Enhancing multi-asset portfolios future performance by applying different covariance estimates

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Abstract

The purpose of this thesis is to test whether it is possible to enhance future portfolio performance, under the context of the mean variance framework, by addressing the estimation errors on the vital input the covariance matrix. The tested methodologies are the Gerber Statistic, the Shrinkage Method and the historical covariance computation method. Furthermore, the performance of the mean variance optimized portfolio is compared to the 1/N asset allocation. The study uses a well-diversified portfolio of 10 assets, ranging in 5 different asset classes, over a time period of 22 years (January 2000 - December 2021). With a evaluation period of 20 years. This paper empirically find that, for almost all scenarios considered, the Gerber statistic's returns are higher compared to those achieved by both historical covariance and by the shrinkage method. Here the real discussion is on the magnitude of the results. As the performance of the Gerber Statistic indeed is higher, but the results do not show a substantial difference. In conclusion the selected covariance estimation methodologies do not differentiate significantly in generating better future performances benchmarked against each other. Though the Gerber Statistic proved to generate the best performances compared to the selected computation methods on the dataset used in this paper. Furthermore, the 1/N approach did not outperform the optimal allocated portfolios at even levels of standard deviation. However the performance was very similar and based on the empirical results of this paper, the ongoing discussion in the literature is reasonable.

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1 Introduction

In 1952 Harry Markowitz introduced the mean-variance optimization framework to approach portfolio construction. When the framework was developed it pioneered the portfolio theory and gave a more standard procedure for financial managers, as a dominant framework was not developed prior. The framework reflects investors' desire for both higher return and lower risk which made it very popular and still used today. Despite this framework being a pioneering piece of work, there is issues regarding estimation of the means, variances and covariances of securities. Here models for selecting portfolios according to the mean variance framework criteria did not account for the simultaneous effect of the errors in estimating the means, variance and covariance. Jobson and Korkie (1980) document these errors in their article, which will be detailed in the literature review. An important input in the mean-variance portfolio construction is the covariance. The estimation of this input has historically been one of the stickiest points in portfolio construction and research on the estimation of this has been done numerous times. Here the standard procedure in application, relied heavily on the availability of the covariance matrix between asset and the sample covariance matrix was often used as an estimate of the actual covariance matrix. Standard statistical method of computing a covariance matrix, was to use historical stock returns and simply compute their sample covariance matrix. The Pearson correlation coefficient inputs the sample covariance of asset I and j and the sample standard deviation of the assets and therefore also the sample mean of the assets. The weakness of this is, that the coefficient, by definition, is calculated over all data points. This is done regardless of whether the points are a result of meaningful co-movement or due to pure noise. Hence, this method tends to be very sensitive to small co-movements created by noise alone. As a results of this, numerous extensions have been developed and tested throughout the literature. This literature shows contradictory empirical results. Some authors like (Demiguel et al, 2009) claimed with empirical evidence, that the out-of-sample Sharpe ratio of the sample-based mean-variance framework was "much lower" than the 1/N strategy. Indicating that the errors in estimating means and co-variances erupt the gains from the optimal allocation relative to the naïve diversification. Other empirical studies, like studies performed by Gerber et al. (2021) and Ledoit and Wolf (2004) indicates it is possible to enhance the performance of a portfolio by using a more robust approach to calculating co-variances.

2 Aim of project

There are well known and documented problems embedded in the estimation of the mean returns, variances and covariances. The aim of this paper is to look specific to the limitations of previous work in terms of mean variance portfolio construction, with a focus on the estimation of the covariances and address these limitations with extensions developed in the literature. Thereby, this paper focuses on how and if portfolio performance can be improved by implementing newly developed extensions to the estimation of covariance according to the mean variance framework criteria.

Research question:

- How well does different covariance estimations methodologies apply in generating better future performance according to the mean variance framework.

  Does the mean variance framework outperform the less complex 1/N approach in terms of future portfolio performance.

The research question is addressed by backtesting how the specific extension models the Gerber Statistic and the Shrinkage Estimator performs relative to the naïve historical covariance computation method. Furthermore, this paper will compare these portfolios, to a portfolio constructed using the most naïve approach to asset allocation the 1/N approach.

3 Literature review

The literature regarding portfolio theory have grown substantially since Markowitz (1952), introduced the mean-variance approach to portfolio selection. This is a result of the ongoing discussion on the required input: an estimation of means, variances and covariances on returns. The issue is that former models for selecting portfolios according to the mean-variance framework criteria, did not account for the simultaneous effect of error in estimating the means, variances and covariances of securities (Frankfurter, Phillips, and Seagle, 1971). As described in the introduction the standard historical covariance computation method are not very robust as the calculation, by definition, includes all data points. This is done regardless of whether the points are a result of meaningful co-movement or due to pure noise. This computation method creates problems that are well documented by Jobson and Korkie (1980): When the number of stocks is large relative to the amount of historical return observations, the sample covariance is estimated with a lot of error. Here the most extreme coefficients in the matrix estimated tend to take on extreme values, not because these are true, but because the extreme values contain an extreme amount of error. These extreme values, that are very unreliable, the mean-variance software will place its biggest weights on. Michaud (1989) call this phenomenon "error maximization" and is the last thing financial managers want. In short, the pearsons correlation coefficient tends to take on extreme values not because this is actually the real-life case, but because they tend to contain extreme amounts of error. But as early as Sharpe (1963) various models have been used to ease the computational burden and to improve statistical properties of the covariance matrix estimates. However, many covariance matrix estimators employ product-moment-based estimates that inherently are not robust. Work on the robust estimators by Tukey (1960), Huber and Ronchetti (2009) and Hampel (1968, 1974) have largely overcome this problem. Nevertheless, financial time series contain characteristics that make even robust techniques unsuitable, as financial times series are very noisy and this noise can easily be misinterpreted as information. The result of this is that the covariance estimates can be disorted if the series contains extremely large observations, both positive or negative and the covariance matrix estimates often have non-zero entries corresponding to series that in fact have no meaningful correlation (Ledoit and Wolf, 2004).

So, practical application of the Markowitz theorem is complicated with estimation errors of the covariance matrix and expected returns. As a result of this, mean-variance portfolios are often outperformed by equal weight portfolios, where all asset weights are equal to 1/n according to (Demiguel et al, 2009) and (Duchin and Levy, 2009). Demiguel et al compared the performance of 14 models of optimal asset allocation, relative to the benchmark 1/N policy. The research was performed on seven different empirical datasets and the authors concluded, with empirical evidence, that the out-of-sample Sharpe ratio of the sample-based mean-variance framework was "much lower" than the 1/N strategy. Indicating that the errors in estimating means and covariances erupt the gains from the optimal allocation relative to the naïve diversification. Furthermore, the authors implemented various extensions, that have been proposed in the literature, to the mean variance framework. The proposed extensions to deal with the estimation errors, did typically not outperform the 1/N benchmark for the seven empirical datasets. Duchin and Levy (2009) came to the conclusion that the 1/N portfolios outperform the mean variance framework for individual small portfolios out of sample, but for large portfolios, i.e. institutional investors, the mean-variance framework was superior. Thus, the advantage of the 1/N rule is the absence of the exposures to estimation errors and according to Demiguel et al (2009) there is no single model that consistently delivers a Sharpe ratio that is higher than what's given by the 1/N portfolio.

Numerous extensions to the mean variance framework have been developed, throughout the literature, to minimize these beforementioned estimation errors. This research has been conducted for years and replacing Pearson's correlation with other measures of co-movement is not a new phenomenon. To highlight some of the recent: DeMiguel et al. (2013) used implied volatility, Nadler and Schmidt (2004) offered another way of increasing portfolio diversity by replacing Pearson's correlation with partial correlations conditioned on the state of the economy. The Partial correlation was used in other financial implications by Shapira et al (2009) and others. The motivation for introducing the partial correlation as an alternative is that according to the capital asset pricing model, most individual equity assets, follow the common equity market trend and thereby is the Pearson's correlation increased. The Partial correlation focus on the excess return and thereby is the partial correlation often lower than the standard method. Since the partial correlation in theory strips out the small co-movements that are due to market conditions (Cai and Schmidt, 2020).

Another extension to the mean variance framework is a method called shrinkage. Shrinkage is not a new concept in statistics today, but was revolutionary when it was first introduced by Charles Stein Professor at Stanford University (1955). This method was used in various studies and the first attempts to use shrinkage in portfolio selection were conducted by Frost and Savarino (1986) and Jorion (1986). The authors elaborate on the problem of estimation risk. In their work they try to solve this problem by creating an informative prior that effectively reduces estimation error by drawing the sample estimates towards the grand average of those parameters, much similar to the Shrinkage Estimator by Ledoit and Wolf (2004). Frost and Savarino claim with empirical evidence that the results of their simulation imply that their empirical procedure, select portfolios whose ex-ante performance were substantially superior to those provided by the standard historical estimation. The short falls of their specific shrinkage methods were when the number of stocks exceeded the number of historical return observations included in the data set. Recently a new development in the shrinkage method regarding portfolio construction have been made. Ledoit and Wolf (2004) suggested using a matrix obtained from the sample covariance through the transformation called shrinkage. The idea behind this were to pull the most extreme coefficient towards more central values. Hence, the extreme positive values in the estimated coefficients, in the sample covariance matrix, will be pulled downward to compensate for positive errors. Furthermore, a compensation for the negative errors with extremely low estimates of coefficients will be pulled upwards. Thereby, the shrinkage methods theoretically should systematically reduce, some of the beforementioned, estimations errors. The main advantages of this approach are the ease of computation and the property of being unbiased (its expected value is equal to the true covariance matrix). The main disadvantage is that the estimator contains a lot of estimations error when the number of data points is comparable or smaller than the number of individual stocks, which is a common situation in financial applications (Ledoit and wolf 2004).

