Copulas and Value-at-Risk:
Risk Estimation of Portfolios

Charlotte Kargo Lauridsen, Henriette Rønfeldt Pedersen & Thea Lund Jørgensen

Aalborg University - Master Thesis

June 2, 2022

AALBORG UNIVERSITY
STUDENT REPORT

Mathematics-Economics
Skjernvej 4A
9220 Aalborg Øst
http://www.aau.dk

Title:
Copulas and Value-at-Risk:
Risk Estimation of Portfolios

Project Period:
Spring Semester 2022

Project Group:
Group 1.204a

Participants:
Charlotte Kargo Lauridsen
Henriette Rønfeldt Pedersen
Thea Lund Jørgensen

Supervisor:
J. Eduardo Vera-Valdés

Date of Completion:
June 2, 2022

Page Count: 65

Abstract:

Dette projekt analyserer, hvordan *Value-at-Risk* kan prædikteres en dag frem ved brug af ARMA-GARCH modeller, copulaer og Monte Carlo simulationer. I projektet præsenteres en Monte Carlo procedure, som anvendes til at udføre rullende prædiktioner henover to år. Der betragtes to porteføljer med fem aktier i hver fra perioden 2010-2021, hvor perioden 2020-2021 prædikteres *out-of-sample.* Disse aktier er valgt for at repræsentere aktivklasserne *global equity* og *emerging market equity.* Ved brug af backtesting og statistiske test konkluderes det, at *Value-at-Risk* prædiktionerne er tilfredsstillende. De prædikterede *Value-at-Risks* sammenlignes med *Value-at-Risk* baseret på information fra Rådet for Afkastforventninger. *Value-at-Risk* baseret på information fra førnævnte råd under- og overestimerer for henholdsvis *global equity* og *emerging market equity.* Til sidst undersøges det, om de 5 valgte aktier er repræsentative for deres respektive markeder, hvor det konkluderes, at de to porteføljer viser de samme tendenser som de to indekser for markedet.

Preface

This master thesis is written in the spring semester of 2022 by group 1.204a, which consists of Charlotte Kargo Lauridsen, Henriette Rønfeldt Pedersen and Thea Lund Jørgensen. The students study Mathematics and Economics at the School of Engineering and Science at Aalborg University. To write this project the programs **Overleaf** and **RStudio** are used.

The group would like to give a special acknowledgement to,

- J. Eduardo Vera-Valdés, supervisor and Associate Professor at the Department of Mathematical Sciences at Aalborg University, for his guidance and help throughout the forming of this project.
- Bo Mølgaard Nielsen, head of department at the Middle Office for Ringkjøbing Landbobank, for his help and insight into financial institutions.

Reading Guide

Throughout this report the references will appear with labels [1], [2] and so forth. All the references can be found as a numbered list in the bibliography at page 59. The citations in the text will have a reference to the bibliography, where the books are given by author, title, publisher and year, while the internet references are given by author, title, when it is last accessed, year and link. References are primarily given in the start of a chapter or section, when the source is used throughout the chapter or section. Otherwise, the source is given directly where it is used.

Signatures

Charlotte Kargo Lauridsen
clauri17@student.aau.dk

Henriette Rønfeldt Pedersen
hrpe17@student.aau.dk

Thea Lund Jørgensen
tjarge17@student.aau.dk

Contents

1 Introduction

The financial market is concerned with trading of assets and in this regard a useful concept is risk. Investors construct portfolios with the intent of generating profit and avoid loss, however the outcome of a portfolio is uncertain and thus it is beneficial to investigate the risk exposure of a portfolio. In order to construct models that are able to describe the risk exposure of a portfolio, the models need to reflect the risk factors of the data. For this reason it is beneficial to understand the data in the portfolio.

Financial time series often exhibit some stylised facts, which are empirically observed tendencies that are consistently present in financial data. These stylised facts stated in [18, p. 79-80] include

- Log-return series are not independent and identically distributed.
- The squared log-returns show serial correlation.
- The volatility of a log-return series varies over time.
- The log-return series exhibit volatility clustering.
- Log-return series are heavy tailed.

These stylised facts indicate there are dependencies and correlation in the log-return series, which can then be used for prediction purposes. In order to incorporate these stylised facts and construct a model that reflects these tendencies seen in financial data an ARMA-GARCH model is used to address the autoregressive and heteroskedastic behaviour of the log-return series. Such a model can then be used to evaluate the risk exposure of a portfolio. Furthermore, portfolios often consist of numerous assets, which may not exhibit the same tail heaviness or distribution. Thus, copulas are introduced to allow for more flexibility in describing the univariate log-returns and combine them in a joint distribution. Different types of copulas exist and they are used to describe the dependencies between the univariate margins in a multivariate distribution. After the ARMA-GARCH models are fitted in a multivariate manner using copula theory, the model can be used in risk assessment of a portfolio. One method to measure the risk is Value-at-Risk. This method calculates the highest expected loss to occur with a specified probability. Value-at-Risk describes a bound, the loss will not exceed with a specified probability, which is typically a 95% or a 99% confidence level. In other words, Value-at-Risk describes a maximum loss to occur e.g. 5% or 1% of the time and can be used by investors to determine if a portfolio is too risky or to determine the amount of capital an investor should posses in case of a loss. There are different methods to calculate the Value-at-Risk, which include non-parametric estimation, parametric estimation and Monte Carlo simulations. The focus of this project is to investigate the performance of a Monte Carlo method.

When Danish pension companies and financial institutions assess an investment product in regard to risk and return, their calculations are based on the common expectations published by the Council for Return Expectations. The council publishes the expected returns, expected standard deviations and expected correlations for different groups of assets. Two of these groups are global equity and emerging markets equity. In this project, the focus will be on these two groups. Global equity consists of assets from developed markets, which are countries such as the United States, Denmark and Japan. Emerging markets equity consists of assets from emerging markets, which are countries such as Mexico, Korea and Taiwan. A portfolio for each of these groups are constructed with five assets to represent the two groups. The Council for Return Expectations publishes their expectations twice a year and for different holding periods. The council published their first expectations in January 2020. Expectations on the short term are anticipated to be realised after a year. For this reason, only the reports for 2020 are considered, in order to be able to evaluate the realised performance of these reports in risk assessment. These reports are published shortly before or during the development of the corona pandemic, which may affect the accuracy of the predictions.

1.1 Statement of Intent

This leads to the following statement of intent:

How can one-day-ahead Value-at-Risk be forecasted for portfolios using copulas and Monte Carlo simulations and how can these forecasts be evaluated? How does the forecasts perform compared with the common expectations of return and standard deviation published by the Council for Return Expectations and are the portfolios representative for the respective groups?

Risk and Model Theory 2

In this chapter, fundamental theory to asses risk of a portfolio and model data to obtain accurate one-day-ahead forecasts of risk is presented. The method to calculate risk is called Value-at-Risk and is used to determine the highest expected loss to occur with some probability. Then a mean-variance model called ARMA-GARCH is introduced to capture information in the data and obtain a fitted model. Then copula theory is introduced in order to allow for flexibility in the distribution of residuals from multiple ARMA-GARCH models. Next, a Monte Carlo procedure is presented with the purpose of forecasting Value-at-Risk using an ARMA-GARCH model with copulas. Lastly, backtesting methods are introduced in order to evaluate the forecasted Value-at-Risks.

2.1 Value-at-Risk

This section is based on [1], [18] and [20].

In this section, a method to measure risk called *Value-at-Risk* (VaR) is introduced. This method is commonly used amongst financial institutions to determine the risk exposure of a portfolio. The risk measure VaR is the highest expected loss to occur with a specified probability over a specified holding period. The method has two parameters, namely the time horizon denoted T and the confidence level denoted α, where $\alpha \in [0, 1]$ and typical values of α are 0.95 or 0.99. The VaR describes a bound, where the probability of having a loss greater than this bound over the time horizon is smaller than $1 - \alpha$. The VaR is defined as

$$\mathrm{VaR}_\alpha = \inf\{x \;:\; \mathbb{P}(\mathcal{L} > x) \leq 1 - \alpha\},$$

where $\mathcal{L}$ is the loss over the holding period T. To exemplify the VaR, suppose the holding period is a year, the confidence level is 95% and the VaR is 1 million. Then there is a 5% risk of having a loss greater than 1 million. Thus, VaR measures a potential loss of an investment with a specified probability and can be used to determine the amount of extra capital an investor should posses to avoid bankruptcy in case of a loss.

2.1.1 Estimation of Value-at-Risk

There are different methods to estimate VaR, which is non-parametric estimation, parametric estimation and Monte Carlo estimation. These methods are described next in the case where the portfolio consists of one asset.

In non-parametric estimation the loss is not assumed to belong to a parametric family, such as the normal distribution or the t-distribution. Instead the VaR is calculated from historical data. The confidence level, α, is found by estimating the α-quantile of the return distribution. This quantile is estimated as the α-quantile of a sample of historical returns, which is denoted as $\hat{q}(\alpha)$. Let S denote the size of the current position. The non-parametric estimate of VaR is defined as

$$\widehat{\text{VaR}}_\alpha^{\text{np}} = -S \times \hat{q}(\alpha).$$

Note, the minus sign means the potential loss is returned rather than potential revenue. Since non-parametric estimation uses historical data and is build on the idea that the history will repeat itself, it is preferable to have a large data set to ensure there is enough information in the data to represent different outcomes in time, such as financial crises. In addition, the quantiles are used to calculate the VaR, so the data set should be reasonably large and α should not be too high to avoid inference on outliers. The non-parametric approach has the advantage of not assuming the distribution of data, which means the distribution of the data cannot be misspecified. If the distribution is misspecified it could lead to overestimation or underestimation of the VaR. The problems of misspecification are especially noteworthy in times of financial crises. However, non-parametric estimation needs large data sets to be accurate and thus also heavier computations. For this reason the parametric approach is introduced.

In parametric estimation the loss is assumed to belong to a parametric family, such as the normal distribution or the t-distribution. Assume $F_{\mathcal{L}}(x) = \mathbb{P}(\mathcal{L} \leq x)$ is the loss distribution with mean μ and variance σ^2. Let ϕ be the standard normal distribution function and let $\phi^{-1}(\alpha)$ be the α-quantile of ϕ. Then the normal parametric VaR can be calculated as

$$\widehat{\text{VaR}}_\alpha^{\text{par, Ga}} = -S \times (\hat{\mu} + \hat{\sigma}\phi^{-1}(\alpha)). \tag{2.1}$$

A point x_0 is called the α-quantile of the distribution function F if $F(x_0) \geq \alpha$ and $F(x) < \alpha$ for all $x < x_0$. This method of calculating the VaR is proved for $S = 1$ by showing $F_{\mathcal{L}}(\text{VaR}_\alpha) = \alpha$ as

$$\begin{aligned}
\mathbb{P}(\mathcal{L} \leq \text{VaR}_\alpha) &= \mathbb{P}(\mathcal{L} \leq \mu + \sigma\phi^{-1}(\alpha)) \\
&= \mathbb{P}\Big(\frac{\mathcal{L} - \mu}{\sigma} \leq \phi^{-1}(\alpha)\Big) \\
&= \mathbb{P}\Big(\phi\Big(\frac{\mathcal{L} - \mu}{\sigma}\Big) \leq \alpha\Big) \\
&= \phi(\phi^{-1}(\alpha)) = \alpha.
\end{aligned}$$

Note, in the second last equality Proposition 2.2 is used. Assume for the loss function that $\frac{\mathcal{L}-\mu}{\sigma}$ has a standard t-distribution with ν degrees of freedom, where the loss distribution is denoted as $\mathcal{L} \sim t(\nu, \mu, \sigma^2)$. Note, for the t-distributed losses $\mathbb{E}(\mathcal{L}) = \mu$ and $\text{var}(\mathcal{L}) = \sigma^2 \frac{\nu}{\nu-2}$, thus σ is not the standard deviation, but instead a scale parameter. The parameter is found by isolation, as

$$\text{var}(\mathcal{L}) = \sigma^2 \frac{\nu}{\nu - 2} \Leftrightarrow \sigma = \sqrt{\text{var}(\mathcal{L}) \frac{\nu - 2}{\nu}}. \tag{2.2}$$

The VaR is then calculated as

$$\widehat{\text{VaR}}_\alpha^{\text{par, t}} = -S \times (\hat{\mu} + \hat{\sigma} t_\nu^{-1}(\alpha)), \tag{2.3}$$

where t_ν is the standard t-distribution function with ν degrees of freedom. The Monte Carlo approach uses simulations to imitate possible outcomes, and from these simulations find the

VaR. The idea is to simulate a number of log-returns, which represent the possibly and likely outcomes. These log-returns are sorted in increasing order and the α-quantile of these simulated log-returns is the VaR.

The three methods can be extended to the case where a portfolio consists of multiple assets. This extension is presented for the parametric method in the next section. The extension of the Monte Carlo approach is presented in Section 2.4.

2.1.2 Estimation of Value-at-Risk for a Portfolio

In this section, the parametric estimation method to calculate VaR is presented for a portfolio with multiple assets. The parametric estimation in this case is based on the assumption that returns of assets are multivariate normal or t-distributed, where the return of the portfolio consisting of the assets is univariate normal or t-distributed [20]. This assumption is relaxed by use of copulas, which allow returns of different assets to have different distributions. Copula theory is presented in Section 2.3.

The following theory is true for portfolios containing only stocks and no other types of assets. If other types of assets are included the estimation of VaR becomes more complex. For a portfolio consisting only of stocks, when the means are estimated, the expected return of the portfolio is calculated as

$$\hat{\mu}_P = \sum_{i=1}^{d} w_i \mu_i = \boldsymbol{w}^\top \boldsymbol{\mu},$$

where d is the number of assets, w_i is the weight for asset i, $\boldsymbol{w}$ is the vector of portfolio weights such that $\mathbf{1}^\top \boldsymbol{w} = 1$, μ_i is the expected return of asset i and $\boldsymbol{\mu}$ is the vector of the expected returns. Further, the variance of the return of the portfolio is calculated using the estimated variance-covariance matrix Σ as

$$\hat{\sigma}_P = \boldsymbol{w}^\top \Sigma \boldsymbol{w}.$$

Then, assuming normally distributed returns of the portfolio, VaR is estimated as

$$\widehat{\text{VaR}}_{P,\alpha}^{\text{par, Ga}} = -S \times (\hat{\mu}_P + \phi^{-1}(\alpha)\hat{\sigma}_P),$$

where S is the current position of the portfolio and $\phi^{-1}(\alpha)$ is the normal quantile. If the returns of the portfolio follow a t-distribution instead with scale parameter σ_P given in (2.2), mean μ_P, and tail index ν, then

$$\widehat{\text{VaR}}_{P,\alpha}^{\text{par, }t} = -S \times (\hat{\mu}_P + t_\nu^{-1}(\alpha)\hat{\sigma}_P),$$

is the estimated VaR of the portfolio. Further, if the stocks have a joint normal or a joint t-distribution, then VaR can be calculated by (2.1) or (2.3) since the returns of the portfolio then follow a univariate normal or t-distribution with the same tail index.

2.2 Mean-Variance Model

This section is based on [20].

In this section, the ARMA-GARCH model is briefly introduced in order to capture autoregressive and heteroskedastic behaviour in the data and fit the best model, which in terms will produce more accurate forecasts of VaR.

An *Autoregressive Moving Average* (ARMA) model consists of two terms, which is an autoregressive (AR) part and a moving average (MA) part. An AR(p) model for Y_t is defined as

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t, \quad \forall t,$$

where ε_t is a white noise process, $\varepsilon_t \sim \mathrm{wn}(0, \sigma_\varepsilon^2)$, and ϕ_i for $i = 1, \ldots, p$ are parameters that depend on the past. Information from the past is incorporated into the model through $\phi_i Y_{t-i}$, where larger values of ϕ_i incorporate more information from the past.

Consider the AR(1) model. If the process is stationary, then Y_t and Y_{t-1} have the same variance, denoted σ_Y^2. Then the variance of the AR(1) model can be written as

$$\sigma_Y^2 = \phi^2 \sigma_Y^2 + \sigma_\varepsilon^2.$$

In order for Y_t and Y_{t-1} to have the same variance $|\phi|$ must be smaller than 1, which equates to the process being stationary. If $|\phi| > 1$ the process has an explosive behaviour. When the process Y is repeatedly inserted into the AR(1) model for $|\phi| < 1$, it yields the *infinite moving average*, denoted MA(∞), representation of the process, which is

$$Y_t = \mu + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots = \mu + \sum_{h=0}^{\infty} \phi^h \varepsilon_{t-h}.$$

The MA(∞) process is a weighted average of all past values of the white noise process. This process can be approximated with fewer past values since $\phi^h \to 0$ for $h \to \infty$, which mean for large values of h the weight on the past value is small and thus the past value will have almost no effect on the process Y_t. The MA(q) model is defined as

$$Y_t = \mu + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \quad \forall t.$$

An ARMA(p,q) model is a combination of the AR process and the MA process, defined as

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \quad \forall t.$$

In the ARMA model, the process Y_t depends on both lagged values of itself and lagged values of the noise process. The AIC and BIC values can be used to determine which model is preferred. Furthermore, the ACF, QQ-plot and time series plots of the residuals can be used to analyse if the assumptions on the error term are satisfied. Such an analysis will often show heavy tails in the QQ-plot and volatility clustering in the time series plot for financial data. These problems can be addressed using a *Generalised Autoregressive Conditional Heteroskedasticity* (GARCH) model on the residuals. The GARCH(p,q) model is defined as

$$\varepsilon_t = a_t \sigma_t,$$

where

$$\sigma_t = \sqrt{\omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2},$$

and a_t is a white noise process with $\sigma_a^2 = 1$. Further, ω, α_i for $i = 1, \dots, p$ and β_j for $j = 1, \dots, q$ are constants, where $\omega > 0$. The sum satisfies that $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ where $\alpha_i, \beta_j \geq 0$.

The term $\sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2$ contains past values of the process, ε_t, which mean the process depends on its past. The term $\sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$ contains past volatilities and allows the conditional standard deviation to exhibit periods with more persistent volatility compared to an ARCH model, which is a special case of the GARCH model defined with $\sigma_t = \sqrt{\omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2}$.

2.3 Copulas

This section is based on [10], [11], [18], [19] and [20].

In this section, copula theory is introduced to allow for more flexibility in the fitting of a model. Copulas can evaluate a number of univariate processes in a multivariate distribution, where the univariate processes can assume different distributions. Copulas characterise the dependence between these univariate components in the joint distribution and is useful in risk assessment of portfolios. As an example, the univariate assets in a portfolio may have weakly correlated returns but their largest losses may occur in the same periods and be dependent on each other. Copulas are used to describe these extreme behaviour dependencies. Furthermore, when copulas are used to forecast VaR parametrically a number of problems are alleviated, which include there is no longer an assumption of a joint multivariate distribution. A copula is defined as follows.

Definition 2.1 (Copula)
A copula is a multivariate cumulative distribution function (CDF)[1]*, which consists of standard uniform univariate marginal distributions with notation* $U(0, 1)$.

When using copulas it is often useful to transform the probability function, thus the following proposition is presented.

Proposition 2.2 (Probability Transformation)
If Y *has a continuous CDF denoted* F_Y*, then the probability transformation of* Y *denoted* $F_Y(Y)$ *has a uniform distribution,* $U(0, 1)$*. If* F_Y *is strictly increasing, then* F_Y^{-1} *exists such that*

$$\mathbb{P}\{F_Y(Y) \leq y\} = \mathbb{P}\{Y \leq F_Y^{-1}(y)\} = F_Y\{F_Y^{-1}(y)\} = y,$$

holds true [20, p. 675].

Assume $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ has a multivariate CDF, denoted F_Y, with continuous marginal univariate CDFs, denoted $F_{Y_1}, \dots, F_{Y_d}$. Then by Proposition 2.2, $F_{Y_1}(Y_1), \dots, F_{Y_d}(Y_d) \sim U(0, 1)$. For that reason, the CDF of $\{F_{Y_1}(Y_1), \dots, F_{Y_d}(Y_d)\}$ is a copula. It is called the copula of $\boldsymbol{Y}$ and is denoted by C_Y. The copula, C_Y, contains information about the dependence between the components of $\boldsymbol{Y}$ and no information about the marginal CDFs of $\boldsymbol{Y}$. The next theorem shows the multivariate distribution can be defined as a copula.

[1]See Definition A.1 in Appendix A.1.

Theorem 2.3 (Sklar's Theorem)
Let $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ be a random vector with joint CDF, F_Y, and marginal univariate CDFs $F_{Y_1}, \ldots, F_{Y_d}$. Then, there exists a d-dimensional copula function $C : [0,1]^d \to [0,1]$ such that

$$F_Y(y_1, \ldots, y_d) = C(F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)), \tag{2.4}$$

for all $y_1, \ldots, y_d \in \mathbb{R}^d$. If the marginal univariate CDFs are continuous, then C is unique. Otherwise, C is uniquely determined only on $Ran(F_{Y_1}) \times \cdots \times Ran(F_{Y_d})$, where $Ran(F_{Y_i})$ denotes the range[2] of F_{Y_i} for $i = 1, \ldots, d$.

Conversely, let C be a copula and $F_{Y_1}, \ldots, F_{Y_d}$ be marginal univariate CDFs. Then, F_Y is a multivariate CDF with marginal univariate CDFs defined as in (2.4) *[10, p. 2].*

Proof.
For pointers on the general proof of Sklar's Theorem we refer to [18, p. 223]. ■

In the following the existence and uniqueness properties of a copula from Sklar's Theorem in the case of continuous marginal distributions $F_{Y_1}, \ldots, F_{Y_d}$ are shown.

