactly embedded in $L^2(\Omega)$, due to Lemma 4.12. As the states are uniformly bounded by $\|u_{k'}\|_U \le C\|f\|_{L^2(\Omega)}$, for some $C > 0$ from Corollary 3.22, we know $\{u_{k'}\}_{k' \in \mathbb{N}}$ is relatively compact in $L^2(\Omega)$. Hence we may use the same argumentation as in Lemma 2.6 to extract further subsequences, not relabeled, for which there exists some $\tilde{u} \in U$ satisfying

$$\begin{aligned} u_{k'} &\to \tilde{u} \text{ in } L^2(\Omega), \\ u_{k'}(x) &\to \tilde{u}(x) \text{ a.e. in } \Omega. \end{aligned}$$

We will use the a.e. pointwise convergence of $\{u_{k'}\}_{k' \in \mathbb{N}}$ (and hence also of $\{G u_{k'}\}_{k' \in \mathbb{N}}$) to control (4.18). By continuity in $L^2(\Omega)$ we have

$$\lim_{k' \to \infty} \int_\Omega f u_{k'} \, \mathrm{d}x = \int_\Omega f \tilde{u} \, \mathrm{d}x. \tag{4.19}$$

As for the quadratic term in $J(u_{k'}; \sigma_{k'})$, we claim that

$$\int_\Omega \int_\Omega \sigma |G\tilde{u}|^2 \, \mathrm{d}x' \, \mathrm{d}x \le \liminf_{k' \to \infty} \int_\Omega \int_\Omega \sigma_{k'} |G u_{k'}|^2 \, \mathrm{d}x' \, \mathrm{d}x. \tag{4.20}$$

To this end, we define

$$\mu(A) = \int_\Omega \int_\Omega \chi_A \, \sigma_{k'} \, \mathrm{d}x' \, \mathrm{d}x, \quad \mu_{k'}(A) = \int_\Omega \int_\Omega \chi_A \, \sigma_{k'} \, \mathrm{d}x' \, \mathrm{d}x, \quad \forall k' \in \mathbb{N},$$

and prove that $\lim \mu_{k'}(A) = \mu(A)$ for all measurable $A \subset \Omega \cup \Omega_I$. Indeed for such subsets we have $\chi_A \in L^1(\Omega \cup \Omega_I)$ and therefore weak $*$-convergence of the conductivity distributions gives us the setwise convergence of measures. We now realize that (4.20) is obtained by the generalized Fatou's lemma [61, Section 11.4]. Therefore, we may combine (4.19)-(4.20) with (4.18) to get

$$J(u; \sigma) \leq \lim_{k \to \infty} J(u_{k'}; \sigma_{k'}) = \liminf_{k \to \infty} J(u_k; \sigma_k).$$

At last, since $J(u; \sigma) \leq J(u; \sigma)$ due to the nonlocal Dirichlet principle, we may recall (4.17) and conclude that $\lim_{k \to \infty} J(u_k; \sigma_k) = J(u; \sigma)$. Finally, returning to (4.16), we see that

$$0 \leq \limsup_{k \to \infty} \_ \|u_k - u\|_U^2 \leq \lim_{k \to \infty} \Big[2J(u; \sigma_k) - 2J(u_k; \sigma_k) \Big] = 0,$$

and we have the result. □

Note that this approach follows the same idea as the one-dimensional framework in Chapter 2. In particular, we secure an embedding in a fractional Sobolev space $W^{s,2}(\Omega)$, which in turn gives us a compact embedding in $L^2(\Omega)$. The compact embedding is then used to obtain pointwise convergence of states, which due to the nonlocal framework also provides pointwise convergence of their nonlocal gradients. With pointwise convergence established, a continuity property is proved using a Fatou-type lemma and the direct method becomes applicable. The current approach will use the following compactness result.

Proposition 4.14. *The set of admissible controls $\mathcal{A}$ is weakly compact with respect to the weak $*$-topology on $L^\infty(\Omega \cup \Omega_I)$.*

Proof. We appeal to the Banach-Alaoglu theorem [18, Theorem 3.16], which states that the closed unit ball in $L^\infty(\Omega \cup \Omega_I)$ is weakly compact with respect to the weak $*$-topology. Note that $\mathcal{A}$ is merely an affine transformation of the closed unit ball, and hence also weakly compact in the weak $*$-topology on $L^\infty(\Omega \cup \Omega_I)$. □

4.2.1 Identification

We are now ready to study the nonlocal identification problem. As we previously mentioned, both the heat source $f \in L^2(\Omega)$ and the target temperature state $u^* \in L^2(\Omega)$ are assumed to be known. The problem is seeking the optimal conductivity distribution σ, with corresponding temperature state minimizing the distance from the observed state. Therefore, we study

$$\begin{aligned} \min_{u, \sigma} \; & J(u; \sigma) = \frac{1}{2}\|u - u^*\|_{L^2(\Omega)}^2, \\ \text{s.t.} \; & -\mathcal{D}(\sigma \mathcal{G} u) = f \quad \text{in } \Omega, \\ & u = 0 \quad \text{in } \Omega_I. \end{aligned} \tag{4.21}$$

We invoke the conductivity-to-state operator. Instead of considering (4.21), we look for a minimizer of the reduced functional $\hat{J}(\sigma) = J(S\sigma; \sigma)$, i.e.

$$\min_{\sigma \in \mathcal{A}} \hat{J}(\sigma), \tag{4.22}$$

with $= A$ or $= A$ for the first or second approach, respectively. Having done the groundwork, we are ready to state and prove that either approach admits optimal controls.

Theorem 4.15. *Assume either*

(i) $= A$ *for some* > 0, *or*

(ii) $= A$ *and* (4.14) *holds.*

Then there exists an optimal control to (4.22).

Proof. We apply the direct method in both cases. Note that in either case the reduced functional $J : \;\to U$ is proper and bounded below by zero. Hence there exists a minimizing sequence $\{\;_k\}_{k\in\mathbb{N}}$ with $J(\;_k) \to \inf J$. We will now prove that $\{\;_k\}_{k\in\mathbb{N}}$ has a limit $\;\in\;$ in a specified topology, and that the associated state $u = S$ is the strong $L^2(\;)$-limit of the states $u_k = S\;_k$, possibly for some subsequence. As such, we can invoke continuity of the norm to show that

$$\inf J = \lim_{k\to\infty} J(\;_k) = \lim_{k\to\infty} \frac{1}{2}\|u_k \;\; u\|^2_{L^2(\;)} = \frac{1}{2}\|u \;\; u\|^2_{L^2(\;)} = J(\;);$$

which proves that the limit point is a solution.

For (i) we note that $= A$ is compact in $L^\infty(\;)$ due to Proposition 4.11. Therefore there exists a subsequence $\{\;_{k'}\}_{k'\in\mathbb{N}}$ that converges to some $\;\in A$ strongly in $L^\infty(\;)$. The corresponding states $u_{k'} = S\;_{k'}$ all satisfy their variational formulations

$$\int\!\!\int \;_{k'} G\, u_{k'} G\, v\, dx'\, dx = \int f v\, dx; \quad \forall v \in U\;; \quad \forall k' \in \mathbb{N}. \tag{4.23}$$

Again, the a priori stability bound from Corollary 3.22 yields boundedness of the subsequence $\{u_{k'}\}_{k'\in\mathbb{N}}$ in U. Therefore, we may extract a further subsequence, not relabelled, which converges weakly to some $u \in U$, since U is a reflexive Banach space. We now wish to prove that $u = S$. To this end we exploit the linearity of the variational formulation and rewrite (4.23) as

$$\begin{aligned}\int f v\, dx = \int\!\!\int \; G\, u G\, v\, dx'\, dx + \int\!\!\int \; G\,(u_{k'} \;\; u) G\, v\, dx'\, dx \\ + \int\!\!\int (\;_{k'} \;\;)G\, u_{k'} G\, v\, dx'\, dx.\end{aligned} \tag{4.24}$$

Let us now study the last two terms on the right-hand side of (4.24) in the limit as $k' \to \infty$. Weak convergence of states in U implies

$$\lim_{k'\to\infty} \int\!\!\int \; G\, u_{k'} G\, v\, dx'\, dx = \int\!\!\int \; G\, u G\, v\, dx'\, dx; \quad \forall v \in U\;;$$

since $G\, v \in L^2(\;)$. On the other hand, the Hölder and Cauchy-Schwarz inequalities implies

$$\left|\int\!\!\int (\;_{k'} \;\;)G\, u_{k'} G\, v\, dx'\, dx\right| \;\; \|\;_{k'} \;\;\|_{L^\infty(\;)} \|u_{k'}\|_U \|v\|_U\;; \quad \forall v \in U\;:$$

Since the states are uniformly bounded, the strong convergence $_{k'} \to$ asserts that taking the limit in (4.24) gives

$$\int \int G\, uG\, v\, \mathrm{d}x' \mathrm{d}x = \int f v\, \mathrm{d}x; \quad \forall v \in U;$$

proving $u = S$. Strong continuity of the conductivity-to-state operator, Proposition (4.10), then implies the strong convergence $u_{k'} \to u$ in U , and by the nonlocal Poincaré inequality, also in $L^2(\ )$.

For (ii), Proposition 4.14 implies A is weakly compact with respect to the weak - topology on $L^1(\ )$. Therefore we may extract a subsequence, without relabelling, for which $_k \overset{*}{\rightharpoonup} \ \in A$. For each conductivity distribution we have corresponding states which we again denote $u = S$, and $u_k = S \ _k$ for all $k \in \mathbb{N}$. We now recall the continuity result of Proposition 4.13, which implies that $u_k \to u$ in $L^2(\ )$ when $_k \overset{*}{\rightharpoonup}$ in A . □

4.2.2 Compliance minimization

We now turn our attention to the compliance minimization problem. Our goal is to minimize the weighted average of the equilibrium temperature state with respect to a given heat source $f \in L^2(\ )$. Hence, we consider

$$\begin{aligned} \min_{u;} \ & J(u; \ ) = \int f u\, \mathrm{d}x; \\ \text{s.t.} \quad & D(\ G\, u) = f \quad \text{in} \ ; \\ & u = 0 \quad \text{in} \ : \end{aligned} \tag{4.25}$$

The functional J is known in the literature as the *compliance*. Recalling the variational formulations of the state equation, the reduced functional of (4.25) takes many forms

$$J(\ ) = \int f S \ \mathrm{d}x = \int \int |G\, S \ |^2 \mathrm{d}x' \mathrm{d}x = 2J(S \ ; \ ); \tag{4.26}$$

where J is the Dirichlet energy functional. In particular, note that the reduced functional is bounded below by zero on A , and that it is a bounded linear functional with respect to the state. Consequently, we may prove the existence of a optimal control to (4.25) following the exact same arguments presented for the identification problem. For completeness, we present the resolution of the reduced problem

$$\min_{\in} J(\ ): \tag{4.27}$$

Theorem 4.16. *Assume either*

(i) $= A$ *for some* < 1, *or*

(ii) $= A$ *and* (4.14) *holds.*

Then there exists a solution to (4.27).

Proof. See the proof of Theorem 4.15. □

Dual formulation

In view of the nonlocal Dirichlet principle, the compliance minimization problem may actually be stated as a saddle-point problem. Indeed inserting (4.26) into (4.27), we see that it is equivalent to

$$\max_{\sigma\in\mathcal{A}}\min_{u\in U} J(u;\sigma). \tag{4.28}$$

This is the nonlocal equivalent of the saddle-point problem introduced in [22]. We will now present the reformulation of (4.28) introduced in [39]. Recall that the nonlocal Kelvin principle is the dual problem to the nonlocal Dirichlet principle, see Theorem 3.34. Their strong duality gave us the equality

$$J(S_\sigma;\sigma) + J^*(q;\sigma) = 0, \tag{4.29}$$

where $q \in \mathcal{Q}(f)$ is the nonlocal flux associated to S_σ, and J^* is the complementary energy functional. Inserting (4.29) into (4.28), we see that it is equivalently stated as a single minimization problem

$$\min_{(\sigma;q)\in\mathcal{A}\times\mathcal{Q}(f)} J^*(q;\sigma). \tag{4.30}$$

Note that we state (4.30) to seek conductivity distributions $\sigma \in \mathcal{A}$, without further regularity assumptions. In the present case, convexity of the complementary energy functional J^*, in both arguments, is sufficient to find a solution.

Theorem 4.17. *There exists a solution* $(\sigma^*;q^*) \in \mathcal{A}\times\mathcal{Q}(f)$ *to* (4.30)*, which equivalently induces a solution* $(\sigma^*;S_{\sigma^*}) \in \mathcal{A}\times U$ *of* (4.28)*. In either case, we obtain an optimal control* $\sigma^* \in \mathcal{U}$ *to* (4.27).

Proof. The functional J^* is proper and bounded below by zero. Hence, we can extract a minimizing sequence $\{\sigma_k;q_k\}_{k\in\mathbb{N}} \in \mathcal{A}\times\mathcal{Q}(f)$ for which the sequence of functional values $J^*(q_k;\sigma_k) \searrow \inf_{\mathcal{A}\times\mathcal{Q}(f)} J^*$ monotonically. Recalling that

$$J^*(q;\sigma) = \frac{1}{2}\int_\Omega\int_\Omega \sigma^{-1}|q|^2\,\mathrm{d}x'\,\mathrm{d}x \geq \frac{1}{2}\overline{\sigma}^{-1}\|q\|^2_{L^2(\Omega\times\Omega)}, \quad \forall(\sigma;q)\in\mathcal{A}\times\mathcal{Q},$$

we see that J^* is coercive in its second argument with respect to $L^2(\Omega\times\Omega)$, for all conductivity distributions in $\mathcal{A}$. Since $\mathcal{D}^*q = f$ for all $q \in \mathcal{Q}(f)$, coercivity in $\mathcal{Q}(f)$ is equivalent to coercivity in $L^2(\Omega\times\Omega)$. Now, boundedness of $\{J^*(q_k;\sigma_k)\}_{k\in\mathbb{N}}$ implies $\{q_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{Q}(f)$. Since $\mathcal{Q}(f)$ is a closed subset of a reflexive Banach space, there exists a weakly convergent subsequence $\{q_{k'}\}_{k'\in\mathbb{N}}$. Using Proposition 4.14, the weak compactness of $\mathcal{A}$ in the weak*-topology implies that we can extract an additional subsequence, not relabelled, for which $\sigma_k \overset{*}{\rightharpoonup} \sigma^* \in \mathcal{A}$. However, we argue that the weak* limit in $L^\infty(\Omega\times\Omega)$ is also a weak limit in $L^2(\Omega\times\Omega)$. To this end let $\varphi \in L^2(\Omega\times\Omega)$, and note that $\varphi \in L^1(\Omega\times\Omega)$ since $\Omega\times\Omega$ is bounded. Hence, the weak* convergence of conductivities gives

$$\lim_{k'\to\infty}\langle\sigma_{k'},\varphi\rangle_{\Omega\times\Omega} = \langle\sigma^*,\varphi\rangle_{\Omega\times\Omega},$$

which proves it is a weak $L^2(\quad)$ limit as well. In total we have a subsequence $\{\ _{k'}; q_{k'}\}_{k'\in\mathbb{N}}$ and some $(\ ; q) \in A \quad Q(f)$ with

$$\ _{k'} \rightharpoonup \quad \text{in } L^2(\quad);$$
$$q_{k'} \rightharpoonup q \quad \text{in } Q(f).$$

We equip $L^2(\quad) \quad Q(f)$ with any appropriate choice of norm such that we can apply Mazur's lemma [18, Corollary 3.8]. Consequently, there exists a sequence of convex combinations described by a function $N : \mathbb{N} \to \mathbb{N}$

$$\{\ (k')_\ell \in [0,1] \mid \ell = k', \ldots, N(k')\}_{k'\in\mathbb{N}}, \quad \text{with} \sum_{\ell=k'}^{N(k')} \ (k')_\ell = 1, \quad \forall k' \in \mathbb{N},$$

for which we get the following strong convergence as $k' \to \infty$

$$\sum_{\ell=k'}^{N(k')} \ (k')_\ell (\ _\ell; q_\ell) \to (\ ; q), \quad \text{in } L^2(\quad) \quad Q(f).$$

Extracting a further subsequence, not relabelled, we have pointwise convergence almost everywhere on . Fatou's lemma will complete the argument. Indeed by nonnegativity and convexity of J in both arguments, we obtain

$$\begin{aligned}
J(q; \ ) &\quad \liminf_{k'\to\infty} J\left(\sum_{l=k'}^{N(k')} \ (k')_l q_l; \sum_{l=k'}^{N(k')} \ (k')_l \ _l\right)\\
&\quad \liminf_{k'\to\infty} \sum_{l=k'}^{N(k')} \ (k')_l J(q_l; \ _l)\\
&\quad \liminf_{k'\to\infty} J(q_{k'}; \ _{k'}) \sum_{l=k'}^{N(k')} \ (k')_l\\
&= \liminf_{k'\to\infty} J(q_{k'}; \ _{k'})\\
&= \inf_{A \quad Q(f)} J.
\end{aligned}$$

Here, the third inequality follows by the monotone convergence of the minimizing sequence. This proves that $(\ ; q)$ is a solution of (4.30). The remaining points follow from the strong duality (4.29). □

Upon examining the current problem formulation, it becomes apparent that the non-uniqueness of solutions can be understood analytically. The functional being minimized is strictly convex with respect to each variable individually, but not jointly. As a result, multiple solutions can arise. Indeed, note that the quadratic term becomes zero as the flux vanishes. When this happens the conductivity distribution may be arbitrarily determined, as it does not change the complimentary energy. However, we will prove that the optimal state, and thus also the optimal flux, is uniquely determined for optimal controls.