In 2022 a new extension of the mean variance framework was published by Sander Gerber, Harry Markowiz et al. The Gerber Statistic focuses on creating and applying more robust estimators to the mean variance framework. The Gerber Statistic is a robust co-movement measure which ignores fluctuations below a certain threshold and limiting the effects of extreme movements. This computation method is designed to recognize co-movement between securities when the movements are substantial and to be insensitive to small co-movements that are due to white noise. The Gerber Statistics is a generalization of Kendall's Tau (1938) which measures correlation between two data groups, in this case returns of two assets, as the ratio between the number of Concordant and Discordant pars in the data set, dived by the sum of the number of concordant and discordant pairs in the set. The notation on this will be formalized in notation section 4.1. The Gerber Statistic generalizes this by including thresholds such that only co-movements which exceed the upper and lower thresholds will be recognized as either concordant or discordant. The key advantage of the Gerber Statistics it that it does not rely on the sample covariance matrix as input, in contrast to the beforementioned computation methods. In short, the Gerber Statistic strips out noisy data, as the GS only includes the subset of the data containing points which corresponds to meaningful co-movement. As the observations need to pierce the thresholds to be "counted". In theory the Gerber Statistic strips out noisy data and thereby, the Gerber statistic do not incorporate all data points as opposed to the standard Pearson correlation. This is the key difference and reason why the Gerber statistic is a more robust co-movement measure than the standard Pearson correlation. Additionally, the formulation of the Gerber statistic does not require any estimates of movement.

This paper will examine the new extensions to the mean variance framework: the Gerber statistic and the Shrinkage Estimator compared to the naïve historical correlation method and the 1/N portfolio. The literature on this field is contradictory and the aim is to test which method performs best for the data set and time frame given in this paper. The Gerber Statistic is selected as one of the extensions to be examined, as this is a new and fairly untested approach, which do not rely on the sample covariance matrix as and input in comparison to several of the alternative extensions. Additionally, the shrinkage estimator is selected as an extension to be examined. This approach is selected, as the method theoretically should systematically reduce, some of the beforementioned, estimations errors. The main disadvantage of the shrinkage approach is, as beforementioned, when the number of stocks is large relative to the amount of historical return observations, the sample covariance is estimated with a lot of error. This disadvantage should be eliminated in this paper as the data is on broad indices with a 20-year time frame with weekly observations. Hence, the number of assets is comparably small relative to the amount of historical return observations and thereby this approach should perform well theoretically. The Gerber statistic and Shrinkage Estimator have some conceptual differences, as the shrinkage method directly inputs the sample covariance matrix and the Gerber statistic does not rely on the sample covariance matrix as input. Instead, the GS computes concordant and discordant pair counts. The two extensions also have some form of similarities as the Shrinkage Estimator method uses a shrinkage constant δ (formalized in section 4.2.2), that controls the degree to which the sample covariance matrix is shrunk. Similar as the value c in the GS, which determines the magnitude of the beforementioned threshold Hk. The notation on the Gerber Statistic and the Shrinkage Estimator will be formalized in section 4.1 and 4.2 below.

Furthermore, the historical covariance computation method is used as benchmark to evaluate if the extensions yield a substantially better performance or if the selected estimation methods are insignificant.

Alternatively, to the above computation methods, one might consider an estimator with a lot of structure like the single factor model popularized by Sharpe in 1963. An estimator like that contains relatively small estimation error but, tend to be misspecified and can be heavily biased. Bottom line is that all successful risk models find a compromise between the sample covariance matrix and a highly structured estimator. The standard in the industry is to use multifactor models. The idea behind this is to incorporate more relevant factors and thereby make the models become more flexible and thereby reduce the bias. Thus, this leads to an increase in the estimation error. Therefore, financial managers look to find an optimal tradeoff by deciding the nature and the number of factors included in the model. One approach could be to include a combination of industry factors and risk indices like the Barra U.S equity model. Another approach could be to include statistical factors such as principal components. A financial manager offering risk models based on statistical factors is APT. The philosophy of the approaches used in this paper is different and for the purpose of this paper these methods will not be touched upon.

4 Notation and Methodology

The following section formulates the notation of the Gerber Statistic and the corresponding Gerber Correlation Matrix. This correlation matrix will then be converted to Gerber covariance matrix that will be used in the mean-variance portfolio optimizer (formalized in section 4.3). Furthermore, the section formulates the notation on the Shrinkage Estimator which contains a notation of the Shrinkage Target and the Shrinkage Constant.

4.1 The Gerber statistic

The notation of the Gerber Statistic is based on the article "The Gerber Statistic: a robust co-movement measure for portfolio optimization" by Gerber et al (2021). For each pair of securities (i, j) of asset for each time t, the return observation of each pair (r_{ti}, r_{tj}) is converted to a joint observation defined as $M_{ij}(t)$. This is given by:

$$m_{ij}(t) = \begin{cases} +1 \; if \; r_{ti} \geq +H_i \; and \; r_{tj} \geq +H_j \\ +1 \; if \; r_{ti} \leq -H_i \; and \; r_{tj} \leq -H_j \\ -1 \; if \; r_{ti} \geq +H_i \; and \; r_{tj} \leq -H_j \\ -1 \; if \; r_{ti} \leq -H_i \; and \; r_{tj} \geq +H_j \\ 0 \; otherwise \end{cases} \tag{1}$$

In the formulation above, H_k is a threshold for security k that is calculated:

$$H_k = cs_k \tag{2}$$

Here c is some fraction, typically 0.5 but can be increased to 0.7 or 0.9. S_k is the sample standard deviation of the return of security k. Equation (1) above indicates three observations:

- $m_{ij}(t)$ (the joint observation of a pair of returns) is set to + 1 if the series i and j pierce their thresholds in the same direction at time t simultaneously.
- $m_{ij}(t)$ (the joint observation of a pair of returns) is set to –1 if the series i and j pierce their thresholds in opposite directions at time t simultaneously.
- $m_{ij}(t)$ (the joint observation of a pair of returns) is set to 0 if a least one of the series does not pierce its threshold at time t.

Hence, for every pair of assets the Gerber Statistic converts return observations, whose absolute value is greater than the threshold for both securities, to –1 or +1 and drops all other observations to 0. A pair which pierces their thresholds in the same direction is referred to as a concordant pair and those pairs who pierce their thresholds while moving in opposite directions is defined as discordant pairs. Given the above formulation Gerber et al. defines the Gerber statistic for a pair of assets to be:

$$g_{ij} = \frac{\sum_{t=1}^{T} m_{ij}(t)}{\sum_{t=1}^{T} |m_{ij}(t)} \tag{3}$$

If $n_{i\,j}^{c}$ is set as the number of concordant pairs for series i and j and if $N_{i\,j}^{d}$ is the number of discordant pairs, the above equation is equivalent to:

$$g_{ij} = \frac{n_{ij}^c - n_{ij}^d}{n_{ij}^c + n_{ij}^d} \tag{4}$$

Which is identical to Kendall's Tau if the threshold is set to zero for all K. The idea behind this Gerber Statics in equation (4) is that because the GS relies on the number of simultaneous thresholds pierced, it is insensitive to extreme movements in one security. Furthermore, since the returns series must pierce the threshold to be "counted", the Gerber statistic is also insensitive to small movements that may be due to pure noise.

4.1.1 The Gerber matrix

Now, the Gerber statistic g_{ij} is defined. The next step is the formulation of the Gerber matrix **G**. The matrix contains the Gerber statistic in the i-th row and j-th column. Gerber et al. (2021) defines the matrix of returns as $\mathbf{R} \in R^{TxK}$ with entry R_{tk} in its t-th row and k-th column. Letting **U** be an indicator matrix with the same size as **R** for returns piercing the upper threshold, having entries u_{tj} so:

$$u_{tj} = \begin{cases} 1 & if\ r_{tj} \geq +H_j \\ 0 & Otherwise \end{cases}$$

With the above definition the matrix for the number of samples that exceeds the upper threshold is:

$$N^{UU} = U^T U \tag{5}$$

Here the ijth element $n_{ij}^{UU} of N^{UU}$ is the number of samples where both time series i exceeds the upper threshold and for which time series j simultaneously exceeds the upper threshold. Letting **D** be an indicator matrix, again with the same size as **R**, for returns falling below the lower threshold, having entries d_{tj} so:

$$d_{tj} = \begin{cases} 1 & \text{if } r_{tj} \leq -H_j \\ 0 & Otherwise \end{cases}$$

With the above definition the matrix for the number of samples that is falling below the lower threshold is:

$$N^{DD} = D^T D \tag{6}$$

Again here the ijth element $n_{ij}^{DD} of N^{DD}$ is the number of samples where both time series i exceeds the upper threshold and for which time series j simultaneously exceeds the upper threshold.
Using equation (5) and (6) the matrix that contains the concordant pairs is defined as:

$$N_{CONC} = \ N^{UU} + N^{DD} = U^T U + \ D^T D \tag{7}$$

And the matrix that contains the discordant pairs is defined as:

$$N_{Disc} = U^T D + \ D^T U \tag{8}$$

Now it is time to define the Gerber Matrix G corresponding to the Gerber Statistic written in equation 4 in matrix form:

$$\mathbf{G} = (\mathbf{N_{CONC}} - \mathbf{N}_{DISC}) \oslash (\mathbf{N}_{CONC} + \mathbf{N}_{DISC}),$$

The symbol $\oslash$ is the Hadamard elementwise division. The corresponding Gerber Covariance matrix Σ_{GS} is defined as:

$$\sum_{GS} = diag\,(\sigma)\,\mathbf{G} diag(\sigma) \tag{9}$$

Here σ is a N $\times$ 1 vector of sample standard deviation of the historical asset returns. However, a covariance matrix of securities must be positive semidefinite. A square symmetric matrix A ϵR^{TxK} is positive semi-definite if:

$$v^T Hv \ \geq 0, \ \ \forall v \ \in R^n \tag{10}$$

The matrix R is positive semidefinite if -R is negative semidefinite and matrix is PSD if the eigenvalues of a matrix A are non-negative. To check if a form is PSD computing the eigenvalue decomposition of the underlying symmetric matrix. The eigenvalues are a special set of scalars associated with linear system of equations i.e., a matrix equation. The basic equation is:

$$Ax = \ \lambda x \tag{11}$$

The value λ is the eigenvalue of A.