Suppose all random variables have CDFs which are continuous and strictly increasing, meaning the CDFs are assumed to be increasing on their support. These assumptions are reasonable in many financial applications, since a continuous distribution is considered. Further, the assumption of continuity entails the probability will increase strictly. Recall C_Y is the CDF of $\{F_{Y_1}(Y_1), \ldots, F_{Y_d}(Y_d)\}$, and by the definition of the CDF the following is obtained

$$\begin{aligned} C_Y(u_1, \ldots, u_d) &= \mathbb{P}\{F_{Y_1}(Y_1) \leq u_1, \ldots, F_{Y_d}(Y_d) \leq u_d\} \\ &= \mathbb{P}\{Y_1 \leq F_{Y_1}^{-1}(u_1), \ldots, Y_d \leq F_{Y_d}^{-1}(u_d)\} \\ &= F_Y\{F_{Y_1}^{-1}(u_1), \ldots, F_{Y_d}^{-1}(u_d)\}, \end{aligned} \tag{2.5}$$

which shows the uniqueness. Let $u_j = F_{Y_j}(y_j)$ for $j = 1, \ldots, d$, then

$$F_Y(y_1, \ldots, y_d) = C_Y\{F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\}, \tag{2.6}$$

which is the identity (2.4) in Sklar's Theorem 2.3 and thus the existence is shown.

The probability density function of C_Y is introduced for later usage in Section 2.3.3 to find the estimates in maximum likelihood. Let c_Y be the probability density function associated with C_Y, then

$$c_Y(u_1, \ldots, u_d) = \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C_Y(u_1, \ldots, u_d).$$

The probability density function of $\boldsymbol{Y}$ is found by differentiating (2.6), which is

$$f_Y(y_1, \ldots, y_d) = c_Y\{F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\} f_{Y_1}(y_1) \cdots f_{Y_d}(y_d), \tag{2.7}$$

where $f_{Y_1}, \ldots, f_{Y_d}$ are the univariate marginal probability density functions of $Y_1, \ldots, Y_d$, respectively.

Copulas are invariant to strictly increasing transformations of the marginals, meaning they do not change after such transformations are used. This is stated in the following proposition.

[2] See Definition A.2 in Appendix A.1.

Proposition 2.4
Let $g_1, \dots, g_d$ be strictly increasing functions and let $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ be a random vector with copula C and continuous marginal distributions. Then, $(g_1(Y_1), \dots, g_d(Y_d))$ also has the copula C.

Proof.
Assume the transformation g_j is strictly increasing and $X_j = g_j(Y_j)$ for $j = 1, \dots, d$. The CDF of $\boldsymbol{X} = (g_1(Y_1), \dots, g_d(Y_d))$ is

$$\begin{aligned} F_X(x_1, \dots, x_d) &= \mathbb{P}\{g_1(Y_1) \le x_1, \dots, g_d(Y_d) \le x_d\} \\ &= \mathbb{P}\{Y_1 \le g_1^{-1}(x_1), \dots, Y_d \le g_d^{-1}(x_d)\} \\ &= F_Y\{g_1^{-1}(x_1), \dots, g_d^{-1}(x_d)\}, \end{aligned} \tag{2.8}$$

where the CDF of X_j is given as

$$F_{X_j}(x_j) = F_{Y_j}\{g_j^{-1}(x_j)\}.$$

This implies that

$$\begin{aligned} F_{X_j}^{-1}(u_j) &= g_j\{F_{Y_j}^{-1}(u_j)\} \\ g_j^{-1}\{F_{X_j}^{-1}(u_j)\} &= F_{Y_j}^{-1}(u_j). \end{aligned} \tag{2.9}$$

Using (2.5), (2.8) and (2.9) the copula of $\boldsymbol{X}$ is

$$\begin{aligned} C_X(u_1, \dots, u_d) &= F_X\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\} \\ &= F_Y[g_1^{-1}\{F_{X_1}^{-1}(u_1)\}, \dots, g_d^{-1}\{F_{X_d}^{-1}(u_d)\}] \\ &= F_Y\{F_{Y_1}^{-1}(u_1), \dots, F_{Y_d}^{-1}(u_d)\} \\ &= C_Y(u_1, \dots, u_d), \end{aligned}$$

which mean $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same copula. ■

2.3.1 Types of Copulas

In this section, different copulas are introduced, which are the Gaussian copula, the t-copula and four types of Archimedean copulas. Copulas are used to model dependencies in the data and these different types of copulas represent different dependency structures, where the best fitting copula is selected after evaluation.

First two examples of implicit copulas are defined, which are the Gaussian and the t-copula. Implicit copulas refer to copulas extracted from multivariate distributions by use of Sklar's Theorem. Such copulas do not necessarily have a simple closed form expression. Next, four examples of explicit copulas are presented, which are the Gumbel copula, the Clayton copula, the Frank copula and the Joe copula. Explicit copulas refer to copulas that have a simple closed form expression and follow a construction known to yield copulas.

The Gaussian and the t-copula are obtained from multivariate distributions using Sklar's Theorem 2.3. In order to define these two copulas, the following relation between the variance-covariance and correlation matrices is presented. Let Σ denote the variance-covariance matrix.

Two useful operations on the variance-covariance matrix are

$$\Delta(\Sigma) := \text{diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{dd}}),$$
$$\wp(\Sigma) := (\Delta(\Sigma))^{-1}\Sigma(\Delta(\Sigma))^{-1},$$

where $\Delta(\Sigma)$ extracts a diagonal matrix of standard deviations from Σ and $\wp(\Sigma)$ extracts the correlation matrix, P. Then the relation between the variance-covariance matrix Σ and the correlation matrix P of the d-dimensional random vector of returns $\boldsymbol{X}$ is

$$P = \wp(\Sigma).$$

A copula is called a *Gaussian copula* if $\boldsymbol{Y} \sim \mathbb{N}_d(\boldsymbol{\mu}, \Sigma)$ is a multivariate normal random vector. This copula does not depend on the univariate normal marginal distributions but on the dependencies within $\boldsymbol{Y}$. The marginal distributions are standardised using strictly increasing transformations, thus Proposition 2.4 can be used, which implies the copula of $\boldsymbol{Y}$ is the same as the copula of $\boldsymbol{X} \sim N_d(\boldsymbol{0}, P)$, where $P = \wp(\Sigma)$ is the correlation matrix of $\boldsymbol{Y}$. This copula is by (2.5) given as

$$\begin{aligned} C_P^{\text{Ga}}(\boldsymbol{u}) &= \mathbb{P}(\Phi(X_1) \le u_1, \dots, \Phi(X_d) \le u_d) \\ &= \boldsymbol{\Phi}_P(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)), \end{aligned}$$

where Φ is the standard univariate normal distribution function and $\boldsymbol{\Phi}_P$ is the joint distribution function of $\boldsymbol{X}$. There are $\frac{1}{2}d(d-1)$ parameters in the correlation matrix and when $d = 2$ the notation C_P^{Ga} is converted to C_ρ^{Ga}, where $\rho = \rho(X_1, X_2)$.

For the Gaussian copula if $P = I_d$ *the independence copula*[3] is obtained and if $P = J_d$, meaning the $d \times d$ matrix consists only of ones, *the co-monotonicity copula*[3] is obtained. If $d = 2$ and $\rho = -1$ then *the counter-monotonicity copula*[3] is obtained. Thus, the Gaussian copula for $d = 2$ can be considered as a dependence structure where the parameter ρ indicates the strength of the dependence.

The d-dimensional t-copula is given by

$$C_{\nu,P}^t(\boldsymbol{u}) = t_{\nu,P}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)),$$

where P is the correlation matrix, $t_{\nu,P}$ is the joint distribution function of the vector $\boldsymbol{X} \sim t_d(\nu, \boldsymbol{0}, P)$ and t_ν is the distribution function of the standard univariate t-distribution with ν degrees of freedom. The ν parameter affects both the univariate marginal distributions and the tail dependence between the components. The amount of tail dependence of the random vector with a t-copula is determined by ν which is described in Section 2.3.2. As for the Gaussian copula, if $P = J_d$, the t-copula equals the co-monotonicity copula. However, if $P = I_d$ the independence copula is not obtained since the uncorrelated multivariate t-distributed random variables are not independent when assuming $\nu < \infty$.

Note, the method used to extract a copula from the multivariate normal distribution and multivariate t-distribution can be used to extract other copulas from any other distribution with continuous marginal distribution functions.

[3]See Section A.2 in Appendix A for a description of the independence copula, the co-monotonicity copula and the counter-monotonicity copula.

Examples of copulas with simple closed form expressions are *Archimedean copulas.* Such copulas are referred to as explicit copulas. An Archimedean copula with a strict generator is defined as

$$C(\boldsymbol{u}) = \varphi^{-1}\{\varphi(u_1) + \cdots + \varphi(u_d)\}, \tag{2.10}$$

where φ is the generator function, which satisfies the conditions,

(1) φ is continuous, strictly decreasing and convex, which maps $[0, 1]$ onto $[0, \infty]$,

(2) $\varphi(0) = \infty$,

(3) $\varphi(1) = 0$.

The generator function φ is not unique, i.e. if φ is multiplied with any positive constant a, it generates the same copula as φ. If assumption (2) is relaxed, the generator is no longer strict and the construction of the copula is more complicated. Note, if $\boldsymbol{u}$ is permuted in (2.10), the value of $C(\boldsymbol{u})$ is unchanged, which means the distribution is called exchangeable. Archimedean copulas are most useful, when all univariate components are expected to have similar dependencies. In this project, four types of Archimedean copulas are considered, namely the Gumbel, Clayton, Frank and Joe copula. The copulas are presented with their generator function, inverse generator function and multivariate definition derived using (2.10).

The Gumbel copula has the generator function $\varphi_{\text{Gu}}(u|\theta) = (-\log(u))^\theta$, where $\theta \in [1, \infty)$. Thereby, the inverse generator function is given by $\varphi_{\text{Gu}}^{-1}(u|\theta) = \exp(-u^{1/\theta})$ and then by (2.10) the Gumbel copula is defined as

$$\begin{aligned} C_\theta^{\text{Gu}}(\boldsymbol{u}) &= \exp\left(- \left(\sum_{i=1}^{d} (-\log u_i)^\theta \right)^{1/\theta} \right) \\ &= \exp\left(- \left((-\log u_1)^\theta + \ldots + (-\log u_d)^\theta \right)^{1/\theta} \right). \end{aligned}$$

In the case where $\theta = 1$ the independence copula is obtained. Further, in the case where $\theta \to \infty$ the co-monotonicity copula is obtained. This means the Gumbel copula cannot have a negative dependence and will interpolate between independence and perfect positive dependence, where the amount of dependence is determined by the value of θ. The Clayton copula has the generator function $\varphi_{\text{Cl}}(u|\theta) = \frac{1}{\theta}(u^{-\theta} - 1)$, where $\theta \in (0, \infty)$. The inverse generator function is given by $\varphi_{\text{Cl}}^{-1}(u|\theta) = (1 + \theta u)^{-1/\theta}$. The Clayton copula is then defined as

$$\begin{aligned} C_\theta^{\text{Cl}}(\boldsymbol{u}) &= \left(\sum_{i=1}^{d} (u_i^{-\theta} - 1) + 1 \right)^{-1/\theta} \\ &= (u_1^{-\theta} + \ldots + u_d^{-\theta} + 1 - d)^{-1/\theta}. \end{aligned}$$

Note, the Clayton copula is not defined for $\theta = 0$, thus the limit as θ approaches zero is considered instead. In the case where $\theta \to 0$, it is the independence copula and in the case where $\theta \to \infty$ it is the co-monotonicity copula. This means, the Clayton copula will interpolate between independence and perfect positive dependence, where the strength of the dependence is determined by θ. Further, the Frank copula has the generator function $\varphi_{\text{Fr}}(u|\theta) = -\log\left(\frac{\exp(-\theta u)-1}{\exp(-\theta)-1}\right)$, where $\theta \in \mathbb{R}\backslash\{0\}$ and the inverse generator function is

$\varphi_{\text{Fr}}^{-1}(u|\theta) = -\frac{1}{\theta}\log(1+\exp(-u)(\exp(-\theta)-1))$. The Frank copula is defined as

$$C_\theta^{\text{Fr}}(\boldsymbol{u}) = -\frac{1}{\theta}\log\left(1+\frac{\prod_{i=1}^d(\mathrm{e}^{-\theta u_i}-1)}{(\mathrm{e}^{-\theta}-1)^{d-1}}\right).$$

For $\theta \to 0$ the Frank copula is the independence copula. As $\theta \to -\infty$ the Frank copula converges to the counter-monotonicity copula, and as $\theta \to \infty$ it converges to the co-monotonicity copula. This means, the Frank copula interpolates between perfect negative dependence and perfect positive dependence. Lastly, the Joe copula has the generator function $\varphi_{\text{Joe}}(u|\theta) = -\log(1-(1-u)^\theta)$, where $\theta \in [1,\infty)$. Here, the inverse generator function is $\varphi_{\text{Joe}}^{-1}(u|\theta) = 1-(1-\exp(-t))^{1/\theta}$. The Joe copula is defined as

$$C_\theta^{\text{Joe}}(\boldsymbol{u}) = 1-\left(1-\prod_{i=1}^{d}\left(1-(1-u_i)^\theta\right)\right)^{1/\theta}.$$

For $\theta = 1$ the Joe copula is the independence copula, and as $\theta \to \infty$ it converges to the co-monotonicity copula. Similar to the Gumbel copula, the Joe copula cannot have negative dependence.

2.3.2 Tail Dependence

In this section, the concept of tail dependence is introduced with the purpose of being able to describe extreme dependencies between the components of the joint distribution. Tail dependence between random variables is a measure of their association in the tails of the distribution. This measure depends only on the copula of the random variables in the continuous case. The coefficients of tail dependence are given by the limits of the conditional probabilities exceeding the quantile q. Note, the tail dependence is only presented for the bivariate case, but can be extended to the multivariate case [17, p. 291-292].

Let (Y_1, Y_2) be a random vector with copula C_Y. The *coefficient of lower tail dependence* is defined as

$$\begin{aligned}
\lambda_l &= \lim_{q\downarrow 0}\mathbb{P}\{Y_2 \le F_{Y_2}^{-1}(q) \mid Y_1 \le F_{Y_1}^{-1}(q)\} && (2.11)\\
&= \lim_{q\downarrow 0}\frac{\mathbb{P}\{Y_1 \le F_{Y_1}^{-1}(q), Y_2 \le F_{Y_2}^{-1}(q)\}}{\mathbb{P}\{Y_1 \le F_{Y_1}^{-1}(q)\}}\\
&= \lim_{q\downarrow 0}\frac{\mathbb{P}\{F_{Y_1}(Y_1) \le q, F_{Y_2}(Y_2) \le q\}}{\mathbb{P}\{F_{Y_1}(Y_1) \le q\}} && (2.12)\\
&= \lim_{q\downarrow 0}\frac{C_Y(q,q)}{q}.
\end{aligned}$$

Here, the extreme left tail is considered, since the limit is taken as $q \downarrow 0$. If Y_1 and Y_2 are independent, then $\mathbb{P}\{Y_2 \le y_2 \mid Y_1 \le y_1\} = \mathbb{P}\{Y_2 \le y_2\}$ for all y_1 and y_2. This means the conditional probability in (2.11) is equal to the unconditional probability $\mathbb{P}\{Y_2 \le F_{Y_2}^{-1}(q)\}$, which is converging to 0 as $q \downarrow 0$. Thus, $\lambda_l = 0$, which implies Y_1 and Y_2 behave as if they are independent in the extreme left tail. Consider (2.12), the numerator is the definition of the copula, and the denominator equals q, since $F_{Y_1}(Y_1) \sim U(0,1)$.

The *coefficient of upper tail dependence* is defined as

$$\begin{aligned}
\lambda_u &= \lim_{q\uparrow 1} \mathbb{P}\{Y_2 > F_{Y_2}^{-1}(q) \mid Y_1 > F_{Y_1}^{-1}(q)\} \\
&= \lim_{q\uparrow 1} \frac{1 - \mathbb{P}\{Y_1 \le F_{Y_1}^{-1}(q)\} - \mathbb{P}\{Y_2 \le F_{Y_2}^{-1}(q)\} + \mathbb{P}\{Y_1 \le F_{Y_1}^{-1}(q), Y_2 \le F_{Y_2}^{-1}(q)\}}{1 - \mathbb{P}\{Y_1 \le F_{Y_1}^{-1}(q)\}} \\
&= \lim_{q\uparrow 1} \frac{1 - \mathbb{P}\{F_{Y_1}(Y_1) \le q\} - \mathbb{P}\{F_{Y_2}(Y_2) \le q\} + \mathbb{P}\{F_{Y_1}(Y_1) \le q, F_{Y_2}(Y_2) \le q\}}{1 - \mathbb{P}\{F_{Y_1}(Y_1) \le q\}} \\
&= \lim_{q\uparrow 1} \frac{1 - q - q + C_Y(q,q)}{1-q} \\
&= \lim_{q\uparrow 1} \frac{1 - 2q + C_Y(q,q)}{1-q} \\
&= \lim_{q\uparrow 1} \frac{2 - 2q - 1 + C_Y(q,q)}{1-q} \\
&= \lim_{q\uparrow 1} \frac{2(1-q)}{1-q} - \frac{1 - C_Y(q,q)}{1-q} \\
&= \lim_{q\uparrow 1} 2 - \frac{1 - C_Y(q,q)}{1-q} \\
&= 2 - \lim_{q\uparrow 1} \frac{1 - C_Y(q,q)}{1-q}.
\end{aligned}$$

If $\lambda_l \in (0,1]$, C has lower tail dependence, which means Y_1 and Y_2 are lower tail dependent. If $\lambda_l = 0$ then C has no lower tail dependence, which means Y_1 and Y_2 are lower tail independent. This is analogue for λ_u.

Any bivariate Gaussian copula, C^{Ga}, with $\rho \neq 1$ does not have tail dependence, that is $\lambda^{\text{Ga}} = 0$ [20, p. 197]. For a bivariate t-copula C^t with correlation ρ and degrees of freedom ν, the coefficient of lower tail dependence is given by

$$\lambda_l^t = 2F_{t,\nu+1}\left\{ -\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}} \right\},$$

where $F_{t,\nu+1}$ is the CDF of the t-distribution with $\nu + 1$ degrees of freedom. The t-copula converges to the Gaussian copula when $\nu \to \infty$, since $F_{t,\nu+1}(-\infty) = 0$ meaning that $\lambda_l^t \to 0$ as $\nu \to \infty$. Further, $\lambda_l^t \to 0$ when $\rho \to -1$, since λ_l^t measures positive tail dependence. For both the Gaussian copula and the t-copula, it holds true that $\lambda_u = \lambda_l$.

For the Archimedean copulas the tail dependence coefficients are listed in Table 2.1.

Copula	Upper tail	Lower tail
Gumbel	$2 - 2^{1/\theta}$	0
Clayton	0	$2^{-1/\theta}$
Frank	0	0
Joe	$2 - 2^{1/\theta}$	0

Table 2.1. Coefficients of tail dependence for the Archimedean copulas [11, 18].

The Clayton copula is the only copula with lower tail dependence. When lower tail dependence is present in a loss distribution it suggests a great loss for one asset is likely to occur simultaneous with a great loss for another asset in the portfolio. Upper tail dependence will instead suggest a great return for one asset will occur simultaneous with a great return for another asset in the portfolio. The Gumbel and the Joe copulas have the same upper tail dependence but the Joe copula allows for even stronger upper tail dependence than the Gumbel copula [20, p. 192]. The Frank copula does not take tail dependence into account.

The concept of tail dependence is essential for risk management, since it indicates whether the risk of an extreme negative return in the portfolio is low or high. The risk of simultaneous negative returns is small if there is no lower tail dependency in the returns of the assets in the portfolio, which suggests less extreme risk. The risk of simultaneous negative returns in the portfolio can be high if there is lower tail dependency. Thus, it is beneficial to consider tail dependencies when assessing the diversification and risk of a portfolio.

2.3.3 Estimation of Parameters

In this section, estimation of an unknown copula is presented for the parametric setting. Here, methods to estimate marginal distribution parameters, $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_d$, and copula parameters, $\boldsymbol{\theta}_C$, are presented.

Let $\boldsymbol{Y}_{1:n} = \{(Y_{i,1}, \dots, Y_{i,d}) : i = 1, \dots, n\}$ be a random sample of a d-variate random vector $\boldsymbol{Y}$ with distribution function, F_Y, and univariate marginal distributions $F_{Y_1}, \dots, F_{Y_d}$, which are continuous. Then Sklar's Theorem 2.3 states there exists a unique d-dimensional copula function $C : [0,1]^d \to [0,1]$ such that

$$F_Y(y_1, \dots, y_d) = C(F_{Y_1}(y_1), \dots, F_{Y_d}(y_d)).$$

In order to estimate the copula parameters the marginal distributions are needed, wherefore these are estimated first. In the parametric setting it is assumed that $F_{Y_1}, \dots, F_{Y_d}$ belong to absolutely continuous parametric families of univariate distribution functions. In order to estimate the parameters of $F_{Y_1}, \dots, F_{Y_d}$ and the copula, the two methods maximum likelihood and pseudo-maximum likelihood are presented.