Maximum principle

Let us proceed by considering a necessary condition for optimal controls. For the problem at hand, we present a maximum principle, which parallels the maximum principle described in [22, Theorem 3.1] for the local case.

Theorem 4.18. *Let $\in A$ be an optimal control to* (4.27) *and assume $u = S$ denotes its associated state. Then the pair $(\; ; u)$ satisfies the following maximum principle:*

$$\int \int \tilde{} \, |G \; u|^2 \, dx' \, dx \qquad \int \int \quad |G \; u|^2 \, dx' \, dx; \quad \forall \tilde{} \in A \; : \tag{4.31}$$

Proof. Assume $\in A$ is an optimal control with associated state $u = S$, and let $\tilde{} \in A$ be arbitrary. Fix some $\in [0; 1]$ and consider the control $+ \; (\tilde{} \quad)$ and its associated state which we denote $u + w = S(\; + \; (\tilde{} \quad))$. We utilize the variational formulation of states to see that

$$\int f v \, dx = \int \int (\; + \; (\tilde{} \quad)) G \; (u + w) G \; v \, dx' \, dx; \quad \forall v \in U \; :$$

Expanding the right-hand side yields for all $v \in U$

$$\begin{aligned} \int f v \, dx = \int \int & \quad G \; (u + w) G \; v \, dx' \, dx \\ + \int \int & (\tilde{} \quad) G \; (u + w) G \; v \, dx' \, dx: \end{aligned}$$

Explicitly writing the variational formulation of u now gives us for all $v \in U$

$$\begin{aligned} 0 = \int \int & \quad G \; (u + w) G \; v \, dx' \, dx + \int \int (\tilde{} \quad) G \; (u + w) G \; v \, dx' \, dx \\ & \int \int \quad G \; u G \; v \, dx' \, dx: \end{aligned}$$

Testing this with $v = u + w$ yields

$$\begin{aligned} 0 = \int \int & \quad |G \; (u + w)|^2 \, dx' \, dx + \int \int (\tilde{} \quad) |G \; (u + w)|^2 \, dx' \, dx \\ & \int \int \quad G \; u G \; (u + w) \, dx' \, dx: \end{aligned} \tag{4.32}$$

We leave this inequality for now, and now wish to show that

$$\int f w \, dx \quad 0: \tag{4.33}$$

Indeed, since is assumed optimal we have

$$0 \quad J(\; + \; (\tilde{} \quad)) \quad J(\;) = \int f(u + w) \quad f u \, dx = \int f w \, dx:$$

Similarly, the Dirichlet principle gives us the inequality

$$J(u;\ )\quad J(u+w;\ ).$$

Expanding both sides and rearranging we get the inequality

$$\int\int |G\ u|^2\,dx'\,dx \quad \int\int |G\ (u+w)|^2\,dx'\,dx \quad 2\int f w\,dx.$$

Since we have (4.33), we can write

$$\int\int |G\ u|^2 + \int f w\,dx \quad \int\int |G\ (u+w)|^2\,dx'\,dx \quad \int f w\,dx$$
$$\int\int |G\ (u+w)|^2\,dx'\,dx;$$

which we by the variational formulation of u means

$$\int\int G\ u G\ (u+w) \quad \int\int |G\ (u+w)|^2\,dx'\,dx.$$

But for (4.32) to hold, this implies that

$$\int\int (\tilde{\ }\ \ )|G\ (u+w)|^2\,dx'\,dx \quad 0;$$

which rearranged gives

$$\int\int \tilde{\ }|G\ (u+w)|^2\,dx'\,dx \quad \int\int |G\ (u+w)|^2\,dx'\,dx. \tag{4.34}$$

Recalling the strong continuity of the conductivity-to-state operator given in Proposition 4.10, we know that $u+w \to u$ in U as $\searrow 0$. Indeed, there exists some $C > 0$ such that

$$\|u \quad (u+w)\|_U \quad C\|\quad (\quad + (\tilde{\ }\quad))\|_{L^1(\ )}$$
$$C\|\tilde{\ }\quad \|_{L^1(\ )} \to 0.$$

By Proposition 3.18, the functional

$$U \ni v \mapsto \int\int |G\ v|^2\,dx'\,dx$$

is continuous for all conductivities $\in A$. Hence (4.34) also holds in the limit, and we get

$$\int\int \tilde{\ }|G\ u|^2\,dx'\,dx \quad \int\int |G\ u|^2\,dx'\,dx;$$

which is what we wanted. □

The maximum principle (4.31) was utilized in [22] to construct an algorithm for the numerical approximation of optimal controls to (4.27). This was extended to the nonlocal case in [4]. Another use of the maximum principle is to assert the uniqueness of optimal states.

Proposition 4.19. *Let* $\ ,\tilde{\ } \in A$ *and assume they and their equilibrium states* $(u, w) = (S\ , S\tilde{\ })$ *both satisfy the maximum principle* (4.31)*. Then* $u = w$*. Particularly, if both* $\ $ *and* $\tilde{\ }$ *are optimal controls to* (4.27)*, then their states coincide.*

Proof. Let us use the nonlocal Dirichlet principle. We see that

$$\min_{v \in U} J(v; \tilde{\ }) \quad J(u; \tilde{\ }) \quad J(u; \ ) = \min_{v \in U} J(v; \ ) \quad J(w; \ ) \quad J(w; \tilde{\ }), \tag{4.35}$$

where the equality and the second inequality follow from the nonlocal Dirichlet principle and maximum principle for $(\ , u)$, respectively. The last inequality follows from the maximum principle for $(\tilde{\ }, w)$, and its nonlocal Dirichlet principle implies

$$J(w; \tilde{\ }) = \min_{v \in U} J(v; \tilde{\ }).$$

Consequently, everything in (4.35) holds with equality. In particular w satisfies the Dirichlet principle

$$\min_{v \in U} J(v; \ ) = J(w; \ ),$$

which by uniqueness means $u = w$. Since optimal controls satisfy the maximum principle, the last statement follows. □

As an easy corollary to Proposition 4.19, we find that the maximum principle is actually a sufficient optimality condition.

Corollary 4.20. *Let* $\ \in A$ *and let* $u = S\ $ *be its corresponding state. If* $(\ , u)$ *satisfies the maximum principle* (4.31)*, then* $\ $ *is an optimal control to* (4.27)*.*

Proof. We know that (4.27) admits an optimal control $\tilde{\ } \in A$ with associated state $w = S\tilde{\ }$. Applying Proposition 4.19 we find that $u = w$, which implies that

$$J(\ ) = \int f u \, dx = \int f w \, dx = J(\tilde{\ }).$$

This proves that $\ $ minimizes the reduced functional J. □

4.3 Obstacle problems

The last problem we introduce is the nonlocal equivalent of the famous local obstacle problem. Our formulation can be seen as a prototype for a problem that models the equilibrium position of an elastic membrane under the influence of a vertical force, constrained by an obstacle within the domain. First, let's recall the local problem. Given an obstacle function $g : \ \to \mathbb{R}$, and assuming the equilibrium state $u \in H^1_0(\ )$ is subjected to a load $f \in L^2(\ )$, we impose the condition that the equilibrium state lies above the obstacle. This leads to the definition of the set of ***admissible states*** as

$$A(g) = \{v \in H^1_0(\ ) \mid v \quad g, \text{ a.e. in } \quad g. \tag{4.36}$$

Note that without further assumptions, the set of admissible states can be empty. This is the case if the obstacle admits positive values on the boundary $\ $, which are unattainable by states in $H_0^1(\Omega)$. Thus, if an admissible state must both vanish on the boundary and satisfy the unilateral constraint $v \geq g$ in (4.36), then we need $g \leq 0$ on $\partial\Omega$. Assuming $g \in H_0^1(\Omega)$ is a simple solution, since it implies that the obstacle itself is an admissible state. However, such an assumption prohibits obstacles with jump discontinuities, which is symptomatic of local theory. For further discussions on the unilateral constraint and more general obstacles, we refer to [37] for regular obstacles and [6] for a capacity theory-based approach.

The local obstacle problem supposes that the equilibrium state satisfies a minimum energy principle. Hence we wish to find the admissible state which minimizes the Dirichlet energy. We consider

$$\min_{v \in \mathcal{A}(g)} I(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} f v \, dx. \tag{4.37}$$

The nonlocal analog of (4.37) is simply obtained by constraining the nonlocal Dirichlet principle to the corresponding set of nonlocal admissible states

$$\mathcal{A}_\omega(g) = \{v \in U_\omega \mid v \geq g, \text{ a.e. in } \Omega\}.$$

Consequently, we consider

$$\min_{v \in \mathcal{A}_\omega(g)} J(v) = \frac{1}{2}\int_{\Omega_\omega}\int_{\Omega_\omega} |\mathcal{G}_\omega v|^2 \, dx' \, dx - \int_{\Omega} f v \, dx. \tag{4.38}$$

In contrast to the local case, assuming that the obstacle itself is an admissible state, i.e., $g \in U_\omega$, may not be restrictive in the nonlocal setting. The elements of the nonlocal state space U_ω are generally not continuous. We consider two cases as presented in [44]. In the first case, under certain assumptions on the nonlocal kernel, we find that the norm on U_ω is equivalent to the norm on the fractional Sobolev space $W^{s,2}(\Omega)$ for fractional exponents $s < 1/2$. We will simply say that U_ω is equivalent to $W^{s,2}(\Omega)$. In this case, the admissible states may exhibit jump discontinuities, as shown in [48, Lemma 6.1]. In the second case, under different assumptions, we find that U_ω is equivalent to $L^2(\Omega)$. In this case, the admissible states and permitted obstacles can be even more irregular. However, the solution to (4.38) can still be obtained straightforwardly, as demonstrated in the following results.

Lemma 4.21. *Let $g \in U_\omega$ be an obstacle. Then the set of nonlocal admissible states $\mathcal{A}_\omega(g)$ is weakly closed in U_ω.*

Proof. We prove that $\mathcal{A}_\omega(g)$ is closed, and convex in U_ω. To show convexity, let $u, v \in \mathcal{A}_\omega(g)$, and consider the sets of zero measure $N_u, N_v \subset \Omega$ such that $u(x) \geq g(x)$ for all $x \in \Omega \setminus N_u$ and $v(x) \geq g(x)$ for all $x \in \Omega \setminus N_v$. For any $\lambda \in [0, 1]$, we have

$$\lambda u(x) + (1-\lambda)v(x) \geq \lambda g(x) + (1-\lambda)g(x) = g(x), \quad \forall x \in \Omega \setminus (N_u \cup N_v),$$

which implies that $\lambda u + (1-\lambda)v \geq g$ almost everywhere in Ω. Hence, $\mathcal{A}_\omega(g)$ is convex. To show closedness, let u_k be a sequence in $\mathcal{A}_\omega(g)$ that converges strongly to $u \in U_\omega$. By the nonlocal Poincaré inequality, we have $u_k \to u$ in $L^2(\Omega)$. Therefore, there exists a subsequence $u_{k'}$ that converges pointwise on $\Omega \setminus N_1$, where N_1 is a zero measure set in Ω. For each k', there exists further zero measure sets $N_{k'}$ such that $u_{k'}(x) \geq g(x)$ for all $x \in \Omega \setminus N_{k'}$. Hence, we have

$$\lim_{k'} u_{k'}(x) = u(x) \geq g(x); \quad \forall x \in \Omega \setminus \left(\bigcup_{k' \in \mathbb{N}} N_{k'} \cup N_{\infty} \right).$$

Since a countable union of zero measure sets has zero measure, we have shown that $u \geq g$ almost everywhere in Ω, and therefore $u \in A(g)$. Thus, $A(g)$ is closed. In conclusion, $A(g)$ is closed, and convex, which implies that it is weakly closed. □

Theorem 4.22. *Let $g \in U$ be an obstacle. Then there exists a unique nonlocal admissible state $u \in A(g)$ solving the obstacle problem* (4.38).

Proof. The obstacle problem (4.38) is clearly the nonlocal Dirichlet principle constrained to the nonempty, weakly closed set of nonlocal admissible states $A(g)$. Following the direct method as in the proof of Theorem 3.20, we find that there exists a minimizing sequence in $A(g)$ which converges weakly in U. However, since $A(g)$ is weakly closed, the minimizing sequence converges in $A(g)$ and the result follows. Uniqueness of the solution follows from Lemma 3.17, since the Dirichlet energy functional is strictly convex. □

Optimality conditions and projections

Since the nonlocal obstacle problem is a constrained form of the Dirichlet principle, the necessary and sufficient optimality condition is readily found.