An example of a model A, $A^2, A^3, \ldots$ of a matrix. Think of you need the hundredth power A^{100}. The A^{100} can be found by using the eigenvalues of A instead of multiplying 100 matrices. Hence the eigenvalues are a way to see into the heart of a matrix. (Analysis of data, 2021)

When Gerber et al. worked with the above notation on real data, the authors found that the covariance matrix (equation 9) was often not positive semidefinite. Therefore, an alternate notation of the Gerber Statistic, which yields a positive semidefinite covariance matrix, was developed. To show this consider the graphical illustrations in table 1 below, which shows the relationship between two securities.

UD	UN	UU
ND	NN	NU
DD	DN	DU

Figure 1: Graphical relationship between two securities. Source: Own production inspiration Gerber et al (2021)

- The letter U represent the case where a securities return lies above the upper threshold.
- The letter N represent the case where a securities return lies between the upper and lower threshold.
- The letter D represent the case where a securities return lies below the lower threshold.

In table 1 above the rows is a categorization of security i and the columns is a categorization of security j. The boundaries between the rows and the columns are the beforementioned thresholds. To give an example of the observation process: if at time t the return of security i is above the upper threshold this observation would lie in the top row (UD UN UU). If at the same time t return of security j is below the lower threshold the observation lies in the left side of the table (UD ND DD). Hence the specific observation would lie in UD.
Across time, t = 1,...,T, observations will be spread over the nine observation types. Letting n_{ij}^{pq} be the number of observations where the returns of securities i and j lie in regions p and q, respectively, for $p, q \epsilon \{U, N, D\}$. With this notation Gerber et al. writes and equivalent expression to the statistic presented in equation 4 as:

$$g_{ij} = \frac{n_{ij}^{UU} + n_{ij}^{DD} - n_{ij}^{UD} - n_{ij}^{DU}}{n_{ij}^{UU} + n_{ij}^{DD} + n_{ij}^{UD} + n_{ij}^{DU}} \tag{12}$$

As discussed, the denominator in equation (4) must be altered to obtain a Gerber matrix which yields a corresponding covariance matrix in positive semidefinite form. The alternate notation is given in equation 13 below:

$$g_{ij} = \frac{n_{ij}^{UU} + n_{ij}^{DD} - n_{ij}^{UD} - n_{ij}^{DU}}{T - n_{ij}^{NN}} \tag{13}$$

In the empirical studies performed by Gerber et al. (2021), for all cases of Gerber thresholds c considered, the authors observed the covariance matrix corresponding to the Gerber statistics (eq. 13) to be positive semidefinite in all cases. In this paper the same was observed. Hence the modified Gerber statistic, from now on referred to as the Gerber statistic, provides a covariance matrix corresponding to the Gerber statistic in a positive semi definite form. Because the denominator in equation 13 is non-negative by definition. The numerator is positive if the sum of $n_{ij}^{UU} and\ n_{ij}^{DD}$ exceeds the sum of $n_{ij}^{UD}\ and\ n_{ij}^{DU}$. The numerator is zero if the sums are equal and negative otherwise. The choice of the Gerber Statistic's denominator in equation (13) is based on the observation: As c, the fraction multiplied by the standard deviation in the threshold equation (2), becomes larger more datapoint will be included in the NN region in table 1 above. This is the region where none of the asset returns in time t pierce the lower or upper threshold. The denominator in equation (13) subtracts these points and leads the statistic to be more robust and less sensitive to “noise” in the data. The authors Gerber et al. (2021) refers to this artifact of the Gerber Statistic as “stripping noise” from the data.

4.1.2 Illustration of The Gerber Statistic

The following section illustrate how the Gerber Statistic in equation (11) is calculated on returns between a pair of assets. In this example the Gerber Statistic is calculated over 26 weekly returns of the two securities S&P 500 and the Gold XAU index. This is done from the period January 2021 until June 2021 to illustrate how the manual calculation of the Gerber Statistic is performed. Table 2 below displays three different thresholds of H_k as notated in equation (2). We consider the different values of c: $c = 0.5$, $c = 0.7$ and $c = 0.9$.

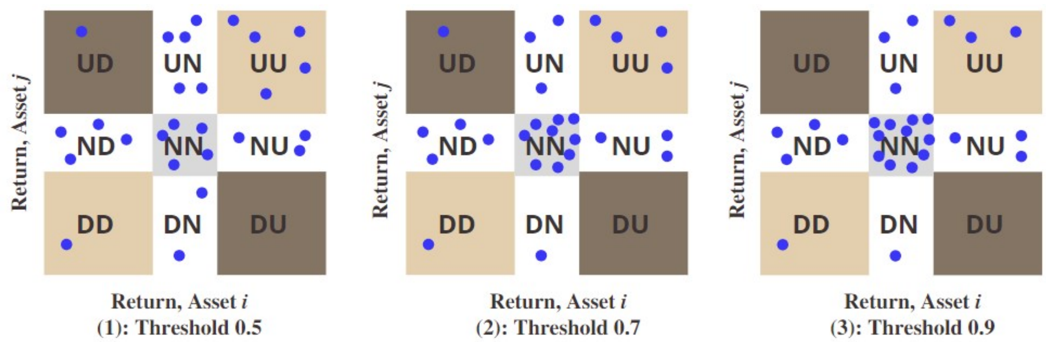

Figure 2: Illustration of pairwise returns between two securities.

Every blue dot in the table indicates a pairwise weekly return. The points in the brown zone is discordant pairs and the point in the sand zones correspond to concordant pairs. Here the return series between asset i and j, in this example the S&P 500 and the gold XAU index. These return series are transformed to -1, 0, 1 by using the upper and lower thresholds calculated in equation (2). See appendix 1 for threshold calculation and pairwise division.

The Gerber Statistic is then calculated by counting the points that falls into each area of the table. As mentioned, there is three cases to calculate with $c = 0.5$, $c = 0.7$ and $c = 0.9$.

- Figure 2 (1): $c = 0.5$. The counts for the 9 regions are $n_{ij}^{UD} = 1, n_{ij}^{UN} = 5, n_{ij}^{UU} = 5, n_{ij}^{ND} = 4, n_{ij}^{NN} = 5, n_{ij}^{NU} = 3, n_{ij}^{DD} = 1, n_{ij}^{DN} = 2, n_{ij}^{DU} = 0$. Inserting these values into the Gerber statistic in equation (13):

$$g_{ij} = \frac{5 + 1 - 1 - 0}{26 - 5} = 0.2381$$

- Figure 2 (2): $c = 0.7$. The counts for the 9 regions are $n_{ij}^{UD} = 1, n_{ij}^{UN} = 3, n_{ij}^{UU} = 4, n_{ij}^{ND} = 4, n_{ij}^{NN} = 9, n_{ij}^{NU} = 3, n_{ij}^{DD} = 1, n_{ij}^{DN} = 1, n_{ij}^{DU} = 0$. Inserting these values into the Gerber statistic in equation (13):

$$g_{ij} = \frac{4 + 1 - 1 - 0}{26 - 9} = 0.2352$$

- Figure 2 (3): $c = 0.9$. The counts for the 9 regions are $n_{ij}^{UD} = 0, n_{ij}^{UN} = 3, n_{ij}^{UU} = 3, n_{ij}^{ND} = 4, n_{ij}^{NN} = 11, n_{ij}^{NU} = 3, n_{ij}^{DD} = 1, n_{ij}^{DN} = 1, n_{ij}^{DU} = 0$. Inserting these values into the Gerber statistic in equation (13):

$$g_{ij} = \frac{3 + 1 - 0 - 0}{26 - 11} = 0.2667$$

The above calculations are done over a period of T=26 weeks. To obtain a more precise Gerber Statistic the time periods should be expanded. As the above section, only is an example and illustration of the Gerber statistic, the calculated values are not used in the empirical study. In the empirical study a lookback window of T=104 weeks (2 years) is employed. This computation on all returns series is done through code run in Phyton. See appendix 5 for Python code.

4.2 The Shrinkage Estimator

The following notation section are based on study by Ledoit and Wolf (2004) in the article "Honey, I Shrunk the Sample Covariance Matrix" Consider the sample covariance matrix S and a highly structures estimator, denoted by F. The authors find a compromise between the two by computing a convex linear combination:

$$\delta F + (1 - \delta) S$$

Where δ is a number between 0 and 1 and is referred to as the shrinkage constant.