Maximum Likelihood

Assume the marginal CDFs are given by parametric models $F_{Y_1}(\cdot \mid \boldsymbol{\theta}_1), \dots, F_{Y_d}(\cdot \mid \boldsymbol{\theta}_d)$ and a parametric model $c_Y(\cdot \mid \boldsymbol{\theta}_C)$ is given for the density of the copula. The log-likelihood is obtained by taking the logarithm of (2.7),

$$\begin{aligned}\log\{L(\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C)\} = \sum_{i=1}^{n} \Bigg(&\log\Big[c_Y\big\{F_{Y_1}(Y_{i,1} \mid \boldsymbol{\theta}_1), \dots, F_{Y_d}(Y_{i,d} \mid \boldsymbol{\theta}_d) \,\big|\, \boldsymbol{\theta}_C\big\}\Big] \\ &+ \log\{f_{Y_1}(Y_{i,1} \mid \boldsymbol{\theta}_1)\} + \dots + \log\{f_{Y_d}(Y_{i,d} \mid \boldsymbol{\theta}_d)\}\Bigg). \end{aligned} \tag{2.13}$$

Maximum likelihood estimation (MLE) finds the maximum of (2.13) with respect to the set of parameters $(\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_d, \boldsymbol{\theta}_C)$. Nevertheless, there are two problems with this type of estimation. First, for large values of d, it can be computationally burdensome to maximise (2.13) numerically, because of the curse of dimensionality. Second, the method requires parametric models for both the univariate marginal distributions and the copula. If these are misspecified it may cause the estimated parameters to be biased. The former issue can be solved using the pseudo-maximum likelihood explained below, so the computational burden is reduced compared to the maximum likelihood estimator. The latter issue can be avoided by estimating the univariate marginal distributions $F_{Y_1}, \dots, F_{Y_d}$ non-parametrically.

In the next section, pseudo-maximum likelihood is introduced, which can handle both the parametric and the non-parametric approach. The parametric approach is also called *Inference Functions for Margins* (IFM). The non-parametric approach will not be studied in this project.

Pseudo-Maximum Likelihood

Pseudo-maximum likelihood estimation consists of two steps. The first step is to estimate the d univariate marginal distributions individually by assuming parametric models $F_{Y_1}(\cdot \mid \boldsymbol{\theta}_1), \ldots, F_{Y_d}(\cdot \mid \boldsymbol{\theta}_d)$ for the univariate marginal CDFs. Thereafter, estimation of the unknown marginal parameters $\boldsymbol{\theta}_j$ for $j = 1, \ldots, d$ is performed by maximum likelihood using the data $Y_{1,j}, \ldots, Y_{n,j}$, hence $\hat{F}_{Y_j}(\cdot) = F_{Y_j}(\cdot \mid \hat{\boldsymbol{\theta}}_j)$. The second step maximises the following expression with respect to $\boldsymbol{\theta}_C$,

$$\sum_{i=1}^{n} \log[c_Y\{\hat{F}_{Y_1}(Y_{i,1}), \ldots, \hat{F}_{Y_d}(Y_{i,d}) \mid \boldsymbol{\theta}_C\}].$$

This method avoids optimisation with a high dimension by estimating the parameters in the univariate marginal distributions and in the copula separately. The values $\hat{F}_{Y_j}(Y_{i,j})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d$ should be approximately uniformly distributed and therefore they are called the uniform-transformed variables.

Recall, the pseudo-maximum likelihood estimator has a computational advantage over the maximum likelihood estimator. Nevertheless, a disadvantage is that the pseudo-maximum likelihood estimator is less efficient than the maximum likelihood estimator [13]. For both estimation methods one should be aware that misspecified marginal distributions may cause a bias in the estimation.

2.4 Monte Carlo Forecasts

This section is based on [16].

In this section, a Monte Carlo procedure to forecast one-day-ahead VaR is presented. Here, only equally weighted portfolios consisting of d assets are considered. From this point forward the notation for the number of observations n is changed to T, since time observations are considered. The daily log-return of asset j is defined as

$$r_t^j = \log\left(\frac{P_t^j}{P_{t-1}^j}\right) = \log(P_t^j) - \log(P_{t-1}^j), \quad \text{for } j = 1, \ldots, d,$$

where P_t^j is the price of asset j at time t. The log-return of the portfolio is given by

$$r_t^P = \frac{1}{d} r_t^1 + \cdots + \frac{1}{d} r_t^d.$$

In order to estimate the VaR of a portfolio, the joint distribution of the vector of log-returns, $(r_t^1, \ldots, r_t^d)$, is found. The vector of log-returns is modelled using a mean-variance model introduced in Section 2.2 and copula theory presented in Section 2.3. Hereafter, the VaR is forecasted using Monte Carlo simulations. In order to forecast VaR, the data is first fitted using an ARMA-GARCH model, where two different density functions for the standardised residuals are assumed: The normal distribution and the t-distribution. Note, when $\nu \to \infty$ the t-distribution reduces to the normal distribution.

The profit and loss (P&L) function of the portfolio composed of d assets is denoted L_t and is given by

$$\begin{aligned} L_t &= \frac{1}{d}P_t^1 + \cdots + \frac{1}{d}P_t^d - \left(\frac{1}{d}P_{t-1}^d + \cdots + \frac{1}{d}P_{t-1}^d\right) \\ &= \frac{1}{d}P_{t-1}^1(\exp(r_t^1) - 1) + \cdots + \frac{1}{d}P_{t-1}^d(\exp(r_t^d) - 1). \end{aligned} \tag{2.14}$$

The procedure is based on T observations of log-returns of d assets. In the following procedure an ARMA-GARCH model and copulas are used to forecast one-day-ahead VaR at a 95% and a 99% confidence level.

(1) Use T observations to fit an ARMA-GARCH model to each return series and estimate the marginal distributions for each of the standardised residual processes.

(2) Forecast one-step means, $\hat{r}_{T+1}^j$, and variances, $\hat{\sigma}_{T+1}^j$, at time $T+1$ for $j = 1, \dots, d$.

(3) Estimate the copula parameters by the probability integral transforms $u_t^1, \dots, u_t^d$ of the standardised residuals $a_t^1, \dots, a_t^d$.

(4) Simulate k random variables $(u_{T+1}^{1,k}, \dots, u_{T+1}^{d,k})$ for $k = 1, \dots, N$ from the copula[4].

(5) Obtain the simulated standardised residuals $a_{T+1}^{j,k}$ for $j = 1, \dots, d$ using the inverse functions of the estimated marginal distributions as

$$(a_{T+1}^{1,k}, \dots, a_{T+1}^{d,k}) = \left(F_{1,T+1}^{-1}(u_{T+1}^{1,k}; \hat{\boldsymbol{\theta}}_1), \dots, F_{d,T+1}^{-1}(u_{T+1}^{d,k}; \hat{\boldsymbol{\theta}}_d)\right).$$

(6) Obtain the simulated asset log-returns by

$$(r_{T+1}^{1,k}, \dots, r_{T+1}^{d,k}) = \left(\hat{r}_{T+1}^1 + a_{T+1}^{1,k} \cdot \sqrt{\hat{\sigma}_{T+1}^1}, \dots, \hat{r}_{T+1}^d + a_{T+1}^{d,k} \cdot \sqrt{\hat{\sigma}_{T+1}^d}\right).$$

(7) Calculate the values of L_{T+1}^k for $k = 1, \dots, N$ by (2.14).

(8) Sort the N values of L_{T+1}^k in increasing order and calculate the 95% VaR and the 99% VaR as:

   (i) 95% VaR is the absolute value of the $N \times (1 - 0.95)$ ordered value in L_{T+1}^k.

   (ii) 99% VaR is the absolute value of the $N \times (1 - 0.99)$ ordered value in L_{T+1}^k.

(9) Repeat steps (1)-(8) M times by rolling over the daily returns. In other words, after forecasting the next trading day, the log-return and price of the next day are added to the data set and the first log-return and price in the data set are deleted. This ensures the data set always consists of T observations.

The procedure results in M rolling forecasts of VaR with N Monte Carlo simulations. These forecasts are one-day-ahead out-of-sample forecasts, which can be evaluated with methods called backtesting introduced in Section 2.5.

[4]See [11, p. 88, 90] for simulation of copulas.

2.5 Backtesting

This section is based on [4], [5] and [16].

In this section, backtesting methods are presented, which are methods used to evaluate the predictive performance of a model using historical data. Such methods are introduced to assess the performance of the forecasted VaR.

First a hit series is defined, which has the value 1 when the loss exceeds VaR and 0 otherwise. This is written as

$$I_t = \begin{cases} 1 & \text{if } L_t < -\text{VaR}_t \\ 0 & \text{if } L_t \geq -\text{VaR}_t \end{cases}, \quad \text{for } t = 1, \dots, T.$$

It is expected that the VaR is violated with a probability of $1-\alpha$ and the number of times a loss exceeds the VaR is calculated as

$$Z = \sum_{t=1}^{T} I_t.$$

The ratio of VaR exceedances is calculated as Z/T. With this notation, Kupiec's unconditional coverage test is introduced, which is a likelihood ratio test statistic defined as

$$\text{LR}_{\text{UC}} = -2\log\left[\alpha^{T-Z}(1-\alpha)^Z\right] + 2\log\left[(1-Z/T)^{T-Z}(Z/T)^Z\right],$$

where $T-Z$ is the successes and Z is the failures and the term $\alpha^{T-Z}(1-\alpha)^Z$ is the probability of having $T-Z$ successes. Thus, it is tested if it holds true that $1-\alpha = Z/T$. In other words, the test statistic compares the predicted number of times VaR is exceeded against the observed number of times VaR is exceeded. This test is asymptotically distributed as $\chi^2(1)$ under the null hypothesis, which states the observed number of exceedings statistically equals the predicted number of exceedings. Note, $(1-\alpha)$ is the probability for the losses to exceed the VaR which is $(1-\alpha)\cdot 100$ percent and if the losses exceed the VaR more frequently it underestimates the risk of the portfolio and conversely if the losses exceed the VaR less frequently the risk of the portfolio is overestimated. Kupiec's method is able to reject the model for having both too many and too few failures, however it is very simple and has a number of disadvantages. Kupiec's test requires a large sample size to be accurate and it only focuses on the number of failures and does not consider the nature of the losses. For this reason Christoffersen's independence test is presented, which further examines if the failures and successes are independent. This test is defined as

$$\text{LR}_{\text{CC}} = -2\log\left[\alpha^{T-Z}(1-\alpha)^Z\right] + 2\log\left[(1-\pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1-\pi_{11})^{n_{10}}\pi_{11}^{n_{11}}\right],$$

where n_{ij} is the number of observations with value i followed by value j in the hit series I_t for i,j = 0,1 and $\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$ is the corresponding probability. This test is asymptotically distributed as $\chi^2(2)$ under the null hypothesis, which is the probabilities of VaR exceedings are independent. These probabilities are independent if $\pi_{01} = \pi_{11} = \alpha$. Where Kupiec's unconditional coverage test restricts the number of violations allowed, Christoffersen's independence test further restricts on the way the violations occur. The reasoning behind this test is VaR should take clustering into account, where periodic low or high volatilities are reflected in the expected probability of failures and successes. Thus, Christoffersen's independence test is further able to reject a model for having too many or too few clustered violations.

Kupiec's and Christoffersen's statistical tests compare the calculated and observed number of exceedings of VaR to determine the accuracy or reliability of a model. These methods do not address the magnitude of the violations, for which reason it is beneficial to additionally consider a loss function. Such methods can be used to evaluate the number of violations and further distinguish between models under the alternative hypothesis, where the magnitude of the exceedings is assessed. The first loss function proposed by Lopez equals the hit series and is defined as

$$C_t^{L1} = \begin{cases} 1 & \text{if } L_t < -\text{VaR}_t \\ 0 & \text{if } L_t \geq -\text{VaR}_t \end{cases}. \tag{2.15}$$

This loss function ignores the magnitude of the losses. For this reason Lopez proposes another loss function, which has an added term $(|L_t| - \text{VaR}_t)^2$ that incorporates the magnitude of the exceedings, where larger failures are penalised. It is defined as

$$C_t^{L2} = \begin{cases} 1 + (|L_t| - \text{VaR}_t)^2 & \text{if } L_t < -\text{VaR}_t \\ 0 & \text{if } L_t \geq -\text{VaR}_t \end{cases}. \tag{2.16}$$

Another loss function proposed by Blanco and Ihle incorporates the average size of the exceedings of VaR and is defined as

$$C_t^{BI} = \begin{cases} \frac{|L_t| - \text{VaR}_t}{\text{VaR}_t} & \text{if } L_t < -\text{VaR}_t \\ 0 & \text{if } L_t \geq -\text{VaR}_t \end{cases}. \tag{2.17}$$

There are different methods to evaluate the outcome of the loss functions. One method is to evaluate the sample average of the loss functions,

$$\hat{C} = \frac{1}{T} \sum_{t=1}^{T} C_t. \tag{2.18}$$

The sample average of different models can then be compared, where the model with the smallest $\hat{C}$ is preferred because it indicates a better goodness-of-fit.

In practice, the two statistical tests from Kupiec and Christoffersen are used to select the best model. Next, the loss functions are used to compare the costs of different admissible choices.

Modelling 3

In this chapter, data processing and modelling are presented. First, the data is presented for two categories of assets, which are global equity assets and emerging markets equity assets. Then the data is modelled using a mean-variance model from Section 2.2 and copula theory from Section 2.3, after which the VaR is forecasted using the Monte Carlo procedure in Section 2.4 on an equally weighted portfolio containing five assets. These forecast are then evaluated using the backtesting methods from Section 2.5.

3.1 Data Description

The data used in this project is collected from Yahoo Finance. As described in Chapter 1 the focus is on two groups of assets, which are *global equity* and *emerging markets equity*. To represent the two groups, two portfolios are considered containing five assets which are selected from different countries and different sectors of industry. The two portfolios are referred to as group 5 for the portfolio representing global equity and group 6 for the portfolio representing emerging markets equity. The observations are given in United States dollar (USD). For group 5 the chosen assets are:

- Apple (AP) from the US, which is information technology.
- Adidas (AD) from Germany, which is clothing, accessories and luxury goods.
- Novo Nordisk (NN) from Denmark, which is from the pharmaceutical industry.
- Sony Group (SG) from Japan, which is cyclical consumption.
- Prudential Financial (PF) from England, which is from the financial sector.

For group 6 the chosen assets are:

- Fomento Económico Mexicano (FM) from Mexico, which is mineral water and beverage.
- Korea Electric Power Corporation (KE) from South Korea, which is electric utility.
- PLDT Inc. (PH) from the Philippines, which is telecommunication.
- Taiwan Semiconductor Manufacturing (TS) from Taiwan, which is information technology.
- United Overseas Bank (UO) from Singapore, which is from the financial sector.

These assets are divided into three samples with one observation each day. The full sample is from 2010-01-04 to 2021-12-30 containing 3020 observations. The estimation sample is from 2010-01-04 to 2019-12-31 containing 2516 observations. The forecasting sample is from 2020-01-02 to 2021-12-30 containing 504 observations, which is concurrent with the corona pandemic. Further, the adjusted close price is considered in the analysis and is referred to as the price. The prices of the equally weighted portfolios for group 5 and 6 are shown in Figure 3.1.

Figure 3.1. The equally weighted portfolio prices for group 5 and 6. The dashed line represents the split between the estimation sample and the forecasting sample.

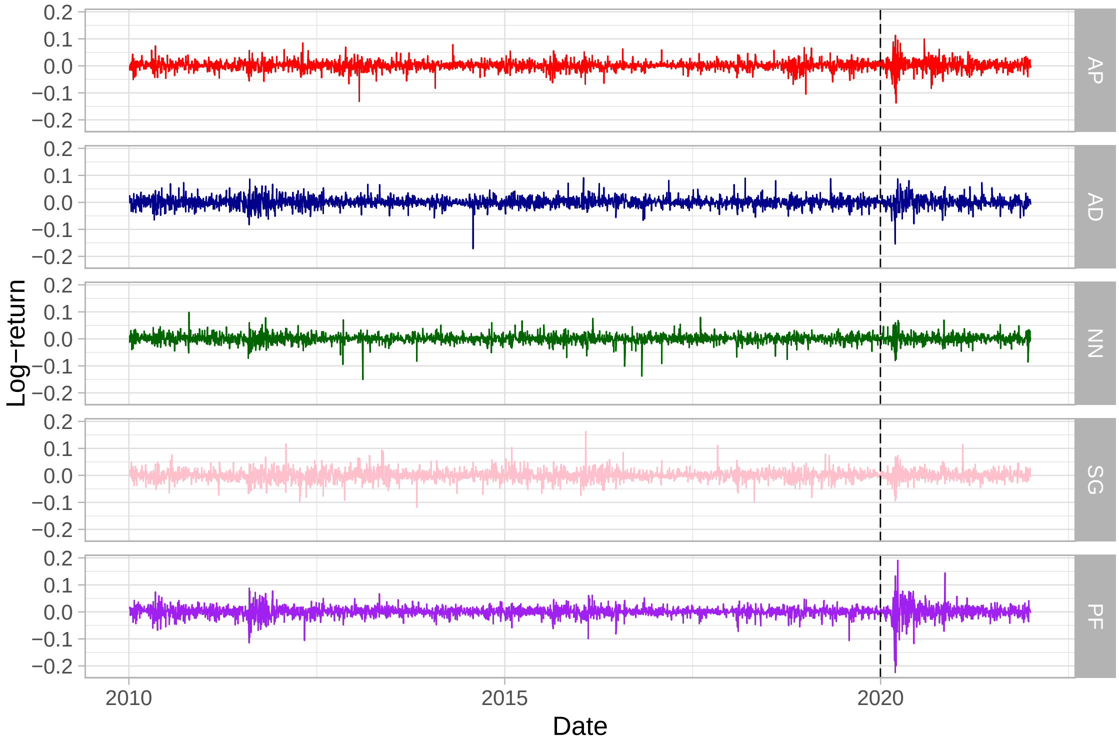
Note, the scales of the prices for the two groups are different. The prices are increasing throughout the period. However, in the beginning of 2020 there is a relatively high decrease in the price, which is probably caused by the corona pandemic. To further investigate the portfolios, the correlations between the two groups for the three periods are calculated and the results are shown in Table 3.2.

Correlations between portfolio prices for group 5 and 6			
	Full sample	Estimation sample	Forecasting sample
Correlation	0.85	0.71	0.96

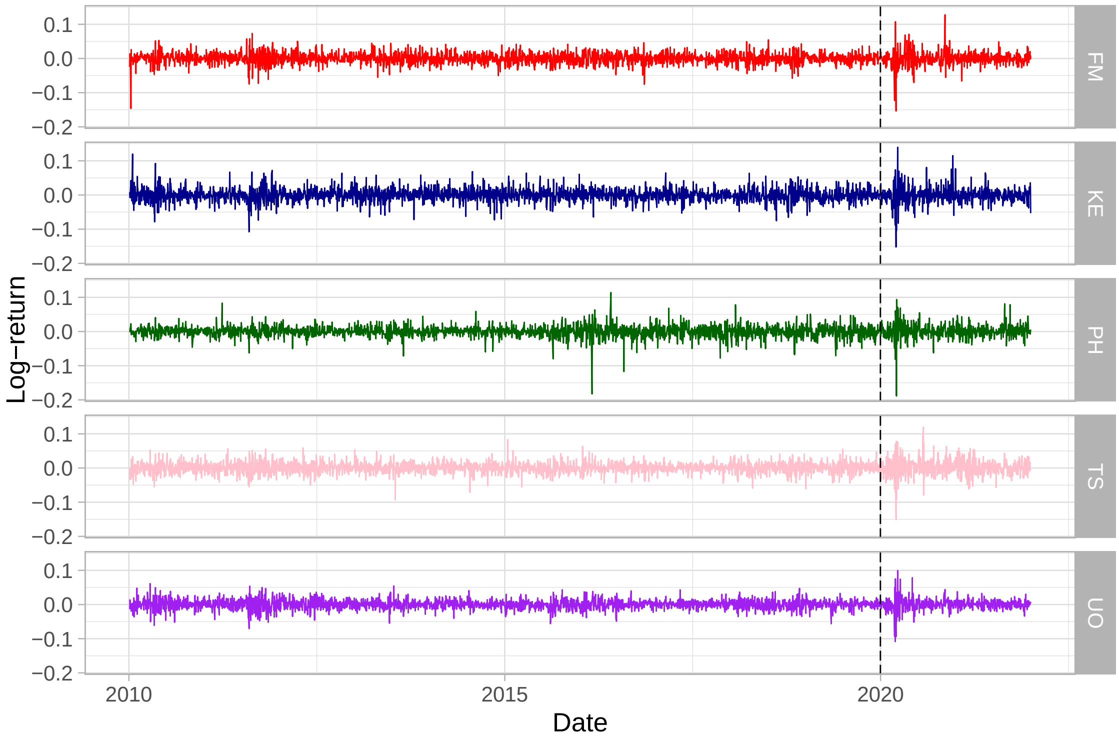
Table 3.2. Correlations.

There is a high correlation between the two groups for all time periods and the correlation is especially high in the forecasting sample.

To model a mean-variance model of each asset, the price processes need to be stationary and therefore, the price processes are converted into daily log-returns. The full samples of log-returns are plotted in Figure 3.3 for the respective assets in group 5 and 6.



(a) The log-returns of the assets from group 5.



(b) The log-returns of the assets from group 6.

Figure 3.3. Illustration of the log-returns from the respective groups, where the dashed black line represents the date that splits the estimation sample and the forecasting sample.

The figure shows the time series are approximately stationary, but with some volatility clustering. The corona pandemic starts in the beginning of 2020 where a considerable clustering appears for most of the time series. The estimation sample is used to estimate the model and the forecasting sample is used for out-of-sample predictions of the models.

