Tournament Schedules Using Combinatorial Design Theory

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Abstract:

This thesis considers organization of tournament schedules with certain requirements on symmetry from a combinatorial perspective. To describe this, block designs are introduced along with several properties and results. Most important of these are Fisher's inequality and the Bruck-Ryser-Chowla theorem, both of which excludes the existence of certain designs. Fisher's inequality holds for any design, while the Bruck-Ryser-Chowla theorem holds for symmetric designs.

Resolvable designs and difference systems are introduced to construct a tournament of $2n$ teams where each team meet once. Firstly, it is constructed such that there are $2n-2$ breaks in the pattern of home and away games and it is then extended to include a second half with venues interchanged where there are $6n-6$ breaks and no consecutive breaks.

Secondly, a flaw in this construction is described, and to remove this flaw, a construction where there are no teams x and y that both play team z immediately after playing team w, is presented.

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List of Abbreviations

- θ: Primitive element of $\mathbb{F}_q$
- $(X, \mathcal{A})$: A design with set of points X and set of blocks $\mathcal{A}$
- $(G, +)$: An additive group
- (v, k, λ)-BIBD: A balanced incomplete block design with parameters v, k, and λ
- $\mathbb{F}_q$: The finite field of order q
- $\mathbb{Z}/n\mathbb{Z}$: The integers modulo n
- BRC-Theorem: The Bruck-Ryser-Chowla theorem
- HAP: Home-Away-pattern
- I_n: The unit $n \times n$ matrix
- J_n: The $n \times n$ matrix with all entries 1
- PBD: Pairwise balanced design
- $PG_d(q)$: d-dimensional projective geometries of order q

Preface

This is a master's thesis written by a student of the Department of Mathematical Sciences at Aalborg University. The thesis revolves around combinatorial design theory and is readable to anyone of interest in the subject, however, some familiarity with basic abstract algebra, linear algebra and number theory is recommended.

Definitions, theorems, or the likes, follows the same numbering throughout the thesis, while examples, equations, figures and tables follows separate numbering. The first number indicates the chapter while the second indicates the order of which they appear such that example 3.5, for instance, means the fifth example of the third chapter. Furthermore, throughout the thesis the symbol ■ indicates the end of a proof, while ◃ indicates the end of a remark.

Throughout the thesis there are references to various sources of literature, and these are collected in a bibliography at the end of the thesis. References to this literature follows the Vancouver-standard such that they are referenced using [#], where # is determined by the order of which they appear in the bibliography. At the start of each chapter the main source of literature for the chapter will be mentioned and if further materials are used aside from that, a direct reference to the material will appear.

Finally, I want to thank my supervisor throughout the thesis, Oliver Wilhelm Gnilke, for helping with the frame of the thesis, and for his supervision and constructive criticism throughout the semester as well.

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Chapter 1

Introduction

A subject of mathematics which has seen rapid advances in the later years is design theory [1]. It originates from the context of mathematical puzzles and brain teasers from the eighteenth and nineteenth century, however, as a mathematical subject it really began in the twentieth century from the work of statisticians F. Yates and R. A. Fisher on design and analysis of statistical experiments [1]. Combinatorial design theory revolves around the question on whether it is possible to arrange elements from a finite set into subsets such that certain balance properties are satisfied [1]. The term of balance originates with Fisher and Yates as well, and is beneficial in many cases due to the fact that it achieves fairness since all comparisons between pairs of elements then occur equally often [2].

Since the original works of Fisher and Yates applications in a wide variety of areas has been found. This includes, but is not limited to, group testing, mathematical biology and chemistry, cryptography, and tournament scheduling. Many of such applications revolves around existence and non-existence, that is, does a design of a given type exist and if so under which conditions. Such conditions have been found in general, however, it still not possible to guarantee the existence of certain designs. Instead only an exclusion of the existence of ones that do not meet certain conditions can be made [1].

The subject of this thesis is to use combinatorial design theory to schedule games of a season in a sports league under certain constraints. When scheduling a season there are several constraints to consider such that the tournament becomes as fair to all teams as possible. Such a constrain could consider the geography of the participating teams, see for instance [3] for a detailed view on this. The focus of this thesis, however, is to consider the home-away pattern in the schedules such that as few teams as possible have multiple home or away matches in a row, while making sure that different teams have these breaks when such breaks in the pattern are unavoidable. Another constraint on the tournament schedule examined in this thesis is the carry-over effect. This is to ensure that as few teams as possible is scheduled to meet the same teams in a certain order after each other.

Chapter 2

Balanced Incomplete Block Designs

2.1 Preliminary Definitions & Results

In this section, based on [1, pp. 2-5] unless otherwise specified, some of the basic definitions and results in design theory will be presented. The first thing to be introduced is the formal definition of a *design*.

Definition 2.1: Design

A *design* is a pair $(X, \mathcal{A})$ such that:

1) X is a finite set of elements called *points*

2) $\mathcal{A}$ is a finite collection of nonempty subsets of X called *blocks*

When two blocks in a design are identical they are said to be *repeated blocks*, and if a design contains no repeated blocks, it is said to be a *simple design*. Note that in case of multiplicity of the elements of X it will be denoted using []. That is, for instance:

$$[3,7,4,3] \neq \{3,7,4,3\} = \{3,7,4\}$$

One of the most studied type of designs are the balanced incomplete block designs (BIBDs), and as mentioned in the introduction the study of these began with the work of Fisher and Yates in the 1930's.

Definition 2.2: Balanced Incomplete Block Design (BIBD)

Let v, k and λ be positive integers such that $2 \leq k < v$. A (v, k, λ)-BIBD is a design $(X, \mathcal{A})$ such that:

1) $|X| = v$

2) Each block contains exactly k points

3) Every pair of distinct points is contained in exactly λ blocks

The term of balance originates from the third property in definition 2.2, that is, every pair of points is compared equally many times. The block design is said to be *incomplete* since $k < v$. Furthermore if $\lambda > 1$, the BIBD can contain repeated blocks. The notations of the variables for the BIBD is summarized in the following table, inspired by [4, p. 885]:

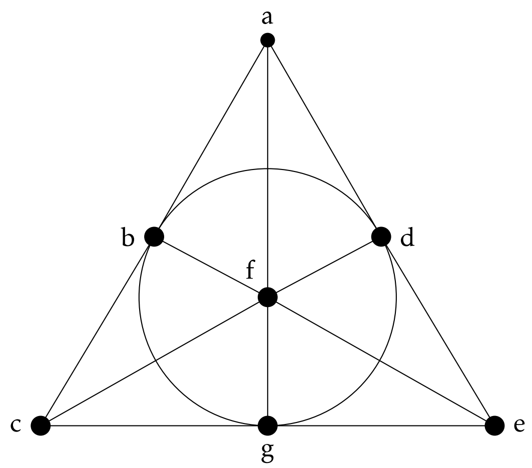
Variable	Description
v (order)	Number of elements in X
b (block number)	Number of elements in $\mathcal{A}$
r (replication number)	Number of blocks containing a given point
k (block size)	Number of points in a block
λ (index)	Number of blocks to which every pair of distinct points belong

Example 2.1
Consider a BIBD given by $X = \{a,b,c,d\}$ and with blocks $\mathcal{A} = \{abc, abd, acd, bcd\}$. For this design the order of X is 4, the block size is 3, while the index number is 2. Therefore, this is a (4,3,2)-BIBD.

Example 2.2 (The Fano plane)
Consider the design $(X, \mathcal{A})$ with:

$$X = \{a,b,c,d,e,f,g\} \text{ and } \mathcal{A} = \{abc, ade, afg, bfe, cfd, cge, bdg\}$$

In this design the order of X is 7 and $|\mathcal{A}| = 7$ as well. The block size k is 3 and the replication number r is 3 as well. Finally, each pair of distinct points appear together in one block, so λ is 1. Hence this is a (7,3,1)-BIBD. This is a special design known as the *Fano plane* and can be seen as a diagrammatic representation in the following way, which shall be described in further detail in section 3.2:



Notice that a (v,k,λ)-BIBD is sometimes referred to as a (v,b,r,k,λ)-BIBD, taking account for all five parameters included in the BIBD. Although all five parameters may be mentioned when their value need to be emphasized, this proves unnecessary due to the following propositions.

Proposition 2.3:

For a (v, k, λ)-BIBD every point occurs in exactly

$$r = \frac{\lambda(v-1)}{k-1}$$

Proof
Let $(X, \mathcal{A})$ be a (v, k, λ)-BIBD and suppose that $x \in X$. Furthermore let r_x denote the number of blocks containing x and define:

$$I := \{(y, B) \mid y \in X, y \neq x, B \in \mathcal{A}, \{x, y\} \subseteq B\}$$

Since $|X| = v$ and $x \in X$ there are $v-1$ ways to choose $y \in X$ such that $y \neq x$. For each such y there are a total of λ blocks, B, such that $\{x, y\} \subseteq B$. Therefore $|I| = \lambda(v-1)$.

Furthermore there are r_x ways to choose a block B such that $x \in B$. Since there are a total of k points in B there are $k-1$ ways to pick $y \in B$ for $x \neq y$. That is, $|I| = r_x(k-1)$. In conclusion:

$$\lambda(v-1) = |I| = r_x(k-1)$$

■

Proposition 2.4:

The number of blocks in a (v, k, λ)-BIBD is exactly:

$$b = \frac{vr}{k}$$

Proof
Let $(X, \mathcal{A})$ be a (v, k, λ)-BIBD, let $|\mathcal{A}| = b$ and define

$$I := \{(x, B) \mid x \in X, B \in \mathcal{A}, x \in B\}$$

Since $|X| = v$, it is possible to choose $x \in X$ in v different ways. For each such x, there are r blocks, B, such that $x \in B$. That is $|I| = vr$.

Furthermore, there are b ways to choose a block $B \in \mathcal{A}$ and since each of these blocks contain k elements, there are k ways to choose a $x \in B$. Hence $|I| = bk$ and therefore:

$$vr = |I| = bk$$

■

Notice that by the definition of b and r they must be integers, which allows the exclusion of the existence of certain designs. For instance a $(16, 5, 3)$-BIBD cannot exist since the replication number is not an integer:

$$r = \frac{3(16-1)}{5-1} = \frac{45}{4} = 11,25$$

However, these conditions do not guarantee the existence of BIBDs, they can only exclude ones that do not meet the criteria. One of the main goals in combinatorial design theory is to determine necessary and sufficient conditions on the existence of (v,k,λ)-BIBDs. This is very difficult in general and for many parameter sets the answer remains unknown to this day. As an example it is currently unknown if a $(22,8,4)$-BIBD exist.

2.2 Incidence Matrices

Often it can be convenient to represent a BIBD using an incidence matrix, which is what is described in this section, based on [1, pp. 6-8].