This technique is called shrinkage as the sample covariance matrix is shrunk towards the structured estimators. The shrinkage constant δ measures the weight that is given to the structured estimator. The idea behind the shrinkage principle is by properly combining to extreme estimators a compromise can be obtained and thereby an estimator that performs better than either of the extreme. Any shrinkage estimator has three ingredients:

- Estimator with no structure
- Estimator with a lot of structure
- Shrinkage constant

In our case is the estimator with no structure is the sample covariance matrix. The structured estimator (shrinkage target) and the shrinkage constant will we be formalized in section 4.2.1 and section 4.2.2

4.2.1 Shrinkage target

The aim of the shrinkage target is to fulfill two requirements at the same time: The target should only involve a small number of free parameters, hence a lot of structure, but it should at the same time reflect important characteristics of the unknown quantity that is estimated. In the working paper published in 2003 by Ledoit and Wolf they suggest the single-factor matrix of Sharpe (1963) as the shrinkage target. In this paper and in the author's paper published in 2004 another suggestion is made. Ledoit and Wolf (2004) suggest a "constant correlation model". The model defines that all the pairwise correlations are identical. The average of all the sample correlations is the estimator of the common constant correlation. This number together with the vector of sample variances implies the shrinkage target, denoted by F in further notation.

To show this equation some additional notation is needed. Let y_{it}, $1 \le i \le N, 1 \le t \le T$, denote the return on security i at time t. The empirical study of this method assumes that the returns are independent and identically distributed over time and have finite fourth moments. The sample average of the return are given by:

$$\hat{y}_{i.=} T^{-1} \sum_{t=1}^{T} y_{it.}$$

Where Σ denote the population (or true) covariance matrix and S denote the sample covariance matrix. Entries of the matrices Σ and S are denoted by σ_{ij} and S_{ij} respectively.

The population and sample correlations between security i and j are given by:

$$\varrho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \qquad \text{and} \qquad r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}}$$

And the average population and sample correlations between return on security i and j are given by:

$$\bar{\varrho} = \frac{2}{(N-1)N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \varrho_{ij} \qquad \text{and} \qquad \bar{r} = \frac{2}{(N-1)N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} r_{ij}$$

Ledoit and Wolf (2004) Defines the population constant correlation matrix Φ by means of the population variances and the average population correlation:

$$\phi_{ii} = \sigma_{ii} \qquad \text{and} \qquad \phi_{ij} = \bar{\varrho}\sqrt{\sigma_{ii}\sigma_{jj}}$$

Correspondingly, Ledoit and Wolf (2004) define the sample constant correlation matrix F by means of the sample variances and the average sample correlation:

$$f_{ii} = s_{ii} \qquad \text{and} \qquad f_{ij} = \bar{r}\sqrt{s_{ii}s_{jj}}$$

(14)

This matrix f is the shrinkage target.

4.2.2 Shrinkage Constant

A practical problem is which value to choose for the shrinkage constant. According to Ledoit and Wolf (2004) any choice of δ strictly between 0 and 1 would yield a compromise between S and F. However, this results in infinitely many possibilities. There is an optimal shrinkage constant, which minimizes the expected distance between the shrinkage estimator and the true covariance matrix. Ledoit and Wolf refer to this as δ^* and the formula for estimating is given by:

$$\hat{\delta}^* = \max\left\{0, \min\left\{\frac{\hat{\kappa}}{T}, 1\right\}\right\}$$

(15)

The notation on how to derive at this above equation is given in appendix B Ledoit and Wolf 2004 article “Honey, I Shrunk the Sample Covariance Matrix”. In short what the authors do is finding the shrinkage parameter by minimizing the Frobenius norm between the true covariance matrix Σ and the shrinkage estimator Σ_{SM}. The notation on how to derive at this formula is numerous pages long filled with equations. This is deemed out of scope for the project as the notation is less significant as long as the end equations if used correct.

The operational shrinkage estimator of the covariance matrix regarding the Shrinkage method is now ready for use. The equation is given below:

$$\hat{\Sigma}_{Shrink} = \hat{\delta}^* F + (1 - \hat{\delta}^*) S$$

(16)

4.3 Portfolio optimization: Mean-Variance framework

The portfolio optimization which is considered in this paper is that of the mean-variance optimization (MVO) developed by Harry Markowitz (1952). This mean-variance optimization framework finds an optimal asset allocation (portfolio weights ω_i) under certain risk and return restrictions, given that future asset characteristics as expected return μ_i, variance $\sigma_{i}{}_{i}^{2}$ and covariance σ_{ij} are known for each asset $i \epsilon$ (1,. . . ,K). The expected return on a portfolio and its variance can be derived as $\mu_p = \omega^T \mu$ and $\sigma_P^2 = \omega^T \Sigma \omega$, where $\omega = \omega_1, ..., \omega_K$ is a vector of portfolio weights for the K assets. $\mu = \mu_1, ..., \mu_K$ is a vector of expected returns and σ is a covariance matrix of the asset returns.

The long-only mean variance optimization with transactions cost included is formulated as the following:

$$Maximize: \omega^T \mu - \psi 1^T |\omega - \omega_0|$$

$$Subject\ to: \omega^T \sum \omega \leq \sigma_{target}^2$$

$$\omega^T 1 = 0$$

$$0 \leq \omega_k \leq 1, \forall k = \{1, 2, \ldots, N\}. \tag{17}$$

The above portfolio optimization maximizes the expected return of a portfolio $\omega^T\ \mu$ subtracted by a transaction cost: $1^T|\omega - \omega_0|$ This transaction term tracks the proportional cost of portfolio weight changes. Hence, the absolute deviation of new weight vector ω from the previous vector ω_0. The symbol ψ is a fixed proportional transaction cost which is selected to be 10 basis points (0.1%) in this paper. The transaction cost of 10 basis point is selected on what transaction cost would be on trading platform available for the retail/private investor. This transaction cost is added, as the portfolio is rebalanced monthly, the results will be more precise and real life like. Furthermore, it helps to regulate the turnover of the portfolio. The portfolio weight vector ω is subject to the standard constraint, no short sales allowed, given in Markowitz (1952). Mean-variance portfolio with unconstrained weights can yield extreme long and short positions that can be unattractive to investors (Jacobs et al. 2013). Furthermore, Jagannathan and Ma (2003) proved with empirical evidence that mean-variance optimizers are already implicitly applying some form of shrinkage to the sample covariance matrix when the not short sales constrain is used. This should generally be beneficial in terms of improving the stability of the weights.
The framework yields optimal portfolio weights with risk constraint σ_{target}^2 and turnover penalty ψ by solving the optimization problem given in equation (17). Given the turnover constraint ψ the set of optimal solutions of $\omega \epsilon [0, 1]^N$ for each of the determined risk levels $\sigma_{target}^2 \epsilon R^+$ constitutes the efficient frontier. Every point on the frontier determines an efficient portfolio with a maximization of the return given a predetermined risk target.

The missing piece is to provide estimates for estimation for the vector of estimated asset returns μ and the covariance matrix of asset returns Σ. The expected return at time t for asset i, written as μ_{ti} is estimated based on the sample means on the historical returns given a lookback window of T-months also written as: $\mu_{ti} = \sum_{d=t-1}^{t-T} r_{di}$. The estimations of the covariance matrix Σ, is the focus point of the paper. Here the Gerber Statistic is benchmarked and compared to the two competing methods The historical covariance and the Shrinkage Method. Furthermore, an equal weighted portfolio is included to highlight if there is a substantial advantage on using the extensions on the mean variance optimization framework compared to the most naive allocation method. Summery of competing methods:

- Historical covariance (HC)

- The shrinkage method (SM)
- The Gerber statistic (GS)
- Equal weights (1/N)

4.4 Methodology

This paper will backtest the beforementioned computation methods to benchmark the performance of the different covariance estimators under the context of portfolio optimization. The study will use 20 years' worth of data ranging from 1th of January 2002 – December 2021. Because two years' worth of weekly returns are required to initialize the first portfolio, the performance evaluation ranges from the period January 2002 to December 2021. The period from January 2000 – December 2002 is the initial window of T = 104 weeks which is utilized to estimate the expected return vector μ and the covariance matrix. When this is done, the quadratic optimizer is used to solve for an optimal portfolio given different applied risk targets. The portfolios are held for one month. At the end of the month the realized return of the portfolio is computed. After the portfolios being held for one month the portfolios are re-balanced. The process will be repeated by moving the in-sample period one month forward and computing the updated efficient portfolio for the next month. This rolling-window procedure offers the advantage of being more adaptive to market structural changes, than the expanding window as data accumulates.

The backtest methodology are employed by using code i python. The Code are based on work done by Yinsen (2021) in relation to the Gerber Statistic article. This code is then modified to fit the methodology and data of this paper.

5 Data

To apply the methodology presented in section 4.4 the following data is collected. This paper will use a total of 5 different asset classes ranging from risky securities to fixed income securities to ensure a well-diversified portfolio. The data is collected over the time-period January 2000 to December 2021. This time-frame is selected to evaluate how the covariance methods perform on a long time horizon and in different market conditions.

5.0.1 Equities

MSCI Europe Large Cap Index:

- The index captures large cap companies across 15 developed markets in Europe. The index covers approximately 70 % of the free float-adjusted equity market capitalization across the developed markets in Europe. The index contains 197 constituents. The index is neutral weighted in terms of the stock factor characteristics value, momentum, quality and yield. Additionally, the index provides a small overweight in stocks with factor characteristic low volatility and an underweight of low size companies (MSCI Inc, 2022).

    Exposure: Europe, Large Cap, Developed Markets.

MSCI Europe Small Cap index

- The index tracks the performance of 1,059 small cap constituents across the 15 developed markets countries in Europe. The index covers approximately 14 % of the free float adjusted market capitalization in the European equity markets. As the index name suggest, the index provides a large overweight in low size companies. Additionally, the other factor characteristics are weighted from neutral to a small underweight (MSCI Inc, 2022).

    Exposure: Europe, Small Cap, Developed Markets.

MSCI Emerging Markets index

- The index tracks the performance of large and mid cap companies across 25 Emerging Markets countries. The index contains 1,420 constituents and covers approximately 85 % of the free float adjusted market capitalization in each country. Most of the factor characteristics are weighted neutral with a small underweight in low size companies (MSCI Inc, 2022).