In Table 3.4 and 3.5 the descriptive statistics for the log-returns of the five assets are presented for the three periods for the assets in group 5 and 6, respectively. In Table 3.4 it is seen the mean of AP is higher than the mean of the remaining assets for the three periods, while the standard deviations are similar for all five assets. Note, the mean of AD in the forecasting sample is negative. The 95% VaR and the 99% VaR are calculated from historical returns and for both levels of VaR, the values in the forecasting sample are higher than the values in the estimation sample for all assets except for SG. It is expected that the VaR is higher in the forecasting sample since the corona pandemic is emerging in this time period. The tendency seen for SG can be caused by the corona pandemic as well, where their products and services are more desired during the corona pandemic. Note, the lower number of observations in the forecasting sample may cause a less reliable historical VaR estimation. The skewness of SG is positive in the three periods, while the four other assets exhibit negative skewness. Further, all assets exhibit excess kurtosis. The Jarque-Bera test is used to test for normality, which is described in Appendix A.3. The null hypothesis states that the data is normally distributed with kurtosis equal to 3 and no skewness. All of the test statistics are relatively high and all the p-values are less than 0.05, which mean the test rejects the null hypothesis of normality for the three periods for all assets. The Augmented Dickey-Fuller (ADF) test is a unit root test, which tests the null hypothesis of non-stationarity of the time series. Here, the p-values of all tests are less than 0.05, which indicates all assets are stationary. The Ljung-Box test is an autocorrelation test, which is described in Appendix A.4. The null hypothesis of the Ljung-Box test states there is no autocorrelation. This test is conducted on the log-returns at lag 25, which means the statistic is based on 25 autocorrelation coefficients. The p-values for AP, AD and PF are less than 0.05 in the full sample and the p-values are less than 0.05 for AP and PF in the estimation sample. The p-values for AD in the estimation sample, and for NN and SG for both the full and the estimation sample are greater than 0.05, which mean the time series are not autocorrelated. All p-values are less than 0.05 in the forecasting sample.

In Table 3.5 it is seen that the mean of TS is higher than the mean of the four other assets and the mean of KE is negative for the three periods. Further, the mean of PH is negative in the estimation sample and the mean of FM is negative in the forecasting sample. The standard deviation varies for all assets, but are highest in the forecasting sample. The 95% VaR of all assets is lower for the estimation sample than for the forecasting sample. This is also true for the 99% VaR for all assets except for PH where the 99% VaR is higher in the estimation sample than in the forecasting sample. All assets exhibit negative skewness and excess kurtosis, except the skewness of KE, which is positive in the three periods. The test statistics of the Jarque-Bera test are relatively high and the p-values are less than 0.05, thus the assets exhibit non-normality in the three periods. The p-values of the ADF test are less than 0.05, which indicates all assets are stationary. Further, the p-values of the Ljung-Box test with lag 25 are less than 0.05, except for FM for both the full sample and the estimation sample. A p-value less than 0.05 indicates the assets are autocorrelated.

Group 5: Global Equity	Full sample					Estimation sample					Forecasting sample				
	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF
Mean	0.0011	0.0006	0.0008	0.0005	0.0004	0.0010	0.0008	0.0007	0.0004	0.0004	0.0018	-0.0002	0.0014	0.0012	0.0005
St. Dev.	0.0177	0.0188	0.0163	0.0203	0.0209	0.0162	0.0178	0.0159	0.0204	0.0177	0.0236	0.0232	0.0178	0.0195	0.0325
95% VaR	0.0263	0.0283	0.0230	0.0313	0.0314	0.0253	0.0263	0.0226	0.0322	0.0289	0.0342	0.0347	0.0247	0.0261	0.0437
99% VaR	0.0474	0.0494	0.0450	0.0539	0.0556	0.0441	0.0462	0.0436	0.0551	0.0490	0.0676	0.0664	0.0499	0.0455	0.0821
Skewness	-0.3113	-0.3661	-0.7099	0.1744	-0.8000	-0.3416	-0.1276	-0.8390	0.1780	-0.5259	-0.2722	-0.8332	-0.2558	0.1687	-0.8602
Excess kurtosis	6.1290	7.0224	8.2706	4.6496	14.773	4.8297	5.7892	9.5578	4.6589	4.1756	5.5026	7.6048	3.7260	4.5565	12.141
Jarque-Bera test	4774	6271	8858	2735	27774	2493	3519	9868	2288	1943	641	1270	296	438	3151
P-value	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16
ADF test	-13.8080	-14.2100	-15.3696	-14.1420	-14.2263	-12.8507	-13.6287	-14.2116	-13.1830	-13.4793	-8.6363	-7.9391	-9.5647	-7.6520	-7.9208
P-value	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01
Ljung-Box test	99.7200	43.9729	34.5611	33.0310	106.9408	41.9626	34.7092	25.7546	21.0756	75.5991	123.2763	53.0311	51.2867	58.7080	68.8543
P-value	7e-11	0.0109	0.0964	0.1303	4e-12	0.0181	0.0936	0.4208	0.6884	5e-07	6e-15	0.0009	0.0015	0.0002	6e-06

Table 3.4. Descriptive statistics of the log-returns of group 5.

Group 6: Emerging Markets Equity	Full sample					Estimation sample					Forecasting sample				
	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO
Mean	0.0002	-0.0001	0.0001	0.0009	0.0002	0.0003	-0.0000	-0.0002	0.0008	0.0002	-0.0003	-0.0004	0.0015	0.0015	0.0002
St. Dev.	0.0168	0.0195	0.0173	0.0176	0.0144	0.0155	0.0184	0.0164	0.0157	0.0138	0.0219	0.0246	0.0214	0.0251	0.0174
95% VaR	0.0255	0.0298	0.0255	0.0267	0.0219	0.0241	0.0290	0.0252	0.0244	0.0217	0.0296	0.0346	0.0276	0.0347	0.0230
99% VaR	0.0404	0.0524	0.0481	0.0430	0.0390	0.0384	0.0495	0.0481	0.0391	0.0370	0.0636	0.0671	0.0456	0.0612	0.0504
Skewness	-0.5063	0.0468	-0.7673	-0.0922	-0.2400	-0.4688	0.0566	-0.7244	-0.0658	-0.1647	-0.4892	0.0403	-0.9026	-0.1562	-0.4105
Excess kurtosis	8.2026	4.9981	11.9082	4.3817	4.7891	4.6200	2.9129	9.8451	1.7842	1.7733	10.3303	6.9033	13.5172	4.1022	9.7741
Jarque-Bera test	8593	3143	18134	2419	2914	2329	891	10377	335	341	2260	1013	3899	359	2017
P-value	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16
ADF test	-14.3657	-14.1328	-14.6651	-14.3714	-14.8659	-14.3839	-13.3128	-13.4410	-13.3729	-13.8552	-7.1610	-8.5054	-7.7797	-7.9314	-7.8263
P-value	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01
Ljung-Box test	31.8279	45.8863	52.6746	105.8372	100.8842	33.0631	42.3518	64.7302	40.7769	47.1151	38.6571	52.7911	51.8378	106.4369	133.2609
P-value	0.1631	0.0066	0.0010	6.4e-12	4.5e-11	0.1295	0.0165	2.3e-05	0.0242	0.0048	0.0399	0.0010	0.0013	5.1e-12	< 2.2e-16

Table 3.5. Descriptive statistics of the log-returns of group 6.

In Table 3.6 the correlation matrices based on the Pearson correlation coefficient for group 5 are shown for the three periods.

Group 5: Global Equity Correlations															
	Full sample					Estimation sample					Forecasting sample				
	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF
AP	1.00	0.34	0.28	0.36	0.42	1.00	0.32	0.24	0.31	0.40	1.00	0.39	0.44	0.56	0.47
AD	0.34	1.00	0.33	0.32	0.46	0.32	1.00	0.33	0.30	0.43	0.39	1.00	0.35	0.42	0.55
NN	0.28	0.33	1.00	0.28	0.30	0.24	0.33	1.00	0.25	0.31	0.44	0.35	1.00	0.40	0.30
SG	0.36	0.32	0.28	1.00	0.41	0.31	0.30	0.25	1.00	0.41	0.56	0.42	0.40	1.00	0.46
PF	0.42	0.46	0.30	0.41	1.00	0.40	0.43	0.31	0.41	1.00	0.47	0.55	0.30	0.46	1.00

Table 3.6. Correlation matrices.

The correlations between the assets are positive, which mean there is a positive degree of linear dependence. In general, the correlation coefficients are slightly higher in the forecasting sample. In Table 3.7 the correlation matrices based on the Pearson correlation coefficient for group 6 are shown for the three periods.

Group 6: Emerging Markets Equity Correlations															
	Full sample					Estimation sample					Forecasting sample				
	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO
FM	1.00	0.32	0.23	0.37	0.43	1.00	0.29	0.24	0.36	0.38	1.00	0.42	0.22	0.38	0.57
KE	0.32	1.00	0.22	0.34	0.35	0.29	1.00	0.19	0.31	0.29	0.42	1.00	0.30	0.41	0.49
PH	0.23	0.22	1.00	0.25	0.27	0.24	0.19	1.00	0.24	0.25	0.22	0.30	1.00	0.27	0.31
TS	0.37	0.34	0.25	1.00	0.46	0.36	0.31	0.24	1.00	0.45	0.38	0.41	0.27	1.00	0.51
UO	0.43	0.35	0.27	0.46	1.00	0.38	0.29	0.25	0.45	1.00	0.57	0.49	0.31	0.51	1.00

Table 3.7. Correlation matrices.

In group 6, the correlations between the assets are positive. Again, slightly higher correlations are seen in the forecasting sample except for the correlation between FM and PH, where the correlation in the forecasting sample is lowest compared with the two other periods.

3.2 Modelling of Marginal Distributions

In this section, the mean-variance model introduced in Section 2.2 is fitted. As seen in Table 3.4 and 3.5 each asset is stationary, but autocorrelation is present in almost all series and thus it is suitable to fit an ARMA-GARCH model to the data. Here, the observations in the estimation sample are modelled. The models ARMA(0,0)-GARCH(1,1) and ARMA(1,1)-GARCH(1,1) are compared assuming two different marginal distributions, namely the normal distribution and the t-distribution. Note, only a GARCH(1,1) model is considered, since this order is sufficient in most cases [9]. The results for group 5 are shown in Table 3.8.

Group 5: The ARMA(0,0)-GARCH(1,1) model										
	Normally distributed standardised residuals					*t*-distributed standardised residuals				
	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF
LL	-6895	-6610	-6846	-6284	-6745	-7036	-6776	-7125	-6460	-6866
AIC	-5.4795	-5.2533	-5.4412	-4.9938	-5.3608	-5.5910	-5.3843	-5.6624	-5.1331	-5.4564
BIC	-5.4703	-5.2441	-5.4319	-4.9846	-5.3515	-5.5794	-5.3727	-5.6508	-5.1215	-5.4448

(a) Model selection criteria of the model with ARMA(0,0) for group 5.

Group 5: The ARMA(1,1)-GARCH(1,1) model										
	Normally distributed standardised residuals					*t*-distributed standardised residuals				
	AP	AD	NN	SG	PF	AP	AD	NN	SG	PF
LL	-6895	-6613	-6848	-6285	-6746	-7037	-6778	-7128	-6462	-6869
AIC	-5.4784	-5.2538	-5.4412	-4.9930	-5.3598	-5.5902	-5.3843	-5.6631	-5.5902	-5.4571
BIC	-5.4645	-5.2399	-5.4273	-4.9791	-5.3459	-5.5740	-5.3681	-5.6469	-5.5740	-5.4409

(b) Model selection criteria of the model with ARMA(1,1) for group 5.

Table 3.8. The log-likelihood, AIC and BIC values for different orders of ARMA-GARCH models for group 5.

The best models are selected by comparing the log-likelihood, AIC and BIC values. The analysis yields the best model for AP is ARMA(0,0)-GARCH(1,1) and for the remaining assets the ARMA(1,1)-GARCH(1,1) is best. Furthermore, all of the preferred models assume *t*-distributed standardised residuals. To check whether it is reasonable to fit a GARCH model, the residuals of the ARMA models are investigated for heteroskedasticity and thus GARCH effects, using the Ljung-Box test on the squared residuals where the results are shown in Table 3.9.

Group 5: Ljung-Box test on the squared residuals of the ARMA model										
	AP ARMA(0,0)		AD ARMA(1,1)		NN ARMA(1,1)		SG ARMA(1,1)		PF ARMA(1,1)	
	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value
Lag 5	36.1434	8.9e-07	28.9162	2.3e-06	12.8800	0.0049	29.7529	1.6e-06	338.3136	$< 2e-16$
Lag 15	94.1037	1.7e-13	42.8566	4.7e-05	15.1855	0.2959	47.0656	9.4e-06	515.4989	$< 2e-16$
Lag 25	119.6249	2.6e-14	61.0355	2.7e-05	19.3666	0.6798	62.4980	1.7e-05	668.2209	$< 2e-16$

Table 3.9. Test for heteroskedasticity in the residuals of the ARMA models.

A p-value less than 0.05 rejects the null hypothesis, which is no GARCH effects. The table shows there is GARCH effects of orders 5, 15 and 25 present in all the squared residuals for all assets at the 5% significance level, except for NN. The test for NN shows there is GARCH effects present at lag 5, but not at lags 15 and 25. Thus, the GARCH model is incorporated to capture the heteroskedasticity in the residuals.

The log-likelihood, AIC and BIC values of the ARMA-GARCH estimation for group 6 are shown in Table 3.10.

Group 6: The ARMA(0,0)-GARCH(1,1) model										
	Normally distributed standardised residuals					*t*-distributed standardised residuals				
	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO
LL	6988	6525	6884	6915	7344	7060	6650	7040	6980	7385
AIC	-5.5539	-5.1855	-5.4714	-5.4956	-5.8371	-5.6102	-5.2842	-5.5944	-5.5465	-5.8692
BIC	-5.5446	-5.1762	-5.4621	-5.4863	-5.8278	-5.5986	-5.2726	-5.5828	-5.5349	-5.8576

(a) Model selection criteria of the model with ARMA(0,0) for group 6.

Group 6: The ARMA(1,1)-GARCH(1,1) model										
	Normally distributed standardised residuals					*t*-distributed standardised residuals				
	FM	KE	PH	TS	UO	FM	KE	PH	TS	UO
LL	6989	6525	6894	6920	7346	7060	6652	7047	6983	7388
AIC	-5.5529	-5.1842	-5.4773	-5.4983	-5.8366	-5.6088	-5.2839	-5.5987	-5.5478	-5.8696
BIC	-5.5390	-5.1703	-5.4634	-5.4844	-5.8227	-5.5925	-5.2677	-5.5824	-5.5316	-5.8534

(b) Model selection criteria of the model with ARMA(1,1) for group 6.

Table 3.10. The log-likelihood, AIC and BIC values for different orders of ARMA-GARCH models for group 6.

The log-likelihood, AIC and BIC values in the table indicate the best models are ARMA(0,0)-GARCH(1,1) for FM and KE, and ARMA(1,1)-GARCH(1,1) for the remaining assets. Again, these models assume *t*-distributed standardised residuals. The results of the Ljung-Box test conducted on the squared residuals of the chosen ARMA models are shown in Table 3.11.

Group 6: Ljung-Box test on the squared residuals of the ARMA model										
	FM ARMA(0,0)		KE ARMA(0,0)		PH ARMA(1,1)		TS ARMA(1,1)		UO ARMA(1,1)	
	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value
Lag 5	40.4877	1.2e-07	40.6360	1.1e-07	32.0920	5e-07	28.9397	2.3e-06	230.8579	$< 2e\text{-}16$
Lag 15	85.5052	6.8e-12	70.8666	3.1e-09	55.0577	3.9e-07	56.2952	2.4e-07	491.6890	$< 2e\text{-}16$
Lag 25	116.9596	7.6e-14	89.8282	3.1e-09	70.6998	9.5e-07	77.7983	7.2e-08	620.1089	$< 2e\text{-}16$

Table 3.11. Test for heteroskedasticity in the residuals of the ARMA models.

The Ljung-Box test indicates there are GARCH effects of orders 5, 15 and 25 present in all time series. Thus, it is appropriate to use a GARCH model on the residuals to capture the heteroskedasticity.

The parameter estimates for the marginal distributions for group 5 are listed in Table 3.12.

Group 5: Parameter estimates for marginal distributions and statistic tests										
	AP ARMA(0,0)		AD ARMA(1,1)		NN ARMA(1,1)		SG ARMA(1,1)		PF ARMA(1,1)	
	Value	P-value	Value	P-value	Value	P-value	Value	P-value	Value	P-value
μ	0.0015	0.0000	0.0008	0.0047	0.0010	0.0000	0.0005	0.1499	0.0011	0.0000
ϕ			0.6855	0.0003	0.8397	0.0000	-0.7129	0.0001	0.5463	0.0450
θ			-0.7134	0.0001	-0.8641	0.0000	0.6878	0.0002	-0.5867	0.0256
ω	0.0000	0.0000	0.0000	0.1394	0.0001	0.0002	0.0000	0.2288	0.0000	0.0000
α	0.1054	0.0000	0.0339	0.0000	0.1071	0.0001	0.0345	0.0000	0.0926	0.0000
β	0.8464	0.0000	0.9519	0.0000	0.6687	0.0000	0.9626	0.0000	0.8758	0.0000
ν	4.3587	0.0000	4.8110	0.0000	3.9458	0.0000	4.2946	0.0000	4.9784	0.0000
	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value
Ljung-Box test on the standardised residuals										
Lag 5	5.1582	0.3969	3.6168	0.3059	2.1867	0.5346	3.3611	0.3392	5.0182	0.1705
Lag 15	19.9352	0.1744	14.7849	0.3210	9.2697	0.7523	13.8709	0.3830	26.0063	0.0170
Lag 25	32.6064	0.1412	25.8046	0.3102	20.3519	0.6206	17.7059	0.7732	37.8929	0.0262
Ljung-Box test on the squared standardised residuals										
Lag 5	3.8034	0.5781	0.8335	0.8414	1.3949	0.7067	2.8029	0.4230	3.6663	0.2998
Lag 15	9.3577	0.8581	2.8231	0.9985	4.6137	0.9827	8.4551	0.8127	8.7071	0.7947
Lag 25	14.6315	0.9496	5.1851	1.0000	10.7147	0.9859	13.3819	0.9432	12.2316	0.9667

Table 3.12. Estimation results for group 5 with the chosen models.

The table shows the mean parameter μ is significant for all models except for SG. Here, AP has the highest mean estimate which is consistent with Table 3.4. The constant from the GARCH model, which is the parameter ω, is significant for AP, NN and PF but not for AD and SG. However, the only ω estimate different from zero is for NN. The remaining parameters ϕ, θ, α and β are all significant for all assets, which indicates the ARMA-GARCH model is an appropriate choice of model. Further, the degree of freedom parameter ν is significant for all models, and the values are around 4-5. The Ljung-Box test is used to inspect the standardised residuals for autocorrelation and the squared standardised residuals for GARCH effects. The test does not reject the null hypothesis of no autocorrelation in the standardised residuals at lags 5, 15 and 25 at the 5% significance level, except the test for PF, which rejects the null hypothesis at lags 15 and 25. Further, the test conducted on the squared standardised residuals does not reject the null hypothesis of no autocorrelation at lags 5, 15 and 25. Thus, the marginal distributions seem to be adequately fitted for all of the assets.

In Table 3.13 the parameter estimates are shown for the marginal distributions of group 6.

Group 6: Parameter estimates for marginal distributions and statistic tests										
	FM ARMA(0,0)		KE ARMA(0,0)		PH ARMA(1,1)		TS ARMA(1,1)		UO ARMA(1,1)	
	Value	P-value	Value	P-value	Value	P-value	Value	P-value	Value	P-value
μ	0.0005	0.0414	-0.0002	0.5500	0.0002	0.5740	0.0007	0.0000	0.0004	0.0511
ϕ					-0.0026	0.9875	0.9938	0.0000	0.3503	0.2206
θ					0.0757	0.6504	-1.0000	0.0000	-0.3941	0.1595
ω	0.0000	0.0000	0.0000	0.0013	0.0000	0.0430	0.0000	0.0057	0.0000	0.0923
α	0.0614	0.0000	0.0296	0.0000	0.0285	0.0000	0.0328	0.0000	0.0675	0.0000
β	0.8946	0.0000	0.9570	0.0000	0.9637	0.0000	0.9553	0.0000	0.9109	0.0000
ν	6.2919	0.0000	4.1064	0.0000	4.4720	0.0000	6.1126	0.0000	7.3999	0.0000
	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value	Q-stats.	P-value
Ljung-Box test on the standardised residuals										
Lag 5	12.7825	0.0255	5.2032	0.3916	9.0979	0.0280	4.7286	0.1928	4.2569	0.2350
Lag 15	20.7479	0.1451	18.8445	0.2209	16.1224	0.2426	16.5120	0.2226	30.4350	0.0041
Lag 25	27.8071	0.3168	31.8624	0.1620	25.0983	0.3452	34.8820	0.0534	40.7468	0.0127
Ljung-Box test on the squared standardised residuals										
Lag 5	0.2776	0.9980	7.7870	0.1684	10.6159	0.0140	6.9593	0.0732	12.1560	0.0069
Lag 15	3.4458	0.9991	10.4135	0.7930	15.6135	0.2706	10.8895	0.6201	16.1224	0.2426
Lag 25	7.3692	0.9998	17.4709	0.8640	22.2098	0.5076	14.7369	0.9038	27.5986	0.2313

Table 3.13. Estimation results for group 6 with the chosen models.

The mean parameter μ is only significant for the model for FM and the model for TS. The ARMA parameters ϕ and θ are only significant for the model of TS. The parameter ω is significant for all models, except the model for UO, but all the estimates are zero. The GARCH parameters α and β are significant for all models. In addition, the degree of freedom parameter ν is significant for all models, and the values are around 4-7, which is slightly higher than for group 5. The Ljung-Box on the standardised residuals does not reject the null hypothesis of no autocorrelation at the 5% significance level for most of the lags and assets. At lag 5 for FM and PH and lags 15 and 25 for UO the test rejects the null hypothesis of no autocorrelation. The Ljung-Box test on the squared standardised residuals does not reject the null hypothesis of no GARCH effects at lags 15 and 25 at the 5% significance level. The test rejects the null hypothesis at lag 5 for PH and UO.