Definition 2.5: Incidence Matrix

Let $(X,\mathcal{A})$ be a design with $X = \{x_1,\ldots,x_v\}$ and $\mathcal{A} = \{B_1,\ldots,B_b\}$. Then the incidence matrix of $(X,\mathcal{A})$ is the $v \times b$ binary matrix M, constructed using that:

$$m_{ij} = \begin{cases} 1 \text{ if } x_i \in B_j \\ 0 \text{ if } x_i \notin B_j \end{cases}$$

The incidence matrix of a (v,b,r,k,λ)-BIBD satisfies:

- Every column contains a 1 exactly k times
- Every row contains a 1 exactly r times
- Two distinct rows both contain a 1 in exactly λ columns

Example 2.3
Consider the $(4,4,3,3,2)$-BIBD from example 2.1 with $X = \{a,b,c,d\}$ and $\mathcal{A} = \{abc,abd,acd,bcd\}$. The incidence matrix of this design is a 4×4 matrix given by:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Proposition 2.6:

Let M be a binary $v \times b$ matrix and let $2 \leq k < v$. Then M is the incidence matrix of a (v,b,r,k,λ)-BIBD if and only if both of the following equations are satisfied;

$$MM^T = \lambda J_v + (r-\lambda)I_v \tag{2.1}$$

$$\boldsymbol{u}_v M = k\boldsymbol{u}_b \tag{2.2}$$

where J_v is the $v \times v$ matrix where every entry is 1 and $\boldsymbol{u}_v$ is the vector of dimension v with every coordinate equal to 1.

Proof
Let $(X, \mathcal{A})$ be a (v, b, r, k, λ)-BIBD with $X = \{x_1, \ldots, x_v\}$ and $\mathcal{A} = \{B_1, \ldots, B_b\}$, and let M denote the incidence matrix for this design. The $(i, j)th$ entry of MM^T is given by:

$$\sum_{h=1}^{b} m_{ih} m_{jh} = \begin{cases} r \text{ if } i = j \\ \lambda \text{ if } i \neq j \end{cases} \tag{2.3}$$

In other words, every diagonal entry of MM^T is equal to r while every other entry in MM^T is λ. That is:

$$MM^T = \lambda J_v + (r - \lambda) I_v$$

Now consider $\boldsymbol{u}_v M$. The ith entry of this matrix vector product is equal to the total number of entries in column i of M that has a 1 in it. Finally, since every column of M contains a 1 exactly k times, it means that:

$$\boldsymbol{u}_v M = k \boldsymbol{u}_b$$

Conversely, suppose that M is a binary $v \times b$ binary matrix such that equations (2.1) and (2.2) are satisfied. Let $(X, \mathcal{A})$ be the design corresponding to the incidence matrix M, then $|X| = v$ and $|\mathcal{A}| = b$. Furthermore by equation (2.2) it follows that every block of $\mathcal{A}$ contains k points. Finally by equation (2.1) every pair of points occur in exactly λ blocks and every point occurs in r blocks. Therefore the matrix M is an incidence matrix of a (v, b, r, k, λ)-BIBD ■

The constructions from equations (2.1) and (2.2) may be a bit abstract to realize in the general case, so for clarity consider the following example showcasing these.

Example 2.4
Consider the $(4, 4, 3, 3, 2)$-BIBD from example 2.1. It is seen that $MM^T = \lambda J_v + (r - \lambda) I_v$ since:

$$MM^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \lambda J_v + (r - \lambda) I_v$$

Finally, $\boldsymbol{u}_v M = k \boldsymbol{u}_b$ since:

$$\boldsymbol{u}_v M = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = 3 \boldsymbol{u}_b$$

An incidence matrix that satisfies only equation (2.1) is equivalent to a certain type of design called a *pairwise balanced design (PBD)*.

Definition 2.7: Parwise Balanced Design (PBD)

Let $\lambda \in \mathbb{N}^+$. A design $(X, \mathcal{A})$ such that every pair of distinct points is contained in exactly λ blocks is called a pairwise balanced design. Furthermore if every point $x \in X$ occurs in exactly r blocks in $\mathcal{A}$ for $r \in \mathbb{N}^+$, then the PBD is said to be *regular*.

Note that a PBD $(X, \mathcal{A})$ can contain complete blocks, that is, blocks of size $|X|$, and if the PBD consist only of complete blocks it is said to be *a trivial PBD*. However, if $(X, \mathcal{A})$ contains no complete blocks it is said to be a *a proper PBD*.

It turns out that proposition 2.6 does not hold for PBDs since equation (2.2) generally is not satisfied for PBDs as seen in the following example.

Example 2.5
Consider the following 8×18 matrix:

$$\begin{bmatrix} 1&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&1&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \end{bmatrix}$$

This is the incidence matrix for the design $(X, \mathcal{A})$ with

$X = \{a,b,c,d,e,f,g,h\}$ and $\mathcal{A} = \{abcd, efgh, ae, af, ag, ah, be, bf, bg, bh, ce, cf, cg, ch, de, df, dg, dh\}$

For this design $v = 8$, $b = 18$, $r = 5$ and $\lambda = 1$. This is a regular PBD since every pair of distinct points are contained in 1 block and every point occurs in 5 blocks. However, it is not a BIBD which can be seen in two equivalent ways:

1) the incidence matrix: Every column does not contain the same number of 1s

2) $\mathcal{A}$: Every block does not contain the same number of points

It can be verified that equation (2.1) is satisfied, however equation (2.2) is not since:

$$\boldsymbol{u}_8 M = \begin{bmatrix} 4&4&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix} \neq k\boldsymbol{u}_{18} \text{ for any } k \in \mathbb{N}^+$$

2.3 Fisher's Inequality

With the use of propositions 2.3 and 2.4 some criteria to help exclude the existence of certain (v,k,λ)-BIBDs have been made. In this section, based on [1, pp. 16-18], another necessary condition for the existence of a (v,k,λ)-BIBD will be proven. Once more it must be clarified that this does not give conditions on the existence of such designs, but only excludes the existence on those that do not meet the criteria.

Theorem 2.8: Fisher's Inequality

Any (v,b,r,k,λ)-BIBD satisfies that $b \geq v$.

Proof
Let $(X, \mathcal{A})$ be a (v, b, r, k, λ)-BIBD with $X = \{x_1, \ldots, x_v\}$ and $\mathcal{A} = \{B_1, \ldots, B_b\}$. Furthermore let M be the incidence matrix for this BIBD and define $\boldsymbol{s}_j$ to be the vector corresponding to the j^{th} row of M^T and equivalently $\boldsymbol{s}_j{}^T$ for M. Then $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_b$ are all vectors in $\mathbb{R}^v$.

Define $I := \left\{\boldsymbol{s}_j \mid 1 \leq j \leq b\right\}$, then the subspace of $\mathbb{R}^v$ spanned by these $\boldsymbol{s}_j$ is given by:

$$S := \left\{\sum_{j=1}^{b} c_j \boldsymbol{s}_j \mid c_1, \ldots, c_b \in \mathbb{R}\right\}$$

If it can be shown that $S = \mathbb{R}^v$ the proof will be concluded, since this would mean that the b vectors in I spans the vector space $\mathbb{R}^v$, and because $\mathbb{R}^v$ has dimension v and is spanned by a set of b vectors, then $b \geq v$ as desired.

The vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_v$ form a basis for $\mathbb{R}^v$, that is, every vector in $\mathbb{R}^v$ can be expressed as a linear combination of these vectors. It is now sufficient to show that $\boldsymbol{e}_i \in S$ for $1 \leq i \leq v$ since this means that every $\boldsymbol{e}_i$ can be expressed as a linear combination of vectors in I. Observe that:

$$\sum_{j=1}^{b} \boldsymbol{s}_j = \begin{bmatrix} r & \ldots & r \end{bmatrix} \Leftrightarrow \sum_{j=1}^{b} \frac{1}{r} \boldsymbol{s}_j = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \tag{2.4}$$

Fix an i where $1 \leq i \leq v$, then:

$$\sum_{\{j \mid x_i \in B_j\}} \boldsymbol{s}_j = (r - \lambda)\boldsymbol{e}_i + \begin{bmatrix} \lambda & \ldots & \lambda \end{bmatrix} = (r - \lambda)\boldsymbol{e}_i + \lambda \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \tag{2.5}$$

It follows from proposition 2.3 that $\lambda(v-1) = r(k-1)$ and by the definition of BIBD that $v > k$. This indicates that $r > \lambda$ and therefore $r - \lambda > 0$. Next combine equations (2.4) and (2.5) to obtain:

$$\sum_{\{j \mid x_i \in B_j\}} \boldsymbol{s}_j = (r - \lambda)\boldsymbol{e}_i + \lambda \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} = (r - \lambda)\boldsymbol{e}_i^T + \lambda \sum_{j=1}^{b} \frac{1}{r} \boldsymbol{s}_j$$

Finally solve for $\boldsymbol{e}_i$ to obtain

$$\boldsymbol{e}_i = \sum_{\{j \mid x_i \in B_j\}} \frac{1}{r - \lambda} \boldsymbol{s}_j - \sum_{j=1}^{b} \frac{\lambda}{r(r - \lambda)} \boldsymbol{s}_j$$

which expresses $\boldsymbol{e}_i$ as a linear combination of $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_b$ as desired. ■

Remark 1. By proposition 2.4 Fisher's inequality can be stated equivalently as $r \geq k$. ◁

In the proof of Fisher's inequality, it is never used that all blocks have the same size which means that the theorem can be generalized to regular PBDs for which $r > \lambda$. Since a trivial PBD indicates that the blocks have the same size, it means that a regular PBD satisfies that $r > \lambda$ if and only if it is a nontrivial PBD. That is:

Theorem 2.9:

Any nontrivial regular PBD satisfies that $b \geq v$

Here the PBD is assumed to be regular, which ensures that every point occurs in equally many blocks. Actually, Fisher's inequality holds for all nontrivial PBDs and not just regular ones, however this is beyond the scope of this thesis and refer to [1, pp. 193-194] for a proof of this.

Chapter 3

Finite Geometry

In this chapter the theory of a finite analogue to the Euclidean geometry will be presented. These are known as projective and affine planes and help to determine further exclusion of certain designs. The only known projective planes to exist to this day are those of order of a prime power. In order to introduce the finite geometry, the notion of symmetry must be presented first, including the most important theorem on the two necessary conditions on the existence of symmetric designs known together as the Bruck-Ryser-Chowla (BRC) theorem. Finally, resolvability will be introduced for the purpose of the applications described in the next chapter as well as briefly tying this term together with the finite geometry.

3.1 Symmetry

This section is based on [1, pp. 23-25], and serves as an introduction to the term symmetric BIBDs. If the extreme case of Fisher's inequality is satisfied, that is, $b = v$, then the design is said to be *symmetric*. This terminology does not mean that the corresponding incidence matrix is symmetric though. For instance, the Fano plane has $b = v = 7$, however, the corresponding incidence matrix is not symmetric. Instead, the terminology is due to the symmetry between the blocks and points in the design relying on the following proposition. Note that the proof of this proposition follows the same notation as the proof of Fisher's inequality.