Proposition 4.23. *Let $g \in U$ be an obstacle. Then the nonlocal admissible state $u \in A(g)$ solves* (4.38) *if and only if it satisfies the linear variational inequality*

$$\int_\Omega \int_\Omega G u \, G(v - u) \, \mathrm{d}x' \, \mathrm{d}x \geq \int_\Omega f (v - u) \, \mathrm{d}x; \quad \forall v \in A(g). \tag{4.39}$$

Proof. We consider the first variation of the Dirichlet energy functional. If $u \in A(g)$ solves (4.38) then

$$J(u) \leq J(v); \quad \forall v \in A(g). \tag{4.40}$$

Since $A(g)$ is convex, we may insert $u + t(v - u) \in A(g)$ into (4.40) for $t \in [0; 1]$ and $v \in A(g)$. Hence we obtain

$$\frac{1}{2} \int_\Omega \int_\Omega \frac{1}{t} \left(|G u + t G(v - u)|^2 - |G u|^2 \right) \mathrm{d}x' \, \mathrm{d}x - \int_\Omega f (v - u) \, \mathrm{d}x \geq 0. \tag{4.41}$$

Following the proof of Theorem 3.21 we take the limit $t \searrow 0$ and obtain

$$\int_\Omega \int_\Omega G u \, G(v - u) \, \mathrm{d}x' \, \mathrm{d}x - \int_\Omega f (v - u) \, \mathrm{d}x \geq 0; \quad \forall v \in A(g);$$

which is exactly (4.39). This proves the necessity. To obtain sufficiency we consider the usual convexity argument. Indeed, the Dirichlet energy functional is convex, see Lemma 3.17, and hence for $t \in [0; 1]$ and $v \in A(g)$ we have

$$J(u + t(v - u)) \leq J(u) + t \left(J(v) - J(u) \right).$$

Rearranging gives

$$\frac{1}{t}\Big(J(u+t(v-u)) - J(u)\Big) \le J(v) - J(u).$$

Note that the left-hand side is exactly the left-hand side of (4.41). Hence, as $t \searrow 0$ we must have $J(u) \le J(v)$ if (4.39) is assumed to hold. □

The linear variational inequality allows us to characterize the solutions of certain obstacle problems in terms of projections. To this end, we introduce the Hilbert space projection $P_{\mathcal{A}(g)} : \mathcal{U} \to \mathcal{A}(g)$ onto the closed and convex set of admissible states $\mathcal{A}(g) \subset \mathcal{U}$. We recall its definition

$$P_{\mathcal{A}(g)}(v) = \arg\min_{w \in \mathcal{A}(g)} \frac{1}{2}\|v - w\|_{\mathcal{U}}^2, \quad \forall v \in \mathcal{U}. \tag{4.42}$$

We can now state and prove the following formula for solutions.

Corollary 4.24. *Let $g \in \mathcal{U}$ be an obstacle, and assume that $f = 0$. Then the admissible state $u \in \mathcal{A}(g)$ solving the obstacle problem* (4.38) *is the projection*

$$u = P_{\mathcal{A}(g)}(0).$$

Proof. Recall that for $p = 2$, $\mathcal{U}$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{\mathcal{U}} = \int_\Omega \int_\Omega \mathcal{G}u\,\mathcal{G}v \,\mathrm{d}x' \,\mathrm{d}x, \quad \forall u, v \in \mathcal{U}.$$

Since $f = 0$ and $u \in \mathcal{A}(g)$ solves the obstacle problem, we see from (4.39) that

$$\langle u, v - u\rangle_{\mathcal{U}} \ge 0, \quad \forall v \in \mathcal{A}(g). \tag{4.43}$$

Since $u = u - 0$, we have

$$\langle u - 0, v - u\rangle_{\mathcal{U}} \ge 0, \quad \forall v \in \mathcal{A}(g),$$

which is the optimality condition for (4.42) characterizing $u = P_{\mathcal{A}(g)}(0)$. □

Corollary 4.24 disguises the the complexity of the problem into the computation of a projection operator, for which there is no explicit formula in general. Hence, for practical reasons, we consider a reformulation of the nonlocal obstacle problem. To this end, we define for an obstacle $g \in \mathcal{U}$ the set

$$\hat{\mathcal{A}}(g) = \{v \in L^2(\Omega) \mid v \ge g, \text{ a.e. in } \Omega\},$$

where the unilateral constraint considers the $L^2(\Omega)$-representative of the obstacle. To be specific, we denote the continuous embedding $K : \mathcal{U} \to L^2(\Omega)$, and we see that (4.38) is equivalent to

$$\min_{v \in \mathcal{U}} J(v) + I_{\hat{\mathcal{A}}(g)}(Kv), \tag{4.44}$$

where χ_M is the convex indicator function of a convex subset $M \subseteq L^2(\Omega)$ defined by

$$\chi_M(v) = \begin{cases} 0, & \text{if } v \in M, \\ \infty, & \text{else}, \end{cases} \qquad \forall v \in L^2(\Omega).$$

Remark that this reformulation leads us to enforce the unilateral constraint in $L^2(\Omega)$ instead of the state space U_Ω. Such reformulations are often to be found in the local literature, since they may be more tractable for numerical iterative methods. In fact, common iterative methods for the local obstacle problem, such as penalty methods [64], or fixed-point methods [43], are formulated by a sequence of L^2-projections onto the obstacle. In the next chapter, we will see that similar methods are applicable to the nonlocal problem. The following formula will become useful.

Proposition 4.25. *Let $g \in U_\Omega$ be an obstacle. Then the projection of $v \in L^2(\Omega)$ onto $\hat{\mathcal{A}}(g)$ can be represented by*

$$P_{\hat{\mathcal{A}}(g)}(v)(x) = P_{[g(x),\infty)}(v(x)), \quad \text{for a.e. } x \in \Omega. \tag{4.45}$$

Proof. Since the projection is now done in $L^2(\Omega)$, we see that

$$P_{\hat{\mathcal{A}}(g)}(v) = \operatorname*{arg\,min}_{w \in \hat{\mathcal{A}}(g)} \frac{1}{2}\|v - w\|^2_{L^2(\Omega)} = \operatorname*{arg\,min}_{w \in \hat{\mathcal{A}}(g)} \int_\Omega (v - w)^2 \, dx. \tag{4.46}$$

We now realize that minimum can be found by minimizing the integrand of (4.46) pointwise. If we fix a L^2-representative of the obstacle g, we pointwise want

$$w(x) = \operatorname*{arg\,min}_{w \in [g(x),\infty)} (v(x) - w)^2, \quad \forall x \in \Omega.$$

This proves the formula $w(x) = P_{[g(x),\infty)}(v(x))$. To ensure that such a function w resides in $\hat{\mathcal{A}}(g)$, we need to show that $w \in L^2(\Omega)$. This is however immediate by writing the pointwise projection explicitly. We find

$$w(x) = v(x) - \min\{v(x) - g(x), 0\}, \quad \forall x \in \Omega,$$

and consequently, by the triangle inequality, we have

$$\|w\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} + \|v - g\|_{L^2(\Omega)} \leq 2\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} < \infty,$$

which finishes the proof. □

Chapter 5

Numerical approximation

In the upcoming chapter, we will delve into the numerical approximation techniques for solving the variational problems discussed in this thesis. Most importantly, we will consider the variational problem governing the nonlocal diffusion law. Again, we only consider the case of linear diffusion, since its nonlocal Dirichlet principle provide a variational formulation suitable for the Galerkin method. In particular, we consider approximations using the finite element method (FEM). FEM has been successfully applied to local PDE problems for decades, and a mature mathematical theory has been developed to analyze its performance [14, 16, 16]. On the other hand, the numerical approximation of solutions to nonlocal problems is still in its infancy. Fortunately, as we have seen, elements of the PDE methodology extend to the nonlocal case, and this is also the case for FEM approximations. We refer to the discussions in [7, 30, 33]. First, this chapter discusses the derivation of a finite element method for approximating nonlocal equilibrium states. In particular, we prove convergence and error estimates, and we explore aspects of the implementation. Lastly, we perform a series of numerical experiments. We will consider both the numerical approximation of nonlocal equilibrium states, and the numerical solutions to variational problems posed in Chapter 4.

5.1 Nonlocal FEM

Let us first recall the problem at hand. Our goal is to numerically approximate the equilibrium temperature state satisfying a nonlocal diffusion law. For a given volumetric heat source $f \in L^2(\Omega)$, the nonlocal Dirichlet principle states that the equilibrium temperature state satisfies a minimum energy principle, which equivalently is the unique solution to the linear variation problem

$$\text{Find } u \in U : \int_\Omega \int_\Omega \mathcal{G} u \mathcal{G} v \, \mathrm{d}x' \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in U. \tag{5.1}$$

Recalling the bilinear form and linear functional

$$a(u, v) = \int_\Omega \int_\Omega \mathcal{G} u \mathcal{G} v \, \mathrm{d}x' \, \mathrm{d}x \quad \forall u, v \in U,$$
$$\ell(v) = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in U,$$

we see that (5.1) may be written as

$$\text{Find } u \in U_\delta : \quad a(u, v) = \ell(v), \quad \forall v \in U_\delta . \tag{5.2}$$

In the numerical search for an approximate solution to (5.2), we will apply the Galerkin method. We briefly introduce it for the sake of completeness. To this end we assume that $\{U_{\delta,h}\}_{h>0}$ is a sequence of finite-dimensional subspaces of U_δ and we consider the *Galerkin scheme*

$$\text{Find } u_h \in U_{\delta,h} : \quad a(u_h, v_h) = \ell(v_h), \quad \forall v_h \in U_{\delta,h}. \tag{5.3}$$

For each $h > 0$ the problem in (5.3) is a finite-dimensional variant of the one in (5.2), and the corresponding solution $u_h \in U_{\delta,h}$ is an approximation of the actual solution. It is important to mention that the solution $u_h \in U_{\delta,h}$ exists, since the unique existence theory is inherited from (5.2), due to the finite dimensionality of $U_{\delta,h}$. Remark that $U_{\delta,h}$ admits a finite basis, and thus (5.3) is merely a linear system of equations, which we call the *Galerkin system*. Herein lies the cleverness of the Galerkin method. It is clear that the approximate solution u_h can be different from the original solution u. However, its quality as an approximation entirely depends on the choice of the subspace $U_{\delta,h}$. We will return to this point later. Thus, the Galerkin method boils down to systematically choosing good approximation spaces $\{U_{\delta,h}\}_{h>0}$, so that the corresponding Galerkin scheme admits increasingly better solution approximations as $h \searrow 0$. The role of $h > 0$ will soon be apparent.

5.1.1 FEM discretization

We choose our approximation spaces using the finite element method. For practical reasons we limit our discussion to the two-dimensional case of polygonal domains. Specifically, we consider the unit square $\Omega = (0, 1)^2$. FEM requires us to construct increasingly finer meshes of the corresponding interaction domain Ω_δ, which is clearly not polygonal. To circumvent this difficulty, we extend the nonlocal boundary Γ_δ such as in Figure 5.1a. We denote the closure of the extended polygonal domain by $\overline{\Omega}_{\delta,h}$. We can now uniformly decompose $\overline{\Omega}_{\delta,h}$ into regular quadrilateral elements E with no elements crossing the boundary of Ω. See Figure 5.1b. Thus

$$\overline{\Omega}_{\delta,h} = \cup_{E \in \mathcal{T}_h} E,$$

where $\mathcal{T}_h$ denotes the triangulation of the mesh, and $h > 0$ denotes the diagonal length of the elements.

On each element of the mesh, we will approximate the solution using polynomials. Specifically, we will utilize the first-order quadrilateral Lagrangian basis $\mathbb{Q}_1$. Recall that each element of U_δ implicitly vanishes outside of Ω. Therefore, we identify the mesh vertices $\{x_{h,i}\}_{i=1}^N$ in the interior of Ω as the nodes (or degrees of freedom) of our Lagrangian basis $\{\phi_{h,i}\}_{i=1}^N$. The Lagrangian basis is defined by restricting each basis function to be an element-wise bilinear polynomial satisfying the nodal relation

$$\phi_{h,i}(x_{h,j}) = \delta_{ij}, \quad \forall i, j = 1, \ldots, N.$$

Here δ_{ij} denotes the Kronecker delta. Note that in the present case, each basis function is supported in exactly four neighbouring elements. Their corresponding

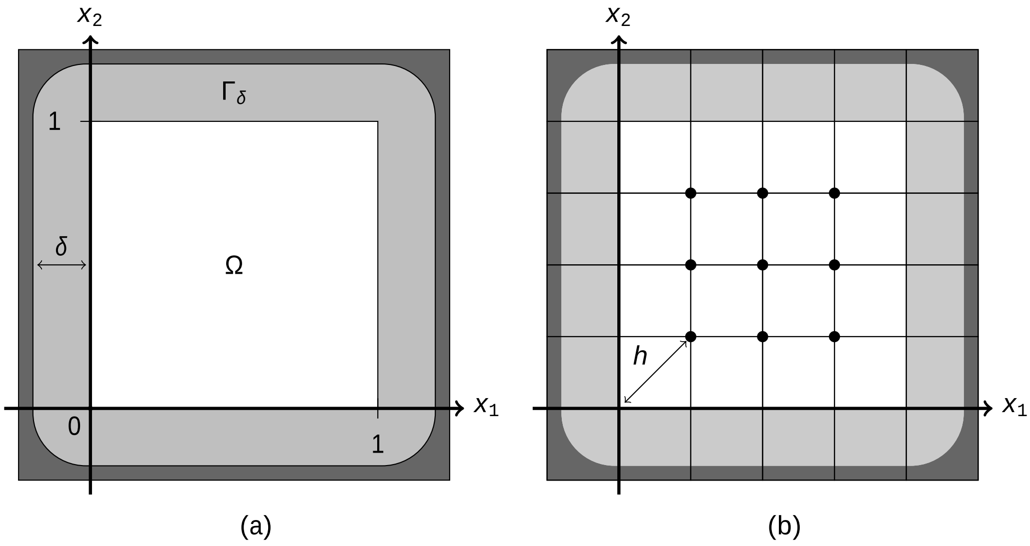


Figure 5.1: The mesh generation process for the domain $\Omega = (0,1)^2$. The nonlocal boundary is extended so that a regular quadrilateral mesh can exactly cover the interaction domain, with no elements straddling across the boundary. The interior vertices are marked.

finite element approximation space is simply defined as their span

$$U_{\delta,h} = \mathrm{span}\{\phi_{h,i}\}_{i=1}^{N},$$

and hence every $u_h \in U_{\delta,h}$ is a continuous element-wise bilinear polynomial with basis expansion

$$u_h(x) = \sum_{i=1}^{N} u_h(\xi_{h,i})\,\phi_{h,i}(x), \quad \forall x \in \Omega.$$

Inserting the basis expansion into (5.3) and iterating over the Lagrangian basis, we obtain the Galerkin system

$$A_h U_h = F_h, \tag{5.4}$$

defined by

$$\begin{aligned} U_h &= [u_h(\xi_{h,i})]_{i=1}^{N} \in \mathbb{R}^N, \quad F_h = [\ell(\phi_{h,i})]_{i=1}^{N} \in \mathbb{R}^N, \quad \text{and} \\ A_h &= [A_{h,i,j}]_{i,j=1}^{N} = [a(\phi_{h,i}, \phi_{h,j})]_{i,j=1}^{N} \in \mathbb{R}^{N\times N}. \end{aligned} \tag{5.5}$$

We refer to F_h as the *source vector* and A_h as the *stiffness matrix*.

5.1.2 Convergence and error estimates

Next, we will analyze the convergence of our Galerkin scheme and establish that the analysis for the nonlocal problem is analogous to that of the corresponding local problem. To this end, we start by recalling a fundamental estimate for the Galerkin method. First, we note that our choice of finite elements is conformal, since we have the following string of inclusions

$$U_{\delta;h} \subset H_0^1(\Omega) \subset U_\delta .$$

Additionally, we remember that the coercivity of the functional

$$U_\delta \ni v \mapsto \int_{\Omega} \int_{\Omega_\delta} |\mathcal{G}_\delta v|^2 \,\mathrm{d}x' \,\mathrm{d}x,$$

was established in Lemma 3.19. This implies the coercivity of the bilinear form $a : U_\delta \times U_\delta \to \mathbb{R}$, i.e. there exists a constant $\alpha > 0$, such that

$$a(u; u; \delta) \geq \alpha \|u\|_{U_\delta}^2 , \quad \forall u \in U_\delta .$$

In addition, we can use the Cauchy-Schwarz and nonlocal Poincaré inequalities to find a $C > 0$ such that the bilinear form a is continuous

$$|a(u; v; \delta)| \leq C \|u\|_{U_\delta} \|v\|_{U_\delta} , \quad \forall u, v \in U_\delta .$$

Both of these properties allows one to derive the following a priori error estimate.