3.3 Modelling of Copulas

In this section, the copula parameters are estimated after having modelled the marginal distributions. Here, The Normal copula, the t-copula, and the four Archimedean copulas introduced in Section 2.3.1 are estimated. Table 3.14 shows the estimation results based on t-distributed marginals for all assets in group 5.

Group 5: Copula estimation with t-distributed marginals							
Copula	Parameter	DF	LL	AIC	BIC	Upper tail	Lower tail
Normal	0.5976		33863	-67725	-67719	0.0000	0.0000
Student's t	0.5732	4.2829	40208	-80412	-80401	0.2822	0.2822
Clayton	0.7542		31022	-62041	-62035	0.0000	0.3989
Gumbel	1.778		27853	-55704	-55699	0.5233	0.0000
Frank	8.649		13876	-27751	-27745	0.0000	0.0000
Joe	1.24		12289	-24576	-24570	0.2514	0.0000

Table 3.14. Results from fitting different copulas for group 5.

The preferred copula is the t-copula based on the log-likelihood, AIC and BIC values. This supports the conclusion made in [16, p. 349], which states a t-copula often yields a better fit for multivariate financial return data. The second best is the Normal copula followed by the Clayton and the Gumbel copula. The worst copula is the Joe copula followed by the Frank copula. The Clayton copula has lower tail dependence, while the Gumbel copula and the Joe copula have upper tail dependence. In Table 3.15 the results of copula estimation are shown for group 6 based on t-distributed marginals.

Group 6: Copula estimation with t-distributed marginals							
Copula	Parameter	DF	LL	AIC	BIC	Upper tail	Lower tail
Normal	0.655		32020	-64038	-64032	0.0000	0.0000
Student's t	0.6326	6.6073	34547	-69090	-69079	0.2289	0.2289
Clayton	0.5492		24245	-48487	-48481	0.0000	0.2831
Gumbel	1.788		26394	-52786	-52780	0.5265	0.0000
Frank	7.091		12241	-24480	-24474	0.0000	0.0000
Joe	1.384		13253	-26504	-26498	0.3498	0.0000

Table 3.15. Results from fitting different copulas for group 6.

The same observations as for group 5 are seen for group 6, where the preferred copula is the t-copula based on the log-likelihood, AIC and BIC values. The t-copula is followed in order by the Normal copula, the Gumbel copula and the Clayton copula. The worst copula is the Frank copula followed by the Joe copula. In the next section, VaR is estimated based on the four best copulas for groups 5 and 6.

3.4 Estimation of Value-at-Risk

In this section, one-day-ahead out-of-sample VaR is forecasted using the Monte Carlo procedure introduced in Section 2.4. In the implementation of the procedure $N = 10.000$ Monte Carlo simulations and $M = 504$ one-day-ahead out-of-sample forecast are used. These forecasts are evaluated using the backtesting methods presented in Section 2.5, which include both statistical tests and loss functions. Table 3.16 shows the forecasting performances of the procedure based on different copulas with t-distributed standardised residuals for group 5, which includes the ratio of VaR exceedances, Kupiec's unconditional coverage test and Christoffersen's independence test.

Group 5: Backtesting of VaR forecasts with statistical tests						
Copulas	95% VaR			99% VaR		
	Z/T	LR_{UC}	LR_{CC}	Z/T	LR_{UC}	LR_{CC}
Normal	0.0656	4.3718	5.7915	0.0159	0.2291	0.2934
P-value		0.0365	0.0553		0.6322	0.8636
Student's t	0.0755	12.2748	12.2811	0.0159	4.8617	4.8657
P-value		0.0005	0.0022		0.0275	0.0878
Clayton	0.0696	7.8881	8.4776	0.0139	4.8617	4.8657
P-value		0.0050	0.0144		0.0275	0.0878
Gumbel	0.0736	11.1007	11.1325	0.0278	0.1780	0.3231
P-value		0.0009	0.0038		0.6731	0.8508

Table 3.16. Results from evaluating the one-day-ahead out-of-sample VaR forecasts for group 5.

The table shows the procedure based on the Normal copula is closest to the desired ratio of VaR exceedances for the 95% VaR and the procedure based on the Clayton copula is closest for the 99% VaR. This result is conflicting with Table 3.14, where the t-copula is the best choice. Recall, the null hypothesis of Kupiec's unconditional coverage test states the observed number of exceedings statistically equals the predicted number of exceedings, and the null hypothesis of Christoffersen's test states the VaR exceedings are independent. Thus, a p-value greater than 0.05 is needed to not reject a models reliability. The null hypothesis is rejected for all models with a 95% confidence level, except for the Normal copula tested with Christoffersen's test. When the tests are performed for the 99% confidence level, Kupiec's unconditional coverage test rejects the null hypothesis for the t-copula and Clayton copula but does not reject the null hypothesis for the Normal copula and the Gumbel copula. Christoffersen's test does not reject the null hypothesis for any of the models. This backtesting analysis indicates the Normal copula is the preferred copula for modelling the assets in group 5. In Table 3.17 the results from backtesting are shown for group 6.

Group 6: Backtesting of VaR forecasts with statistical tests						
Copulas	95% VaR			99% VaR		
	Z/T	LR_{UC}	LR_{CC}	Z/T	LR_{UC}	LR_{CC}
Normal	0.0577	1.8149	0.1780	0.0139	2.5426	0.3231
P-value		0.1779	0.6731		0.2805	0.8508
Student's t	0.0557	2.9671	0.0002	0.0119	4.0140	0.1008
P-value		0.0850	0.9893		0.1344	0.9509
Clayton	0.0596	4.3718	0.1780	0.0119	4.4472	0.3231
P-value		0.0365	0.6731		0.1082	0.8508
Gumbel	0.0596	3.6388	1.5021	0.0179	3.7784	1.7612
P-value		0.0564	0.2203		0.1512	0.4145

Table 3.17. Results from evaluating the one-day-ahead out-of-sample VaR forecasts for group 6.

The table shows the t-copula is closest to the ratio of VaR exceedances for both the 95% VaR and the 99% VaR. This is consistent with Table 3.15. Note, the ratios of VaR exceedances are all quite close to 5% and 1% for the respective VaR levels. This indicates all models can adequately represent the 95% and the 99% VaR. The results of Kupiec's unconditional coverage test and Christoffersen's independence test are that the null hypothesis is not rejected for all copulas for the 95% VaR and the 99% VaR, which also suggests the models are reliable in producing accurate VaR forecasts. The only exception is the Clayton copula for the 95% VaR, which for the Kupiec's unconditional coverage test has a p-value less than 0.05, which means the Clayton copula statistically fails to accurately predict the number of observed exceedings. This analysis shows the t-copula is preferred for modelling the assets in group 6. Further, based on the backtesting analysis, the VaR forecasts are better for group 6 than for group 5

Next, the loss functions from Section 2.5 are used to determine which copula is better. Here, the two loss functions proposed by Lopez and the one proposed by Blanco and Ihle are considered. The results of the loss functions for group 5 are shown in Table 3.18.

Group 5: Backtesting of VaR forecasts with loss functions						
Copulas	95% VaR			99% VaR		
	C_t^{L1}	C_t^{L2}	C_t^{BI}	C_t^{L1}	C_t^{L2}	C_t^{BI}
Normal	0.0716	0.4812	0.1085	0.0080	0.2284	0.0274
Student's t	0.0875	0.5018	0.1160	0.0020	0.2245	0.0267
Clayton	0.0795	0.4970	0.1146	0.0020	0.2052	0.0224
Gumbel	0.0855	0.5050	0.1185	0.0119	0.3018	0.0422

Table 3.18. Results from evaluating the violations of VaR for group 5.

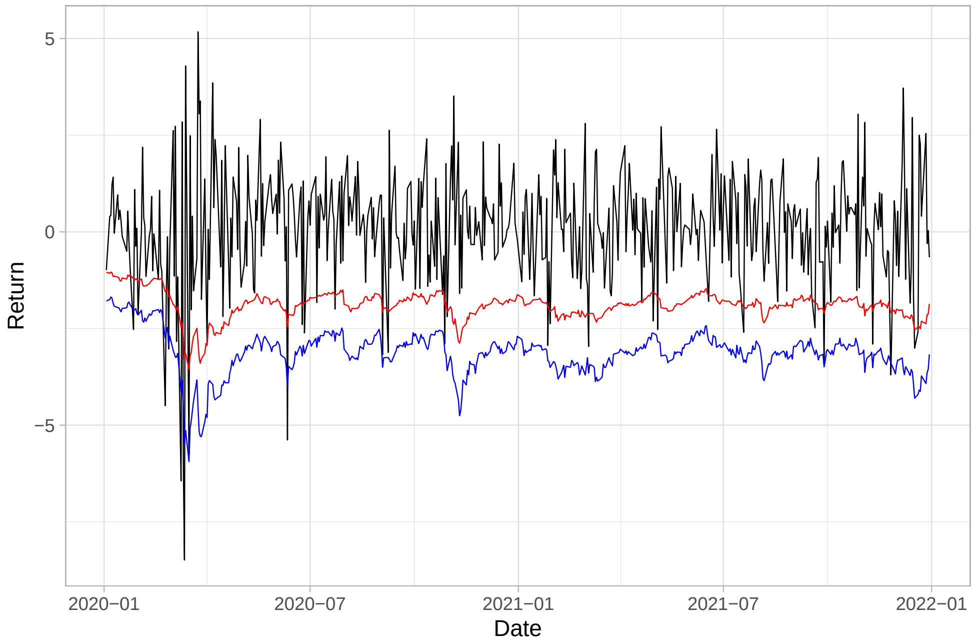
The loss functions are used to evaluate the violations and further distinguish between models. Note, in C_t^{L1} the loss function equals the hit series in (2.15), for C_t^{L2} the loss function in (2.16) is used and in C_t^{BI} the loss function in (2.17) is used. The three loss functions are evaluated with the sample average of the loss functions in (2.18). Table 3.18 shows the Normal copula is again preferred for the 95% VaR based on the three loss functions. For the 99% VaR the Clayton copula is preferred. In Table 3.19 the results of the loss functions are shown for group 6.

Group 6: Backtesting of VaR forecasts with loss functions						
Copulas	95% VaR			99% VaR		
	C_t^{L1}	C_t^{L2}	C_t^{BI}	C_t^{L1}	C_t^{L1}	C_t^{BI}
Normal	0.0636	0.0935	0.0297	0.0119	0.0285	0.0058
Student's t	0.0676	0.0926	0.0309	0.0099	0.0256	0.0057
Clayton	0.0716	0.0996	0.0360	0.0119	0.0270	0.0062
Gumbel	0.0696	0.0975	0.0331	0.0159	0.0350	0.0077

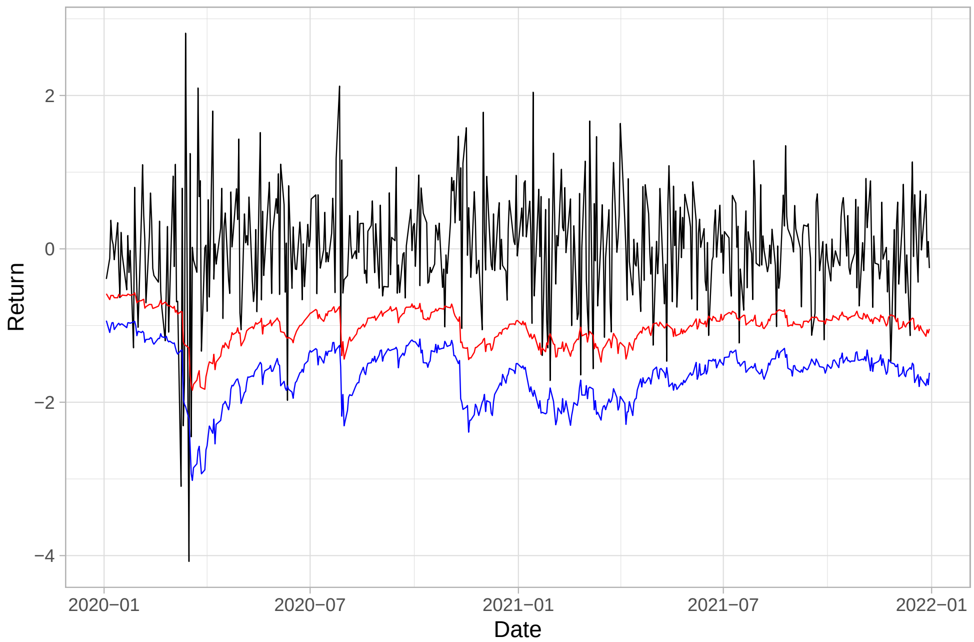
Table 3.19. Results from evaluating the violations of VaR for group 6.

The table shows the Normal copula and the t-copula for the 95% VaR yield the best goodness-of-fit. The t-copula is best in C_t^{L2}, which takes the magnitude of the exceedings into account where larger failures are penalised. For the 99% VaR the t-copula violates the VaR the least for all loss functions.

For an illustration of the predictive performance of the VaR for group 5 and 6, the portfolio returns, the 95% VaR and the 99% VaR are plotted in Figure 3.20. The figure shows the forecasts based on the Monte Carlo procedure with the Normal copula for group 5 and the t-copula for group 6.



(a) Forecasting performance of group 5 with the Normal copula.



(b) Forecasting performance of group 6 with the *t*-copula.

Figure 3.20. Depiction of returns for group 5 and 6 with their respective VaR forecasts from the Monte Carlo procedure, where the red curve represents the 95% VaR and the blue curve represents the 99% VaR.

Figure 3.20 shows how the VaR fluctuates with the portfolio returns for both groups. Here, the ratios of VaR exceedances shown in Table 3.16 and 3.17 are the percentage of returns that exceeds the 95% VaR and the 99% VaR.

Comparison of Value-at-Risk 4

In this chapter, the analysis conducted in Chapter 3 is evaluated by comparing the findings with the common expectations for return and standard deviation published by the Council for Return Expectations. In addition, two index funds that represent group 5 and 6 are introduced with the purpose of determining if the findings for the portfolios consisting of five assets are representative for the respective groups.

4.1 Council for Return Expectations

This section in based on [3].

In this section, the Council for Return Expectations is introduced along with their published expectations of returns, standard deviations and correlations for different categories of assets. These expectations are then compared with the analysis and forecasts conducted in Chapter 3. The expectations from the Council for Return Expectation are published on the cite `www.afkastforventninger.dk`[1]. These published expectations are used by various different financial institutions, e.g. pension companies and banks.

The Council for Return Expectations was established in 2018 by Insurance & Pension Denmark and Finance Denmark and the first report was published in 2020. The council consists of three independent experts who are appointed their position for three years at the time. Currently, the council consists of chairman Jesper Rangvid, who is a professor at Copenhagen Business School and head of the Pension Research Centre, Torben M. Andersen, who is a professor at Aarhus University and chairman of ATP's Council of directors and former president of the Danish Economic Councils, and lastly Peter Engberg Jensen, who is the chairman of the council of directors for Finansiel Stabilitet and former group CEO at the bank Nykredit.

The purpose of the council is to determine the common expectations of returns to be used by pension companies and financial institutions when calculating pension prognoses and return expectations from investments. The expectations are determined twice a year. The motive behind having all danish pension companies and financial intuitions conduct their analysis on the same return expectations is to ensure realistic forecasts and thus reliable investments products for the consumer.

[1] The links to the reports from the Council for Return Expectations are given here:
`https://www.afkastforventninger.dk/media/1431/samfundsforudsaetninger-2020-3.pdf`
`https://www.afkastforventninger.dk/media/1440/samfundsforudsaetninger-andet-halvaar-2020.pdf`
`https://www.afkastforventninger.dk/media/1453/samfundsforudsaetninger-1-halvaar-2021-1.pdf`
`https://www.afkastforventninger.dk/media/1483/samfundsforudsaetninger-2-halvaar-2021-fejlret tet.pdf`

The council calculates the expectations on different time horizons, which are 1-5 years, 1-10 years, 6-10 years and 11+ years. However, the report from the first half of 2020 only contains expectations for 1-10 years and 11+ years. Nonetheless, this project will focus on the short term and the expectations in Table 4.1 are for 1-10 years for the 1st half of 2020 and 1-5 years for the remaining reports. The expectations are calculated for 10 different categories of assets. This project focuses on the two categories global equity and emerging markets equity, which are group 5 and 6, respectively. The expectations for all categories of stocks can be seen in Appendix A.5. The expectations for the categories in focus are given in Table 4.1.

	5. Global Equity		6. Emerging Markets Equity	
	Return	St. Dev.	Return	St. Dev.
1st half of 2020[2]	5.5%	11.0%	9.5%	28.4%
2nd half of 2020	6.0%	13.5%	9.5%	29.9%
1st half of 2021	5.6%	13.5%	8.5%	25.1%
2nd half of 2021	5.4%	13.9%	7.7%	24.5%

Table 4.1. Common return expectations for global equity and emerging markets equity on the short term.

Table 4.1 shows the expected return and standard deviation for an asset, whose holding period approximately spans from start January or start July for 2020 and 2021 and then either 1-5 years or 1-10 years ahead, depending on the report. It appears stocks from group 5 are expected to have a return around 5.5% and this expectation rises in the second half of 2020. In 2021 the expected return slightly reduces. The standard deviation increases from 11.0% in the first half of 2020 to 13.9% in the second half of 2021. These evolutions might be caused by the corona pandemic, which started in the beginning of 2020. If the expectations for group 6 are considered, it shows the expected return decreases from 9.5% to 7.7% in the period from 2020 to 2021 and the standard deviation decreases from 28.4% to 24.5%. If the two groups are compared, stocks from group 5 have a smaller standard deviation than stocks from group 6. This indicates stocks from group 5 are less risky. The expected return from stocks in group 5 are also smaller compared with group 6. The higher return for emerging markets can be caused by the potential for economical growth in the countries. The higher standard deviation can be caused by an unstable government, a volatile currency and lack of labor and materials in the countries. For a more detailed explanation, see [8]. Developed countries will typically have a more mature and robust economy [22].

The expectations for the correlation between the two groups global equity and emerging markets equity published by the Council for Returns Expectations are given in Table 4.2.

Correlation between group 5 and 6				
	1st half of 2020[2]	2nd half of 2020	1st half of 2021	2nd half of 2021
Correlation	0.8	0.8	0.7	0.7

Table 4.2. Correlation expectations between global equity and emerging markets equity on the short term.

[2]The expectations for 1-10 years are listed for the 1st half of 2020. For the remaining reports, the expectations for 1-5 years are listed.

The table shows the expected correlations for the 1st half and the 2nd half are equal in the respective years and are relatively high. The correlation in 2020 decreases from 0.8 to 0.7 in 2021.

In order to compare the VaRs calculated in Section 3.4 with the expectations provided by the Council for Return Expectations, the parametric method is used to calculate the VaR based on these expectations. The expectations are seen in Table 4.1. The parametric method is introduced in Section 2.1.1 and the formula for VaR with normally distributed losses is repeated as,

$$\widehat{\text{VaR}}_\alpha^{\text{par, Ga}} = -S \times (\hat{\mu} + \hat{\sigma}\phi^{-1}(\alpha)). \tag{4.1}$$

The current position, S, is the value of the portfolio at the start of the period. Thus, S is calculated as the portfolio value at the last banking day in December 2019 for the report published in the first half of 2020 and the last banking day in June for the report published in the second half of 2020. In this project, the focus is on these two reports. In (4.1), $\hat{\mu}$ and $\hat{\sigma}$ are the expected return and standard deviation found in Table 4.1. The α-quantiles are more commonly called z values and are chosen for the 95% and the 99% confidence level in a one-tailed distribution. For the normal distribution, these values are

$$\begin{aligned} z &= -1.65 \quad \text{for the 95\% VaR,} \\ z &= -2.33 \quad \text{for the 99\% VaR.} \end{aligned} \tag{4.2}$$

The values of σ for the normal distribution is the standard deviation for the respective periods, which are the following values

$$\begin{aligned} \sigma &= 0.11 && \text{for the 1st half of 2020 for group 5,} \\ \sigma &= 0.135 && \text{for the 2nd half of 2020 for group 5,} \\ \sigma &= 0.284 && \text{for the 1st half of 2020 for group 6,} \\ \sigma &= 0.299 && \text{for the 2nd half of 2020 for group 6.} \end{aligned} \tag{4.3}$$

For t-distributed losses the formula for VaR is repeated as,

$$\widehat{\text{VaR}}_\alpha^{\text{par, t}} = -S \times (\hat{\mu} + \hat{\sigma} t_\nu^{-1}(\alpha)),$$

The current position S and $\hat{\mu}$ are found in the same way as for the normally distributed losses. Recall, $\hat{\sigma}$ is no longer the standard deviation but a scale parameter defined as

$$\sigma = \sqrt{\text{var}(\mathcal{L})\frac{\nu - 2}{\nu}}.$$

The degree of freedom, ν, determines the corresponding z values for the 95% VaR and the 99% VaR. The analysis conducted in Section 3.2 shows $\nu = 4$ is an appropriate degree of freedom for group 5, where the corresponding z values are

$$\begin{aligned} z &= -2.132 \quad \text{for the 95\% VaR,} \\ z &= -3.747 \quad \text{for the 99\% VaR.} \end{aligned}$$

The analysis shows $\nu = 6$ is an appropriate degree of freedom for group 6, where the corresponding z values are

$$\begin{aligned} z &= -1.943 \quad \text{for the 95\% VaR,} \\ z &= -3.143 \quad \text{for the 99\% VaR.} \end{aligned}$$

The scale parameter σ for the t-distribution is determined by the degrees of freedom and the variance of the losses for the respective periods, which yield the following values

$$\begin{aligned}
\sigma &= 0.11 \cdot \sqrt{(4-2)/4} = 0.0778 && \text{for the 1st half of 2020 for group 5,} \\
\sigma &= 0.135 \cdot \sqrt{(4-2)/4} = 0.0955 && \text{for the 2nd half of 2020 for group 5,} \\
\sigma &= 0.284 \cdot \sqrt{(6-2)/6} = 0.2319 && \text{for the 1st half of 2020 for group 6,} \\
\sigma &= 0.299 \cdot \sqrt{(6-2)/6} = 0.2441 && \text{for the 2nd half of 2020 for group 6.}
\end{aligned} \tag{4.4}$$

The VaRs calculated in percent with the parametric method with these values are given in Table 4.3.