Proposition 3.1:

Let $(X, \mathcal{A})$ be a symmetric (v,k,λ)-BIBD and denote $\mathcal{A} = \{B_1,\dots,B_v\}$. Furthermore let $1 \le i, j \le v$ and $i \ne j$. Then:

$$| B_i \cap B_j | = \lambda$$

Proof

Fix an h such that $1 \le h \le b$. Apply equations (2.4) and (2.5) to obtain:

$$\sum_{\{i\,|\,x_i \in B_h\}} \sum_{\{j\,|\,x_i \in B_j\}} \boldsymbol{s}_j \overset{(2.5)}{=} \sum_{\{i|x_i \in B_h\}} \Big((r-\lambda)\boldsymbol{e}_i + \begin{bmatrix} \lambda & \dots & \lambda \end{bmatrix} \Big) = (r-\lambda)\boldsymbol{s}_h + k \begin{bmatrix} \lambda & \dots & \lambda \end{bmatrix}$$
$$= (r-\lambda)\boldsymbol{s}_h + k\lambda \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \overset{(2.4)}{=} (r-\lambda)\boldsymbol{s}_h + \sum_{j=1}^{b} \frac{\lambda k}{r} \boldsymbol{s}_j$$

However, it is possible to compute this sum in another way as well. By interchanging the order of summation, it is readily seen, that this is the summation of $\boldsymbol{s}_j$ for those i where $x_i \in B_h$ for all

j with $x_i \in B_j$. This corresponds to the summation of all $\boldsymbol{s}_j$ for those i where $x_i \in B_h \cap B_j$ with $j \neq h$. That is:

$$\sum_{\{i \mid x_i \in B_h\}} \sum_{\{j \mid x_i \in B_j\}} \boldsymbol{s}_j = \sum_{\{j \mid x_i \in B_j\}} \sum_{\{i \mid x_i \in B_h\}} \boldsymbol{s}_j = \sum_{j=1}^{b} \sum_{\{i \mid x_i \in B_j \cap B_h\}} \boldsymbol{s}_j = \sum_{j=1}^{b} \mid B_h \cap B_j \mid \boldsymbol{s}_j$$

Thus, combining these two ways of calculating the double sum results in:

$$(r-\lambda)\boldsymbol{s}_h + \sum_{j=1}^{b} \frac{\lambda k}{r} \boldsymbol{s}_j = \sum_{j=1}^{b} \mid B_h \cap B_j \mid \boldsymbol{s}_j \tag{3.1}$$

Since the BIBD is symmetric by assumption, $b = v$, and by proposition 2.4 this implies that $r = k$ as well. Thus:

$$(r-\lambda)\boldsymbol{s}_h + \sum_{j=1}^{v} \lambda \boldsymbol{s}_j = \sum_{j=1}^{v} \mid B_h \cap B_j \mid \boldsymbol{s}_j \tag{3.2}$$

Recall from the proof of theorem 2.8 that;

$$I := \left\{\boldsymbol{s}_j \mid 1 \leq j \leq b\right\} \text{ and } S := \left\{\sum_{j=1}^{b} c_j \boldsymbol{s}_j \mid c_1, \ldots, c_b \in \mathbb{R}\right\}$$

and that it was shown that $\mathbb{R}^v = S$, that is, the b vectors of I spans the vector space $\mathbb{R}^v$. Since $b = v$ by assumption the b vectors of the set I must be the minimal number of vectors to span $\mathbb{R}^v$. This means that I is a basis for $\mathbb{R}^v$ as well and thus the coefficients of any $\boldsymbol{s}_j$ on the left and right side of equation (3.2) must be equal. Therefore, for any $h \neq j$:

$$\mid B_h \cap B_j \mid = \lambda$$

Since h was arbitrarily fixed such that $1 \leq h \leq b$ and because $1 \leq j \leq b$ it follows that this holds for any two different blocks in $\mathcal{A}$. ■

It may be a bit difficult to realize why the equation

$$\sum_{\{i \mid x_i \in B_h\}} \left((r-\lambda)\boldsymbol{e}_i + \begin{bmatrix}\lambda & \ldots & \lambda\end{bmatrix}\right) = (r-\lambda)\boldsymbol{s}_h + k\begin{bmatrix}\lambda & \ldots & \lambda\end{bmatrix}$$

is satisfied so for clarity consider the following example showcasing this.

Example 3.1
Consider the $(4,4,3,3,2)$-BIBD from example 2.1, where $b = v = 4$ so it is symmetric. As seen in example 2.3 the incidence matrix of this design is a 4×4 matrix given by:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M^T$$

Fix $h = 1$ which results in $B_1 = \{x_1, x_2, x_3\}$. Then:

$$\begin{aligned}
\sum_{\{i|x_i \in B_h\}} \left((r-\lambda)\boldsymbol{e}_i + \begin{bmatrix}\lambda & \dots & \lambda\end{bmatrix}\right) &= \sum_{i=1}^{3} \left((r-\lambda)\boldsymbol{e}_i + \begin{bmatrix}\lambda & \dots & \lambda\end{bmatrix}\right) \\
&= \begin{bmatrix}(r-\lambda+\lambda) & \lambda & \lambda & \lambda\end{bmatrix} + \begin{bmatrix}\lambda & (r-\lambda+\lambda) & \lambda & \lambda\end{bmatrix} + \begin{bmatrix}\lambda & \lambda & (r-\lambda+\lambda) & \lambda\end{bmatrix} \\
&= \begin{bmatrix}r+2\lambda & r+2\lambda & r+2\lambda & 3\lambda\end{bmatrix} = (r-\lambda)\begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} + 3\begin{bmatrix}\lambda & \lambda & \lambda & \lambda\end{bmatrix} \\
&= (r-\lambda)\boldsymbol{s}_h + 3\begin{bmatrix}\lambda & \lambda & \lambda & \lambda\end{bmatrix}
\end{aligned}$$

Similar can be made for $h = 2, 3, 4$.

With this proposition it is now possible to conclude the statement on why these BIBD are called symmetric. Using proposition 2.4 with $v = b$ it follows that $r = k$ and thus:

- every block contains k points, and
- every point occurs in k blocks

Finally by proposition 3.1 it follows that:

- every pair of points occur in λ blocks, and
- every pair of blocks intersect in λ points

It is because of the symmetry between blocks and points in these statements that these types of designs are called symmetric.

3.1.1 Residual & Derived BIBDs

Proposition 3.1 provides two methods to construct new BIBDs from existing ones. This is the subject of this subsection based on [1, p. 25-26]. The two methods are known as *residual* and *derived designs* and are defined as follows.

Definition 3.2: Derived & Residual BIBD

Let $(X, \mathcal{A})$ be a symmetric (v, k, λ)-BIBD and let $B_0 \in \mathcal{A}$. Then the *derived* BIBD is given by

$$\mathrm{Der}(X, \mathcal{A}, B_0) := (B_0, \{B \cap B_0 \mid B \in \mathcal{A}, B \neq B_0\})$$

and the *residual* BIBD is given by:

$$\mathrm{Res}(X, \mathcal{A}, B_0) := (X \backslash B_0, \{B \backslash B_0 \mid B \in \mathcal{A}, B \neq B_0\})$$

In other words, a derived BIBD is constructed by deleting all those points that are *not* in a given block B_0 and then deleting the block B_0, while a residual design is constructed by deleting all points in B_0 and as a result the block itself as well. The following propositions show that the derived and the residual designs live up to their names and are in fact BIBDs under certain conditions.

Proposition 3.3:

Suppose that $(X, \mathcal{A})$ is a symmetric (v, k, λ)-BIBD and let $B_0 \in \mathcal{A}$. If $\lambda \geq 2$ then $\text{Der}(X, \mathcal{A}, B_0)$ is a $(k, v-1, k-1, \lambda, \lambda-1)$-BIBD.

Proof
Since every block of the original design consisted of k points, and the derived design is constructed by removing all points except those in a given block, the number of points in the derived design is k. Furthermore since $(X, \mathcal{A})$ is symmetric, $v = b$, and one block is deleted to obtain the derived design. Therefore, the number of blocks in the derived design is $v-1$.

By proposition 3.1 any two blocks of the original design have λ points in common. Thus, after removing all those points not in B_0, the number of points in the remaining blocks is λ. Moreover, every distinct pair of points belong to λ blocks by proposition 3.1 and by removing one of these blocks, B_0, any pair of distinct points now belong to $\lambda - 1$ blocks.

Finally, the only parameter remaining is the replication number. If the derived design is a BIBD then proposition 2.3 can be applied with the parameters of the derived design:

$$r = \frac{(\lambda-1)(k-1)}{\lambda-1} = k-1$$

Thus, all that remains to be shown is that the derived design is in fact a BIBD. For any (v, b, r, k, λ)-BIBD $2 \leq k < v$ by definition. For the derived design this becomes $2 \leq \lambda < k$ by the arguments above. Since the original BIBD was symmetric, $r = k$, and using proposition 2.3

$$r = \frac{\lambda(v-1)}{k-1} \Leftrightarrow k(k-1) = \lambda(v-1) \tag{3.3}$$

Naturally, the number of points in a block, k, cannot exceed the total number of points in X, v, that is $v > k$. Therefore for equation (3.3) to hold, the condition $\lambda < k$ must be satisfied as well. Thus if the derived design has $2 \leq \lambda$ then it is a $(k, v-1, k-1, \lambda, \lambda-1)$-BIBD. ∎

Proposition 3.4:

Suppose that $(X, \mathcal{A})$ is a symmetric (v, k, λ)-BIBD and let $B_0 \in \mathcal{A}$. If $k \geq \lambda + 2$ then $\text{Res}(X, \mathcal{A}, B_0)$ is a $(v-k, v-1, k, k-\lambda, \lambda)$-BIBD.

Proof
The design $(X, \mathcal{A})$ consists of v points and since every block of the original design consists of k points and all points in one of these blocks are deleted, the remaining points in the residual design is $v-k$. Because $(X, \mathcal{A})$ is symmetric, $v = b$, and since one block, B_0 is deleted when all of the points in the block are removed the number of blocks remaining are $v-1$.

By proposition 3.1 any two blocks of the original design have λ points in common so by removing those points from the total of k points in any of the original blocks, the residual design contains $k-\lambda$ points in each block.

Finally, since all points in B_0 are removed, these points are removed from all other blocks as well. Therefore, any point still appears in the same number of blocks as the original design and any pair of points appear together in the same number of blocks as the original design as well. Thus the residual design has parameters $(v-k, v-1, k, k-\lambda, \lambda)$.