  Emerging Markets (China, Taiwan, India etc.), Large cap, Mid cap.

OMX Copenhagen 20 index

- The index tracks the performance of the 20 most actively traded shares on the Copenhagen Stock Exchange. Hence, the index tracks a limited number of constituents, which guarantees excellent liquidity. The index is market weighted price index and is revised twice a year (Nasdaq Global Index Watch, 2022).

  Exposure: Large cap, Danish market.

S&P 500 index

- The index tracks the performance of the largest traded US equities and covers approximately 80 % of the available US market capitalization (S&P Dow Jones Indices, 2022)

  Exposure: Large Cap, US market.

NASDAQ Composite

- The index tracks the performance of all Nasdaq domestic and international based common type stocks listed on the Nasdaq Stock Market. To be eligible for inclusion in the index the securities US listing must be exclusive on the Nasdaq Stock Market, unless it was dually listed on another US exchange prior to 2004 (Nasdaq Global Index Watch, 2022).

  Exposure: US market, all capitalization's, overweight in information technology.

5.0.2 Real estate

FTSE EPRA NAREIT Developed Europe index

- The index tracks the performance of listed real estate companies and Real Estate Investment Trust. The index provides investors with diversification as real estate is low correlated with other asset classes. The index tracks real estate investment across all Europe, with highest weights in Germany, UK and Sweden (FTSE Russell, 2022).

  Exposure: Real estate, Europe.

5.0.3 Cash and Cash equivalents

XAU Gold/Silver Sector Index

- The index is a capitalization-weighted index composed of companies involved in the gold or silver mining industry (Nasdaq Global Index Watch, 2022).

  Exposure: Gold, Silver, Global

5.0.4 Commodities

Bloomberg Commodity Index

- The index tracks the performance of commodities and reflects commodity futures price movements. The index is rebalanced annually and weighted 2/3 by trading volume and 1/3 by world production. Weight-caps are applied at the commodity, sector and group level for diversification (Bloomberg, 2022).

  Exposure: Commodities

5.0.5 Fixed income

Bloomberg Euro Aggregate Bond Index

- The index tracks the performance of the fixed rate, investment-grade Euro-denominated bond market. U.S. Treasuries represent nearly 40% of the index, hence the index is euro denominated. The remaining components represent the debt of major industries as real estate, industrial companies, financial institutions, and utilities (Bloomberg, 2022).

  Exposure: Fixed income, Europe.

The data from the above asset classes is collected as the total weekly return from 2000-2021 and is collected from Factset. Each Asset contain 1448 observations over the time period. The total return accounts for cash distributions such as dividends and stock splits. Commodities are added as several studies find that the efficient frontier is improved by including commodities (Anson 1999).

5.1 Summery statistics

Table 1 highlight the summery statistics for the 10 assets. The descriptive statistics are calculated using weekly data from January 2000 to December 2021.

Summery statistics:			
Index	**Aritmetic Return (%)**	**Geometric Return (%)**	**Annualized SD (%)**
MSCI Europe Large Cap	5.45%	3.56%	18.83%
MSCI Europe Small cap	8.28%	6.44%	18.32%
MSCI Emerging Markets	8.71%	6.46%	20.40%
OMX C20	11.61%	9.42%	19.74%
S&P 500	8.83%	6.89%	18.91%
Nasdaq COMP	9.78%	6.53%	15.99%
Bloomberg Euro Aggregate bond	4.34%	4.29%	3.23%
FTSE EPRA/NAREIT Developed Europe	6.18%	4.21%	19.10%
Gold XAU	8.48%	2.48%	33.56%
Bloomberg Commodity	2.52%	1.33%	15.26%

Table 1: Summery Statistics on the assets in the dataset

The 10 assets are a part of five different asset classes. Hence, the portfolio should be well diversified and therefor a strong correlation between the asset classes is not anticipated. This is confirmed on figure 3 below which displays a correlation matrix between the total weekly returns of the nine assets from the period January 2000 to December 2021.

	EU Large Cap	Emerging Markets	OMX C20	S&P 500	Euro Aggregate bond	Real estate	XAU	Commodity	Nasdaq COMP
EU small cap	0.87894686	0.736428508	0.7637369	0.688167	-0.065217147	0.81215832	0.2725651	0.3668457	0.623165654
EU Large Cap		0.716848299	0.7537573	0.777763	-0.109091515	0.744603165	0.2249488	0.3493137	0.684901105
Emerging Markets			0.6002318	0.640939	-0.10921391	0.590163345	0.3091006	0.4457931	0.586286076
OMX C20				0.583929	-0.052496915	0.628510257	0.2328336	0.2776293	0.536222837
S&P 500					-0.145771922	0.555686681	0.1351552	0.3789363	0.889719625
Euro Aggregate bond						0.049663493	0.0325242	-0.135342	-0.15305164
Real estate							0.2293707	0.3047286	0.444653675
XAU								0.3044785	0.120795002
Commodity									0.322654884

Figure 3: Correlation between the 10 assets

Table two indicates a high correlation between securities in the same assets class, i.e. The equities are highly correlated.

6 Empirical results

The following section will present the empirical results based on the dataset and timeframe presented in section 5, the transaction cost presented in section 4.3 and the backtesting methodology presented in section 4.4. This procedure will be tested on the three different methodologies on calculation covariance: The Gerber Statistic (GS), The Shrinkage Method (SM) and the historical covariance (HC). As the main objective is to backtest the Gerber Statistic, benchmarked against the competing methods, the empirical results section is divided into results based on the different thresholds c values (equation 2): $c = 0.5$, 0.7 and 0.9.

6.1 Gerber Statistic with $c = 0.5$

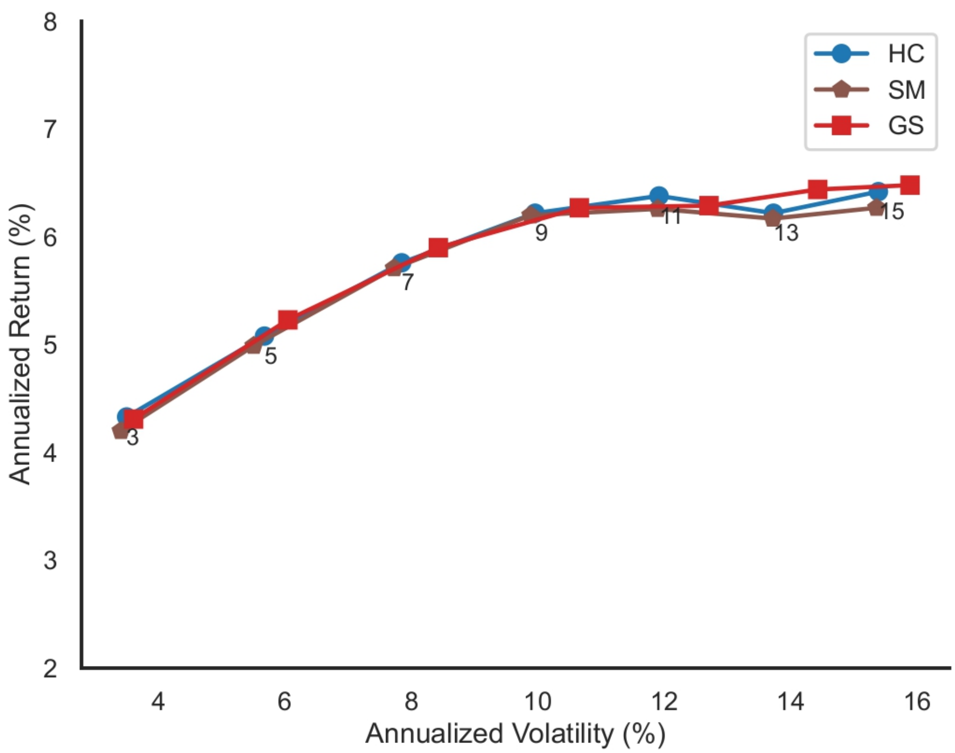

Figure 4: The ex-post efficient frontiers.

Figure 4 illustrates the realized performance in terms of annualized return and annualized volatility of portfolios with different targets of risk ranging from 3% - 15%. The blue frontier shows the ex-post performance of the historical covariance (HC). The brown frontier shows the ex-post performance of the Shrinkage Method (SM) and the red frontier shows the ex-post performance of Gerber Statistic. At the threshold c = 0.5 the Gerber statistic yields a slightly higher geometric return than both HC and SM for all risk levels except the 3 % risk level. Here the geometric returns are differentiated by 0.2 basis points. The portfolios have similar values of portfolio turnover, skewness and kurtosis. The performances are summarized on the first part of the table on page 25 at the end of this section. The table are calculated using excel files from appendix folder two. See the appendix for detailed performance.

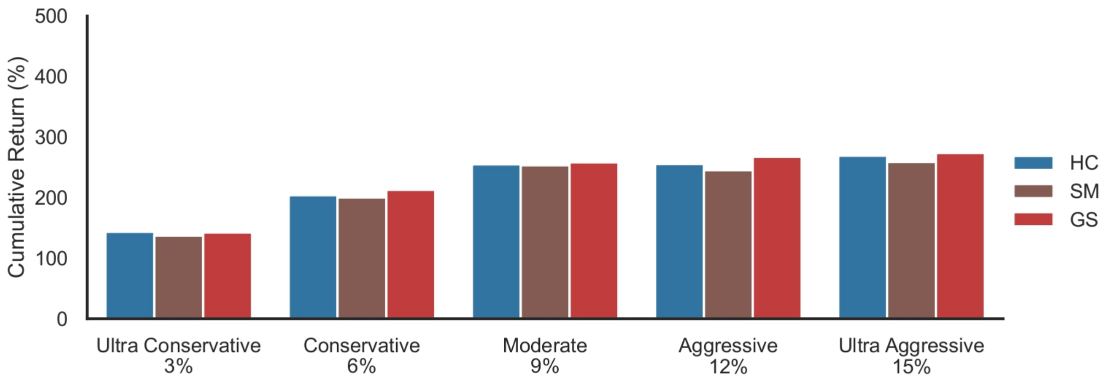

Figure 5: Cumulative returns in percentage for time period (2002-2001).