5. Global Equity				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	12.65%	16.275%	9.6130%	12.5478%
99% VaR	20.13%	25.455%	18.9468%	24.0029%

(a) The VaR calculated in percent for group 5 with the five assets introduced in Section 3.1.

6. Emerging Markets Equity				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	37.36%	39.835%	35.5552%	37.9349%
99% VaR	56.672%	60.167%	63.3812%	67.2308%

(b) The VaR calculated in percent for group 6 with the five assets introduced in Section 3.1.

Table 4.3. Annualised VaR calculated for the respective assets in group 5 and 6.

In Table 4.3 the current position for the first half of 2020 is calculated with the prices on the banking day 2019-12-31. The portfolio value for group 5 is $S = 87.7915$ and the portfolio value for group 6 is $S = 42.3278$. The current position for the second half of 2020 is calculated with the prices on the banking day 2020-06-30. The portfolio values for group 5 and 6 are respectively $S = 81.6282$ and $S = 34.3751$.

Table 4.3 shows the VaR increases from the first half of 2020 to the second half of 2020. If the VaR for group 5 is considered, it appears the VaR is smaller for the t-distribution compared with the normal distribution for both the 95% VaR and the 99% VaR. If the VaR for group 6 is considered, it appears the VaR with a 95% confidence level is smaller for the t-distribution compared with the normal distribution. For the 99% confidence level the opposite appears to be true, where the VaR is larger for the t-distribution compared with the normal distribution. This tendency is not expected as the t-distribution normally has heavier tails and thus VaR is expected to be greater for the t-distribution. The reason for the tendency seen in Table 4.3 is examined. First, compare the z values for the normal distribution and the t-distribution.

Here, the absolute value of z is greater for the t-distribution with the respective confidence levels, which indicates a larger VaR. If the respective values for σ are considered in (4.3) and (4.4), it appears the values are lower for the t-distribution for both groups. These σ values appear to cause the tendency seen in Table 4.3 and the same tendencies are seen in other literature including [12].

In order to compare the Monte Carlo VaR with the VaR based on the expectations from the Council for Return Expectations, the latter VaR needs to be calculated for daily returns and standard deviations rather than annual. Thus, the annual returns and standard deviations must be converted. The formulas for converting an annual return to a daily return and for converting an annual standard deviation to a daily standard deviation are derived from formulas stated in [14, 21]. Since only the trading days are considered, the formulas are rewritten with 250 days. The formulas are given by

$$\begin{aligned} R_{\text{daily}} &= \sqrt[250]{R_{\text{annual}}/100 + 1} - 1, \\ \sigma_{\text{daily}} &= \frac{\sigma_{\text{annual}}}{\sqrt{250}}. \end{aligned}$$

Here, R is the return and 250 is the approximate number of trading days in a year. Note, the expectations from the Council for Return Expectations apply for the holding period between 1-5 and 1-10 years, depending on the report. Thus, the number of trading days could be scaled to the number of trading days in a 5 or 10 year period instead of a 1 year period. With these conversions the daily VaR is calculated and given in Table 4.4.

5. Global Equity	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	0.9890	1.1309	0.8203	0.9385
99% VaR	1.4042	1.6049	1.3386	1.5299

(a) The daily VaR calculated with the parametric method for group 5.

6. Emerging Markets Equity	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	1.2639	1.0600	1.1908	1.0188
99% VaR	1.7560	1.5021	1.9357	1.6557

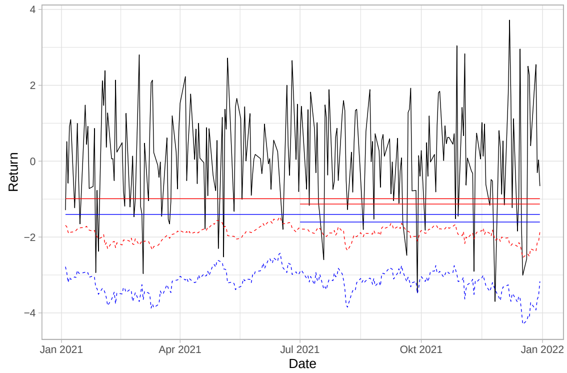
(b) The daily VaR calculated with the parametric method for group 6.

Table 4.4. The daily VaR calculated for group 5 and 6 with the parametric method for the normal and the t-distribution. The daily returns and standard deviations are calculated by converting the annual returns and standard deviations published by the Council for Return Expectations.

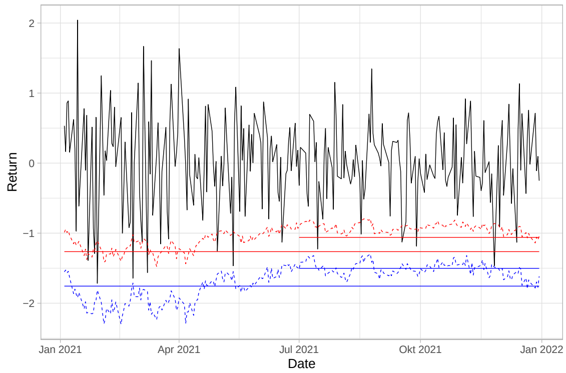
In Table 4.4, the daily VaR is calculated in losses rather than percentages. The same tendencies that describe Table 4.3 are applicable for these daily VaR.

4.1.1 Comparison with Monte Carlo Value-at-Risk Forecasting

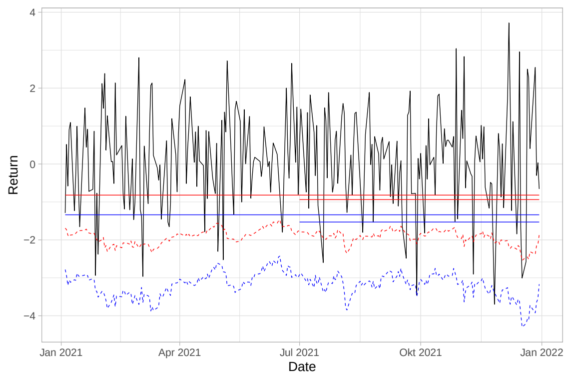
In this section, the VaR forecasts calculated using the Monte Carlo procedure in Section 3.4 are compared with the parametric VaR based on the expected returns and standard deviations published by the Council for Return Expectations. In Figure 4.5 these values are plotted for group 5 and 6.



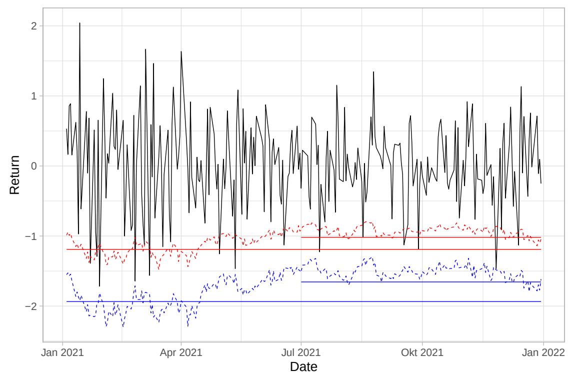
(a) Forecasting performance of group 5 with the normal distribution.



(b) Forecasting performance of group 6 with the normal distribution.



(c) Forecasting performance of group 5 with the t-distribution.



(d) Forecasting performance of group 6 with the t-distribution.

Figure 4.5. The dashed lines represent the forecasts based on the Monte Carlo procedure with the Normal copula for group 5 and the t-copula for group 6. The (dashed) red curve represents the 95% VaR and the (dashed) blue curve represents the 99% VaR.

In Figure 4.5 the black line is the observed daily portfolio return, the dashed lines are the forecasted VaR from the Monte Carlo procedure based on the Normal copula for group 5 and the t-copula for group 6 with a 95% and a 99% confidence level and the solid constant lines are the VaR calculated with the parametric approach. Here, the solid line that spans the entire period is calculated from the expectations published in the report from the first half of 2020 and the solid line that spans the period from July 2021 to January 2022 is calculated with the expectations published in the second half of 2020. The daily VaR can be seen in Table 4.4. Note, the blue dashed line and the blue solid line are the 99% confidence level and the red lines are the 95% confidence level.

In Figure 4.5 there is little difference between the forecasts based on different distributions. For group 5 it appears the VaR calculated using the Monte Carlo procedure is larger than the VaR calculated using the parametric approach. Here, the Monte Carlo VaR with a 99% confidence level appears to capture almost all realised losses, whereas the parametric VaR is lower and appears to be violated more frequently. For group 6 the VaR using Monte Carlo and the VaR using the parametric approach seem to be close to each other. However, the parametric VaR with both the normal distribution and the t-distribution is never violated for the 99% confidence level. To investigate the violations further, the ratios of the VaR exceedances are calculated for the parametric VaR, which is seen in Table 4.6. The ratio of VaR exceedances for the Monte Carlo procedure is seen in Table 3.16 for group 5 and Table 3.17 for group 6.

Group 5: Ratio of VaR exceedances				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	0.1952	0.0837	0.2191	0.1036
99% VaR	0.1116	0.0478	0.1155	0.0518

(a) The ratio of VaR exceedances for group 5.

Group 6: Ratio of VaR exceedances				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	0.0279	0.0199	0.0359	0.0199
99% VaR	0.0000	0.0000	0.0000	0.0000

(b) The ratio of VaR exceedances for group 6.

Table 4.6. The ratio of VaR exceedances, Z/T, with the normal distribution and the t-distribution.

Consider the exceedances for group 5 based on the Normal copula in Table 3.16 and the exceedances for group 5 with the normal distribution in Table 4.6 for the 95% confidence level. Here, the Monte Carlo VaR is violated 6.56% of the time, the parametric VaR for the first half of 2020 is violated 19.52% of the time and the parametric VaR for the second half of 2020 is violated 8.37% of the time. Thus, the Monte Carlo VaR is closer to being violated only 5% of the time. Furthermore, it appears the Council for Return Expectations had underestimated the risk in the first half of 2020 but an adjustment in the expectation for the second half of 2020 had improved the level of VaR. If the t-distribution is considered for the 95% confidence level the parametric VaR for the first half of 2020 is violated 21.91% of the time and the parametric VaR for the second half of 2020 is violated 10.36% of the time. Thus, the same tendencies are seen as for the normal distribution. If the 99% confidence level is considered the same pattern appears with lower percentile violations.

Consider the exceedances for group 6 based on the t-copula in Table 3.17 and the exceedances for group 6 with the normal distribution in Table 4.6 for the 95% confidence level. The Monte Carlo VaR is violated 5.57% of the time, the parametric VaR for the first half of 2020 is violated 2.79% of the time and the parametric VaR for the second half of 2020 is violated 1.99% of the time. If the t-distribution is considered, the parametric VaR for the first half of 2020 is violated 3.59% of the time and 1.99% of the time for the second half of 2020. This suggests, the parametric VaR is overestimating the risk whereas the Monte Carlo VaR represents a more accurate 95% VaR. The Monte Carlo VaR is violated 1.19% of the time for the 99% confidence level and the parametric VaR is never violated as seen in Figure 4.5. These observations also indicate parametric VaR based on the expectations published by the Council of Return Expectations overestimates the risk in group 6.

4.2 Comparison with Index Funds

In this section, it is investigated if the portfolios consisting of five assets are representative of group 5 and 6. The assets in the respective portfolios are chosen with the assumption of them being representative of their respective group. However, it is unlikely these 5 assets are fully characteristic for their groups. To get a better picture of how the markets are developing in general, two index funds are introduced to represent group 5 and 6, respectively. The funds are designed to measure the performance of equity securities in the large and mid-capitalization in the respective markets. For group 5 the index fund *iShares MSCI World ETF* (DM) is used and for group 6 the index fund *iShares MSCI Emerging Markets ETF* (EM) is used [7, 6]. The same analysis conducted for the portfolios for group 5 and group 6 in Chapter 3 is performed in this section for the index funds in order to compare them.

The two index funds are divided into three samples as in Section 3.1. The full sample is from 2012-02-01 to 2021-12-30 containing 2496 observations. The estimation sample is from 2012-02-01 to 2019-12-31 containing 1992 observations. Note, the full sample and the estimation sample do not start from 2010 since the DM index fund did not exist until mid-January 2012. The forecasting sample is from 2020-01-02 to 2021-12-30 containing 504 observations. As for the analysis conducted in Chapter 3, the forecasting sample is concurrent with the corona pandemic. Further, the adjusted price is used in the analysis and is referred to as the price. In Figure 4.7 the prices for the index funds are depicted.

Figure 4.7. The prices of the index funds. The dashed line represents the split between the estimation sample and the forecasting sample.

The price of DM increases in the estimation sample with minor fluctuations. In the start of the forecasting period, and thereby in the start of the corona pandemic, a negative fluctuation is seen but after a year the price is back to the same level as before the corona pandemic, where the same increasing tendency is seen. The price of EM rises slowly with notable fluctuations. Here, a negative fluctuation is also seen in the start of the forecasting period. Note, the scale for DM and EM differs. The plots in Figure 4.7 are similar to the plots in Figure 3.1, which are the portfolio prices consisting of five assets for group 5 and 6, respectively. From these plots it seems the two portfolios in Chapter 3 are representative for the two markets, although the portfolios consists of a low number of assets.

To further investigate the movements in the index funds the log-returns are illustrated in Figure 4.8.

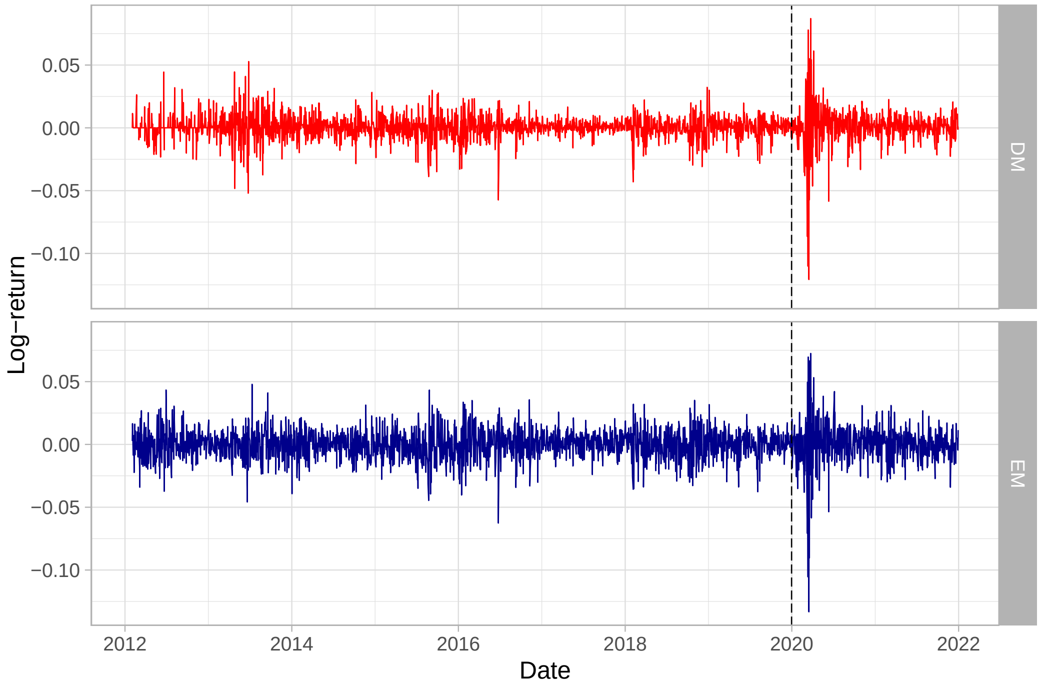


Figure 4.8. The log-returns of the index funds. The dashed line represents the split between the estimation sample and the forecasting sample.

The figure shows clustering in the log-returns for both index funds. Especially in the start of 2020 where the corona pandemic started, more extreme values of the log-returns are seen. The same pattern is present for the assets in the portfolios, which is illustrated in Figure 3.3

In Table 4.9 the descriptive statistics for the log-returns of the two index funds are presented for the three periods. The same methods used in Table 3.4 and 3.5 for the portfolios for group 5 and 6 are used in this table.

Descriptive statistics						
	Full sample		Estimation sample		Forecasting sample	
	DM	EM	DM	EM	DM	EM
Mean	0.0005	0.0001	0.0004	0.0001	0.0007	0.0002
St. Dev.	0.0108	0.0130	0.0092	0.0116	0.0158	0.0173
95% VaR	0.0154	0.0197	0.0141	0.0189	0.0213	0.0235
99% VaR	0.0308	0.0339	0.0273	0.0314	0.0509	0.0534
Skewness	-1.0501	-0.8312	-0.3104	-0.2186	-1.4684	-1.4863
Excess kurtosis	17.8792	8.8836	4.5458	1.1957	16.5955	12.2935
Jarque-Bera test	33753	8510	1752	135	6020	3392
P-value	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16	< 2e-16
ADF test	-11.0070	-10.0806	-9.3010	-8.8794	-5.3454	-4.3022
P-value	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01
Ljung-Box test	369.1604	114.8658	71.0103	29.2865	279.9802	161.5437
P-value	< 2.2e-16	1.8e-13	2.7e-06	0.2521	< 2.2e-16	< 2.2e-16

Table 4.9. Descriptive statistics of the log-returns of the index funds.

The table shows DM has the highest mean and it also varies the most across the three sample periods. The mean is higher for the forecasting sample, especially for DM. This is also seen in the reports from the Council for Return Expectations in Table 4.1, where the expected return is increased from the 1st half of 2020 to the 2nd half of 2020 for global equity. The standard deviation is higher for EM than for DM and the standard deviations are higher for the forecasting sample. This is consistent with the reports from the Council for Return Expectations and the plots of the prices for the index funds in Figure 4.7. The 95% VaR and the 99% VaR are highest for the forecasting sample, which is consistent with the mean and standard deviation. In all periods the index funds are negatively skewed and exhibit excess kurtosis. The Jarque-Bera test has relatively high test statistics and p-values less than 0.05 for DM and EM in all samples which indicates non-normality. For the ADF test all p-values are less than 0.05, which indicates both index funds are stationary. The Ljung-Box test is performed on the log-returns with lag 25. All p-values are less than 0.05, which indicate there is autocorrelation present in the log-returns, with the exception of EM in the estimation sample.

When comparing the descriptive statistics in Table 4.9 for the index funds with the descriptive statistics in Table 3.4 and 3.5 for the portfolios for group 5 and 6, the same tendencies are seen, which include a higher mean, standard deviation, 95% VaR and 99% VaR for the forecasting sample. Furthermore, nearly all assets are negatively skewed and exhibit excess kurtosis. In Table 4.10 the correlation between the two index funds is given for the three sample periods.

Correlations of the index funds			
	Full sample	Estimation sample	Forecasting sample
Correlation	0.72	0.63	0.86

Table 4.10. Correlations.

The correlations between the index funds are all positive and relatively high, especially for the forecasting sample. The correlation for the forecasting sample can be compared to the expected correlations in Table 4.2 published by the Council for Return Expectations. Here, the correlation expectations are 0.8 for both reports which is close to 0.86. Note, the reports from the Council for Return Expectations become effective a year after they are released. Therefore, the forecasting sample cannot be directly compared with the reports. The correlations between the portfolios seen in Table 3.2 have the same tendencies as the index funds. However, the correlations between the index funds are smaller than for the portfolios.

4.2.1 Modelling

In this section, a mean-variance model is fitted in order to conduct one-day-ahead forecasts for the index funds. The order of the ARMA-GARCH model is selected as well as the distribution of the standardised residuals. The analysis is conducted on the estimation sample and the results are shown in Table 4.11.

The ARMA(0,0)-GARCH(1,1) model				
	Normally distributed standardised residuals		*t*-distributed standardised residuals	
	DM	EM	DM	EM
LL	8302	7584	8448	760
AIC	-6.6521	-6.0765	-6.7679	-6.0899
BIC	-6.6427	-6.0671	-6.7563	-6.0782

(a) Model selection criteria of the model with ARMA(0,0) for index funds.

The ARMA(1,1)-GARCH(1,1) model				
	Normally distributed standardised residuals		*t*-distributed standardised residuals	
	DM	EM	DM	EM
LL	8314	7586	8463	7603
AIC	-6.6598	-6.0758	-6.7784	-6.0892
BIC	-6.6458	-6.0618	-6.7621	-6.0728

(b) Model selection criteria of the model with ARMA(1,1) for the index funds.

Table 4.11. The log-likelihood, AIC and BIC values for different orders of ARMA-GARCH models.

The best models for the index funds are selected by considering the log-likelihood, AIC and BIC values. This yields the best models assume t-distributed standardised residuals and an ARMA(1,1)-GARCH(1,1) for DM and an ARMA(0,0)-GARCH(1,1) for EM. The orders of the ARMA model are consistent with the autocorrelation results from the Ljung-Box test on the estimation sample in Table 4.9, where the null hypothesis of no autocorrelation is rejected for DM and not rejected for EM in the estimation sample.