Lemma 5.1 (Céa's lemma). *Let $h > 0$ and assume that $u \in U_\delta$ solves* (5.2) *and $u_h \in U_{\delta;h}$ solves* (5.4)*. Then we have the a priori estimate*

$$\|u - u_h\|_{U_\delta} \leq \frac{C}{\alpha} \min_{v_h \in U_{\delta;h}} \|u - v_h\|_{U_\delta} .$$

Proof. This is the statement of Céa's lemma, see [17, Theorem 2.8.1]. □

Note that both constants are independent of the diagonal length $h > 0$. Hence, Céa's lemma states that the error between the solution and our approximation depends on the *approximation error*

$$\min_{v_h \in U_{\delta;h}} \|u - v_h\|_{U_\delta} .$$

In turn the approximation error depends entirely on how well our finite element space can approximate the original solution. Therefore, we need to investigate the approximation properties of our finite elements. Unfortunately, we run into a problem. The state space U_δ is abstract by definition, and therefore it is a tedious task to analyze the general approximability of functions in U_δ using finite elements. Therefore, we limit our analysis to equilibrium solutions with some regularity.

Lemma 5.2. *Assume $u \in H_0^1(\Omega)$. Then there exists some $C > 0$ such that*

$$\min_{v_h \in U_{\delta;h}} \|u - v_h\|_{U_\delta} \leq C \min_{v_h \in U_{\delta;h}} \|u - v_h\|_{H^1(\Omega)} .$$

Proof. The proof is straightforward. We know from Proposition 3.60 that $H_0^1(\Omega)$ is continuously embedded in U_δ. Hence there exists some $C > 0$ for which

$$\|v\|_{U_\delta} \leq C \|v\|_{H^1(\Omega)} , \quad \forall v \in H_0^1(\Omega). \tag{5.6}$$

Since $U_{\delta;h} \subset H_0^1(\Omega)$, the result follows by inserting $v = u - v_h$ into (5.6) and taking the minimum over all $v_h \in U_{\delta;h}$. □

Lemma 5.2 reduces the convergence analysis for regular equilibrium states to a question of finite element approximations of Sobolev functions. Therefore, we invoke a classical result of Sobolev interpolation theory. To this end, we define the *interpolation operator* $I_h : H^2(\Omega) \to U_{\Omega,h}$ as

$$I_h u(x) = \sum_{i=1}^{N} u(\xi_{h,i})\,\phi_{h,i}(x), \quad x \in \Omega.$$

Note that I_h takes an element of $u \in H^2(\Omega)$ and maps it to the element-wise bilinear polynomial that corresponds with its nodal values $\{u(\xi_{h,i})\}_{i=1}^N$. The properties of such interpolations are well-understood for our triangulations $\mathcal{T}_h$.

Proposition 5.3. *Assume that $u \in U_\Omega$ solving* (5.2) *lies in $H_0^1(\Omega) \cap H^2(\Omega)$. Then there exists some $C > 0$ independent of $h > 0$, for which we have the estimate*

$$\|u - I_h u\|_{H^l(\Omega)} \leq C h^{2-l} |u|_{H^2(\Omega)}, \quad \forall l \in \{0, 1\}.$$

Proof. We omit the proof, but note that it is based upon the famous Bramble-Hilbert lemma. We refer to either [16, Chapter 2] or [17, Chapter 4] for the details. □

We can now establish the following a priori error estimate and convergence result.

Theorem 5.4. *Assuming that equilibrium state $u \in U_\Omega$ is in $H_0^1(\Omega) \cap H^2(\Omega)$. Then there exists some $C > 0$ independent of $h > 0$, such that*

$$\|u - u_h\|_U \leq C h |u|_{H^2(\Omega)}. \tag{5.7}$$

In particular, the approximations $\{u_h\}_{h \searrow 0}$ converge strongly to u in U_Ω as $h \searrow 0$.

Proof. We can apply Lemma 5.2 together with Céa's lemma (Lemma 5.1) to see that

$$\|u - u_h\|_U \leq \frac{C}{\alpha} \min_{v_h \in U_{\Omega,h}} \|u - v_h\|_U \leq \frac{C_1 C}{\alpha} \min_{v_h \in U_{\Omega,h}} \|u - v_h\|_{H^1(\Omega)}, \tag{5.8}$$

for some $C_1 > 0$ independent of $h > 0$. The right-hand side of (5.8) can be overestimated by the interpolation $v_h = I_h u \in U_{\Omega,h}$ and thus Proposition 5.3 with $l = 1$ gives us a constant $C_2 > 0$ independent of $h > 0$ with

$$\|u - u_h\|_U \leq \frac{C_1 C}{\alpha} \|u - I_h u\|_{H^1(\Omega)} \leq \frac{C_1 C_2 C}{\alpha} h |u|_{H^2(\Omega)}.$$

This proves the estimate (5.7), which in turn allows us to conclude that $u_h \to u$ in U_Ω as $h \searrow 0$. □

Theorem 5.4 leaves much to be desired, as it only proves a convergence rate of order $O(h)$. It is however immediate that similar arguments hold in the case of higher-order Lagrangian elements. Hence, by assuming further regularity of the equilibrium state, we obtain the following result.

Corollary 5.5. *Let $m \geq 1$ be an integer and assume $U_{\Omega,h}$ arises from m-th order quadrilateral Lagrangian basis $\mathcal{Q}_m$ on the triangulation $\mathcal{T}_h$. Assume also that the equilibrium state $u \in U_\Omega$ is in $H_0^1(\Omega) \cap H^{m+1}(\Omega)$. Then there exists some $C > 0$ independent of $h > 0$, such that*

$$\|u - u_h\|_U \leq C h^m |u|_{H^{m+1}(\Omega)}.$$

Proof. Under the present assumptions there exists some $C > 0$ independent of $h > 0$, for which

$$\|u - I_h u\|_{H^l(\Omega)} \leq C h^{m+1-l} |u|_{H^{m+1}(\Omega)}, \quad \forall l = 0, \ldots, m, \tag{5.9}$$

see e.g. [17, Theorem 4.4.20]. The proof now follows that given for Theorem 5.4 by inserting $l = 1$ in (5.9). □

It is important to note that these results were obtained without further assumptions on the state space U^{δ} and the nonlocal kernel. As we have seen throughout the thesis, there exists many different types of kernel assumptions, each extending the applicability of the nonlocal model in different ways. Let us reconsider the case in which U^{δ} is equivalent to $L^2(\Omega)$. As noted in [33], this assumption allows us to approach the approximation error in $L^2(\Omega)$ since

$$\min_{v_h \in U^{\delta,h}} \|u - v_h\|_{U^{\delta}} \leq C \min_{v_h \in U^{\delta,h}} \|u - v_h\|_{L^2(\Omega)},$$

for some constant $C > 0$.

Corollary 5.6. *Assume U^{δ} is equivalent to $L^2(\Omega)$. Following the assumptions of Corollary 5.5 there exists some $C > 0$ independent of $h > 0$, such that*

$$\|u - u_h\|_{U^{\delta}} \leq C h^{m+1} |u|_{H^{m+1}(\Omega)}.$$

Proof. Since we only need to approximate in $L^2(\Omega)$, we can take $l = 0$ in (5.9). □

Note that in the present case with bilinear elements, i.e. $m = 1$, the equivalence assumption gives us the error estimate

$$\|u - u_h\|_{U^{\delta}} \leq C h^2 |u|_{H^2(\Omega)}. \tag{5.10}$$

Hence, the error of order $O(h^2)$ stands in contrast to the $O(h)$ result of Theorem 5.4. Remark further that when U^{δ} is equivalent to $L^2(\Omega)$, we see that the discontinuous finite elements are conformal. Similarly holds if U^{δ} is equivalent to a fractional Sobolev space $W^{s,p}(\Omega)$ with $s < 1/2$. This was investigated in [70].

5.1.3 Implementation

We now proceed to detail the numerical implementation of the finite element approximation. Before we can solve the Galerkin system introduced in (5.4), we first need to *assemble* it. By this we mean that the entries in the source vector F_h and stiffness matrix A_h defined in (5.5) must be evaluated. Analytical evaluation is, of course, out of the question. So we have to make do with their numerical approximations by means of quadrature rules. The procedure for approximating the source vector is identical to the local case and will therefore not be discussed. On the other hand, the entries of the stiffness matrix consist of an additional integral layer compared to the local case. Its approximation will be discussed in detail.

Elemental stiffness matrices

Instead of computing the entries of A_h entry-by-entry, we use the usual method of element-wise computation. This is possible because each entry of A_h can be decomposed as follows. For any triangulation $\mathcal{T}_h$ of $\Omega_{\delta,h}$, and for any pair of indices $i,j = 1,\dots,N$, we can write

$$A_{h;i,j} = \int_{\Omega} \int_{\Omega} G_\delta \phi_{h,i} G_\delta \phi_{h,j} \, \mathrm{d}x' \, \mathrm{d}x = \sum_{E_k \in \mathcal{T}_h} \sum_{E_l \in \mathcal{T}_h} A(k,l)_{h;i,j}. \tag{5.11}$$

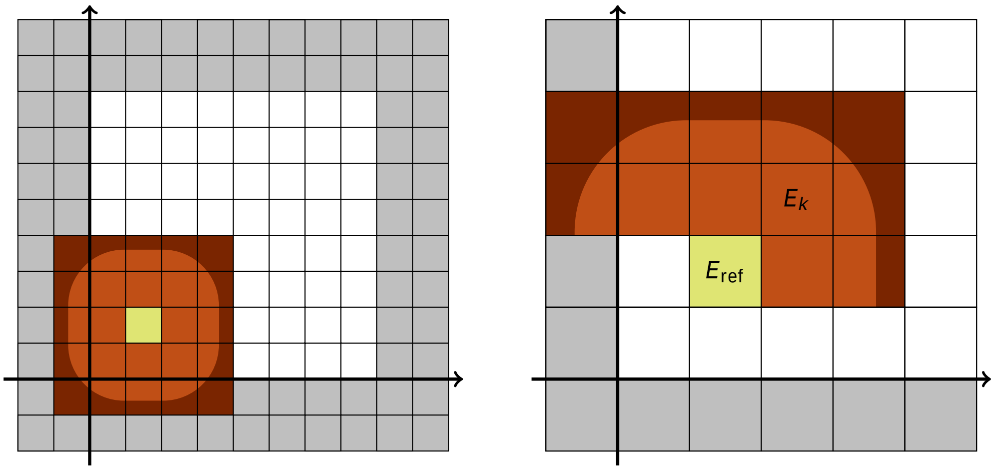
Here, for all pairs of elements $(E_k, E_l) \in \mathcal{T}_h \times \mathcal{T}_h$, we define their ***elemental stiffness matrix*** $A(k,l)_h$ with entries

$$A(k,l)_{h;i,j} = \int_{E_k} \int_{E_l} G_\delta \phi_{h,i} G_\delta \phi_{h,j} \, \mathrm{d}x' \, \mathrm{d}x, \quad \forall i,j = 1,\dots,N.$$

Note that even though the triangulation covers the discretized domain $\Omega_{\delta,h}$, the equality in (5.11) holds because the nonlocal gradients are supported in Ω_δ. Therefore, the previous enlarging of the interaction domain is not an issue. The strategy now is to compute all relevant elemental stiffness matrices, and add their contributions to the *global* stiffness matrix A_h. The number of elements in the mesh scales with $O(h^{-2})$ assuming we drive $h \searrow 0$. The total number of pairs is therefore of order $O(h^{-4})$. However, the interaction radius enforced by the nonlocal kernel limits the number of element pairs that are relevant. In fact, for a fixed element $E_k \in \mathcal{T}_h$ we only have to consider element pairs $(E_k, E_l) \in \mathcal{T}_h \times \mathcal{T}_h$ for which they interact $(E_l + B_\delta) \cap E_k \neq \emptyset$. This is depicted in Figure 5.2a. The number of pairwise interactions per element scales with $O(\delta^2 h^{-2})$. Hence the total number of relevant pairs is of scale $O(\delta^2 h^{-4})$. We can reduce the computational burden by exploiting the symmetry of interactions. Indeed fixing an element pair $(E_k, E_l) \in \mathcal{T}_h \times \mathcal{T}_h$ we may apply Fubini's theorem to exchange the order of integration

$$\begin{aligned}
A(k,l)_{h;i,j} &= \int_{E_k} \int_{E_l} \rho_\delta(x,x') G_\delta \phi_{h,i}(x,x') G_\delta \phi_{h,j}(x,x') \, \mathrm{d}x' \, \mathrm{d}x \\
&= (-1)^2 \int_{E_l} \int_{E_k} \rho_\delta(x',x) G_\delta \phi_{h,i}(x',x) G_\delta \phi_{h,j}(x',x) \, \mathrm{d}x \, \mathrm{d}x' \\
&= \int_{E_l} \int_{E_k} \rho_\delta(x,x') G_\delta \phi_{h,i}(x,x') G_\delta \phi_{h,j}(x,x') \, \mathrm{d}x' \, \mathrm{d}x \\
&= A(l,k)_{h;i,j}.
\end{aligned}$$

Here the second equality follows by Fubini's theorem and the anti-symmetry of the nonlocal gradients, while the third equality is obtained by renaming the variables. The symmetry of element interactions cuts the number of pairwise interactions in half, since each interacting element pair (E_k, E_l) has identical contributions as its mirrored pair (E_l, E_k). The biggest reduction in computation comes from our regular triangulation of $\Omega_{\delta,h}$. All element interactions can be found by computing the pairwise interactions between a suitable reference element $E_{\mathrm{ref}} \in \mathcal{T}_h$ and its neighbors. In essence, the relative mesh positions of a interacting pair (E_{ref}, E_k) fully characterizes which values are present in their elemental stiffness matrix $A(\mathrm{ref}, k)_h$. The computation of the elemental matrices corresponding to the reference element will be referred



(a) (b)

Figure 5.2: Mesh elements interact due to the nonlocality. For a fixed reference element (in yellow), the number of interactions depends on both the element diagonal length h and nonlocal horizon . For the stiffness assembly, only half of the interacting elements need to be considered due to element symmetry.

to as the ***preassembly***. Figure 5.2b illustrates the selection of a reference element and the interactions necessary for the preassembly. Preassembly reduces the number of integrals to $O(\ {}^2h\ {}^2)$, but the computational effort of assembling the global matrix still scales with $O(\ {}^2h\ {}^4)$ since each element has to be considered.

Numerical quadrature

As we mentioned earlier, the computations must be carried out by numerical quadrature. This must be done with care, so that the assembled system does not deviate too far from the Galerkin system (5.4). The local FEM literature contains results formulating sufficient quadrature rules so that the assembled system is well-posed [23]. The nonlocal FEM theory is not yet as mature.