Figure 5 show HC-based portfolio, SM based portfolio and GS based portfolio at five different levels of risk. Given the Gerber threshold c = 0.5. The calculation is constructed so, it is assumed that 200.000 DKK was invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are rebalanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

For all levels of risk, the Gerber Statistic yields a high cumulative return. At 12 % risk target the Gerber Statistic outperforms the HC by more than 10 basis points and SM by more than 20 basis points. Before performing the study and the results was assessed it was assumed that the most standard computation method the historical covariance was outperformed by both the GS and the SM. This is not the case as the SM yields lower cumulative return for all levels of risk compared to both the GS and HC. The performances are summarized on the first part of the table on page 25 at the end of this section. The table are calculated using excel files from appendix folder two.

Method Portfolio risk target	HC	SM	GS
Ultra Conservative (3%)	DKK 487,105.97	DKK 474,319.08	DKK 484,721.00
Conservative (6%)	DKK 608,040.87	DKK 600,461.19	DKK 625,718.43
Moderate (9%)	DKK 709,948.92	DKK 706,846.93	DKK 716,526.71
Aggressive (12%)	DKK 711,083.23	DKK 690,571.31	DKK 734,870.36
Ultra Aggressive (15%)	DKK 738,601.43	DKK 717,834.44	DKK 747,480.51

Table 2: Account DKK value after timeperiod (January 2002 - December 2021)

Table 2 displays the account DKK value in December 2021 for the HC-based portfolio, SM-based portfolios and GS-based portfolios. This is again done at the five different risk targets and given the Gerber threshold c = 0.5. The table are calculated on the same scenario as above with 200.000 DKK invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are re-balanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

6.2 Gerber Statistic with $c = 0.7$

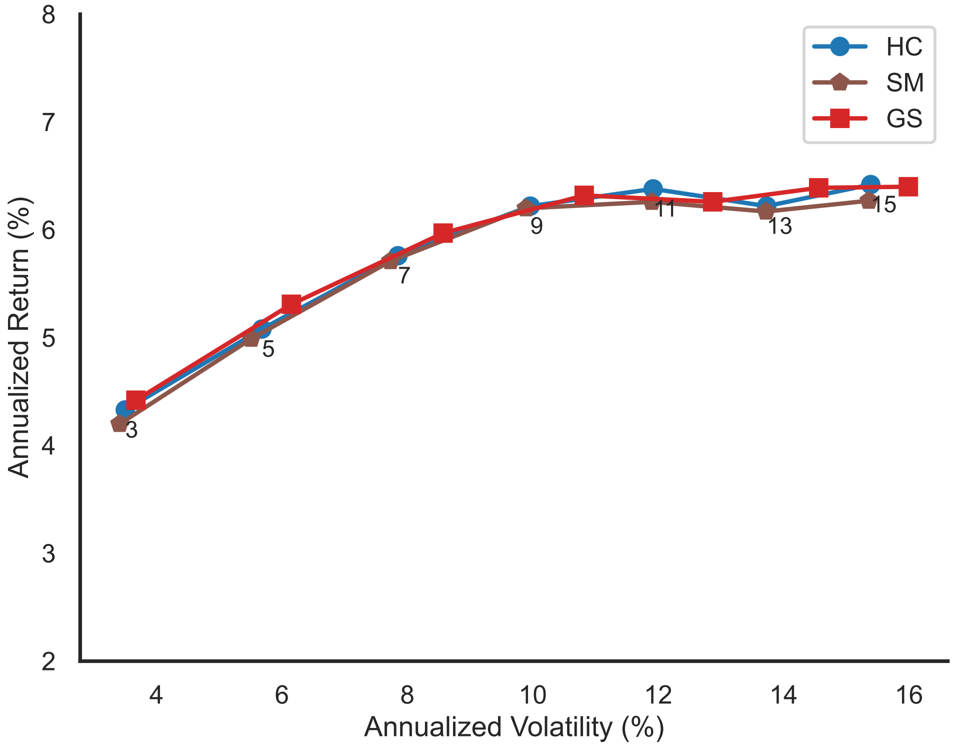

Figure 6: The ex-post efficient frontiers.

Figure 6 illustrates the realized performance in terms of annualized return and annualized volatility of portfolios with different targets of risk ranging from 3% - 15%. The blue frontier shows the ex-post performance of the historical covariance (HC). The brown frontier shows the ex-post performance of the Shrinkage Method (SM) and the red frontier shows the ex-post performance of Gerber Statistic. At the threshold c = 0.7 the Gerber statistic yields a slightly higher geometric return than both HC and SM for all risk levels. The portfolios have similar values of portfolio turnover, skewness and kurtosis. The Sharpe Ratios of the portfolios are very similar and hence, at this threshold the GS portfolio is not able to yield a more favorable risk return profile than the HC and SM.

The performances are summarized on the middle part of the table on page 25 at the end of this section. The table are calculated using excel files from appendix folder three.

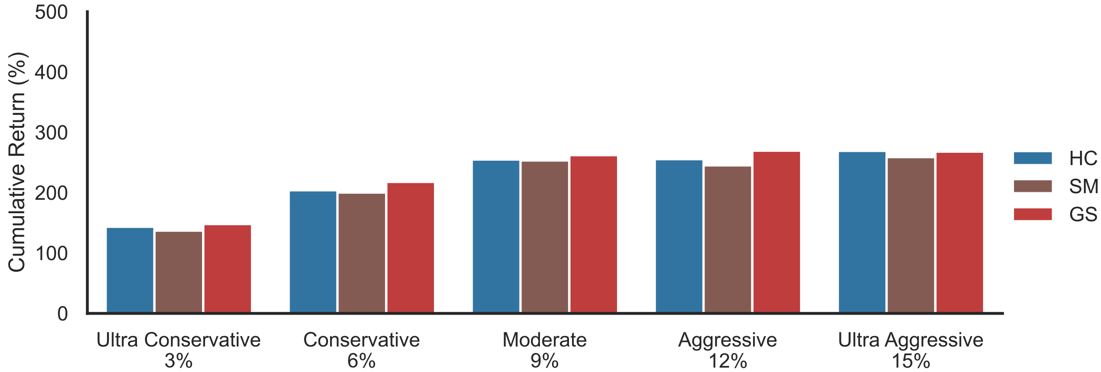

Figure 7: Cumulative returns in percentage for time period (2002-2001).

Figure 7 show HC-based portfolio, SM based portfolio and GS based portfolio at five different levels of risk. Given the Gerber threshold c = 0.7. The calculation is constructed so, it is assumed that 200.000 DKK was invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are rebalanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

For all levels of risk, the Gerber Statistic yields a high cumulative return. At 12 % risk target the Gerber Statistic outperforms the HC by more than 14 basis points and SM by more than 24 basis points. Before performing the study and the results was assessed it was assumed that the most standard computation method the historical covariance was outperformed by both the GS and the SM. This is not the case as the SM yields lower cumulative return for all levels of risk compared to both the GS and HC. The performances are summarized on the middle part of the table on page 25 at the end of this section. The table are calculated using appendix folder three. It should be noted that at the 15 % risk level at threshold c = 0.7 the GS and HC portfolio performance are almost similar.

Table 3 below highlight the account dollar value in December 2021 for the HC-based portfolio, SM-based portfolios and GS-based portfolios. This is again done at the five different risk targets and given the Gerber threshold c = 0.5. The table are calculated on the same scenario as above with 200.000 DKK invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are rebalanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

Method Portfolio	HC	SM	GS
Ultra Conservative (3%)	DKK 487,105.97	DKK 474,319.08	DKK 495,999.10
Conservative (6%)	DKK 608,040.87	DKK 600,461.19	DKK 635,802.74
Moderate (9%)	DKK 709,948.92	DKK 706,846.93	DKK 724,321.44
Aggressive (12%)	DKK 711,083.23	DKK 690,571.31	DKK 739,346.48
Ultra Aggressive (15%)	DKK 738,601.43	DKK 717,834.44	DKK 736,073.53

Table 3: Account DKK value after timeperiod (January 2002 - December 2021)

In the table it is shown that the GS yields a higher cumulative return compared to both the HC and SM and hence, yields a higher account DKK value for all risk targets except the ultra-aggressive portfolio with a 15 % risk target. Here the HC performs at a similar level as the GS. Again, for this data set and the selected time period the SM yields lower returns, at all risk targets, than both the HC and GS.