To check for heteroskedasticity, and thus GARCH effects, the Ljung-Box test is performed on the squared residuals of the ARMA model and the results are seen in Table 4.12.

Ljung-Box test on the squared residuals				
	DM ARMA(1,1)		EM ARMA(0,0)	
	Q-stats.	P-value	Q-stats.	P-value
Ljung-Box test				
Lag 5	177.7048	$< 2e-16$	167.0099	$< 2e-16$
Lag 15	316.9475	$< 2e-16$	291.5252	$< 2e-16$
Lag 25	373.8124	$< 2e-16$	310.6521	$< 2e-16$

Table 4.12. Test for heteroskedasticity in the residuals of the ARMA models.

Table 4.12 shows the Ljung-Box test on the squared residuals of the ARMA models rejects the null hypothesis of no GARCH effects. This indicates there are GARCH effects of orders 5, 15 and 25 present in the index funds. To capture the heteroskedasticity a GARCH(1,1) model is included.

In Table 4.13 the parameters and statistical tests of the selected ARMA-GARCH models are presented.

Parameter estimates and statistic tests				
	DM ARMA(1,1)		EM ARMA(0,0)	
	Value	P-value	Value	P-value
μ	0.0008	0.0000	0.0004	0.0548
ϕ	0.8033	0.0000		
θ	-0.8682	0.0000		
ω	0.0000	0.1515	0.0000	0.0218
α	0.1724	0.0000	0.1073	0.0000
β	0.8266	0.0000	0.8550	0.0000
ν	3.5040	0.0000	13.6392	0.0000
	Q-stats.	P-value	Q-stats.	P-value
Ljung-Box test on the standardised residuals				
Lag 5	10.6039	0.0141	2.8665	0.4127
Lag 15	17.4270	0.1805	5.7016	0.9563
Lag 25	23.6414	0.4239	23.5464	0.4293
Ljung-Box test on the squared standardised residuals				
Lag 5	4.3562	0.2255	1.7766	0.6200
Lag 15	12.0812	0.5210	18.2511	0.1482
Lag 25	19.7651	0.6560	25.1411	0.3431

Table 4.13. Estimation results for the index funds with the chosen models.

The table shows the mean parameter μ is significant for DM but not for EM. The ARMA parameters ϕ and θ are both significant for DM. The constant from the GARCH model, ω, is zero for both index funds and is not significant for DM. The GARCH parameters α and β are all significant. The degree of freedom parameter ν is significant for both index funds. Further, the value of ν for EM is relatively larger than for DM. In Table 3.12 and 3.13, which is the estimation results for the portfolios, it is also seen that the ν parameter is higher for group 6 than for group 5. The Ljung-Box test on the standardised residuals does not reject the null hypothesis of no autocorrelation at lags 5, 15 and 25 for both index funds, except at lag 5 for DM. The Ljung-Box test on the squared standardised residuals does not reject the null hypothesis of no autocorrelation, thus there is no GARCH effects at lags 5, 15 and 25.

4.2.2 Estimation of Value-at-Risk

In this section, VaR is forecasted one-day-ahead for the index funds based on the previous analysis of the orders of the ARMA-GARCH models and distributions of the standardised residuals, which are assumed to be t-distributed. Then, the predicted VaR is tested by statistical tests.

The forecasting procedure for VaR is given by the following steps:

(1) Use the estimation sample with T observations to fit the respective ARMA-GARCH models on the log-returns.

  (i) Use the degree of freedom from Table 4.13 to find the critical value z for the 95% VaR and the 99% VaR.

(2) Forecast one-step means, $\hat{r}_{T+1}$, and variances, $\hat{\sigma}_{T+1}$ at time $T+1$.

(3) Use the values from steps (1)-(2) to find the forecasted VaR by (2.3).

(4) Repeat steps (1)-(3) M times by rolling over the daily returns, where M is the number of days needed to be forecasted.

(5) Convert VaR from log-returns to returns by

$$R_t = P_{t-1}(\exp(r_t) - 1).$$

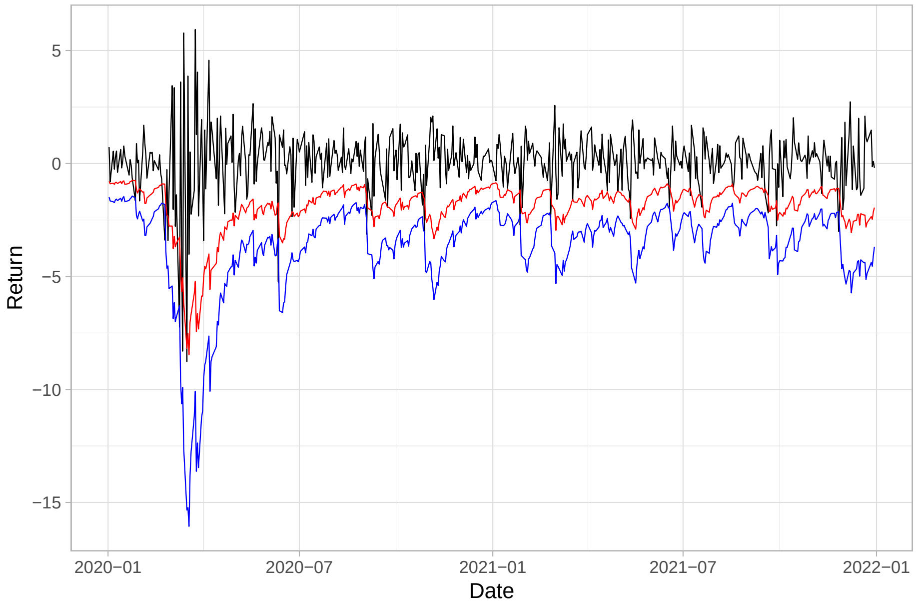
Table 4.14 shows the forecasting performances of the two index funds after the implementation of the aforementioned VaR procedure with $M = 504$.

Backtesting of VaR forecasts with statistical tests						
	95% VaR			99% VaR		
	Z/T	LR_{UC}	LR_{CC}	Z/T	LR_{UC}	LR_{CC}
DM	0.0536	0.2080	10.1307	0.0179	2.4130	10.1307
P-value		0.6483	0.0015		0.2992	0.0063
EM	0.0694	0.0611	0.0003	0.0258	0.0709	0.0806
P-value		0.8048	0.9857		0.9652	0.9605

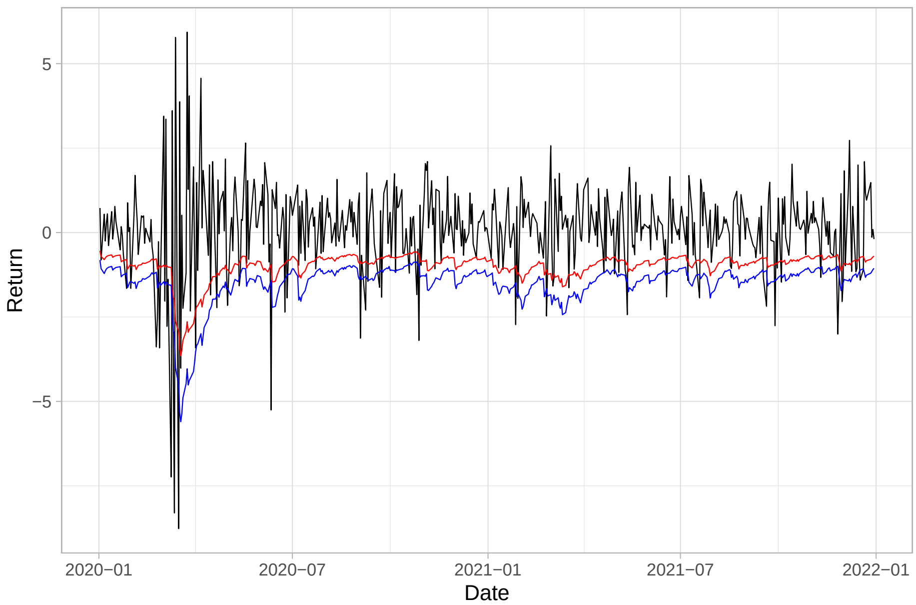
Table 4.14. Results from evaluating the one-day-ahead out-of-sample VaR forecasts.

The table shows DM is closest to the ratio of VaR exceedances for both levels of VaR. The p-values of Kupiec's unconditional coverage test for the index funds are greater than 0.05, which mean the null hypothesis is not rejected. Recall, the null hypothesis states the observed number of exceedings equals the predicted number of exceedings. The p-values of Christoffersen's independence test for DM are less than 0.05, which mean the null hypothesis is rejected. The p-values for EM are greater than 0.05 so the null hypothesis is not rejected.

In Figure 4.15 the performance of the forecasted VaR for the index funds is illustrated.



(a) Forecasting performance of DM.



(b) Forecasting performance of EM.

Figure 4.15. Depiction of returns of the index funds with their respective VaR forecasts, where the red curve represents the 95% VaR and the blue curve represents the 99% VaR.

The figure shows the VaR curves for EM are closer to the returns than the VaR curves for DM. This agrees with the results for Z/T in Table 4.14, since the values for EM are higher and thus more returns are exceeding the VaR compared to DM. Further, the start of 2020 seems to contain a lot of violations for EM, which can be caused by the high fluctuations in the start of the corona pandemic.

4.2.3 Comparison with Value-at-Risk Forecasting

In this section, the performance of the forecasted VaR is compared with the VaR based on information from the Council for Return Expectations. Further, these results are compared with the results in Section 4.1.1 to examine how representative the portfolios are for the two groups. The VaR based on information from the Council for Return Expectations is calculated using the same method as in Section 4.1 where the current position, S, and the degree of freedom, ν, are adjusted to fit the index funds. This adjustment affects the z values and the scale parameter, σ. Recall, the losses are assumed to be normally distributed or t-distributed. For the normal distribution the values for z is given in (4.2) and is repeated as

$$\begin{aligned} z &= -1.65 \quad \text{for the 95\% VaR,} \\ z &= -2.33 \quad \text{for the 99\% VaR.} \end{aligned}$$

Note, these z values for the normal distribution are not affected by the change in ν. For the t-distribution the analysis conducted in Section 4.2.1 indicates an appropriate degree of freedom for DM is 3.5, which yields the following z values,

$$\begin{aligned} z &= -2.22 \quad \text{for the 95\% VaR,} \\ z &= -4.06 \quad \text{for the 99\% VaR.} \end{aligned}$$

The analysis further shows an appropriate degree of freedom for EM is 13.6, which yields the following z values,

$$\begin{aligned} z &= -1.76 \quad \text{for the 95\% VaR,} \\ z &= -2.63 \quad \text{for the 99\% VaR.} \end{aligned}$$

Here, it appears the z values for the t-distribution with $\nu = 13.6$ degrees of freedom is closer to the z values for the normal distribution, and thus it is expected the VaR for EM based on normally distributed losses is closer to the VaR based on t-distributed losses. The σ values for losses assuming the normal distribution is given in (4.3) and is repeated as

$$\begin{aligned} \sigma &= 0.11 \quad \text{for the 1st half of 2020 for DM,} \\ \sigma &= 0.135 \quad \text{for the 2nd half of 2020 for DM,} \\ \sigma &= 0.284 \quad \text{for the 1st half of 2020 for EM,} \\ \sigma &= 0.299 \quad \text{for the 2nd half of 2020 for EM.} \end{aligned} \tag{4.5}$$

For losses assuming a t-distribution the σ values are

$$\begin{aligned} \sigma &= 0.11 \cdot \sqrt{(3.5-2)/3.5} = 0.0471 \quad \text{for the 1st half of 2020 for DM,} \\ \sigma &= 0.135 \cdot \sqrt{(3.5-2)/3.5} = 0.0579 \quad \text{for the 2nd half of 2020 for DM,} \\ \sigma &= 0.284 \cdot \sqrt{(13.6-2)/13.6} = 0.2422 \quad \text{for the 1st half of 2020 for EM,} \\ \sigma &= 0.299 \cdot \sqrt{(13.6-2)/13.6} = 0.2550 \quad \text{for the 2nd half of 2020 for EM.} \end{aligned} \tag{4.6}$$

The current position of the two index funds is found on the trading day 2019-12-31 for the expectations published in the report for the first half of 2020. The portfolio value for DM is $S = 95.5830$, and for EM it is $S = 43.3183$. For the expectations published in the report for the second half of 2020, the current position for DM is $S = 90.0621$ and for EM it is $S = 38.8311$. The daily VaRs with these values are calculated and shown in Table 4.16.

DM				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	1.0767	1.2478	0.9460	1.0966
99% VaR	1.5289	1.7707	1.7470	2.0228

(a) The daily VaR for *iShares MSCI World ETF*.

EM				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	1.2681	1.1975	1.2490	1.1794
99% VaR	1.7972	1.6968	1.8742	1.7695

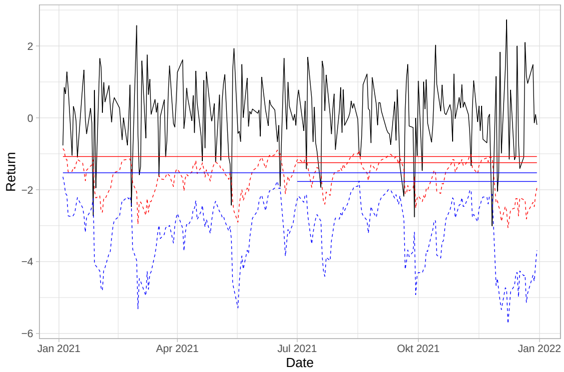
(b) The daily VaR for *iShares MSCI Emerging Markets ETF*.

Table 4.16. The daily VaR calculated with the parametric method, assuming the normal distribution or the t-distribution.

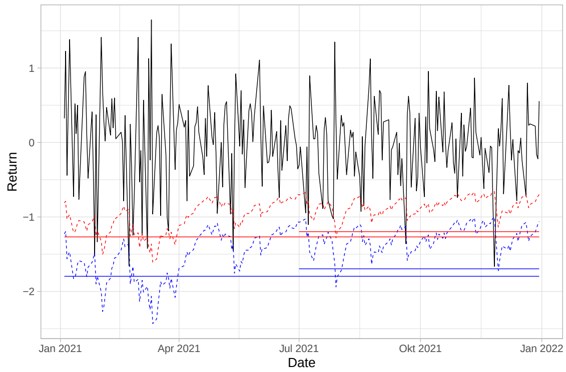
This table shows the 95% VaR for DM with losses assuming a normal distribution is larger compared with losses assuming a t-distribution. For the 99% VaR for DM the opposite is true. For EM the 95% VaR based on losses with a normal distribution and t-distribution appears to be close to each other but slightly lower with the t-distribution. For the 99% VaR, losses assuming a normal distribution is lower compared with losses assuming a t-distribution. The aforementioned tendencies are also seen for the portfolios for group 5 and group 6 in Table 4.4 with the exception of the 99% VaR for DM.

As for Table 4.3 and 4.4, the reason for these tendencies is examined by considering the z and σ values. The absolute z values are greater for the t-distribution for the respective confidence levels, which indicates larger tails and thus larger VaRs. Thus, the tendency is found by comparing the values for σ in (4.5) and (4.6). It is seen the σ values for the t-distribution are lower compared with the values for the normal distribution. This causes the tendency in Table 4.16, where losses assuming the t-distribution have a lower VaR compared with losses assuming the normal distribution for the 95% VaR.

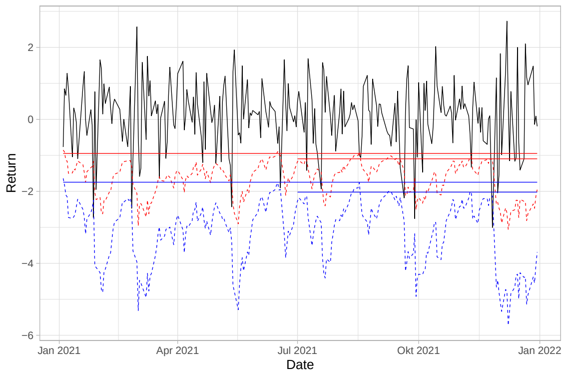
In order to compare the VaR based on information from the Council for Return Expectations in Table 4.16 with the one-day-ahead VaR calculated with the method in Section 4.2.2, both are plotted with the returns of the index funds in Figure 4.17.



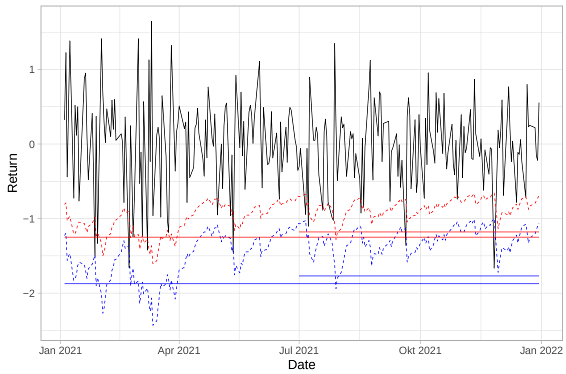
(a) Forecasting performance of DM with the normal distribution.



(b) Forecasting performance of EM with the normal distribution.



(c) Forecasting performance of DM with the t-distribution.



(d) Forecasting performance of EM with the t-distribution.

Figure 4.17. The (dashed) red curve represents the 95% VaR and the (dashed) blue curve represents the 99% VaR.

The figure is interpreted like Figure 4.5 just based on the index funds instead. The figure shows the VaR for DM based on the information from the Council for Return Expectations appears to be lower than the forecasted VaR, which results in more violations. The VaR for EM based on the information from the Council for Return Expectations generally appears to be higher than the forecasted VaR, which results in less violations. Further, no returns violate the 99% VaR for EM based on information from the Council for Return Expectations. The VaR of EM calculated with the common expectations from the report for the first half of 2020 has a lower value than the VaR calculated with the common expectations from the report for the second half of 2020 where the reverse is true for DM.

To further investigate the VaR based on information from the Council for Return Expectations the ratios of VaR exceedances, Z/T, are shown in Table 4.18.

DM: Ratio of VaR exceedances				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	0.1195	0.0866	0.1434	0.1102
99% VaR	0.0518	0.0394	0.0398	0.0315

(a) The ratio of VaR exceedances for DM.

EM: Ratio of VaR exceedances				
	Normal distribution		t-distribution	
	1st half 2020	2nd half 2020	1st half 2020	2nd half 2020
95% VaR	0.0279	0.0157	0.0279	0.0157
99% VaR	0.0000	0.0000	0.0000	0.0000

(b) The ratio of VaR exceedances for EM.

Table 4.18. The ratio of VaR exceedances, Z/T, with the normal distribution and the t-distribution.

The table shows the ratio of VaR exceedances for DM are too high for both the 95% VaR and the 99% VaR. However, the ratio of VaR exceedances for the second half of 2020 is improved compared with the first half of 2020. The change in the ratio of VaR exceedances can be caused by the higher VaR for the second half of 2020 seen in Table 4.16. For EM both levels of VaR have a too low ratio of VaR exceedances. Here, the ratio of VaR exceedances is even lower for the second half of 2020, which might be caused by higher return fluctuations in the first half of 2020 compared to the second half of 2020. The ratio of VaR exceedances for EM based on the normal distribution and the t-distribution are identical, which can be partially caused by the high degree of freedom for the t-distribution. The tendencies seem to be similar to those for the portfolios consisting of 5 assets for the two groups seen in Table 4.6. However, the ratios of VaR exceedances change for the two distributions for the 95% VaR of the portfolio for group 6 in the first half of 2020. This can be caused by the VaR being based on a lower degree of freedom for the portfolios.

Discussion 5

In this chapter, some of the choices and reflections made in this project are discussed. In this project, the risk exposure of a portfolio is investigated through a risk measure called Value-at-Risk. In order to understand this measure, basic theory and different calculations method are presented. Furthermore, an ARMA-GARCH model is introduced to capture autoregressive and heteroskedastic behaviour in the data and different copulas are introduced to combine univariate distributions in a joint distribution with more flexibility. Then, Value-at-Risk is one-day-ahead out-of-sample forecasted by implementing a Monte Carlo procedure that utilises both an ARMA-GARCH model and a copula. These forecasts are then evaluated using backtesting methods. Furthermore, the forecasts are compared with the parametric Value-at-Risk calculated from the common expectations published by the Council for Return Expectations. Moreover, it is investigated if the constructed portfolios are representative of their respective groups by analysing and comparing with two index funds for the groups.

There are different methods to calculate Value-at-Risk. In this project, the Monte Carlo procedure with an ARMA-GARCH model and a copula is used to calculate the one-day-ahead out-of-sample forecasts. This method relies on simulations to calculate Value-at-Risk, which must be reflective of future scenarios in the data. If the data is poorly fitted the procedure will yield inaccurate Value-at-Risk forecasts. This means the Monte Carlo procedure is subject to both model and distribution misspecification. In this project, different orders of ARMA-GARCH are investigated. However, only the GARCH(1,1) is examined, wherefore it could be beneficial to see if different orders would yield a better fit. It could also be beneficial to consider different models, e.g. the TGARCH model as in [16], to examine for a better fit. Furthermore, only the normal distribution and the t-distribution are examined in this project. However, Table 3.4, 3.5 and 4.9 conclude there is skewness present in the log-returns. For this reason, it could be beneficial to consider other types of distributions, such as the skewed t-distribution, in the margins of the copula. This also indicates it could be beneficial to consider a skewed t-copula as well. Note, only the normal copula, the t-copula and four Archimedean copulas are considered in this project. Moreover, the number of Monte Carlo simulations must be high enough to reflect different scenarios. In this project, 10.000 Monte Carlo simulations are used, but other literature such as [5] suggests to use 100.000 simulations, thus it could be beneficial to increase the number of simulations in this project.