In our case, the selection of suitable quadrature rules is challenging due to the presence of double integrals with general kernels. To gain a better understanding of the underlying difficulty, let us explicitly write out the contribution of an interacting pair $(E_k;E_l)\ 2\ T_h\ \ \ T_h$. For a fixed pair of indices $i;j = 1;:::;N$, the entry of the elemental stiffness matrix is

$$A(k;l)_{i;j} = \int_{E_k}\int_{E_l} \ (x;x^0)\,!\ (x\quad x^0)^2 \\ (\ _{h;i}(x)\quad\ _{h;i}(x^0))(\ _{h;j}(x)\quad\ _{h;j}(x^0))\,dx^0\,dx: \tag{5.12}$$

Since the conductivity distribution is assumed positive and symmetric, it's properties are akin to those of the nonlocal kernel. Therefore, for the current discussion, we absorb it into $!$, i.e. we let $\quad 1$. Since the Lagrangian basis is continuous and piecewise polynomial, it is clear that the choice of quadrature rule is dependent on the nonlocal kernel. Different kernels present different difficulties. A good example is the choice

$$\omega_\delta(x) = \frac{c_\delta}{|x|^{1+s}} \chi_{B_\delta}(x), \quad \forall x \in \mathbb{R}^2,$$

for some $s \in (0,1)$ and suitably chosen normalization factor $c_\delta > 0$. Inserting this kernel into (5.12) gives us

$$A(k,l)_{i,j} = c_\delta^2 \int_{E_k} \int_{E_l} \chi_{B_\delta}(x - x') \frac{(\phi_{h,i}(x) - \phi_{h,i}(x'))(\phi_{h,j}(x) - \phi_{h,j}(x'))}{|x - x'|^{2+2s}} \, dx' \, dx. \tag{5.13}$$

Approximating (5.13) well is hard for two reasons. First, the integrand is singular along $x = x'$, and therefore specialized quadrature rules have to be considered. We refer to the discussion in [27]. Second, the integrand has a jump discontinuity if the element E_k does not lie entirely within the interaction region for E_l, i.e. if $E_k \not\subset E_l + B_\delta$. This is the case for the outermost interacting elements in Figure 5.2b. To alleviate this, one can perform the inner integration on $E_l \cap B_\delta(x)$ using specialized quadrature rules as in [30, 63]. Alternatively, one could consider a mollification of the nonlocal kernel to remove the discontinuity [7]. Both of these approaches are beyond the scope of this thesis, but along the lines of the latter, we limit ourselves to the implementation of a continuous cutoff kernel

$$\omega_\delta(x) = c_\delta(\delta - |x|)\chi_{B_\delta}(x), \quad \forall x \in \mathbb{R}^d, \quad \forall \delta, \delta > 0. \tag{5.14}$$

Its choice implies a continuous integrand, for which classical Gauss-Legendre (GL) rules are applicable.

5.2 Numerical experiments

We now begin the numerical experiments. First we verify our implementation and test the nonlocal to local convergence of the approximate equilibrium states. We then explore the numerical approximations of solutions to selected nonlocal problems from Chapter 4. First, we apply the FEM state approximations to numerically solve the optimal distributed control problem. Nonlocal optimal controls are found as solutions to the associated optimality system, which we solve using a descent method. Finally, we consider the numerical solutions of nonlocal obstacle problems. For this purpose, we utilize a fixed-point scheme which alternates between projections and solutions of nonlocal diffusion problems.

5.2.1 State approximations

In our subsequent numerical experiments, we will utilize the implemented method to conduct a series of tests. Initially, we will perform two preliminary experiments to ascertain the most suitable quadrature strategy. Subsequently, we will investigate the convergence of our numerical implementation as the mesh size $h \searrow 0$ approaches zero, comparing the results with the error estimate established in Section 5.1.2. Lastly, we will examine the capability of our implementation to capture the convergence behavior as the nonlocal interaction horizon $\delta \searrow 0$, leading to the limiting local problem.

We assume a homogeneous conductivity distribution $\kappa \equiv 1$, and we fix the nonlocal kernel from (5.14) with constants $\alpha = 1.5$ and $\beta = 2$. Example 3.2 finds the corresponding normalization constant

$$c^2 = \frac{40}{\pi\delta^{10}}.$$

We employ the method of manufactured solutions and fix the analytic solution

$$u_{\text{ana}}(x_1, x_2) = \begin{cases} x_1^2(1-x_1)^2 x_2^2(1-x_2)^2, & \text{if } (x_1, x_2) \in \Omega, \\ 0 & \text{else.} \end{cases} \tag{5.15}$$

Note that $u_{\text{ana}} \in H_0^1(\Omega)$, and it is therefore also in U. In addition, one may argue by continuity of u_{ana} and the cut-off kernel ω that $\mathcal{L}u_{\text{ana}} \in L^2(\Omega)$. Consequently, we can compute the nonlocally corresponding analytic heat source $-\mathcal{L}u_{\text{ana}} = f_{\text{ana}}$ by the formula (3.19). We find

$$\mathcal{L}u_{\text{ana}}(x) = 2\int \omega^2(x - x')[u_{\text{ana}}(x') - u_{\text{ana}}(x)]\,\mathrm{d}x, \quad \forall x \in \Omega. \tag{5.16}$$

We transform the integral in (5.16) to polar coordinates and evaluate it numerically using the adaptive quadrature module in `SciPy` [68]. Similarly to the stiffness matrix, we assemble the corresponding force vector using GL quadrature. Our implementation suffers from large memory requirements due to the complexity of the assembly process. The largest problem we will consider consists of a mesh of size $h = \sqrt{2}/320$ with an interaction horizon of $\delta = 0.2$. Its stiffness matrix requires a total of 77.15 GB of memory to store naively. Therefore we store it in compressed sparse row (CSR) format and solve the assembled system with the black-box Ruge-Stüben algebraic multigrid method. We utilize the implementation found in `PyAMG` [11].

Quadrature strategy

A good quadrature strategy needs to be developed. The number of integrals to evaluate in preassembly grows as $O(\delta^2 h^{-2})$, so very precise quadrature evaluations can become slow. Similarly, the assembly of the force vector requires the evaluation of $O(h^{-2})$ integrals. Even worse is the evaluation of f_{ana}, which itself is an approximation of an integral. A two-dimensional n-point rule will require $O(nh^{-2})$ evaluations during assembly. Therefore it is important to strike a balance between quadrature complexity and precision.

We will first explore how the number of quadrature points affects the accuracy of the integral approximations. We measure this based on the relative error between fixed baseline estimates obtained with $42^4 = 3\,111\,696$ quadrature points for the preassembly and $6^2 = 36$ points for the source assembly. The error is calculated by taking the Euclidean norm over all entries in the preassembled elemental matrices and the assembled force vector, respectively. The results are seen in Figures 5.3-5.4.

We observe that the relative errors generally decrease as the number of quadrature points increases. However, there is a noticeable difference between the preassembly and force assembly. For $\delta = 0.2$ the preassembly errors decay smoothly for all mesh

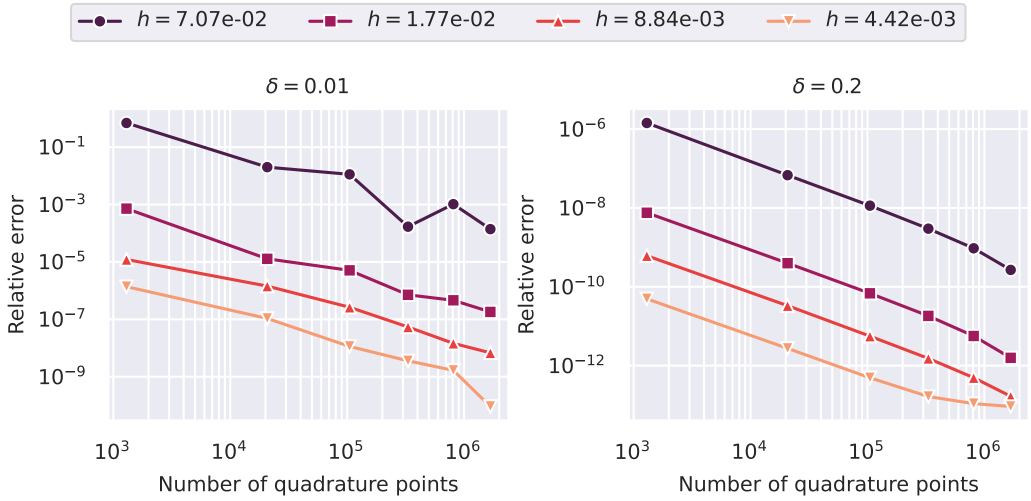


Figure 5.3: Accuracy of the numerical quadrature during preassembly.

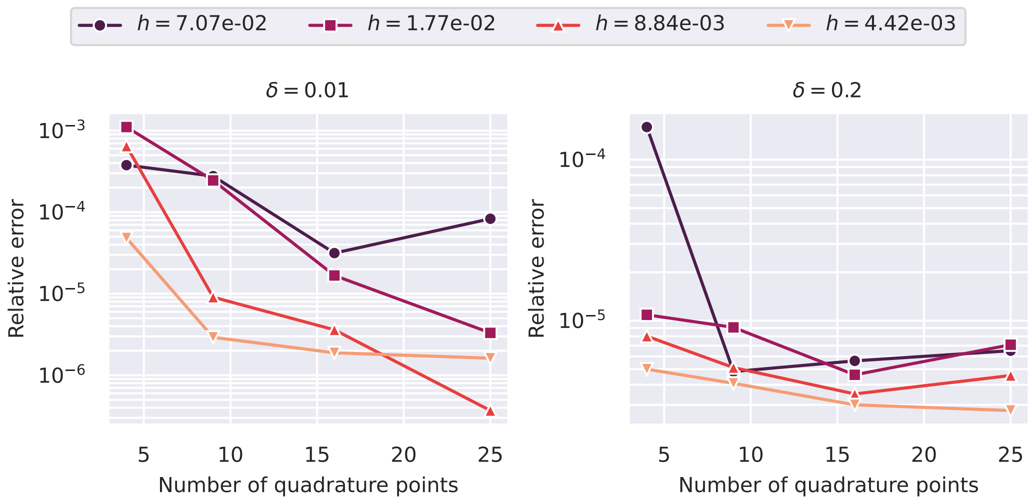


Figure 5.4: Accuracy of the numerical quadrature during force assembly.

sizes, and in particular we see that the finest mesh converges. For $\delta = 0.01$ we see a similar pattern. Similarly holds for the source assembly with the smallest horizon, meanwhile the errors are seemingly independent of quadrature rule when $\delta = 0.2$. Note that the magnitude of the relative errors depends on both the mesh size and the horizon. Looking back at Figure 5.2 it is clear that the approximation of the element interaction depends on how well the elemental interaction domain is approximated by mesh elements. As the mesh size decreases, the approximation becomes better.

As expected, more quadrature points lead to better estimates. However, it is unclear how much better the solution of the assembled system becomes as the number of quadrature points increases. We test this by solving the approximated Galerkin systems for all possible combinations of previous quadrature rules. We report the relative $L^2(\Omega)$ norm errors between the approximated solution and the analytical state u_{ana}. We display the results in Table 5.1.

	h	Relative error
$0{:}01$	$7{:}07 \quad 10^{\ 1}$	$[0{:}18; 2{:}46]$
$0{:}01$	$1{:}76 \quad 10^{\ 1}$	$[5{:}17; 5{:}26] \quad 10^{\ 3}$
$0{:}2$	$7{:}07 \quad 10^{\ 1}$	$[1{:}55; 1{:}57] \quad 10^{\ 1}$
$0{:}2$	$1{:}76 \quad 10^{\ 1}$	$[9{:}59; 9{:}62] \quad 10^{\ 5}$

Table 5.1: Table of intervals containing relative $L^2(\ \ )$ norm errors. For each given and h, the interval contains the solution errors obtained with every quadrature strategy.

For $\ = 0{:}2$ we see that both intervals are small, and that it does not seem to matter which quadrature combination is used. Similarly holds for the finest mesh when $\ = 0{:}01$. However, for the coarsest mesh, the interval spans two orders of magnitude, implying a significant difference in quality of rule combinations. If we exclude combinations with the least complex force assembly then we instead obtain $[1{:}84; 2{:}17] \quad 10^{\ 1}$. Hence we generally observe that the solution errors differ only slightly between quadrature rules. However, they vary by several orders of magnitude between mesh sizes.

Convergence as h & 0

From the previous experiments, we learn that the relative solution error depends mainly on the mesh fineness. Therefore, we set a reasonable $24^4 = 331\,776$ point rule for pre-assembly and a $4^2 = 16$ point rule for force assembly. We now solve the assembled system for different mesh refinements and interaction horizons. The relative solution errors are reported in Figure 5.5.

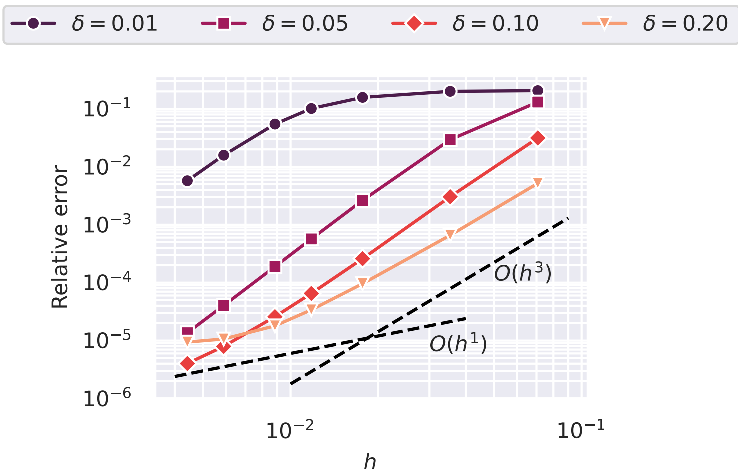


Figure 5.5: Convergence of the approximations u_h with respect to mesh refinement. The dashed lines guide the order of convergence.

For all interaction horizons we observe convergent behavior. However, for the smallest horizon $\ = 0{:}01$ it appears that the solution error does not decrease until the mesh becomes sufficiently fine. Alternatively, for the largest horizon $\ = 0{:}2$, we initially see that the convergence rate is of order $O(h^3)$, but at the last two refinements falls below $O(h)$. It is difficult to gauge the quality of our implementation, since further mesh refinement is not possible because the implementations memory requirements. A stagnant convergence rate is expected, since the expected optimal convergence rate is of order $O(h^2)$. Indeed, the continuity of our cutoff kernel implies that the state space U_δ is equivalent to $L^2(\Omega)$, which in turn validates the error estimates of Corollary 5.6. Finally, we note that the relative error is generally larger for smaller values of δ. This is again expected due to the decreasing approximability of vanishing nonlocal interactions for fixed meshes.

Convergence as $\delta \searrow 0$

For the next experiment, our aim is to numerically verify the localizing property of nonlocal equilibrium states. To this end, we keep the analytical state u_{ana} from (5.15) fixed and now consider its locally corresponding analytic heat source $f_{ana} = -\Delta u_{ana}$. For $(x_1, x_2) \in \Omega$ we find the formula

$$f_{ana}(x_1, x_2) = -2\big((6(x_1 - 1)x_1 + 1)(x_2 - 1)^2 x_2^2 + (6(x_2 - 1)x_2 + 1)(x_1 - 1)^2 x_1^2\big).$$

Our goal now is to investigate how well the state approximations converge towards u_{ana} as the interaction horizon δ approaches zero. To do this, we maintain the previous quadrature strategy and solve the assembled systems for various mesh refinements and interaction horizons. We analyze the relative $L^2(\Omega)$ norm errors between the obtained numerical nonlocal states and the analytical state, and present the results in Figure 5.6.