All that remains to be shown is that the design with these parameters is a BIBD. By definition any (v, b, r, k, λ)-BIBD satisfies that $2 \leq k < v$, and for the parameters of the residual design this results in $2 \leq k-\lambda < v-k$. To show that this hold, suppose that $\lambda + 2 \leq k$. Furthermore let $v-k \leq k-\lambda$, which is equivalent to $v \leq 2k-\lambda$. The symmetry of the BIBD and proposition 2.3 gives the condition that $k(k-1) = \lambda(v-1)$. Since $v \leq 2k-\lambda$ by assumption this results in:

$$k(k-1) = \lambda(v-1) \leq \lambda(2k-\lambda-1) \Leftrightarrow k^2 - k \leq 2k\lambda - \lambda^2 - \lambda \Leftrightarrow (k-\lambda)(k-\lambda-1) \leq 0$$

This condition is satisfied only when $k = \lambda$ or $k = \lambda + 1$ which contradicts the assumption that $k \geq \lambda + 2$. Thus if $k \geq \lambda + 2$ in the residual design then $k - \lambda < v - k$ and hence the residual design is a BIBD. ■

Example 3.2
Consider a symmetric $(11, 5, 2)$-BIBD with $X = \{0, 1, \ldots, 10\}$ and

$$\mathcal{A} = \Big\{\{0,4,7,9,10\},\{0,1,5,8,10\},\{0,1,2,6,9\},\{1,2,3,7,10\},\{1,3,4,5,9\},\{1,4,6,7,8\},$$
$$\{2,4,5,6,10\},\{2,5,7,8,9\},\{0,2,3,4,8\},\{3,6,8,9,10\},\{0,3,5,6,7\}\Big\}$$

The derived design obtained by removing all the points not in the first blocks and deleting the first block becomes a $(5, 10, 4, 2, 1)$-BIBD with $X = \{0, 4, 7, 9, 10\}$ and

$$\mathcal{A} = \Big\{\{0,10\},\{0,9\},\{7,10\},\{4,9\},\{4,7\},\{4,10\},\{7,9\},\{0,4\},\{9,10\},\{0,7\}\Big\}$$

The residual design obtained by removing all points from the first block and then deleting the first block becomes a $(6, 10, 5, 3, 2)$-BIBD with $X = \{1, 2, 3, 5, 6, 8\}$ and

$$\mathcal{A} = \Big\{\{1,5,8\},\{1,2,6\},\{1,2,3\},\{1,3,5\},\{1,6,8\},\{2,5,6\},\{2,5,8\},\{2,3,8\},\{3,6,8\},\{3,5,6\}\Big\}$$

3.2 Affine & Projective Planes

In this section, based on [1, pp. 27-30] unless otherwise specified, the finite analogue to the Euclidean geometry described in the beginning of the chapter will finally be introduced. The first of these is called *a projective plane* and is defined as follows.

Definition 3.5: Projective Plane

An $(n^2 + n + 1, n + 1, 1)$-BIBD with $n \geq 2$ is called a projective plane of order n.

Remark 2. A $(3, 2, 1)$-BIBD does exist, however, for technical reasons this is not seen as a projective plane of order 1. ◁

Projective planes are symmetric which can be realized using proposition 2.3 to see that $r = k$

$$r = \frac{\lambda(v-1)}{k-1} = \frac{1(n^2+n+1-1)}{n+1-1} = \frac{n^2+n}{n} = n+1 = k$$

and from this it follows that $b = v$ using proposition 2.4. Returning to the design presented in example 2.2, the Fano plane, this is an example of a projective plane of order 2, and it is readily seen that the properties mentioned above holds. A geometric interpretation of a projective plane, inspired by [5, pp.59-60], can be described through the following definition:

Definition 3.6: Projective Plane (geometrically)

A projective plane consists of a set of points, a set of lines and an incidence relation that determines which points are on which lines. This incidence relation satisfies that:

- Every two lines intersect in exactly one point
- Every two points have a unique line incident with both of them
- There are at least four points of which no three are on the same line

Notice that the third point excludes the existence of the projective plane of order 1. Furthermore, notice that since there are parallel lines in the Euclidean geometry, the second property is not satisfied and hence the Euclidean geometry is not a projective plane.

Before the proof of the important existence theorem, proposition 3.9, a few lemmas will be necessary for the proof. Lemma 3.7 is inspired by [6, p.125], while lemma 3.8 is an exercise from the same book.

Lemma 3.7:

The number of elements in the vector space ${\mathbb{F}_q}^k$ is exactly q^k.

Proof

Choose a basis, $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$, of ${\mathbb{F}_q}^k$, which is possible since all finite dimensional vector spaces have a basis [7, p. 41]. Then every $\boldsymbol{v} \in \operatorname{span}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$ can be expressed uniquely as

$$\boldsymbol{v} = a_1\boldsymbol{w}_1 + \ldots + a_k\boldsymbol{w}_k \text{ for } a_i \in \mathbb{F}_q \tag{3.4}$$

Since $\mathbb{F}_q$ has q elements, there are q choices for each a_i. Furthermore, due to the uniqueness of equation (3.4) it follows that different choices of a_i results in different elements ${\mathbb{F}_q}^k$. Therefore, it follows that ${\mathbb{F}_q}^k$ contains exactly q^k elements. ■

Lemma 3.8:

The number of different r–dimensional subspaces of the vector space ${\mathbb{F}_q}^n$ is given by:

$$\frac{(q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{r-1})}{(q^r-1)(q^r-q)(q^r-q^2)\ldots(q^r-q^{r-1})} \tag{3.5}$$

Proof
Firstly, $\mathbb{F}_q{}^n$ has q^n elements by lemma 3.7. Next, all possible ordered bases for an $r-$dimensional subspace must be counted. The first basis vector, $\boldsymbol{b_1}$, can be any non-zero vector and therefore there are $q^n - 1$ such choices. The second basis vector can be any vector that is not in the vector space generated by $\boldsymbol{b_1}$, so $\boldsymbol{b_2} \in \mathbb{F}_q{}^n \setminus < \boldsymbol{b_1} >$, and there are $q^n - q$ such choices. A continuation of this process, then gives that the i'th basis vector $\boldsymbol{b_i} \in \mathbb{F}_q{}^n \setminus < \boldsymbol{b_1}, \ldots, \boldsymbol{b_{i-1}} >$ which results in $q^n - q^{i-1}$ choices. Thus it follows that the number of bases for the $r-$dimensional subspace is given by:

$$(q^n - 1)(q^n - q)(q^n - q^2)\ldots(q^n - q^{r-1})$$

However, some of these bases span the same subspace. Therefore the number of different ordered bases of an $r-$dimensional subspace is counted. This is the same idea as above, and hence the first basis vector has $q^r - 1$ choices, the next one has $q^r - q$ choices, and so on. That is, the number of bases that are basis for a given subspace is given by:

$$(q^r - 1)(q^r - q)(q^r - q^2)\ldots(q^r - q^{r-1})$$

Therefore the total number of different $r-$dimensional subspaces is given by equation (3.5). ■

Proposition 3.9:

For every prime power $q \geq 2$, there exists a symmetric $(q^2 + q + 1, q + 1, 1)$-BIBD.

Proof
Assume that q is a prime power. Furthermore let $\mathcal{V}_1$ consist of all the one-dimensional subspaces of $\mathbb{F}_q{}^3$ and let $\mathcal{V}_2$ consist of all the two-dimensional subspaces of $\mathbb{F}_q{}^3$. For each $B \in \mathcal{V}_2$ define

$$A_B := \{C \in \mathcal{V}_1 \mid C \subset B\} \tag{3.6}$$

and

$$\mathcal{A} := \{A_B \mid B \in \mathcal{V}_2\}$$

The claim is now, that the design $(\mathcal{V}_1, \mathcal{A})$ is a projective plane of order q. That is, the points of the design are the one-dimensional subspaces of $\mathbb{F}_q{}^3$ and the blocks are the two-dimensional subspaces of $\mathbb{F}_q{}^3$. Finally, by equation (3.6) the design makes a point incident to a block only if the one-dimensional subspace is contained in the two-dimensional subspace. Therefore, it must be shown that the number of one-dimensional subspaces is given by $|\mathcal{V}_1| = q^2 + q + 1$, and that the block size is given by $|A_B| = q + 1$. Finally it needs to be shown that any pair of distinct one-dimensional subspaces belong to only one block.

Firstly, by lemma 3.8 the total amount of one-dimensional subspaces is given by:

$$|\mathcal{V}_1| = \frac{q^3 - 1}{q - 1} = \frac{(q-1)(q^2 + q + 1)}{q - 1} = q^2 + q + 1$$

Next it must be shown that the size of any block, A_B, is equal to $q + 1$. By lemma 3.7, each two-dimensional subspace B contains q^2 vectors including the zero vector. Each of these $q^2 - 1$ non-zero vectors together with the zero vector defines a one-dimensional subspace of B. Each of these subspaces are counted once for each of the $q - 1$ non-zero vectors inside it. Therefore, the number of one-dimensional subspaces contained in each block is given by:

$$|A_B| = \frac{q^2 - 1}{q - 1} = \frac{(q+1)(q-1)}{q - 1} = q + 1$$

Finally, it needs to be shown that any pair of distinct one-dimensional subspaces belong to only one block. Let C_1 and C_2 be two distinct one-dimensional subspaces of $\mathbb{F}_q{}^3$. Due to the uniqueness of equation (3.4) there exist a unique two-dimensional subspace, B, containing both C_1 and C_2, which results in the unique block A_B. Because this two-dimensional subspace is unique, C_1 and C_2 appear together only in this block A_B. ■

Remark 3. These projective planes of order q are usually denoted $PG_2(q)$ and are considered to be two-dimensional projective geometries. Proposition 3.9 can be generalized to d-dimensional projective geometry $PG_d(q)$ where the points and blocks corresponds to lines and hyperplanes, however, this will not be described further in this thesis. ◃

Example 3.3 (The Fano Plane Revisited)
Consider the vector space $\mathbb{F}_2{}^3$. By proposition 3.9 this results in a symmetric $(7,3,1)$-BIBD. Using the construction from the proof of proposition 3.9 results in the following one- and two dimensional subspaces of $\mathbb{F}_2{}^3$, where the vectors (x_1, x_2, x_3) are written $x_1x_2x_3$ for the sake of simplicity:

$$C_1 = \{000, 101\}, C_2 = \{000, 010\}, C_3 = \{000, 111\}, C_4 = \{000, 100\},$$
$$C_5 = \{000, 001\}, C_6 = \{000, 011\}, C_7 = \{000, 110\}$$

$$B_1 = \{000, 101, 010, 111\}, B_2 = \{000, 101, 100, 001\}, B_3 = \{000, 101, 110, 011\},$$
$$B_4 = \{000, 010, 001, 011\}, B_5 = \{000, 111, 100, 011\}, B_6 = \{000, 111, 001, 110\},$$
$$B_7 = \{000, 010, 100, 110\}$$

But the definition of the blocks given in equation (3.6) this results in the following blocks:

$$A_{B_1} = \{C_1, C_2, C_3\}, A_{B_2} = \{C_1, C_4, C_5\}, A_{B_3} = \{C_1, C_6, C_7\}, A_{B_4} = \{C_2, C_5, C_6\},$$
$$A_{B_5} = \{C_3, C_4, C_6\}, A_{B_6} = \{C_3, C_5, C_7\}, A_{B_7} = \{C_2, C_4, C_7\}$$

By putting $C_1 = a$, $C_2 = b$, $\dots$, $C_7 = g$ this is exactly the Fano plane as presented in example 2.2.

Definition 3.10: Affine Plane

Let $n \geq 2$. A $(n^2, n^2+n, n+1, n, 1)$-BIBD is called an affine plane of order n.

By proposition 3.4 the residual design of a projective plane of order n is an affine plane of order n. Therefore the following proposition is an immediate consequence of proposition 3.9.

Proposition 3.11:

For every prime power $q \geq 2$ there exist a $(q^2, q, 1)$-BIBD.