6.3 Gerber Statistic with $c = 0.9$

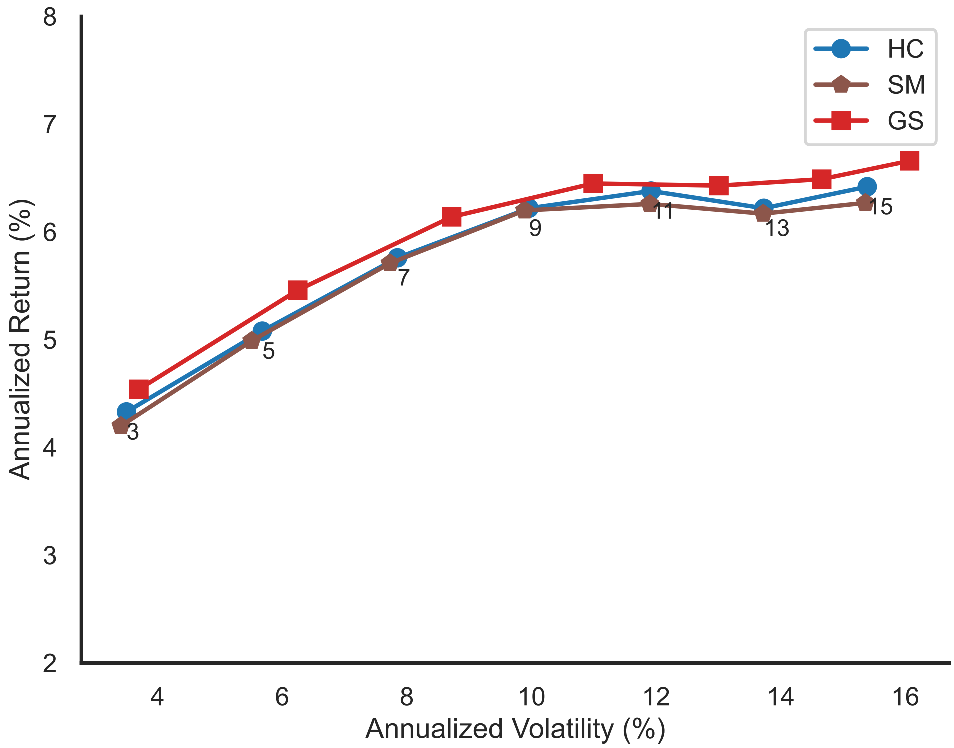

Figure 8: The ex-post efficient frontiers. Source: Own production inspiration Gerber et al (2021).

Figure 8 illustrates the realized performance in terms of annualized return and annualized volatility of portfolios with different targets of risk ranging from 3% - 15%. The blue frontier shows the ex-post performance of the historical covariance (HC). The brown frontier shows the ex-post performance of the Shrinkage Method (SM) and the red frontier shows the ex-post performance of Gerber Statistic. At the threshold c = 0.9 the Gerber statistic yields a slightly higher geometric return than both HC and SM for all risk levels. The portfolios have similar values of portfolio turnover, skewness and kurtosis. The Sharpe Ratios of the portfolios are very similar and hence, at this threshold the GS portfolio is not able to yield a more favorable risk return profile than the HC and SM. See

The performances are summarized on the last and bottom part of the table on page 25 at the end of this section. The table are calculated using appendix folder four.

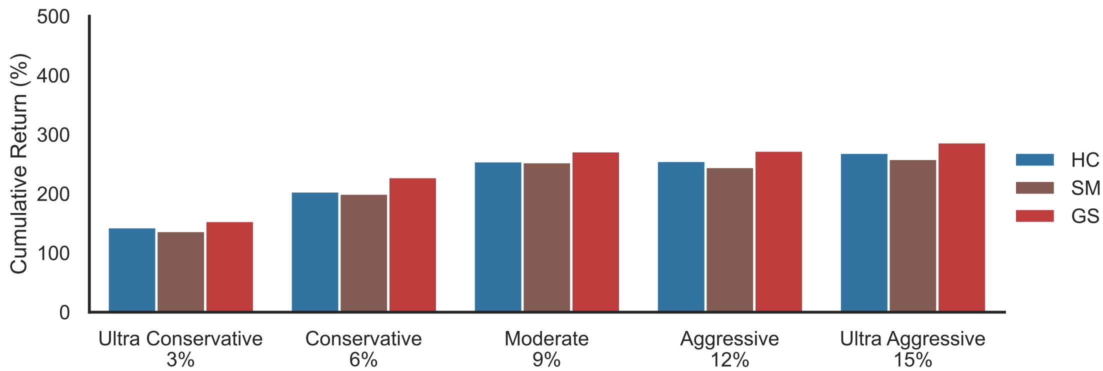

Figure 9: Cumulative returns in percentage for time period (2002-2001).

Figure 9 show HC-based portfolio, SM based portfolio and GS based portfolio at five different levels of risk. Given the Gerber threshold c = 0.9. The calculation is constructed so, it is assumed that 200.000 DKK was invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are rebalanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

For all levels of risk, the Gerber Statistic yields a higher cumulative return. Furthermore, the Gerber statistics yields a higher yearly return. At 12 % risk target the Gerber Statistic outperforms the HC by more than 20 basis points and SM by more than 38 basis points. At 15 % risk target the Gerber Statistic outperforms the HC by more than 35 basis points and SM by more than 45 basis points. Again, the SM yields lower cumulative return and annualized returns for all levels of risk compared to both the GS and HC. The performances are summarized on the last part of the table on page 25 at the end of this section.

Table 4 below highlight the account dollar value in December 2021 for the HC-based portfolio, SM-based portfolios and GS-based portfolios. This is again done at the five different risk targets and given the Gerber threshold c = 0.9. The table are calculated on the same scenario as above with 200.000 DKK invested in January 2002 and left to grow according to the portfolio weights calculated by each of the covariance methods and the given risk targets until December 2021. Note that the portfolio weights are rebalanced monthly based on changes in the covariance between the securities and thereby a new optimal portfolio is constructed.

Method Portfolio	HC	SM	GS
Ultra Conservative (3%)	DKK 487,105.97	DKK 474,319.08	DKK 508,164.97
Conservative (6%)	DKK 608,040.87	DKK 600,461.19	DKK 656,404.98
Moderate (9%)	DKK 709,948.92	DKK 706,846.93	DKK 743,692.90
Aggressive (12%)	DKK 711,083.23	DKK 690,571.31	DKK 745,903.30
Ultra Aggressive (15%)	DKK 738,601.43	DKK 717,834.44	DKK 773,972.77

Table 4: Account DKK value after timeperiod (January 2002 - December 2021)

In the table it is shown that the GS yields a higher cumulative return compared to both the HC and SM and hence, yields a higher account DKK value for all risk targets. The table indicates that the Gerber Statistic with c = 0.9 provides the a most consistent results as the GS yields a higher account value at all risk targets. Again, for this data set and the selected time period the SM yields lower returns, at all risk targets, than both the HC and GS.

6.4 1/N portfolio

Now a comparison of the different covariance computation methods has been conducted. The following section will compare these portfolios made under the mean variance optimization framework to the more “naïve” and easy computable weight allocation, the 1/N portfolio. The portfolio is made on the same data set and under the same timeframe. The portfolio characteristics are highlighted in table 5 below.

1/N - Equal Weights Portfolio	
Sum of Weights	1
Weekly Portfolio Return	0.14 %
Annual Portfolio Return	7.42 %
Weekly Portfolio STD	1.92 %
Annual Portfolio STD	13.86 %
Sharpe Ratio	0.54

Table 5: Equal weights portfolio performance - Time period (January 2002 - December 2021)

6.4.1 1/N portfolio compared to HC portfolio

The HC based portfolio performance differentiate from the different levels of risk.

- At 3 % risk the annual return is 4.44 % with a Sharpe Ratio of 0.94.
- At 6 % risk the annual return is 5.74 % with a Sharpe Ratio of 0.66.
- At 9 % risk the annual return is 6.85 % with a Sharpe Ratio of 0.53.
- At 12 % risk the annual return is 7.23 % with a Sharpe Ratio of 0.41.
- At 15 % risk the annual return is 7.85 % with a Sharpe Ratio of 0.36.

From table at page 25.

At the four lowest levels of risk the 1/N portfolio yields a higher return with a higher Sharpe ratio than the HC based mean variance optimized portfolios. With an annual return of 7.42 % the 1/N yields a slightly lower portfolio return that the HC based at 15 % risk with an annual return of 7.85 %. Based on this comparison is seem that 1/N portfolios perform better to lower levels of risk.

6.4.2 1/N compared to SM portfolio

The SM based portfolio performance differentiate from the different levels of risk.

- At 3 % risk the annual return is 4.30 % with a Sharpe Ratio of 0.93.
- At 6 % risk the annual return is 5.69 % with a Sharpe Ratio of 0.67.
- At 9 % risk the annual return is 6.86 % with a Sharpe Ratio of 0.53.
- At 12 % risk the annual return is 7.12 % with a Sharpe Ratio of 0.40.
- At 15 % risk the annual return is 7.75 % with a Sharpe Ratio of 0.35.

From table at page 25.

Here we see similar results, were the 1/N portfolio yields a higher return at the four lowest levels of risk, than the SM based mean variance optimized portfolios. With an annual return of 7.42 % the 1/N yields a slightly lower portfolio return that the SM based at 15 % risk with an annual return of 7.75 %. Based on this comparison is seem that 1/N portfolios perform better to lower levels of risk.

6.4.3 1/N compared to GS thresholds $c = 0.9$

The GS based portfolios performance differentiate from the different levels of risk and the selected threshold c. The Gerber Statistic performs best at c = 0.9, hence this is the threshold level that will be compared to the 1/N portfolio.

- At 3 % risk the annual return is 4.67 % with a Sharpe Ratio of 0.95.
- At 6 % risk the annual return is 6.20 % with a Sharpe Ratio of 0.65.
- At 9 % risk the annual return is 7.02 % with a Sharpe Ratio of 0.50.
- At 12 % risk the annual return is 7.51 % with a Sharpe Ratio of 0.40.
- At 15 % risk the annual return is 8.02 % with a Sharpe Ratio of 0.35.

See last part of table at page 25

At the three lowest levels of risk the 1/N portfolio yields a higher return. With an annual return of 7.42 % the 1/N yields a slightly lower portfolio return that the GS based at 15 % risk with an annual return of 8.02 %.