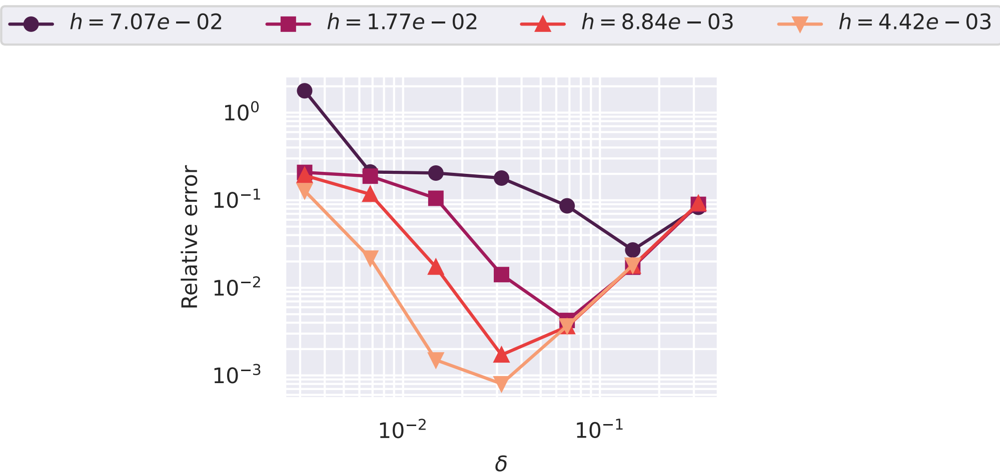


Figure 5.6: Local convergence of the approximations u_h with respect to $\delta \searrow 0$.

It is worth noting that none of the discretizations exhibit convergence as $\delta \searrow 0$. We generally observe that the relative errors initially decrease up to a certain point, after which they begin to increase. The specific point at which this behavior changes depends on the mesh refinement. As the mesh becomes finer, the approximation of the local solution improves. This is not surprising since the nonlocal interactions are poorly approximated for small $\delta > 0$ when the mesh is fixed.

5.2.2 Distributed control

Let us now consider the numerical approximation of optimal controls to the nonlocal distributed control problem. Recall the problem. Given a target equilibrium state $\bar{u} \in L^2(\Omega)$ we seek the optimal volumetric heat source $f \in L^2(\Omega)$ which for some $\lambda \geq 0$ minimizes

$$\min_{f \in L^2(\Omega)} J_\delta(f) = \frac{1}{2}\|S_\delta f - \bar{u}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|f\|^2_{L^2(\Omega)}. \tag{5.17}$$

Here we recall that the control-to-state operator $S_\delta : L^2(\Omega) \to U_\delta$ takes a source $f \in L^2(\Omega)$ and maps it to its corresponding temperature state $u \in U_\delta$ satisfying the nonlocal diffusion law

$$\begin{aligned} \mathcal{L}_\delta u &= f, && \text{in } \Omega, \\ u &= 0, && \text{in } \Omega_I. \end{aligned}$$

As a means to solve (5.17), we employ the optimize-then-discretize methodology. Our wish is to find an optimal control $f \in L^2(\Omega)$ which satisfies the first order optimality condition

$$\nabla J_\delta(f) = S_\delta^*(S_\delta f - \bar{u}) + \lambda f = 0. \tag{5.18}$$

Considering the adjoint state $w = S_\delta^* f - \bar{u} \in U_\delta$, the optimal pair (f, w) satisfies the optimality system

$$\begin{aligned} \mathcal{L}_\delta S_\delta f &= -\lambda^{-1} w, && \text{in } \Omega, & \qquad \mathcal{L}_\delta w &= S_\delta f - \bar{u}, && \text{in } \Omega, \\ S_\delta f &= 0, && \text{in } \Omega_I, & w &= 0, && \text{in } \Omega_I. \end{aligned} \tag{5.19}$$

We seek this pair, by applying a descent method to (5.17). Since we have a formula for the $L^2(\Omega)$ gradient ∇J_δ we can utilize Hilbert space quasi-Newton methods. We initialize a control $f_0 \in L^2(\Omega)$ and consider updates

$$f_{k+1} = f_k + t_k d_k,$$

with step lengths $t_k \in \mathbb{R}$ chosen iteratively for $k \in \mathbb{N}$ such that

$$t_k = \arg\min_{t \in \mathbb{R}} J_\delta(f_k + t d_k), \tag{5.20}$$

and with step directions $d_k \in L^2(\Omega)$ as solutions to

$$B_k d_k = -\nabla J_\delta(f_k).$$

Here the linear operator $B_k : L^2(\Omega) \to L^2(\Omega)$ is initially defined as the $L^2(\Omega)$ identity $B_0 = \mathrm{Id}$ and then iteratively defined by the Hilbert space BFGS updates

$$\begin{aligned} s_k &= f_{k+1} - f_k, \\ z_k &= \nabla J_\delta(f_{k+1}) - \nabla J_\delta(f_k), \\ B_{k+1} g &= B_k g - \frac{\langle B_k s_k, g\rangle}{\langle B_k s_k, s_k\rangle} B_k s_k + \frac{\langle z_k, g\rangle}{\langle z_k, s_k\rangle} z_k, \quad \forall g \in L^2(\Omega); \end{aligned}$$

We refer to [26] for further details. Note that every iterative step requires us to solve three nonlocal equations. First the state equation $u_k = S_\delta f_k$, then the adjoint equation $w_k = S_\delta^*(u_k - \bar{u})$, and finally the step state $S_\delta d_k$ is needed to solve (5.20).

Experiments

Our experiments consider two different target states. For $(x_1, x_2) \in \Omega$ we define

$$u_{\Omega,1}(x_1, x_2) = |\sin(2\pi x_1)| \sin(\pi x_2),$$
$$u_{\Omega,2}(x_1, x_2) = \begin{cases} +x_1(1-x_1)x_2(1-x_2), & \text{if } x_1 < 0.5, \\ -x_1(1-x_1)x_2(1-x_2), & \text{else.} \end{cases}$$

It is important to note that both targets in our study are intentionally designed to be nonsmooth along $x_1 = 0.5$. The first target, denoted as $u_{\Omega,1}$, has a discontinuous derivative, while the second target, denoted as $u_{\Omega,2}$, features a jump discontinuity. See figure 5.7. The choice of these targets serves as an exploration into whether nonlocal optimal states can exhibit more irregularities compared to their local counterparts.

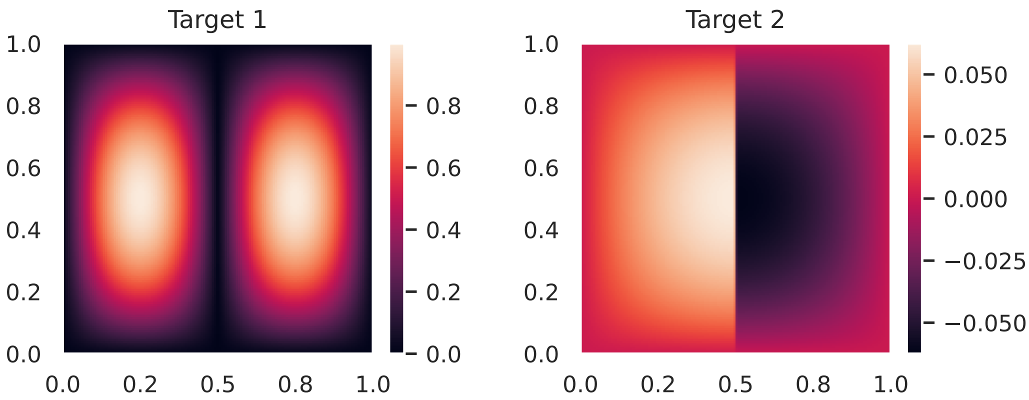


Figure 5.7: Plots of the target states. Target 1 corresponds to the target state $u_{\Omega,1}$, and Target 2 corresponds to the target state $u_{\Omega,2}$.

To conduct our experiments, we set $\alpha = 10^{-8}$ and $h = \sqrt{2}/80$, and we initialize the control as $f_0 = 0$. We consider two different interaction horizons, $\delta = 0.05$ and $\delta = 0.2$, and iterate until the optimality condition (5.18) is satisfied within a tolerance of $\varepsilon = 10^{-8}$, measured in the $L^2(\Omega)$ norm. The resulting full optimal controls and their associated states are presented in Figures 5.9-5.10, while their cross-sections along $x_2 = 0.5$ are shown in Figure 5.8. Additionally, Table 5.2 provides information on the norm of the optimal controls and the relative error between the optimal states and their corresponding targets.

For the first target, we observe that the optimal controls vary in magnitude. Interestingly, the optimal control for $\delta = 0.05$ appears to be closer to the local optimal control compared to the optimal control for $\delta = 0.2$. This observation is supported by their $L^2(\Omega)$ norms. Note that the optimal control for $\delta = 0.2$ has the smallest norm. Additionally, the corresponding states for both cases appear to be similar. However, upon closer examination of their zoomed-in cross-sections, we notice different behaviors at $x_1 = 0.5$. Specifically, for $\delta = 0.2$, the optimal state overlaps with the target state, whereas for $\delta = 0.05$, the optimal state moves towards the local state. Consequently, the optimal state for $\delta = 0.2$ exhibits the least relative error from the target.

	Target	Control norm	Relative error
Local	$u_{;1}$	4.86×10^{1}	7.58×10^{-3}
Local	$u_{;2}$	1.12×10^{1}	1.56×10^{-1}
0.01	$u_{;1}$	4.87×10^{1}	4.58×10^{-3}
0.01	$u_{;2}$	1.14×10^{1}	1.36×10^{-1}
0.2	$u_{;1}$	3.22×10^{1}	1.65×10^{-4}
0.2	$u_{;2}$	5.53	7.79×10^{-4}

Table 5.2: Table of $L^2(\ )$ norm values. For each and target, we report the norm of the optimal control, and the relative norm error between its target and state.

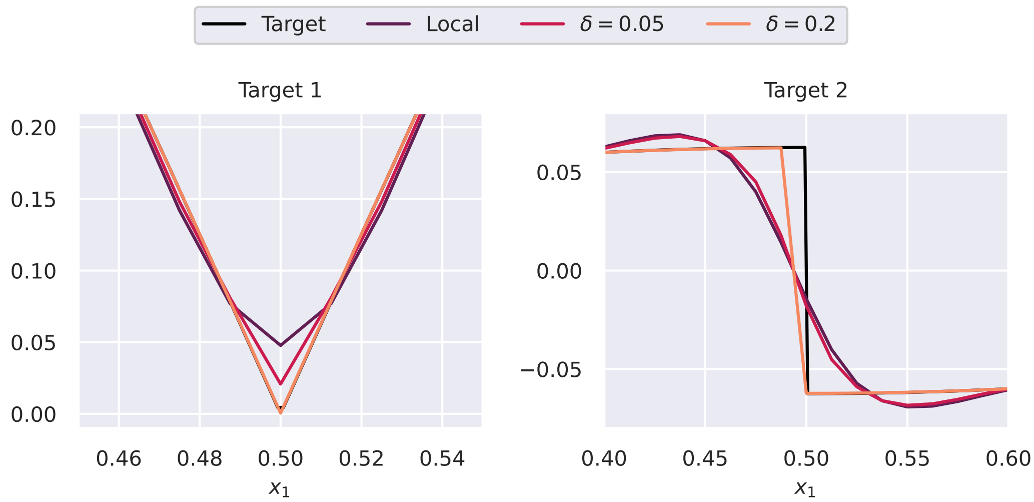


Figure 5.8: Cross sections at $x_2 = 0.5$ of the numerically obtained optimal states and their targets. The state for $\delta = 0.2$ overlaps the first target.

These observations are further supported by the results obtained for the second target. Once again, we observe variations in the magnitudes of the optimal controls, and similarly, the optimal control for $\delta = 0.05$ appears to be closer to the local optimal control compared to the optimal control for $\delta = 0.2$. The $L^2(\ )$ norms also reflect this trend, with the optimal control for $\delta = 0.2$ having the smallest norm. However, in this case, we note that the corresponding states are significantly different. The target state exhibits a discontinuity that is only bridged by a single element for $\delta = 0.2$. On the other hand, the optimal state for $\delta = 0.05$ closely resembles the smoother local optimal state. As before, this disparity is evident from their relative errors.

Overall, our observations suggest that as δ approaches zero, the nonlocal optimal controls move towards the local optimal control. This observation supports the notion that the localizing property of nonlocal equilibrium states extends to nonlocal optimal controls. It is worth mentioning that this convergence was proven true for distributed control problems in [28]. Moreover, it has also been studied for problems of optimal control in the conductivity [5, 38, 39].

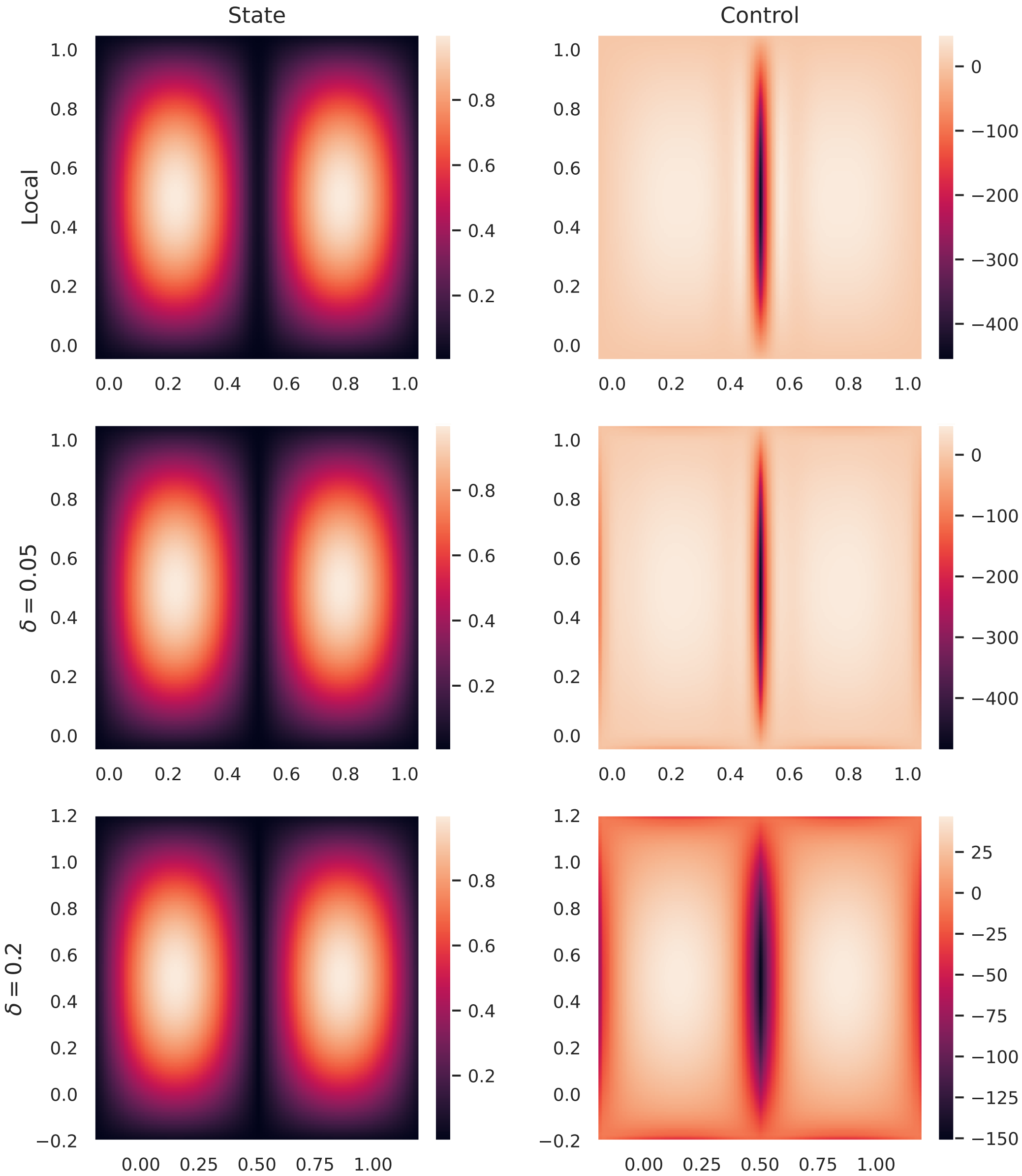


Figure 5.9: First target: Plots of the numerically obtained optimal controls and corresponding states. Note the difference in color scales.