Notice that since $\lambda = 1$ in a projective plane, no derived design of any projective plane can exist. A formal proof of proposition 3.11 using a specific construction similar to that of the projective planes can be found in [1, pp. 102-103]. However, this is beyond the scope of this thesis, and instead only construct affine planes using the residual design of a projective plane.

3.3 The Bruck-Ryser-Chowla Theorem

In this section further results on the existence of certain symmetric designs are presented. The two main results, theorems 3.12 and 3.15, known together as the Bruch-Ryser-Chowla (BRC) theorem are proven and will be related to projective planes as well. The BRC theorem was proven in 1950 and since then no general necessary conditions for the existence of symmetric BIBDs have been proven. Apart from those ruled out by the BRC theorem, the only result is that a projective plane of order 10 does not exist, which was proven in 1989 using a computer. This emphasizes the importance of the results presented in this section, which is based on [1, pp. 30-38] unless otherwise specified.

Theorem 3.12: Bruck-Ryser-Chowla Theorem for v even

Suppose there exists a symmetric (v,k,λ)-BIBD, where v is even. Then $k-\lambda$ is the square of an integer.

Proof
Let v be even and let M denote the incidence matrix of a symmetric (v,k,λ)-BIBD. Then due to the symmetry, $r=k$, and then it follows from proposition 2.6 that

$$MM^T = \lambda J_v + (k-\lambda)I_v$$

and once more due to the symmetry of the design M and M^T are $v\times v$ matrices. For any square matrix $\det\left(MM^T\right) = \det(M)\det\left(M^T\right)$ [7, p. 317]. Since $\det(M) = \det\left(M^T\right)$ it follows that:

$$\det(M)^2 = \det\left(MM^T\right) = \det(\lambda J_v + (k-\lambda)I_v)$$

Because adding or subtracting rows and columns from each other does not change the value of the determinant, these row operations can be performed to obtain an simpler way to calculate the determinant. These operations are performed on the matrix $\lambda J_v + (k-\lambda)I_v$. Firstly subtract the first row from every other row, and secondly add columns $2,3,\ldots,v$ to the first column. This results in the following:

$$\begin{bmatrix} k & \lambda & \lambda & \ldots & \lambda \\ \lambda & k & \lambda & \ldots & \lambda \\ \lambda & \lambda & k & \ldots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \ldots & k \end{bmatrix} \sim \begin{bmatrix} k & \lambda & \lambda & \ldots & \lambda \\ \lambda-k & k-\lambda & 0 & \ldots & 0 \\ \lambda-k & 0 & k-\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda-k & 0 & 0 & \ldots & k-\lambda \end{bmatrix} \sim \begin{bmatrix} k+(v-1)\lambda & \lambda & \lambda & \ldots & \lambda \\ 0 & k-\lambda & 0 & \ldots & 0 \\ 0 & 0 & k-\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & k-\lambda \end{bmatrix}$$

Since this is an upper triangular matrix the determinant is the product of the diagonal entries, that is

$$\det(M)^2 = (k+(v-1)\lambda)(k-\lambda)^{v-1} = k^2(k-\lambda)^{v-1}$$

where the final equality follows from proposition 2.3 and the symmetry of the design. Thus, by taking the square root:

$$\det(M) = k(k-\lambda)^{\frac{v-1}{2}} \tag{3.7}$$

The matrix M has integer entries and therefore the determinant is an integer, and furthermore k is an integer as well. Therefore $(k-\lambda)^{\frac{v-1}{2}}$ must be a rational number for equation (3.7) to hold, but since $v-1$ is odd by assumption, this is only possible if $(k-\lambda)$ is the square of an integer. ■

Before stating the BRC-theorem for odd v, the following two lemmas are required for the proof.

Lemma 3.13:

For any integer $n \geq 0$, there exist integers a_0, a_1, a_2, a_3 such that $n = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2$.

Lemma 3.14:

Suppose that

$$C = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

and let $n = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2$. Then $C^{-1} = \frac{1}{n}C^T$.

Lemma 3.13 is a well-known result from number theory, known as Lagrange's four-square theorem, proven by Langrange in 1770. However, the proof of this lemma is beyond the scope of this thesis and hence it is left out, but can be found in [8, pp. 280-282]. Lemma 3.14, on the other hand, can be easily verified by checking that $C\frac{1}{n}C^T = I$. With these results, the BRC-theorem for odd v is ready to be proven.

Theorem 3.15: Bruck-Ryser-Chowla Theorem for v odd

Suppose there exists a symmetric (v,k,λ)-BIBD with v odd. Then there exist integers x, y, and z, not all zero, such that:

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2 \tag{3.8}$$

Proof

Suppose that $v \equiv 1 (\text{mod } 4)$ and let $v = 4w+1$. Furthermore let M be the incidence matrix of a symmetric (v,k,λ)-BIBD and let $x_1, \ldots, x_v$ be unknown variables to be determined. For $1 \leq i \leq v$ define:

$$L_i := \sum_{j=1}^{v} m_{ji}x_j$$

Since all entries of M are integers, each L_i is a linear function of the x_j with integral coefficients. Next, it is beneficial to show that:

$$\sum_{i=1}^{v} {L_i}^2 = \lambda\left(\sum_{j=1}^{v} x_j\right)^2 + (k-\lambda)\sum_{j=1}^{v} {x_j}^2 \tag{3.9}$$

To show this, firstly realize that if L_i is squared then it is possible to introduce a second index h such that:

$${L_i}^2 = \left(\sum_{j=1}^{v} m_{ji}x_j\right)\left(\sum_{h=1}^{v} m_{hi}x_h\right) = \sum_{j=1}^{v}\sum_{h=1}^{v} m_{ji}m_{hi}x_jx_h$$

Next by summing all these ${L_i}^2$, the following is obtained:

$$\sum_{i=1}^{v} {L_i}^2 = \sum_{i=1}^{v}\sum_{j=1}^{v}\sum_{h=1}^{v} m_{ji}m_{hi}x_jx_h = \sum_{j=1}^{v}\sum_{h=1}^{v}\left(\sum_{i=1}^{v} m_{ji}m_{hi}\right)x_jx_h$$

Since the design is symmetric it follows from proposition 2.4 that $r = k$ and using equation (2.2) from the proof of proposition 2.6 with this fact it follows that:

$$\sum_{h=1}^{v} m_{ji}m_{hi} = \begin{cases}\lambda \text{ if } j \neq h\\ k \text{ if } j = h\end{cases}$$

Using this the following is obtained:

$$\begin{aligned}\sum_{i=1}^{v} {L_i}^2 &= \sum_{\{j\neq h\}} \lambda x_jx_h + \sum_{j=1}^{v} k{x_j}^2\\ &= \sum_{\{j\neq h\}}\sum_{\{j=h\}} \lambda x_jx_h + \sum_{j=1}^{v} k{x_j}^2 - \sum_{\{j=h\}} \lambda x_jx_h\\ &= \sum_{j=1}^{v}\sum_{h=1}^{v} \lambda x_jx_h + \sum_{j=1}^{v} k{x_j}^2 - \sum_{j=1}^{v} \lambda{x_j}^2\\ &= \sum_{j=1}^{v}\sum_{h=1}^{v} \lambda x_jx_h + \sum_{j=1}^{v} (k-\lambda){x_j}^2\\ &= \lambda\sum_{j=1}^{v}\sum_{h=1}^{v} x_jx_h + (k-\lambda)\sum_{j=1}^{v} {x_j}^2\\ &= \lambda\left(\sum_{j=1}^{v} x_j\right)^2 + (k-\lambda)\sum_{j=1}^{v} {x_j}^2\end{aligned}$$

Equation (3.9) is an identity in the variables $x_1,\ldots,x_v$ in which all the coefficients are integers, that is, the equation holds no matter what values of $x_1,\ldots,x_v$ are chosen. Therefore the variables can be transformed into new variables $y_1,\ldots,y_v$ where each y_i is a given integral linear combination of the x_j. By lemma 3.13 there exist integers a_0, a_1, a_2 and a_3 such that ${a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2 = k - \lambda$. Let C be the matrix of lemma 3.14 and for $1 \leq h \leq w$ let

$$\begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h}\end{bmatrix} = \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} C \tag{3.10}$$

Finally let $y_v = x_v$ and let

$$y_0 = \sum_{i=1}^{v} x_i$$

Consider the following equations using equation (3.10) and lemma 3.14:

$$\begin{aligned}
{y_{4h-3}}^2+{y_{4h-2}}^2+{y_{4h-1}}^2+{y_{4h}}^2 &= \begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h}\end{bmatrix}\begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h}\end{bmatrix}^T \\
&= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} C\left(\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} C\right)^T \\
&= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} C\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}^T C^T \\
&= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} CC^T\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}^T \\
&= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} (k-\lambda)I_4\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}^T \\
&= (k-\lambda)\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}\begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}^T \\
&= (k-\lambda)\left({x_{4h-3}}^2+{x_{4h-2}}^2+{x_{4h-1}}^2+{x_{4h}}^2\right)
\end{aligned}$$

Since equation 3.10 holds for $1 \le h \le w$, these calculations do as well, and since $v = 4w+1$ it has been shown that

$$\sum_{j=1}^{v-1} {y_j}^2 = (k-\lambda)\sum_{j=1}^{v-1} {x_j}^2 \tag{3.11}$$

Therefore, it follows that:

$$\begin{aligned}
\sum_{i=1}^{v} {L_i}^2 &= \lambda\left(\sum_{j=1}^{v} x_j\right)^2 + (k-\lambda)\sum_{j=1}^{v} {x_j}^2 \\
&= \lambda\left(\sum_{j=1}^{v} x_j\right)^2 + (k-\lambda)\sum_{j=1}^{v-1} {x_j}^2 + (k-\lambda){x_v}^2 \\
&= \lambda {y_0}^2 + \sum_{j=1}^{v-1} {y_j}^2 + (k-\lambda){y_v}^2
\end{aligned} \tag{3.12}$$

The L_i were originally defined as integral linear combinations of the x_j, however, by using lemma 3.14 this can be rewritten such that each x_j is expressed as a rational linear combination of $y_1, \dots, y_v$ as desired. This is due to the fact that using lemma 3.14 on equation (3.10) results in the following:

$$\begin{aligned}
\begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h}\end{bmatrix} C^{-1} &= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix} CC^{-1} \\
\begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h}\end{bmatrix} \frac{1}{k-\lambda} C &= \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h}\end{bmatrix}
\end{aligned}$$

This results in the following expressions:

$$\begin{aligned}
x_{4h-3} &= \frac{1}{k-\lambda}(a_0 y_{4h-3} + a_1 y_{4h-2} + a_2 y_{4h-1} + a_3 y_{4h}) \\
x_{4h-2} &= \frac{1}{k-\lambda}(-a_1 y_{4h-3} + a_0 y_{4h-2} - a_3 y_{4h-1} + a_2 y_{4h}) \\
x_{4h-1} &= \frac{1}{k-\lambda}(-a_2 y_{4h-3} + a_3 y_{4h-2} + a_0 y_{4h-1} - a_1 y_{4h}) \\
x_{4h} &= \frac{1}{k-\lambda}(-a_3 y_{4h-3} - a_2 y_{4h-2} + a_1 y_{4h-1} + a_0 y_{4h})
\end{aligned}$$

Furthermore, y_v was defined to be x_v and therefore since a_0, a_1, a_2, a_3 are all integers each x_j has been represented as a rational linear combination of $y_1,\dots,y_v$. It follows from this that y_0 can be expressed as such as well since y_0 was originally defined as the sum of all x_i.