The performance of the 1/N portfolio is very similar to the mean variance optimized portfolio for all three computation methods. If the risk targets are set low for the mean variance portfolios the 1/N performance is better. Based on this comparison is seem that 1/N portfolios perform better to lower levels of risk. Here is should be noted that the annual standard deviation of the 1/N portfolio is 13.86 %. Compared to the portfolios with a risk target of 3 % which have a similar standard deviation it is fair to say that the 1/N portfolio is not a low risk portfolio and therefor it is anticipated that this portfolio should outperform at portfolio with a standard deviation around 3 %.

	03pct			06pct			09pct			12pct			15pct		
	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS
Arithmetic Return (%)	4.44	4.30	4.42	5.74	5.69	5.94	6.85	6.86	7.02	7.23	7.12	7.51	7.85	7.75	8.02
Geometric Return (%)	4.33	4.20	4.31	5.44	5.37	5.58	6.22	6.20	6.27	6.23	6.08	6.39	6.42	6.27	6.48
Cumulative Return (%)	143.55	137.16	142.36	204.02	200.23	212.86	254.97	253.42	258.26	255.54	245.29	267.44	269.30	258.92	273.74
Annualized STD (%)	3.50	3.41	3.61	6.77	6.62	7.24	9.96	9.89	10.66	12.86	12.85	13.59	15.39	15.36	15.89
Annualized Skewness	-0.82	-0.72	-1.12	-1.43	-1.41	-1.49	-1.55	-1.54	-1.52	-1.53	-1.54	-1.46	-1.46	-1.45	-1.38
Annualized Kurtosis	7.07	6.21	10.20	11.32	11.00	11.79	12.14	11.74	11.74	12.05	11.69	11.33	11.34	10.84	11.01
Maximum Drawdown (%)	-8.35	-7.42	-8.41	-17.62	-17.56	-19.47	-27.34	-26.62	-29.96	-37.49	-37.12	-40.95	-46.86	-46.64	-49.96
Monthly 95% VaR (%)	-0.70	-0.67	-0.71	-1.35	-1.32	-1.49	-2.06	-2.06	-2.19	-2.59	-2.59	-2.74	-3.09	-3.05	-3.22
Shape Ratio	0.94	0.93	0.91	0.66	0.67	0.64	0.53	0.53	0.50	0.41	0.40	0.40	0.36	0.35	0.35
Turnover	1.38	1.13	1.37	3.93	3.55	3.78	5.88	5.55	5.85	8.09	7.85	8.12	9.57	9.44	9.68

	03pct			06pct			09pct			12pct			15pct		
	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS
Arithmetic Return (%)	4.44	4.30	4.54	5.74	5.69	6.04	6.85	6.86	7.09	7.23	7.12	7.56	7.85	7.75	7.93
Geometric Return (%)	4.33	4.20	4.42	5.44	5.37	5.66	6.22	6.20	6.32	6.23	6.08	6.42	6.42	6.27	6.40
Cumulative Return (%)	143.55	137.16	148.00	204.02	200.23	217.90	254.97	253.42	262.16	255.54	245.29	269.67	269.30	258.92	268.04
Annualized STD (%)	3.50	3.41	3.67	6.77	6.62	7.37	9.96	9.89	10.82	12.86	12.85	13.73	15.39	15.36	15.99
Annualized Skewness	-0.82	-0.72	-1.03	-1.43	-1.41	-1.51	-1.55	-1.54	-1.56	-1.53	-1.54	-1.50	-1.46	-1.45	-1.43
Annualized Kurtosis	7.07	6.21	8.71	11.32	11.00	11.59	12.14	11.74	11.91	12.05	11.69	11.55	11.34	10.84	11.37
Maximum Drawdown (%)	-8.35	-7.42	-9.49	-17.62	-17.56	-20.15	-27.34	-26.62	-30.49	-37.49	-37.12	-41.59	-46.86	-46.64	-50.10
Monthly 95% VaR (%)	-0.70	-0.67	-0.73	-1.35	-1.32	-1.48	-2.06	-2.06	-2.25	-2.59	-2.59	-2.76	-3.09	-3.05	-3.32
Shape Ratio	0.94	0.93	0.93	0.66	0.67	0.64	0.53	0.53	0.50	0.41	0.40	0.40	0.36	0.35	0.34
Turnover	1.38	1.13	1.39	3.93	3.55	3.83	5.88	5.55	5.96	8.09	7.85	8.28	9.57	9.44	9.98

	03pct			06pct			09pct			12pct			15pct		
	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS	HC	SM	GS
Arithmetic Return (%)	4.44	4.30	4.67	5.74	5.69	6.20	6.85	6.86	7.23	7.23	7.12	7.63	7.85	7.75	8.21
Geometric Return (%)	4.33	4.20	4.54	5.44	5.37	5.82	6.22	6.20	6.45	6.23	6.08	6.47	6.42	6.27	6.66
Cumulative Return (%)	143.55	137.16	154.08	204.02	200.23	228.20	254.97	253.42	271.85	255.54	245.29	272.95	269.30	258.92	286.99
Annualized STD (%)	3.50	3.41	3.70	6.77	6.62	7.49	9.96	9.89	10.99	12.86	12.85	13.86	15.39	15.36	16.07
Annualized Skewness	-0.82	-0.72	-1.05	-1.43	-1.41	-1.46	-1.55	-1.54	-1.53	-1.53	-1.54	-1.49	-1.46	-1.45	-1.41
Annualized Kurtosis	7.07	6.21	8.90	11.32	11.00	11.19	12.14	11.74	11.54	12.05	11.69	11.46	11.34	10.84	11.24
Maximum Drawdown (%)	-8.35	-7.42	-9.21	-17.62	-17.56	-20.50	-27.34	-26.62	-30.89	-37.49	-37.12	-42.77	-46.86	-46.64	-50.53
Monthly 95% VaR (%)	-0.70	-0.67	-0.76	-1.35	-1.32	-1.54	-2.06	-2.06	-2.31	-2.59	-2.59	-2.82	-3.09	-3.05	-3.36
Shape Ratio	0.94	0.93	0.95	0.66	0.67	0.65	0.53	0.53	0.50	0.41	0.40	0.40	0.36	0.35	0.36
Turnover	1.38	1.13	1.34	3.93	3.55	3.86	5.88	5.55	6.00	8.09	7.85	8.39	9.57	9.44	9.78

7 conclusion

This paper has now tested whether it was possible to increase future portfolio performance by addressing some of the estimation errors connected to the mean variance framework. As mentioned, the estimation of the covariance has been on of the stickiest points in portfolio historically, with different authors and research yielding contradictory results. Which has led to a discussion on whether the errors in estimating means and covariances erupt the gains from the optimal allocation relative to the naïve diversification.

The conclusion on whether it is possible to enhance future portfolio performance by using extensions on the covariance computation is not so clear. The Shrinkage Method was, in general, outperformed by the Gerber Statistic but surprisingly also the standard historical covariance computation method for this data set with the selected timeframe. The modest performance of the Shrinkage Method led to a speculation on whether the calculations on this method was done correctly and if the statistical properties of the data was as assumed in the methodology. The test on the statistical properties is done though code in python and the data held the assumptions. See appendix 5 for code with description. An interesting exercise could be to perform the same study but with stocks instead of indices as the stocks fluctuate more in price and the Shrinkage Estimator could perform better. The thought behind this is that the indices price fluctuation is relatively flat, compared to stocks, at the extreme values that is "shrunk" is less than if the data was on stocks.

The Gerber Statistic performed more in line with Gerber et al. (2021) own research as the computation method was able to enhance the future performance. Here the question is more regarding the magnitude of the results. The Gerber Statistic performed better than the Shrinkage Method for all levels of risk and all threshold c and performed better than the HC computation method for most levels of risk with few exceptions, where the performance was similar. However, the magnitude of the better performance is not substantial as the biggest difference in the cumulative DKK account value after the 20 years, between HC and GS, was DKK 35.371 corresponding to a cumulative return of 269.30 % at 15 % risk target for the HC based portfolio and a cumulative return of 286.99 % at 15 % risk target for the GS based portfolio at threshold c = 0.9. In conclusion the Gerber Statistic yields in general a better performance, than the historical covariance method and the Shrinkage Method. However, the results don't change dramatically, between using the standard computation method the historical covariance (HC) and the Gerber Statistic, for this time period and using the selected indices. Thereby, performing this study, it is understandable that the empirical work presented in the literature review show contradictory results. As the Gerber Statistic indeed do perform better for this time period and with the selected data, but the magnitude of the results is not overly significant. Furthermore, several authors like Demiguel et al. (2009) brought question to whether the errors in estimating means and co-variances erupt the gains from the optimal allocation relative to the naïve diversification. The performance of the 1/N portfolio in this study was very similar to the performances of the ones gained from the optimal allocation framework. The standard deviation of the portfolio was 13.86 % if we look to the optimal allocated portfolios with a similar risk target 15 % (which is the closest to the 13,86 %) the annual return of the HC, SM and GS portfolios was respectively 7.85 %, 7.75 % and GS at threshold c = 0.5, 0.7 and 0.9 the return was respectively 8.02 %, 7.93 % and 8.21 %. These annual returns compared to the 1/N annual return of 7.42 % is not a significant difference. Nevertheless, the optimal allocated portfolios perform better than the equal weighted portfolio at similar levels of standard deviation for this data set and selected time period.

In conclusion the selected covariance estimation methodologies do not differentiate significantly in generating better future performances benchmarked against each other. Though the Gerber Statistic proved to generate the best performances compared to the selected computation methods on the dataset used in this paper. Furthermore, the 1/N approach did not outperform the optimal allocated portfolios at even levels of standard deviation. However the performance was very similar and based on the empirical results of this paper, the ongoing discussion in the literature is reasonable.

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