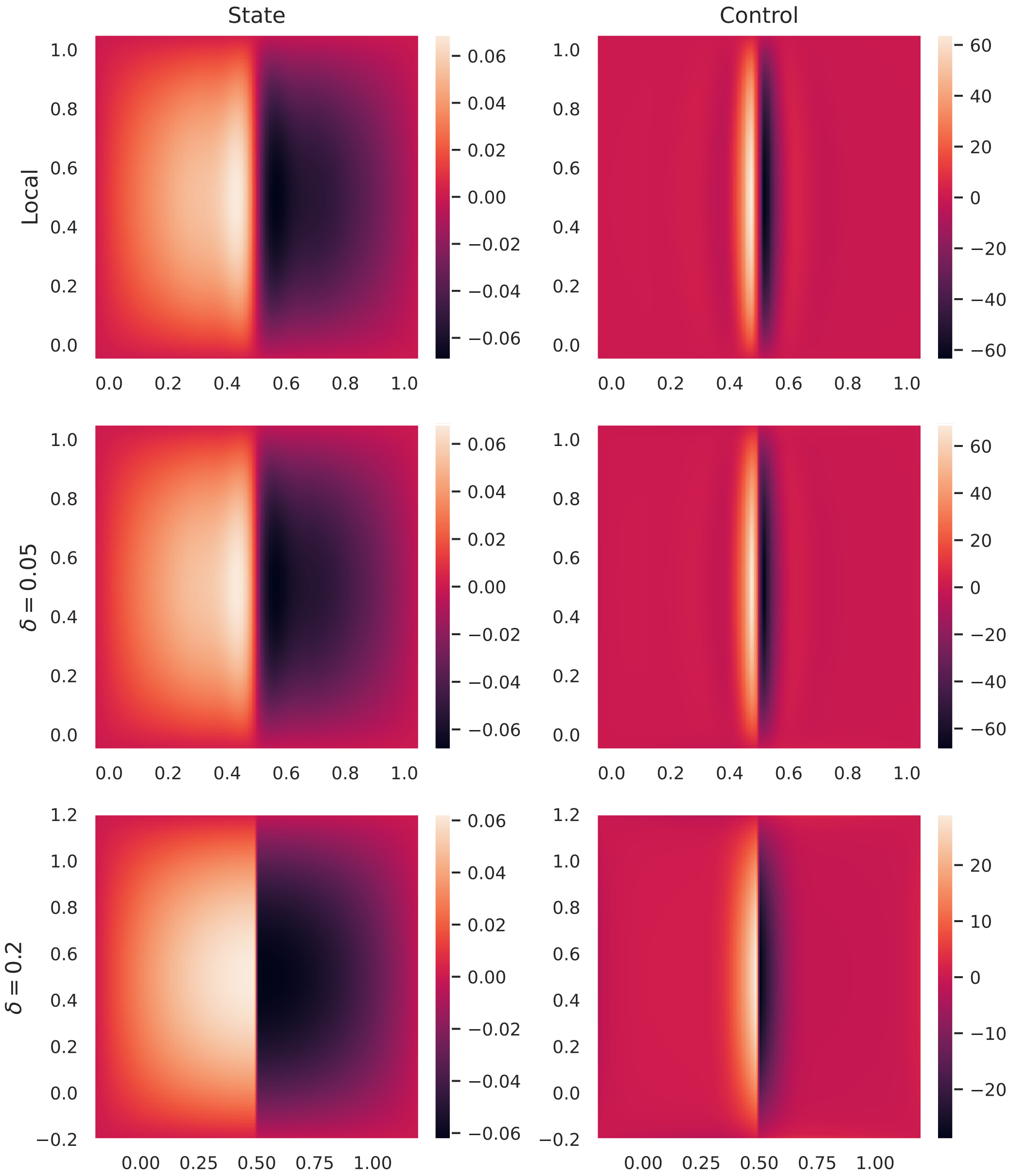


Figure 5.10: Second target: Plots of the numerically obtained optimal controls and corresponding states. Note the difference in color scales.

5.2.3 Obstacle problems

Finally, we turn our attention to the numerical approximation of solutions to nonlocal obstacle problems. We start by reintroducing the embedding $K : U \to L^2(\Omega)$ and the reformulated nonlocal obstacle problem

$$\min_{v \in U} J(v) + \iota_{\hat{A}(g)}(Kv). \tag{5.21}$$

Here, J represents the nonlocal Dirichlet energy with a fixed loading term $f \in L^2(\Omega)$, and the set of admissible states is defined for a given obstacle $g \in U$ as

$$\hat{A}(g) = \{v \in L^2(\Omega) \mid v \geq g, \text{ a.e. in } \Omega\}.$$

We reformulate the problem via an auxiliary variable $y = Kv \in L^2(\Omega)$ for $v \in U$, and rewrite (5.21) as the constrained optimization problem

$$\begin{aligned} \min_{(v,y) \in U \times L^2(\Omega)} \quad & J(v) + \iota_{\hat{A}(g)}(y), \\ \text{s.t.} \quad & Kv = y. \end{aligned} \tag{5.22}$$

The problem in (5.22) is of a form which lends itself to a variety of first-order optimization methods. In our case, we employ a proximal splitting method known as ADLPMM, which is a variant of the well-known ADMM algorithm. We spare the algorithmic details, and refer to [10] for its formulation. After some routine derivations and by introducing two more auxiliary variables, namely $\lambda_k \in L^2(\Omega)$ and $w_k \in U$, the algorithm reduces to a fixed-point scheme $\{(v_k, y_k)\}_{k \in \mathbb{N}}$ which, given a parameter $\rho > 0$, is defined by

$$\begin{aligned} v_{k+1} &= S_\Omega\left(\frac{1}{1+\rho} f - \frac{\rho}{1+\rho} \mathcal{L}_\Omega\left(v_k - \frac{1}{L} w_k\right)\right), \\ y_{k+1} &= P_{\hat{A}(g)}\left(K v_{k+1} + \frac{1}{\rho}\lambda_k\right), \\ \lambda_{k+1} &= \lambda_k + \rho(K v_{k+1} - y_{k+1}), \\ w_{k+1} &= S_\Omega(K v_{k+1} - y_{k+1} + \lambda_{k+1}). \end{aligned}$$

Here $S_\Omega : L^2(\Omega) \to U$ is the control-to-state operator, $\mathcal{L}_\Omega : U \to L^2(\Omega)$ is the nonlocal 2-Laplacian, and $P_{\hat{A}(g)} : L^2(\Omega) \to L^2(\Omega)$ is the projection onto $\hat{A}(g)$. Note that each iteration requires us to perform one projection, and solve for two nonlocal equilibrium states.

Experiments

In the context of obstacle problems, we consider two obstacles, g_1 and g_2, defined for $(x_1, x_2) \in \Omega$ by the following formulas

$$\begin{aligned} g_1(x_1, x_2) &= \left(\sqrt{2}/2 - \sqrt{|x_1 - 0.5|}\right)\left(\sqrt{2}/2 - \sqrt{|x_2 - 0.5|}\right), \\ g_2(x_1, x_2) &= \sqrt{0.2 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2}\, \chi_{B_{0.2}}((x_1 - 0.5, x_2 - 0.5)). \end{aligned}$$

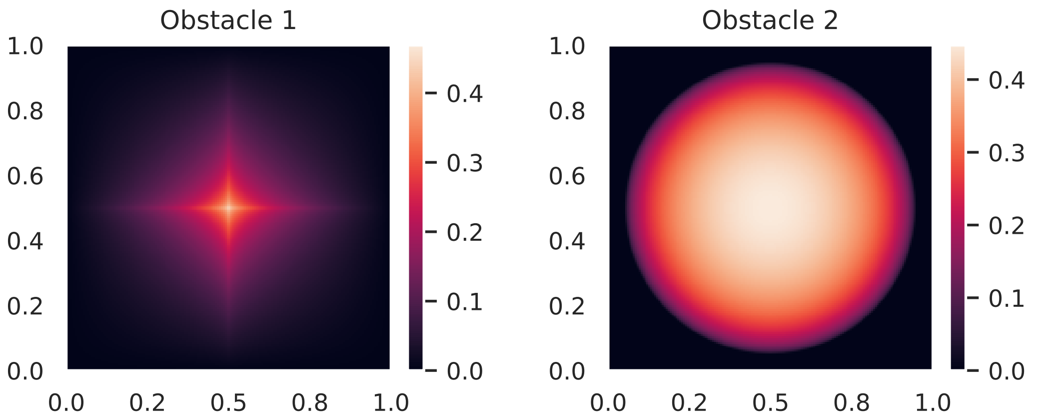


Figure 5.11: Plots of the obstacles. Obstacle 1 corresponds to the obstacle g_1, and Obstacle 2 corresponds to the obstacle g_2.

We refer to their plots in Figure 5.11. Note that the first obstacle, g_1, exhibits a cusp-like behavior along $x_1 = x_2 = 0.5$, while the second obstacle, g_2, features a jump discontinuity.

We fix the loading term as $f = 0$ and set the algorithm parameter to $c = 10$. Again, we consider the mesh with $h = \sqrt{2}/80$, and we proceed to compare the local solutions with the nonlocal solutions using interaction horizons $\delta = 0.05$ and $\delta = 0.2$. We iteratively apply the fixed-point scheme until the auxiliary variables (λ_k, w_k) converge. Specifically, we enforce $\lambda_{k+1} - \lambda_k$ within a tolerance of $\varepsilon = 10^{-8}$, measured in the $L^2(\Omega)$ norm. The solutions are displayed in Figure 5.12, while their corresponding contact sets are shown in Figure 5.13.

Observing the solutions, we note that the solution for $\delta = 0.2$ closely resembles the first obstacle, accurately capturing the sharp cusps along $x_1 = x_2 = 0.5$. On the other hand, the solution for $\delta = 0.05$ lies between the solution for $\delta = 0.2$ and the smoother local solution. This similarity is also evident in their respective contact sets. For $\delta = 0.05$, the contact set is constrained to $x_1 = x_2 = 0.5$, similar to the local contact set. In contrast, the contact set for $\delta = 0.2$ covers a much larger portion of the domain.

Regarding the second obstacle, it is challenging to make qualitative distinctions between their solutions as they appear nearly identical. However, upon considering their contact sets, we observe a contrast between the set obtained for $\delta = 0.2$ and the set for $\delta = 0.05$. For $\delta = 0.05$, the contact set exhibits a rounded square shape, while the contact set for $\delta = 0.2$ is rounder and better resembles the support of the obstacle. Once again, the contact set obtained for $\delta = 0.05$ aligns more closely with the local contact set.

Similar to the nonlocal distributed control case, our numerical results suggest that the localizing property extends to solutions of nonlocal obstacle problems. This conjecture was initially proposed in [44] and holds true for obstacles $g \in H^1_0(\Omega)$, following the arguments presented for equilibrium states in Section 3.5.

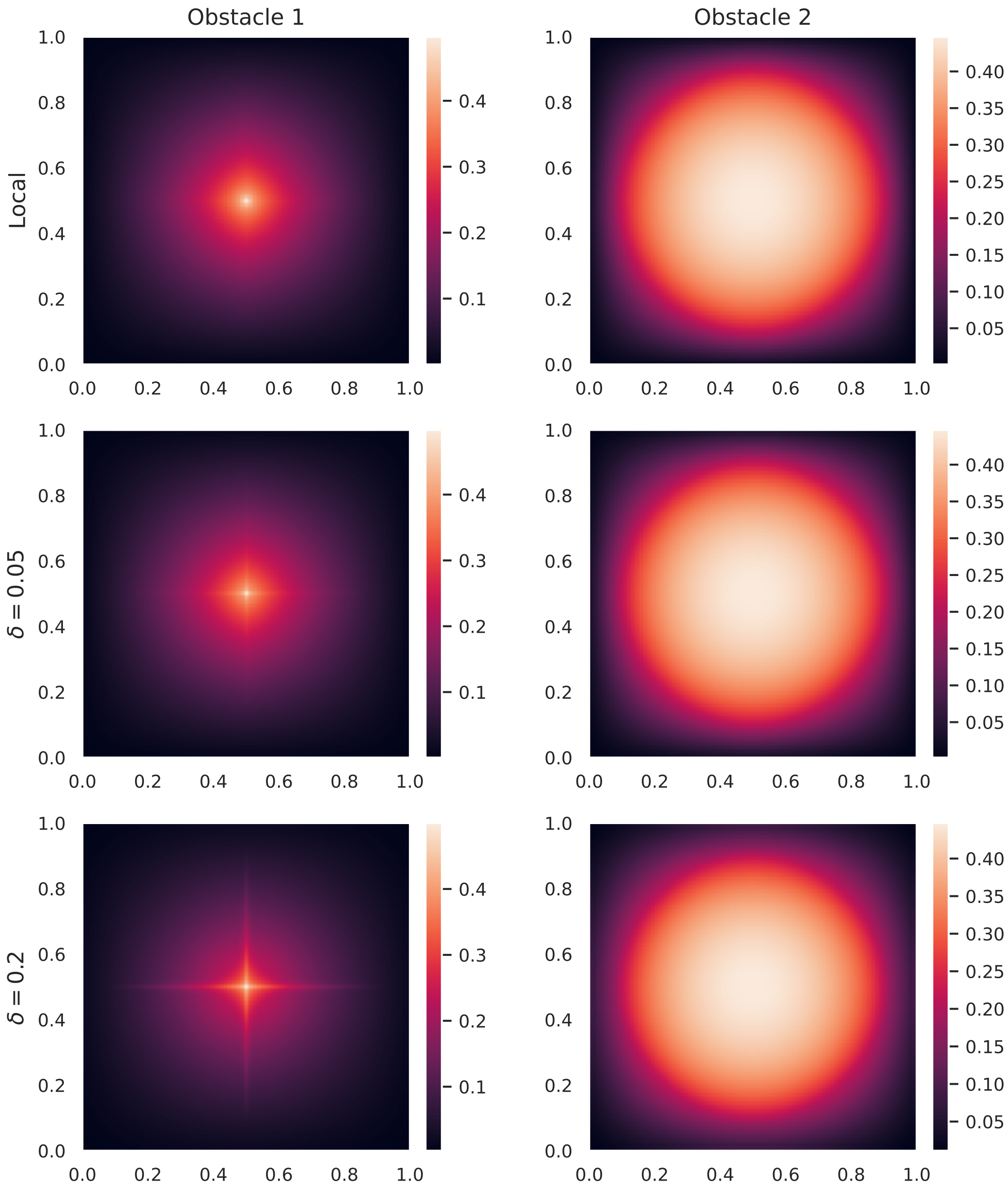


Figure 5.12: Obstacle problem solutions. Plots of the numerically obtained solutions for both obstacles.