Thus to sum up, the L_i were given via equation (3.9) which are polynomials with variables $x_1,\dots,x_v$. These variables are then substituted with new variables while making sure that these new expressions remain as polynomials. This substitution results in equation (3.12), giving a new way to express these polynomials in terms of y_i. Furthermore with the use of lemma 3.14, as seen above, each x_j can be replaced with a rational linear combination of $y_1,\dots,y_v$. Thus the x_j in the definition of L_i can be replaced with this rational linear combination of $y_1,\dots,y_v$, resulting in an equation for polynomials of the variables $y_1,\dots,y_v$ given by equation (3.12).

Next, it will be shown that it is possible to reconstruct equation (3.12) by expressing any of the variables as a rational linear combination of the remaining variables. Let $c_i \in \mathbb{Q}$ and suppose that:

$$L_i = \sum_{i=1}^{v} c_i y_i$$

If $c_1 \neq 1$ then let $L_1 = y_1$, but if $c_1 = 1$ then let $L_1 = -y_1$. For both these cases y_1 can be expressed as a rational linear combination of $y_2,\dots,y_v$ in the following way.

- $c_1 \neq 1 \Rightarrow y_1 = L_1$: $y_1 - c_1 y_1 = c_2 y_2 + \dots + c_v y_v \Leftrightarrow (1-c_1)y_1 = c_2 y_2 + \dots + c_v y_v$ and finally, divide by $(1-c_1)$ on both sides
- $c_1 = 1 \Rightarrow y_1 = -L_1$: $-2y_1 = c_2 y_2 + \dots + c_v y_v$ and finally, divide by -2 on both sides

Furthermore it follows that ${L_1}^2 = {y_1}^2$ and therefore by canceling out ${L_1}^2$ and ${y_1}^2$ in equation (3.12) it results in the following equation:

$$\sum_{i=2}^{v} {L_i}^2 = \lambda {y_0}^2 + \sum_{j=2}^{v-1} {y_j}^2 + (k-\lambda){y_v}^2 \tag{3.13}$$

This process can be continued in the same way, eliminating the variables $y_2,\dots,y_{v-1}$ one at a time while ensuring that each y_j is still a rational linear combination of $y_{j+1},\dots,y_v$ such that ${L_j}^2 = {y_j}^2$. This results in the following equation

$${L_v}^2 = \lambda {y_0}^2 + (k-\lambda){y_v}^2 \tag{3.14}$$

where L_v and y_0 are rational multiples of the variable y_v. Now suppose that $L_v = s y_v$ and $y_0 = t \cdot y_v$, where $s,t \in \mathbb{Q}$, and let $y_v = 1$. Then equation (3.14) results in:

$$(s y_v)^2 = \lambda t \cdot {y_v}^2 + (k-\lambda){y_v}^2 \Leftrightarrow s^2 = \lambda t^2 + k - \lambda \tag{3.15}$$

Next let $s = \dfrac{s_1}{s_2}$ and $t = \dfrac{t_1}{t_2}$ with $s_1, s_2, t_1, t_2 \in \mathbb{Z}$ and $s_2, t_2 \neq 0$. Then equation (3.15) becomes:

$$\left(\frac{s_1}{s_2}\right)^2 = \lambda\left(\left(\frac{t_1}{t_2}\right)^2\right) + k - \lambda \Leftrightarrow (s_1 t_2)^2 = \lambda(s_2 t_1)^2 + (k-\lambda)(s_2 t_2)^2$$

Finally, let $x = s_1 t_2$, $y = s_2 t_2$ and $z = s_2 t_1$. Then an integral solution to equation (3.8) in which at least one of x, y, z is nonzero has been constructed since $s_2, t_2 \neq 0$. Furthermore note that since

$v \equiv 1 \pmod 4$ it follows that $(-1)^{(v-1)/2} = 1$.

The theorem remains to be shown for $v \equiv 3 \pmod 4$, to which the process of the proof is similar to the previous case, but with a few modifications. Let $v = 4w - 1$ and introduce a new variable x_{v+1}. Furthermore add $(k-\lambda){x_{v+1}}^2$ to both sides of equation (3.9) which results in the following equation:

$$\sum_{i=1}^{v} {L_i}^2 + (k-\lambda){x_{v+1}}^2 = \lambda\left(\sum_{j=1}^{v} x_j\right)^2 + (k-\lambda)\sum_{j=1}^{v+1} {x_j}^2 \tag{3.16}$$

For $1 \le h \le w$, let

$$\begin{bmatrix} y_{4h-3} & y_{4h-2} & y_{4h-1} & y_{4h} \end{bmatrix} = \begin{bmatrix} x_{4h-3} & x_{4h-2} & x_{4h-1} & x_{4h} \end{bmatrix} C$$

and finally let $y_0 = \sum_{i=1}^{v} x_i$. Then using the same calculations as for $v \equiv 1 \pmod 4$ and due to the fact that $v = 4w - 1$ it follows that

$$\sum_{j=1}^{v+1} {y_j}^2 = (k-\lambda)\sum_{j=1}^{v+1} {x_j}^2$$

and finally, it follows that

$$\sum_{i=1}^{v} {L_i}^2 + (k-\lambda){x_{v+1}}^2 = \lambda {y_0}^2 + \sum_{j=1}^{v+1} {y_j}^2 \tag{3.17}$$

Next, proceed as in the case of $v \equiv 1 \pmod 4$, letting $L_1 = \pm y_1, \ldots, L_v = \pm y_v$ which results in the following:

$$(k-\lambda){x_{v+1}}^2 = \lambda {y_0}^2 + {y_{v+1}}^2 \tag{3.18}$$

Similar to before, assume that $y_{v+1} = s x_{v+1}$ and $y_0 = t \cdot x_{v+1}$ and let $x_{v+1} = 1$. Then equation (3.18) can be rewritten as follows:

$$(k-\lambda) = \lambda t^2 + s^2 \Leftrightarrow s^2 = k - \lambda - \lambda t^2 \tag{3.19}$$

Next let $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$ with $s_1, s_2, t_1, t_2 \in \mathbb{Z}$ and $s_2, t_2 \neq 0$. Substitute this in equation (3.19) to obtain:

$$(s_1 t_2)^2 = (k-\lambda)(s_2 t_2)^2 - \lambda(s_2 t_1)^2$$

Finally, let $x = s_1 t_2$, $y = s_2 t_2$ and $z = s_2 t_1$. Then an integral solution to equation (3.8) in which at least one of x, y, z is nonzero has been constructed since $s_2, t_2 \neq 0$. Furthermore note that since $v \equiv 3 \pmod 4$ it follows that $(-1)^{(v-1)/2} = -1$. ■

Theorem 3.15 is less straightforward to apply than theorem 3.12 since it involves the existence of integers x, y and z such that the equation (3.8) is satisfied. Therefore, consider the following example.

Example 3.4
The purpose of this example is to show that a symmetric $(43, 7, 1)$-BIBD does not exist. Theorem 3.15 states that if this BIBD exists, then there exist integers x, y and z not all zero such that:

$$x^2 = 6y^2 + (-1)^{21} z^2 \Leftrightarrow x^2 + z^2 = 6y^2 \tag{3.20}$$

Assume that (x, y, z) are all integers and is a solution to equation (3.20). Reducing this modulo 3 results in

$$x^2 + z^2 \equiv 0 \pmod 3 \tag{3.21}$$

Since $x^2 \equiv 0$ or $1 \pmod 3$ for any integer x and similarly for z^2 the only way for equation (3.21) to be satisfied is if both $x^2 \equiv 0 \pmod 3$ and $z^2 \equiv 0 \pmod 3$. Using this rewrite equation (3.20) with $x = 3x_1$ and $z = 3z_1$ with $x_1, z_1 \in \mathbb{Z}$:

$$(3x_1)^2 + (3z_1)^2 = 6y^2 \Leftrightarrow 3{x_1}^2 + 3{z_1}^2 = 2y^2$$

The left-hand side of this equation is divisible by 3 and hence $y \equiv 0 \pmod 3$. Therefore, similarly to x and z rewrite $y = 3y_1$ for $y_1 \in \mathbb{Z}$:

$$3{x_1}^2 + 3{z_1}^2 = 2(3y_1)^2 \Leftrightarrow {x_1}^2 + {z_1}^2 = 6{y_1}^2$$

Therefore if (x, y, z) are all integers and a solution to equation (3.20) then $(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ is also a solution to equation (3.20) where $(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ are all integers. This process can then be repeated indefinitely resulting in a sequence of solutions $(\frac{1}{3})^i(x, y, z)$. These cannot all be integers unless $(x, y, z) = (0, 0, 0)$. Therefore according to theorem 3.15 a $(43, 7, 1)$-BIBD does not exist.

As this example shows, using theorem 3.15 directly can be tedious and hence it is useful to do a bit of work on it, using other results from number theory to create results that are a bit easier to apply. First consider a projective plane of order n, that is, $\lambda = 1$. In this case equation (3.8) becomes:

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2 = ny^2 + (-1)^{(n^2+n)/2}z^2 \tag{3.22}$$

Suppose that $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$. Then equation (3.8) results in $x^2 = ny^2 + z^2$ which has non-trivial solutions $x = z$ and $y = 0$. Thus projective planes of order n that satisfies this, always exist according to theorem 3.15.

Next, suppose that $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ for which $\frac{n^2+n}{2}$ is odd, so equation (3.22) reduces to:

$$x^2 + z^2 = ny^2 \tag{3.23}$$

It is now of interest to see when equation 3.23 has integral solutions (x, y, z), which is the case if and only if

$$x^2 + z^2 = n \tag{3.24}$$

has integral solutions (x, z) since this is equivalent to equation (3.23) with $y = 1$. The following result shows when equation (3.24) has an integral solution. The proof of this result is beyond the scope of this thesis and is therefore omitted here, however, it can be found in [9, pp. 142-145].

Proposition 3.16:

A positive integer n can be expressed as the sum of two integral squares if and only if the prime decomposition contain no term p^k, where p is a prime, $p \equiv 3 \pmod 4$ and k is odd.

Using this proposition along with the discussion above, this results in the following corollary relating theorem 3.15 to projective planes of order n.

Corollary 3.17:

Suppose that $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$. If n cannot be expressed as the sum of two integral squares, then a projective plane of order n does not exist.

Notice once more that this corollary describes the exclusion of certain projective planes, and not the existence. Therefore even if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and n can be expressed as the sum of two integral squares, then it does not mean that the projective plane of order n exists.

Example 3.5
Let $n = 33 = 3 \cdot 11 \equiv 1 \pmod 4$. Here both 3 and 11 are congruent to 3 modulo 4 and since both exponents are odd, the projective plane of order 33 does not exist.