The Monte Carlo procedure is implemented for two portfolios consisting of 5 assets with equal weighting for an ease of computation. However, it could be beneficial to include more assets in the portfolios and optimise the weights to represent their groups more accurately. Additionally, assets from the same country or industry could be added to the portfolio to investigate the effect on the correlations and the resulting Value-at-Risk. It is tried to include the assets Microsoft and Johnson & Johnson in group 5, which is not included in the project, since it did not change the correlations considerable in the group. However, it could be interesting to see the effects of the inclusion of an asset that introduced more correlation in the portfolio.

The Monte Carlo procedure is implemented to forecast one-day-ahead out-of-sample forecasts. It could be beneficial to investigate longer term forecasts such as one-week-ahead or one-year-ahead out-of-sample forecasts. Such longer term forecasts could yield an interesting comparison with the Value-at-Risk calculated from the common expectations published by the Council for Return Expectations. Furthermore, the common expectations are published for the short term, which is 1-5 years and 1-10 years depending on the report. The conversion from these annual expectations to daily expectations may be misrepresenting. Thus, it would be interesting to compare the annual common expectations with longer term Value-at-Risk forecast.

The forecasted Value-at-Risks are evaluated using backtesting methods, which include statistical tests and loss functions. These loss functions are evaluated by a sample average. Another method to evaluate the loss functions is to compare the loss function with a benchmark model, which is the expected score, through a quadratic probability score (QPS) defined as

$$\text{QPS} = \frac{2}{T}\sum_{t=1}^{T}\left(C_t - C_t^*\right)^2 ,$$

where C_t^* is the benchmark model. For the first Lopez loss function, C_t^{L1}, the benchmark model is the expected number of violations. For the second Lopez loss function, C_t^{L2}, the benchmark model needs to be estimated, e.g. using Monte Carlo methods. For the Blanco and Ihle loss function, C_t^{BI}, the benchmark model is the expected difference between the loss and the Value-at-Risk divided with the Value-at-Risk [4].

When analysing the Value-at-Risk based on the Council for Return Expectations it can be expected that the annual returns and standard deviations are determined in a way to allow for unforeseen events, which mean they are constructed with a wide gap. Such a gap could entail larger standard deviations and lower expected returns. In the portfolio based on global equity it is seen the Value-at-Risk is underestimated and in the portfolio for emerging markets equity it is seen the Value-at-Risk is overestimated.

In this project, two index funds are considered to investigate how representative the portfolios are for the two groups. These index funds are modelled and Value-at-Risk is one-day-ahead out-of-sample forecasted using another method, which does not use copulas and Monte Carlo simulations. Thus, it is arguable if it is appropriate to compare these forecasts with the Monte Carlo forecasts. Further, when index funds are considered information regarding how the assets in the respective index funds interacts is not available.

Conclusion 6

Throughout this project, risk assessment is investigated for portfolios with the purpose of forecasting the Value-at-Risk for portfolios by ARMA-GARCH, copulas and Monte Carlo simulations and compare the results with the Value-at-Risk based on the common expectations from the Council for Return Expectations. Furthermore, the portfolios are tested to see if they are representative for the groups global equity and emerging markets equity. The aim of this project is to obtain knowledge to be able to answer the questions in the statement of intent in Section 1.1. The statement of intent is repeated as:

How can one-day-ahead Value-at-Risk be forecasted for portfolios using copulas and Monte Carlo simulations and how can these forecasts be evaluated? How does the forecasts perform compared with the common expectations of return and standard deviation published by the Council for Return Expectations and are the portfolios representative for the respective groups?

In Section 2.1 a method to calculate risk, called Value-at-Risk, is presented, which is the highest expected loss to occur with a specified probability over a specified holding period. Value-at-Risk can be estimated by three different methods, namely non-parametric estimation, parametric estimation and Monte Carlo simulations. The parametric approach and the Monte Carlo approach are extended to consider a portfolio consisting of multiple assets, where the last-mentioned is the focus of this project and is presented in Section 2.4.

In the Monte Carlo procedure, a mean-variance model and copulas are used to model the data. Therefore, in Section 2.2 the ARMA-GARCH model is introduced to capture autoregressive and heteroskedastic behaviour in the data. Hereafter, copulas are presented in Section 2.3 in order to describe the dependence between the components in a portfolio. The advantages of modelling univariate ARMA-GARCH models with copulas compared to modelling multivariate ARMA-GARCH models are the possibility of choosing different distributions for the univariate margins and the number of parameters needed to be estimated is considerable reduced. Multivariate ARMA-GARCH models require increasing numbers of parameters to be estimated when adding a univariate process. Thus, the multivariate ARMA-GARCH model is subject to the curse of dimensionality and large data sets are needed to construct an accurate model. The relationship between the univariate processes that multivariate ARMA-GARCH models capture through the variance-covariance matrix is also indirectly captured with a copula, since the correlation matrix is a transformed variance-covariance matrix. The presented copulas are the Normal copula, the t-copula, the Gumbel copula, the Clayton copula, the Frank copula and the Joe copula. In addition, the tail dependence coefficients are introduced to describe the risk of simultaneous extreme negative returns on a portfolio. Then, the Monte Carlo procedure is presented in Section 2.4 to conduct one-day-ahead out-of-sample forecast.

These forecast are evaluated using backtesting methods introduced in Section 2.5, where the ratio of exceedances, statistical tests and loss functions are presented.

In Chapter 3 the data processing and modelling are presented for portfolios consisting of five assets, that are chosen in order to represent the groups global equity and emerging markets equity. This chapter presents the data, which is collected from Yahoo Finance from 2010-01-04 to 2021-12-30, where the data for 2020 and 2021 is the forecasting sample. This period is concurrent with the corona pandemic. Next, modelling of the marginal distributions and modelling of copulas are presented, where the former shows the t-copula is preferred for the portfolios for both global equity and emerging markets equity, which are group 5 and 6 respectively. Based on the four best copulas, the Monte Carlo procedure is implemented to estimate Value-at-Risk with 10.000 simulations and 504 one-day-ahead out-of-sample forecasts. The backtesting of these forecasts shows the Normal copula and the Clayton copula are preferred for group 5 and the t-copula is preferred for group 6. Furthermore, to determine which copula is preferred for the groups, especially for group 5, the loss functions are used, which conclude the Normal copula is best for group 5 and the Normal copula and the t-copula are best for group 6. In [16], it is stated that the t-copula yields a better fit for financial data, however it is only one of the best for group 6. Intuitively, the t-copula is better for group 6 than for group 5, since the assets in group 6 are more uncertain, which are seen in the higher standard deviations and the heavier tails. In addition, the Clayton copula is preferred for group 5 in backtesting, which can be because it has lower tail dependence and the assets in group 5 might exhibit a negative dependence in the tails. Based on the results in Chapter 3 the Normal copula is chosen for group 5 and the t-copula is chosen for group 6, where the predictive performances of the Monte Carlo procedure with these copulas are plotted in Figure 3.20.

In Chapter 4 parametric Value-at-Risks are calculated with the common expectations for returns and standard deviations published by the Council for Return Expectations. These Value-at-Risks are compared with the Monte Carlo Value-at-Risk forecasts and Value-at-Risk calculated for two index funds, which represent global equity and emerging markets equity. In general, the comparisons show the risk is underestimated for global equity and the risk is overestimated for emerging markets equity based on information from the Council for Return Expectations. Therefore, the Value-at-Risk based on the Monte Carlo procedure and the Value-at-Risk based on the index funds represent more accurate forecasts for both the 95% and the 99% confidence levels. The analysis conducted on the portfolios consisting of five assets chosen in Chapter 3 and the analysis conducted for the index funds show the same tendencies, which indicate the chosen portfolios are representative for global equity and emerging markets equity.

To sum up, the Value-at-Risk forecasted with the Monte Carlo procedure provides the best results and gives the most information on a day to day basis. The information can be useful for investment purposes. The Monte Carlo method forecasts one-day-ahead and incorporates day to day information into the rolling forecasts. The Monte Carlo method produces accurate Value-at-Risk forecasts and is preferable for e.g. day trading. The daily Value-at-Risk based on information from the Council for Return Expectations is calculated with annual expectations that are converted. These expectations do not incorporate day to day realisations and are constant for the holding period. This method might be more accurate and preferable for longer term investments.

Bibliography

[1] Moorad Choudhry. *An Introduction to Value-at-Risk, Fifth Edition*. English. Wiley, 2013. ISBN: 1-118-31672-X.

[2] Jeff B. Cromwell, Walter C. Labys, and Michel Terraza. *Univariate Tests for Time Series Models*. English. Quantitative Applications in the Social Sciences. SAGE Publications Inc, 1994. ISBN: 9780803949911.

[3] Insurance & Pension Denmark and Finance Denmark. *Council of Return Expectations*. Last accessed 2022-04-21. URL: `https://www.afkastforventninger.dk`.

[4] Kevin Dowd. *Measuring Market Risk*. English. Wiley Finance, 2002. ISBN: 0-471-52174-4. URL: `http://dspace.vnbrims.org:13000/xmlui/bitstream/handle/123456789/4963/Kevin%20Dowd%20-%20Measuring%20Market%20Risk%20-%20Copy.pdf?sequence=1&isAllowed=y`.

[5] Dean Fantazzini. *Dynamic Copula Modelling for Value at Risk*. English. Frontiers in Finance and Economics, 2008. URL: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1542608`.

[6] Yahoo Finance. *iShares MSCI Emerging Markets ETF (EEM)*. Last accessed 2022-05-23. 2022. URL: `https://finance.yahoo.com/quote/EEM/`.

[7] Yahoo Finance. *iShares MSCI World ETF (URTH)*. Last accessed 2022-05-23. 2022. URL: `https://finance.yahoo.com/quote/URTH?p=URTH`.

[8] Ameriprise Financial. *Investing in Emerging Markets 101*. Last accessed 2022-05-24. URL: `https://www.ameriprise.com/financial-goals-priorities/investing/emerging-market-investments`.

[9] Peter R. Hansen and Asger Lunde. *A Forecast Comparison of Volatility Models: Does Anything Beat a GARCH(1,1)?* Journal of Applied Econometrics, 2005.

[10] Martin Haugh. *An Introduction to Copulas*. Last accessed 2022-02-16. 2016. URL: `http://www.columbia.edu/~mh2078/QRM/Copulas.pdf`.

[11] Marius Hofert, Ivan Kojadinovic, Martin Mächler, and Jun Yan. *Elements of Copula Modeling with R*. English. Springer, 2018. ISBN: 978-3-319-89634-2. URL: `https://doi.org/10.1007/978-3-319-89635-9`.

[12] Zoran Jeremić and Ivica Terzić. *Empirical Estimation and Comparison of Normal and Student-t Linear VaR on the Belgrade Stock Exchange*. English. Sinteza, 2014. URL: `https://portal.sinteza.singidunum.ac.rs/Media/files/2014/298-302.pdf`.

[13] Harry Joe. *Asymptotic Efficiency of the Two-Stage Estimation Method for Copula-Based Models*. English. Journal of Multivariate Analysis, 2003. URL: `https://core.ac.uk/download/pdf/82185357.pdf`.

[14] Mark Kennan. *How to Convert Daily Returns to Annual*. Last accessed 2022-05-19. URL: https://www.sapling.com/5892755/convert-daily-returns-annual.

[15] G. M. Ljung and G. E. P. Box. *On a Measure of Lack of Fit in Time Series Models*. English. Biometrika, 1978.

[16] Xun Fa Lu, Kin Keung Lai, and Liang Liang. *Portfolio Value-At-Risk Estimation in Energy Futures Markets with Time-Varying Copula-GARCH Model*. English. Springer Science + Business Media, 2011. URL: https://link.springer.com/article/10.1007/s10479-011-0900-9.

[17] Giovanni De Luca and Giorgia Rivieccio. *Advanced Statistical Methods for the Analysis of Large Data-Sets*. English. Springer, 2012. ISBN: 978-3-642-21036-5. URL: https://link.springer.com/content/pdf/10.1007/978-3-642-21037-2.pdf.

[18] Alexander J. McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative Risk Management: Concepts, Techniques, and Tools*. English. Princeton University Press, 2005. ISBN: 978-0-691-16627-8. URL: https://www.noor-book.com/en/ebook-Quantitative-Risk-Management-Concepts-Techniques-and-Tools-pdf.

[19] Roger B. Nelsen. *An Introduction to Copulas, Second Edition*. English. Springer, 2006. ISBN: 978-0387-28659-4. URL: https://link.springer.com/content/pdf/10.1007\%2F0-387-28678-0.pdf.

[20] David Ruppert and David S. Matteson. *Statistics and Data Analysis for Financial Engineering, Second Edition*. English. Springer, 2015. ISBN: 978-1-4939-2613-8. URL: https://ethz.ch/content/dam/ethz/special-interest/math/statistics/sfs/Education/Advanced%20Studies%20in%20Applied%20Statistics/course-material-1921/FinancialData/2710528_1_ruppert.pdf.

[21] Troy Segal. *Common Methods of Measurement for Investment Risk Management*. Last accessed 2022-05-19. 2022. URL: https://www.investopedia.com/ask/answers/041415/what-are-some-common-measures-risk-used-risk-management.asp.

[22] Kent Thune. *Emerging Market Funds vs. International Stock Funds*. Last accessed 2022-05-24. 2021. URL: https://www.thebalance.com/emerging-markets-vs-international-stock-mutual-funds-2466793.

[23] Zach. *How to Conduct a Jarque-Bera Test in R*. Last accessed 2022-05-24. 2019. URL: https://www.statology.org/how-to-conduct-a-jarque-bera-test-in-r/.

Appendix A

A.1 Probability Theory

This section focuses on probability distributions and technical terms which are used in the project.

Definition A.1 (Cumulative Distribution Function)
The cumulative distribution function (CDF) of Y is defined as

$$F_Y(y) = \mathbb{P}\{Y \leq y\}.$$

If Y has a PDF, f_Y, then

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)du.$$

For d random variables $\mathbf{Y} = (Y_1, \dots, Y_d)$ the CDF is given by

$$F_Y(y_1, \dots, y_d) = \mathbb{P}\{Y_1 \leq y_1, \dots, Y_d \leq y_d\},$$

[20, p. 669].

Definition A.2 (Range)
Given a univariate distribution function F, the range of F is given by

$$Ran(F) = \{F(x) : x \in \mathbb{R}\},$$

[11, p. 23].

A.2 Fundamental Copulas

This section is based on [18] and [20].

In this section, three d-dimensional fundamental copulas, which have domain $[0,1]^d$ and range $[0,1]$, are introduced. Fundamental copulas represent a variety of important special dependence structures.

The CDF of d mutually independent $U(0,1)$ random variables is an independence copula C_0. In other words, C_0 is given by

$$C_0(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i, \tag{A.1}$$

where the associated density is uniform on $[0,1]^d$, meaning it is zero everywhere except when $c_0(u_1, \ldots, u_d) = 1$ on $[0,1]^d$. The random variables with continuous distributions are independent if and only if their independent structures are given by (A.1) which is a consequence of Sklar's Theorem 2.3.

The d-dimensional co-monotonicity copula C_+ indicates perfect positive dependence. Let $U \sim U(0,1)$, then C_+ is given as

$$\begin{aligned} C_+(u_1, \ldots, u_d) &= \mathbb{P}(U \leq u_1, \ldots, U \leq u_d) \\ &= \mathbb{P}\{U \leq \min(u_1, \ldots, u_d)\} \\ &= \min(u_1, \ldots, u_d). \end{aligned}$$

Note, the co-monotonicity copula is the CDF of $\boldsymbol{U} = (U, \ldots, U)$, so all d elements of $\boldsymbol{U}$ are equal. Further, the copula is an upper bound of all copula functions, i.e. $C(u_1, \ldots, u_d) \leq C_+(u_1, \ldots, u_d)$ for all $(u_1, \ldots, u_d) \in [0,1]^d$.

The counter-monotonicity copula C_- is a two-dimensional copula defined as the CDF of $(U, 1-U)$ that has perfect negative dependence. Thus,

$$\begin{aligned} C_-(u_1, u_2) &= \mathbb{P}(U \leq u_1, 1 - U \leq u_2) \\ &= \mathbb{P}(1 - u_2 \leq U \leq u_1) \\ &= \max(u_1 + u_2 - 1, 0). \end{aligned} \tag{A.2}$$

The last equation is true since if $1 - u_2 > u_1$ then the probability for the event $\{1 - u_2 \leq U \leq u_1\}$ is 0. Otherwise, the probability is $u_1 + u_2 - 1$ for the interval $(1 - u_2, u_1)$. When $d > 2$ the counter-monotonicity copula does not exist, but when $d = 2$ all copulas are bounded by (A.2).

A.3 Jarque-Bera Test

This section is based on [18], [20], [2] and [23].

In this section the Jarque-Bera test is presented. This test is introduced in order to determine if the data is normally distributed based on the skewness and kurtosis values. The Jarque-Bera test is a goodness-of-fit test and it produces a test statistic which is positive. The test statistic indicates the sample data follows a normal distribution if it is close to zero.

The test statistic is

$$JB = \frac{n}{6} S^2 + \frac{n}{24} (\kappa - 3)^2.$$

Here, S is the sample skewness coefficient and κ is the kurtosis coefficient, which are defined as

$$S = \frac{n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^3}{(n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2)^{3/2}}, \quad \kappa = \frac{n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^4}{(n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2)^2}.$$

The test statistic, JB, is distributed as $\chi^2(2)$ under the null hypothesis of normality. Under normality, S and κ has the values 0 and 3 respectively, which implies JB is zero.

In this project, the Jarque-Bera test is performed using the command `jarque.bera.test` from the package `tsoutliers`.

A.4 Ljung-Box Test

This section is based on [15].

In this section, the Ljung-Box test is presented in order to understand the results from the analysis performed in this project with this test.

The test considers a discrete time series w_t, which is fitted by an autoregressive model. The fit of the model is then analysed by investigating the residuals of the model, $a_1, \ldots, a_n$. The correlations between the residuals are defined as

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{n} a_t a_{t-k}}{\sum_{t=1}^{n} a_t^2}, \quad k = 1, 2, \ldots.$$

The Ljung-Box test is then given by the statistic

$$\tilde{Q}(\hat{\rho}) = n(n+2) \sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k},$$

where m is the number of lags being tested. The null hypothesis states there is no correlation in the residuals and the alternative hypothesis is that the residuals exhibit serial correlation. Thus, this statistic is used to test for autocorrelation in the residuals and can be used to determine if the model is adequately fitted. If there is autocorrelation present in the residuals it suggests there is more information to be extracted, such that only white noise is left in the residuals.

The Ljung-Box test can also be applied to the squared residuals, in which case the test examines if the sizes of the residuals are correlated. If the sizes of the residuals are correlated it suggests there are GARCH effects present in the residuals, and it would be beneficial to include a GARCH model to capture these effects.

In this project, the Ljung-Box test is performed using the command `Box.test` from the package `stats`.

A.5 Council for Return Expectations

In this section, the common expectations published by the Council for Return Expectations are listed. Note, the returns for group 7, 8, 9 and 10 are calculated after net of fees to the funds. These fees includes all fees an investor will pay to the funds.

	Return	St. Dev.
1. Government and mortgage bonds	0.3%	3.1%
2. Investment-grade bonds	1.6%	4.2%
3. High-yield bonds	3.3%	6.5%
4. Emerging markets sovereign bonds	4.4%	8.4%
5. Global equity (developed markets)	5.5%	11.0%
6. Emerging markets equity	9.5%	28.4%
7. Private equity	8.7%	23.9%
8. Infrastructure	5.4%	10.6%
9. Real estate	5.6%	11.2%
10. Hedge funds	4.6%	8.9%

Table A.1. Common return expectations for the 1st half of 2020 for the short term 1-10 years.

	Return	St. Dev.
1. Government and mortgage bonds	-0.3%	3.4%
2. Investment-grade bonds	1.1%	4.6%
3. High-yield bonds	5.0%	10.9%
4. Emerging markets sovereign bonds	3.9%	8.6%
5. Global equity (developed markets)	6.0%	13.5%
6. Emerging markets equity	9.5%	29.9%
7. Private equity	8.5%	23.8%
8. Infrastructure	5.9%	13.2%
9. Real estate	2.8%	6.8%
10. Hedge funds	3.9%	8.5%

Table A.2. Common return expectations for the 2nd half of 2020 for the short term 1-5 years.

	Return	St. Dev.
1. Government and mortgage bonds	-1.2%	3.2%
2. Investment-grade bonds	-0.2%	3.9%
3. High-yield bonds	3.1%	8.0%
4. Emerging markets sovereign bonds	2.4%	6.9%
5. Global equity (developed markets)	5.6%	13.5%
6. Emerging markets equity	8.5%	25.1%
7. Private equity	8.7%	26.4%
8. Infrastructure	5.3%	12.7%
9. Real estate	3.0%	7.8%
10. Hedge funds	3.5%	8.6%

Table A.3. Common return expectations for the 1st half of 2021 for the short term 1-5 years.

	Return	St. Dev.
1. Government and mortgage bonds	-0.7%	3.2%
2. Investment-grade bonds	-0.4%	3.4%
3. High-yield bonds	1.9%	6.0%
4. Emerging markets sovereign bonds	2.7%	7.3%
5. Global equity (developed markets)	5.4%	13.9%
6. Emerging markets equity	7.7%	24.5%
7. Private equity	8.0%	24.1%
8. Infrastructure	4.1%	10.2%
9. Real estate	3.0%	7.8%
10. Hedge funds	2.5%	6.9%

Table A.4. Common return expectations for the 2nd half of 2021 for the short term 1-5 years.