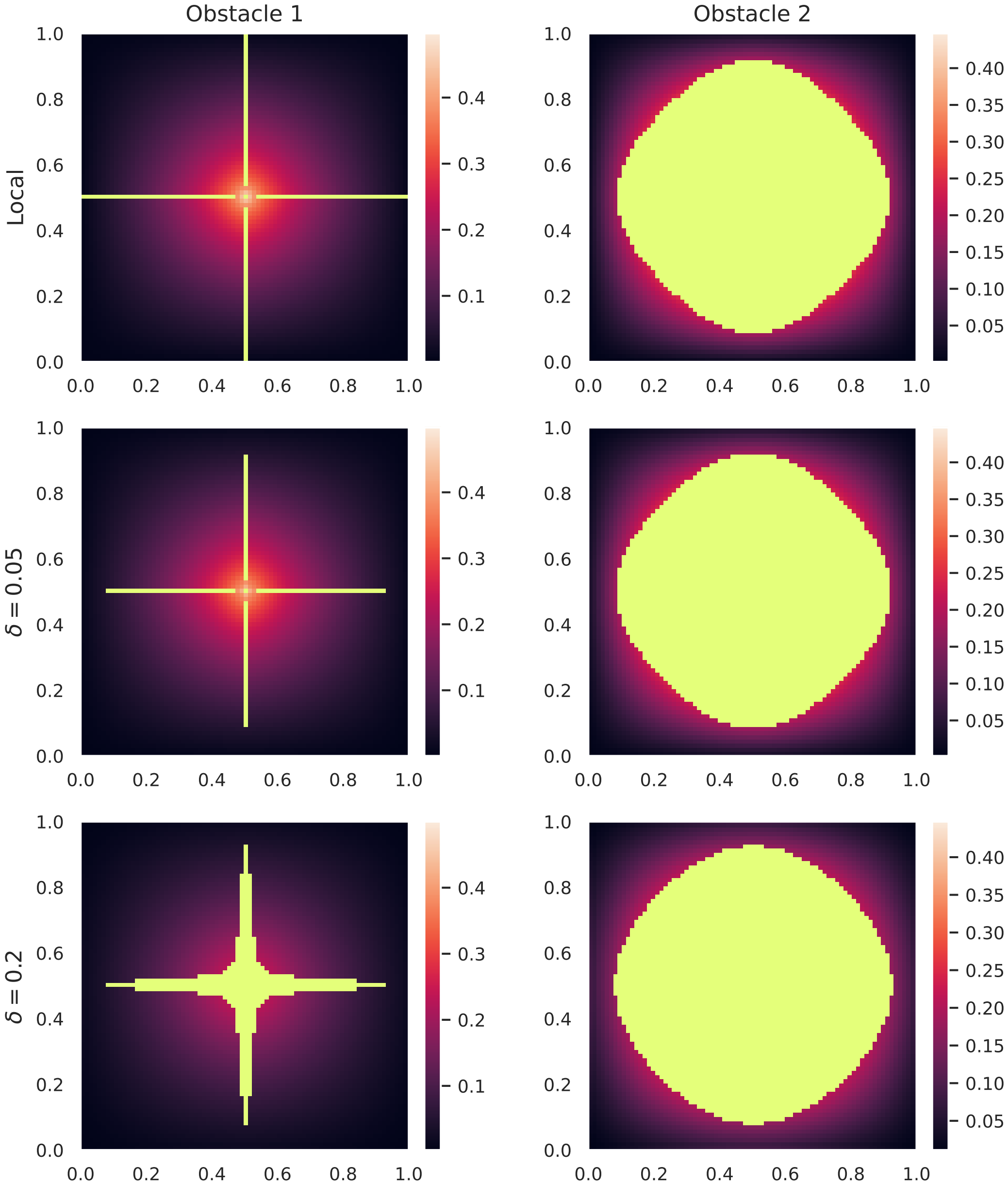


Figure 5.13: Obstacle contact sets. Given a solution u over an obstacle g, we plot the numerical contact set $\{x \in \;\; | \; |u(x) - g(x)| < 10^{-4}\}$ shown in yellow on top of the solution.

Bibliography

[1] G. Allaire, *Shape Optimization by the Homogenization Method*, Applied Mathematical Sciences, Springer, New York, 2001.

[2] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo, *A nonlocal p-Laplacian evolution equation with Neumann boundary conditions*, Journal de Mathématiques Pures et Appliquées, 90 (2008), pp. 201–227.

[3] ——, *A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions*, SIAM Journal on Mathematical Analysis, 40 (2009), pp. 1815–1851.

[4] F. Andrés, D. Castaño, and J. Muñoz, *Minimization of the compliance under a nonlocal p-Laplacian constraint*, Mathematics, 11 (2023).

[5] F. Andrés and J. Muñoz, *Nonlocal optimal design: A new perspective about the approximation of solutions in optimal design*, Journal of Mathematical Analysis and Applications, 429 (2015), pp. 288–310.

[6] H. Attouch, G. Buttazzo, and G. Michaille, *Variational Analysis in Sobolev and BV Spaces*, Society for Industrial and Applied Mathematics, 2014.

[7] E. Aulisa, G. Capodaglio, A. Chierici, and M. D'Elia, *E cient quadrature rules for finite element discretizations of nonlocal equations*, Numerical Methods for Partial Differential Equations, 38 (2021), pp. 1767–1793.

[8] J. Baranger, K. Najib, and D. Sandri, *Numerical analysis of a three-fields model for a quasi-Newtonian flow*, Computer Methods in Applied Mechanics and Engineering, 109 (1993), pp. 281–292.

[9] P. W. Bates and A. Chmaj, *An integrodi erential model for phase transitions: Stationary solutions in higher space dimensions*, Journal of Statistical Physics, 95 (1999), pp. 1119–1139.

[10] A. Beck, *First-Order Methods in Optimization*, MOS-SIAM Series on Optimization, Society for Industrial and Applied Mathematics, Tel-Aviv University, Tel-Aviv, 2017.

[11] N. Bell, L. N. Olson, and J. Schroder, *PyAMG: Algebraic multigrid solvers in Python*, Journal of Open Source Software, 7 (2022), p. 4142.

[12] J. C. Bellido and C. Mora-Corral, *Existence for nonlocal variational problems in peridynamics*, SIAM Journal on Mathematical Analysis, 46 (2014), pp. 890–916.

[13] D. Benson, S. Wheatcraft, and M. M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resources Research, 36 (2004).

[14] D. Boffie, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Series in Computational Mathematics, Springer, New York, 2013.

[15] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces*, in Optimal Control and Partial Differential Equations: A volume in honor of A. Bensoussan's 60th birthday, J. Menaldi, E. Rofman, and A. Sulem, eds., IOS Press, Amsterdam, 2001.

[16] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, Cambridge, 3rd edition ed., 2007.

[17] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, New York, NY, 2008.

[18] H. Brezis, *Function Analysis, Sobolev Spaces and Partial Di erential Equations*, Universitext, Springer, New York, 2010.

[19] C. Bucur and E. Valdinoci, *Nonlocal Di usion and Applications*, Springer, Cham, 2016.

[20] O. Burkovska and M. D. Gunzburger, *Regularity analyses and approximation of nonlocal variational equality and inequality problems*, Journal of Mathematical Analysis and Applications, 478 (2019), pp. 1027–1048.

[21] D. Castaño and J. Muñoz, *Nonlocal optimal control in the source. numerical approximation of the compliance functional constrained by the p-Laplacian equation*. Submitted, 2023.

[22] J. Céa and K. Malanowski, *An example of a max-min problem in partial di erential equations*, SIAM Journal on Control, 8 (1970), pp. 305–316.

[23] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 2002.

[24] B. Cockburn and J. Shen, *A hybridizable discontinuous Galerkin method for the p-Laplacian*, SIAM Journal on Scientific Computing, 38 (2016), pp. A545–A566.

[25] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Applied Mathematical Sciences, Springer, New York, 2nd edition ed., 2008.

[26] J. C. De los Reyes, *Numerical PDE-Constrained Optimization*, SpringerBriefs in Optimization, Springer, Cham, 2015.

[27] M. D'Elia, Q. Du, C. Glusa, M. D. Gunzburger, X. Tian, and Z. Zhou, *Numerical methods for nonlocal and fractional models*, Acta Numerica, 29 (2020), pp. 1–124.

[28] M. D'Elia and M. D. Gunzburger, *Optimal distributed control of nonlocal steady di usion problems*, SIAM Journal on Control and Optimization, 52 (2014), pp. 243–273.

[29] ——, *Identification of the di usion parameter in nonlocal steady di usion problems*, Applied Mathematics & Optimization, 73 (2016), pp. 227–249.

[30] M. D'Elia, M. D. Gunzburger, and C. Vollmann, *A cookbook for approximating Euclidean balls and for quadrature rules in finite element methods for nonlocal problems*, Mathematical Models and Methods in Applied Sciences, 31 (2021), pp. 1505–1567.

[31] Z.-Q. Deng, V. P. Singh, and L. Bengtsson, *Numerical solution of fractional advection-dispersion equation*, Journal of Hydraulic Engineering, 130 (2004), pp. 422–431.

[32] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, 2012.

[33] Q. Du, M. D. Gunzburger, R. B. Lehoucq, and K. Zhou, *Analysis and approximation of nonlocal di usion problems with volume constraints*, SIAM Review, 54 (2012), pp. 667–696.

[34] A. C. Eringen, *Linear theory of nonlocal elasticity and dispersion of plane waves*, International Journal of Engineering Science, 10 (1972), pp. 425–435.

[35] A. C. Eringen and D. G. B. Edelen, *On nonlocal elasticity*, International Journal of Engineering Science, 10 (1972), pp. 233–248.

[36] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements*, Springer, New York, 2004.

[37] L. C. Evans, *Partial di erential equations*, American Mathematical Society, Rhode Island, 2010.

[38] A. Evgrafov and J. C. Bellido, *Nonlocal control in the conduction coefficients: Well-posedness and convergence to the local limit*, SIAM Journal on Control and Optimization, 58 (2020), pp. 1769–1794.

[39] ——, *The nonlocal Kelvin principle and the dual approach to nonlocal control in the conduction coe cients*, arXiv preprint arXiv:2106.06031, (2021).

[40] M. Farhloul, *A mixed finite element method for a nonlinear Dirichlet problem*, IMA Journal of Numerical Analysis, 18 (1998), pp. 121–132.

[41] M. Farhloul and H. Manouzi, *On a mixed finite element method for the p-Laplacian*, Rocky Mountain Journal of Mathematics, 8 (2000), pp. 67–78.

[42] G. Gilboa and S. Osher, *Nonlocal operators with applications to image processing*, Multiscale Modeling & Simulation, 7 (2009), pp. 1005–1028.

[43] R. Glowinski, *Lectures on Numerical Methods for Non-Linear Variational Problems*, Scientific Computation, Springer Berlin, Heidelberg, 2008.

[44] Q. Guan and M. D. Gunzburger, *Analysis and approximation of a nonlocal obstacle problem*, Journal of Computational and Applied Mathematics, 313 (2017), pp. 102–118.

[45] M. D. Gunzburger and R. B. Lehoucq, *A nonlocal vector calculus with application to nonlocal boundary value problems*, Multiscale modeling & simulation, 8 (2010), pp. 1581–1598.

[46] E. Hewitt and K. R. Stromberg, *Real and Abstract Analysis*, Springer, Berlin Heidelberg, 1965.

[47] B. Hinds and P. Radu, *Dirichlet's principle and wellposedness of solutions for a nonlocal p-Laplacian system*, Applied Mathematics and Computation, 219 (2012), pp. 1411–1419.

[48] C. Khor and J. L. Rodrigo, *On sharp fronts and almost-sharp fronts for singular SQG*, arXiv preprint arXiv:2001.10332, (2020).

[49] I. A. Kunin, *Elastic Media with Microstructure I: One-Dimensional Models*, Springer, Berlin Heidelberg, 1982.

[50] ——, *Elastic Media with Microstructure II: Three-Dimensional Models*, Springer, Berlin Heidelberg, 1983.

[51] E. Madenci and E. Oterkus, *Peridynamic Theory and Its Applications*, Springer, New York, 2014.

[52] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter, 2012.

[53] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, Journal of Physics A: Mathematical and General, 37 (2004), p. R161.

[54] J. Muñoz, *Correction to: Generalized Ponce's inequality*, Journal of Inequalities and Applications, 2021 (2021), p. 80.

[55] ——, *Generalized Ponce's inequality*, Journal of Inequalities and Applications, 2021 (2021), p. 11.

[56] ——, *Local and nonlocal optimal control in the source*, Mediterranean Journal of Mathematics, 19 (2022).

[57] P. Pedregal, *On non-locality in the calculus of variations*, SeMA Journal, 78 (2021), pp. 435–456.

[58] A. C. Ponce, *An estimate in the spirit of Poincaré's inequality*, Journal of The European Mathematical Society, 6 (2004), pp. 1–15.

[59] ——, *A new approach to Sobolev spaces and connections to -convergence*, Calculus of Variations and Partial Differential Equations, 19 (2004), pp. 229–255.

[60] S. Rokkam, M. D. Gunzburger, M. Brothers, N. Phan, and K. Goel, *A nonlocal peridynamics modeling approach for corrosion damage and crack propagation*, Theoretical and Applied Fracture Mechanics, 101 (2019), pp. 373–387.

[61] H. L. Royden, *Real Analysis*, Prentice-Hall, New Jersey, 3rd edition ed., 1988.

[62] D. Sandri, *Sur l'approximation numérique des écoulements quasi-Newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau*, ESAIM: Mathematical Modelling and Numerical Analysis, 27 (1993), pp. 131–155.

[63] R. I. Saye, *High-order quadrature methods for implicitly defined surfaces and volumes in hyperrectangles*, SIAM Journal on Scientific Computing, 37 (2015), pp. A993–A1019.

[64] R. Scholz, *Numerical solution of the obstacle problem by the penalty method*, Computing, 32 (1984), pp. 297–306.

[65] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids, 48 (2000), pp. 175–209.

[66] S. A. Silling and R. B. Lehoucq, *Peridynamic theory of solid mechanics*, in Advances in Applied Mechanics, H. Aref and E. van der Giessen, eds., vol. 44, Elsevier, 2010, pp. 73–168.

[67] L. Tartar, *Compensated compactness and applications to partial di erential equations*, Non-linear analysis and mechanics: Heriot-Watt symposium, IV (1979), pp. 136–212.

[68] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. J. Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey, İ. Polat, Y. Feng, E. W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, P. van Mulbregt, and SciPy 1.0 Contributors, *SciPy 1.0: Fundamental algorithms for scientific computing in Python*, Nature Methods, 17 (2020), pp. 261–272.

[69] X. Xu, M. D'Elia, C. Glusa, and J. T. Foster, *Machine-learning of nonlocal kernels for anomalous subsurface transport from breakthrough curves*, arXiv preprint arXiv:2201.11146, (2022).

[70] K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM Journal on Numerical Analysis, 48 (2010), pp. 1759–1780.