Next, let $n = 2250 = 2 \cdot 3^2 \cdot 5^3 \equiv 2 \pmod 4$. Here only $3 \equiv 3 \pmod 4$ and since the exponent is even, the existence of a projective plane of order 2250 cannot be excluded, and indeed $2250 = 15^2 + 45^2$.

For a final application, consider $n = 10 = 2 \cdot 5 \equiv 2 \pmod 4$. Here none of these is congruent to 3 modulo 4 and hence it is conceivable that a projective plane of order 10 exists, and indeed $10 = 1^2 + 3^2$. However, as described in the introduction of this section this is not the case, as proven using a computer in 1989.

Similarly, it is possible to derive an easier to apply corollary of theorem 3.15 with arbitrary λ, however, this is beyond the scope of this thesis and instead refer to [1, pp.37-38] for an explanation on this along with an example on how to apply it. Furthermore [1] describes how to derive corollary 3.17 from this general case.

3.4 Resolvability

This section is based on [2, pp- 11-12], unless otherwise specified, and serves as a brief introduction to resolvability, which is the theoretical approach used in one of the ways to schedule a tournament, described in the next chapter. Furthermore, resolvability relates to the finite geometry in a special way, and since the finite geometry is the theme of this chapter, this relation will be proven later in this section as well.

Definition 3.18: Resolvable BIBD [1, p. 101]

Let $(X, \mathcal{A})$ be a (v, k, λ)-BIBD. A *parallel class* in $(X, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is X. A partition of $\mathcal{A}$ into several parallel classes is called a *resolution*, and $(X, \mathcal{A})$ is said to be *resolvable* if $\mathcal{A}$ has a resolution.

Example 3.6
Consider the Fano plane from example 2.2, that is:

$$X = \{a, b, c, d, e, f, g\} \text{ and } \mathcal{A} = \{abc, ade, afg, bfe, cfd, cge, bdg\}$$

Next, take the residual design of this by removing all points from the first block and deleting this block. This results in a $(4,6,3,2,1)$-BIBD with

$$X = \{d,e,f,g\} \text{ and } \mathcal{A} = \{de, fg, fe, fd, ge, dg\}$$

These can be grouped into three groups consisting of two disjoint blocks each in the following way:

$$\{de, fg\}, \{fe, gd\} \text{ and } \{fd, ge\}$$

Thus these are three parallel classes and the $(4,2,1)$-BIBD is resolvable.

One of the first examples of resolvable designs emerged in relation to Kirkman's schoolgirl problem presented in 1850. It states:

> *"Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily so that no two shall walk abreast twice."*

Kirkman then asked if a resolvable $(15,3,1)$ design existed. He was able to produce a solution, see [2, pp. 148-149], and later it has been proven that the corresponding problem for $6n+3$ schoolgirls always has a solution, however, this is not proven here, but instead refer to [2, pp. 155] for this. Instead, a simple approach will be given in the following example.

Example 3.7 (Kirkman's Schoolgirl Problem)
Label the 15 girls $A, B_1, \ldots, B_7, C_1, \ldots, C_7$ respectively. Firstly, construct day 1 such that all girls are in blocks of three and such that these blocks are disjoint and contain all girls. Next, construct the following day such that A is unchanged, while B_i and C_i are permutated cyclically. This results in the following solution:

- Day 1: AB_1C_1, $B_2B_4C_3$, $B_3B_7C_5$, $B_5B_6C_2$ and $C_4C_6C_7$
- Day 2: AB_2C_2, $B_3B_5C_4$, $B_4B_1C_6$, $B_6B_7C_3$ and $C_5C_7C_1$
- Day 3: AB_3C_3, $B_4B_6C_5$, $B_5B_2C_7$, $B_7B_1C_4$ and $C_6C_1C_2$
- Day 4: AB_4C_4, $B_5B_7C_6$, $B_6B_3C_1$, $B_1B_2C_5$ and $C_7C_2C_3$
- Day 5: AB_5C_5, $B_6B_1C_7$, $B_7B_4C_2$, $B_2B_3C_6$ and $C_1C_3C_4$
- Day 6: AB_6C_6, $B_7B_2C_1$, $B_1B_5C_3$, $B_3B_4C_7$ and $C_2C_4C_5$
- Day 7: AB_7C_7, $B_1B_3C_2$, $B_2B_6C_4$, $B_4B_5C_1$ and $C_3C_5C_6$

Resolvable designs have great use in experimental designs in general as well. Another example of the applications of resolvable designs is in the creation of tournament schedules of for example a football league. Suppose that a league consists of $2n$ teams who play each other once. These fixtures are arranged over $2n-1$ days so that all teams only play a single game each day. Then a tournament like this is a resolvable $(2n,2,1)$ design where the i'th parallel class consists of the pairs of teams playing each other on match day i.

Example 3.8
Consider a league schedule for six teams $A, \ldots, F$ as follows:

- Day 1: A vs B C vs D E vs F

- Day 2: A vs C B vs E D vs F
- Day 3: A vs D C vs E B vs F
- Day 4: A vs E B vs D C vs F
- Day 5: A vs F B vs C D vs E

This corresponds to a resolvable $(6,2,1)$ design on $\{A,B,C,D,E,F\}$ with blocks $\{AB\}, \{CD\}, \ldots, \{DE\}$

This is a subject that will be discussed in further detail in chapter 4, where more detailed schedules will be presented.

3.4.1 Resolvability & Affine Planes

Interpreting resolvable designs geometrically, the blocks can be thought of as lines and each group of disjoint blocks as a collection of parallel lines. Recall that in the projective geometry, every pair of lines intersect, however, the examples above instead reflect the existence of parallel lines. It turns out that all affine planes have such parallel classes and therefore they are sometimes known as Euclidean planes. That is, all affine planes are resolvable. To prove this a few lemmas will be necessary. First off, a result similar to a Euclidean property will be proven. This Euclidean property is that, given a point P not on a line l, a unique line through P parallel to l exists. The following results are based on [1, pp. 104-106].

Lemma 3.19:

Suppose that $(X,\mathcal{A})$ is an affine plane of order n. Let $B \in \mathcal{A}$, $x \in X$ and $x \notin B$. Then there exists exactly one block $A \in \mathcal{A}$ such that $x \in A$ and $B \cap A = \emptyset$.

Proof
Since $(X,\mathcal{A})$ is an affine plane of order n, it is by definition 3.10 a $(n^2, n^2+n, n+1, n, 1)$-BIBD. Thus $\lambda = 1$, which implies that for every point $y \in B$, there exist a unique block B_y such that $\{x,y\} \subset B_y$. Furthermore, $B \cap B_y = \{y\}$ using the fact that $\lambda = 1$ once again.

Next, since the replication number $n+1$, there are exactly $n+1$ blocks containing the point x. But since $|B| = n$, it means that exactly n of the $n+1$ blocks containing the point x intersect with B. Therefore, there exist one more block containing x, denoted A, which satisfies that $B \cap A = \emptyset$ ■

Let $(X,\mathcal{A})$ be an affine plane of order n. To prove that this design is always resolvable, a binary relation $\sim$ on the set of blocks $\mathcal{A}$ is used. It is then proven that this binary relation is also an equivalence relation and finally that each equivalence class of $\sim$ is a parallel class in $(X,\mathcal{A})$. The binary relation $\sim$ is defined as follows for $A, B \in \mathcal{A}$.

$$A \sim B \text{ if } A = B \text{ or } A \cap B = \emptyset \tag{3.25}$$

Lemma 3.20:

Let $(X,\mathcal{A})$ be an affine plane of order n. Then the relation $\sim$, as defined in equation (3.25), is an equivalence relation.

Proof
It needs to be shown that $\sim$ is reflective, symmetric, and transitive. By definition, the relation is reflective, since $A \sim A$ for every $A \in \mathcal{A}$. Furthermore, it is symmetric by definition, since $A \sim B$ if and only if $B \sim A$.

For transitivity suppose that $A \sim B$ and $B \sim C$ for $A, B, C \in \mathcal{A}$. It must now be shown that $A \sim C$. There are four cases to consider.

1) $A = B$ and $B = C$. Then $A = C$ and therefore $A \sim C$.
2) $A = B$ and $B \cap C = \emptyset$. Then $A \cap C = \emptyset$ and hence $A \sim C$
3) $A \cap B = \emptyset$ and $B = C$. Then $A \cap C = \emptyset$ and so $A \sim C$
4) $A \cap B = \emptyset$ and $B \cap C = \emptyset$. If $A = C$ then $A \sim C$ as seen from 1), so instead let $A \neq C$. In this case it will be proven that $A \cap C = \emptyset$. If this does not hold, then since $\lambda = 1$ there would exist a unique point $x \in A \cap C$. But then A and C are two blocks containing the point x and disjoint to B, which contradicts lemma 3.19. Thus, in conclusion $A \cap C = \emptyset$ which means that $A \sim C$

■

Lemma 3.21:

Suppose that $(X, \mathcal{A})$ be an affine plane of order n. Then each equivalence class of $\sim$ is a parallel class in $(X, \mathcal{A})$.

Proof
Let π be an equivalence class of $\sim$ and let $B \in \pi$. Then:

$$\pi := \{A \in \mathcal{A} : A \sim B\}$$

By the definition of $\sim$ any two distinct blocks of π are disjoint. Let $x \in X$. If $x \in B$ then B is a unique block in π such that $x \in B$. On the other hand if $x \notin B$ then by lemma 3.19 there exists exactly one block A such that $x \in A$ and $A \cap B = \emptyset$, which means that $A \in \pi$.

Thus if $x \in B$ then it is contained in the equivalence class π, and if $x \notin B$, then there exists an $x \in A$ such that $A \in \pi$. Therefore π partitions X and any two distinct blocks of π are disjoint. Therefore π is a parallel class in $(X, \mathcal{A})$. ■

With this lemma, the main result of this subsection is ready to be proven.

Proposition 3.22:

Any affine plane is resolvable.

Proof
By lemma 3.21, each equivalence class of $\sim$ is a parallel class as well. Furthermore, by lemma 3.19 every block of the BIBD is contained in exactly one equivalence class of $\sim$. Therefore, the equivalence classes of $\sim$ form a resolution of the affine plane and hence the design is resolvable.

■

Chapter 4

Tournament scheduling

In this chapter some of the theoretical frame presented in the previous chapters will be applied in order to present ways to construct league schedules that consist of $2n$ teams. Two such constructions will be presented in sections 4.3 and 4.4 respectively. However, before this a brief introduction to the *circle method* will be presented along with theoretical background as to why this construction works.

4.1 Difference Methods

As seen in both examples 3.7 and 3.8, these designs had a cyclic structure, starting in one place and simply using addition with modular arithmetic's in order to achieve the following match-ups. However, this has to be constructed with caution since one cannot simply start with anything for this method to work. For instance, consider a schedule of 6 teams and start with matches 0 vs 2, 5 vs 1 and 3 vs 4. Then adding 1 (mod 6) results in matches 1 vs 3, 0 vs 2 and 4 vs 5, and thus a repeated match occurs. In this section, based on [2, pp. 39-41 & 43-45], unless otherwise specified, the construction of such cyclic designs will be presented. First the concept *difference set* will be formally defined, inspired by [1, p. 41].

Definition 4.1: Difference set

Suppose that $(G,+)$ is a finite abelian group of order v with identity element 0. Let k and λ be positive integers such that $2 \leq k < v$. A (v,k,λ)-difference set in $(G,+)$ is a subset $D \subseteq G$ that satisfies:

1) $|D| = k$ with elements $D = \{d_1, \dots, d_k\}$
2) Every $d \in G \backslash \{0\}$ can be expressed on the form $d = d_i - d_j$ in exactly λ ways.

Example 4.1
Consider once again the Fano plane, which is a (7,3,1)-BIBD. Therefore to construct this using a cyclic difference set, a set of 3 elements, $D = \{d_1, d_2, d_3\}$ of $\mathbb{Z}/7\mathbb{Z}$, is necessary. Furthermore each non-zero element $d \in (\mathbb{Z}/7\mathbb{Z})$ must be able to be expressed in the form $d = d_i - d_j$ in exactly one way. Since

$$4\text{-}3 \pmod 7 = 1,\ 6\text{-}4 \pmod 7 = 2,\ 6\text{-}3 \pmod 7 = 3,$$
$$3\text{-}6 \pmod 7 = 4,\ 4\text{-}6 \pmod 7 = 5 \text{ and } 3\text{-}4 \pmod 7 = 6$$

the set $\{3,4,6\}$ is a cyclic (7,3,1)-difference set and gives a simple method of constructing the Fano plane. Simply start with block $\{3,4,6\}$ and add 1 to each point in this block and reduce modulo 7, resulting in $\{4,5,0\}$. Do the same for each point in this new block, and doing so seven times in total, the block returns to the starting block $\{3,4,6\}$.

Note that in the example above, the starting block $\{5,6,1\}$ could have been chosen as a starting block as well, and the example would have been analogue. However, given a starting block D these other potential starting blocks are instead known as *translates*.

Definition 4.2: Translate

If $D = \{d_1,\dots,d_k\}$ is a cyclic (v,k,λ)-difference set, then the set $D+g = \{d_1+g,\dots,d_k+g\}$ for $g \in G$, is called a translate of D.

The collection of all v of these translates is called *the development* of D and is denoted $dev(D)$ [1, p. 42]. Each translate of a (v,k,λ)-difference set is also a difference set, since

$$(d_1+g)-(d_j+g) = d \Leftrightarrow d_i - d_j = d$$

As seen in example 4.1, difference sets gave a simple way to construct the blocks of the Fano plane given a starting block D. This can be generalized to any symmetric (v,k,λ)-BIBD, as follows.

Proposition 4.3:

If $D = \{d_1,\dots,d_k\}$ is a cyclic (v,k,λ)-difference set, then the translates $D, D+1,\dots,D+(v-1)$ are the blocks of a symmetric (v,k,λ)-BIBD, that is, $(G,\mathrm{Dev}(D))$ is a symmetric (v,k,λ)-BIBD.

Proof
By definition, the translates all contain k points. Next it needs to be shown that if $a,b \in G$ then a and b occur together in exactly λ translates. It can be seen that $a = d_i + (a-d_i)$ for all i, and therefore by definition 4.2, the point a occurs in translates $D+(a-d_i)$. Analogously, the point b occurs in the translates $D+(b-d_j)$. Therefore a,b occur together in a translate only when $a-d_i = b-d_j$ for some $i,j \in [1,k]$. However,

$$a-d_i = b-d_j \Leftrightarrow a-b = d_i - d_j$$

and by definition 4.1 there are exactly λ pairs i,j for which this holds. Therefore a,b occur together in exactly λ translates.

Finally, the group G consist of v elements, and there are a total of v translates, so if these translates are the blocks, then $v = b$ and $(G,\mathrm{Dev}(D))$ is symmetric. ∎

Remark 4. Since every (v,k,λ)-difference set results in a symmetric (v,k,λ)-BIBD, these parameters must satisfy the BRC-theorem. ◃

Thus, using proposition 4.3 it is known that the designs obtained from difference set in groups of order v are always symmetric. However, it is also possible to transfer this method to constructing non-symmetric designs.

Definition 4.4: Difference system

Let $D_1,\dots,D_t$ be sets of size k in an additive abelian group G of order v such that the differences arising from the D_i results in each non-zero element of G exactly λ times. Then $D_1,\dots,D_t$ are said to form a (v,k,λ)-difference system in G.

Example 4.2
The sets $\{1,3,4\}$ and $\{1,2,5\}$ form a (9,3,1)-difference system in $\mathbb{Z}/9\mathbb{Z}$, since

2-1 (mod 9)=1, 3-1 (mod 9)=2, 4-1 (mod 9)=3, 5-1 (mod 9)=4,
1-2 (mod 9)=5, 1-3 (mod 9)=6, 1-4 (mod 9)=7, 1-5 (mod 9)=8

Equivalent to how proposition 4.3 showed that the difference sets construct a symmetric BIBD, the following proposition shows that the difference systems generate a BIBD, however, not symmetric.

Proposition 4.5:

Let $D_1,\dots,D_t$ form a (v,k,λ)-difference system in the additive abelian group $G = \{g_0,\dots,g_{v-1}\}$. Then the sets $D_i + g_j$ for $1 \le i \le t$ and $0 \le j \le v-1$, are the blocks of a (v, vt, kt, k, λ)-BIBD

Proof
The points in the design are the elements of G, and this is a group of order v. By definition 4.1, the sets $D_i + g$ are all of order k and since there are t of the D_i and v of the g_i to choose from, there are a total of $t \cdot v$ blocks. Furthermore, by proposition 2.4 it follows that:

$$tv = \frac{vr}{k} \Leftrightarrow r = tk$$

Thus, all that remain to be shown is balance. Let $D_i = \{d_{i1},\dots,d_{ik}\}$ for $1 \le i \le t$ such that $D_i + g_j = \{d_{i1} + g_j,\dots,d_{ik} + g_j\}$ and let $a,b \in G$. Rewrite a such that $a = d_{ih} + (a - d_{ih}$ which implies that a occurs in the translate $D_i + (a - d_{ih})$, $1 \le h \le k$, and equivalently b occurs in the translate $D_i+(b-d_{il})$, $1 \le l \le k$. Therefore the elements a and b occur together in a translate D_i+g_j exactly when $g_j = a - d_{ih} = b - d_{il}$ for $h,l \in [1,k]$. However,

$$a - d_{ih} = b - d_{il} \Leftrightarrow a - b = d_{ih} - d_{il}$$

that is, $d_{ih} - d_{il} \in D_i$, and by definition 4.4 there are exactly λ choices of i,h,l for which this holds. Therefore it follows that the elements $a,b \in G$ occur together in exactly λ translates. ■

Note that if $t = 1$, that is, if the difference system is instead a difference set, the design is symmetric. Hence this proposition gives a very useful tool to construct designs. Furthermore, if the difference sets are allowed to be of different orders, while still maintaining the balance property, this method can construct PBDs as well, however, this will not be expanded further in this thesis.

4.2 Round Robin Tournaments

In this section, based on [2, pp. 13 & 47], unless otherwise specified, round robin tournaments will be described. In a round robin tournament every team play each other once. If the tournament consist of $2n$ teams, there are a total of $n(2n-1)$ matches that take place during $2n-1$ rounds, where each team plays one match each round. This describes a resolvable $(2n,2,1)$-BIBD which exists for each positive integer n due to the following theorem.

Theorem 4.6:

For each positive integer n, there exists a resolvable $(2n,2,1)$-BIBD. That is, there exists a fixture list for a league of $2n$ teams with games scheduled for $2n-1$ days.

Proof
Label the teams $\infty, 1, 2, \ldots, 2n-1$. On match-day i let the fixtures be given by

$$\infty \text{ vs } i, \quad i-1 \text{ vs } i+1, \quad i-2 \text{ vs } i+2, \quad \ldots, \quad i-(n-1) \text{ vs } i+(n-1)$$

where each of these integers are reduced modulo $2n-1$ to be within the interval $[1,2n-1]$. Then team h will play against team k on day i when $h \equiv i-j \pmod{2n-1}$ and $k \equiv i+j \pmod{2n-1}$ for some $j \in [1,n-1]$. However, since the congruence relation is compatible with both addition and subtraction [10, p.242], it follows that:

$$\begin{aligned} h+k &\equiv i-j+i+j = 2i \pmod{2n-1}, \text{ and} \\ k-h &\equiv i+j-(i-j) = 2j \pmod{2n-1} \end{aligned} \tag{4.1}$$

Thus, $h \equiv i-j$ and $k \equiv i+j$ if and only if equation (4.1) is satisfied, but because $2n-1$ is odd, the solutions of i and j in equation (4.1) are unique. Therefore, a design consisting of $2n$ teams has been constructed in such a way that two teams meet each other exactly once, that is, a resolvable $(2n,2,1)$-BIBD. ■

This construction of match-day i results in a simple way to construct the entire fixture list. The first round is given by the matches:

$$\infty \text{ vs } 1, \quad 2 \text{ vs } 2n-1, \quad 3 \text{ vs } 2n-2, \quad \ldots, \quad n \text{ vs } n+1$$

The following matches are then found simply by adding 1 to each number except ∞ and then reducing modulo $2n-1$ and continuing this process until a schedule of $2n-1$ matches are found. This works because the pairs $\{2,2n-1\}, \{3,2n-2\}, \ldots, \{n,n+1\}$ form a difference system in $\mathbb{Z}/(2n-1)\mathbb{Z}$ with differences, organized by the odd and even numbers respectively as follows

- $(2n-1)-2 = 2n-3$, $(2n-2)-3 = 2n-5, \ldots, n+1-n = 1$
- $n-(n+1) = -1 \equiv 2n-2 \pmod{2n-1}$,$\ldots$, $3-(2n-2) = 5-2n \equiv 4 \pmod{2n-1}$ and $2-(2n-1) = 3-2n \equiv 2 \pmod{2n-1}$

where it has been used that:

$$\text{If } a \equiv b \pmod{2n-1} \text{ and } c \equiv d \pmod{2n-1} \text{ then } a-c \equiv b-d \pmod{2n-1}$$

Therefore the differences gives every non-zero element of $\mathbb{Z}/(2n-1)\mathbb{Z}$ exactly once.

Example 4.3 (Tournament for $n = 5$)
Using the construction described in this section to construct a schedule for 10 teams the following is obtained:

1)	∞ vs 1,	9 vs 2,	8 vs 3,	7 vs 4,	6 vs 5
2)	∞ vs 2,	1 vs 3,	9 vs 4,	8 vs 5,	7 vs 6
3)	∞ vs 3,	2 vs 4,	1 vs 5,	9 vs 6,	8 vs 7
4)	∞ vs 4,	3 vs 5,	2 vs 6,	1 vs 7,	9 vs 8
5)	∞ vs 5,	4 vs 6,	3 vs 7,	2 vs 8,	1 vs 9
6)	∞ vs 6,	5 vs 7,	4 vs 8,	3 vs 9,	2 vs 1
7)	∞ vs 7,	6 vs 8,	5 vs 9,	4 vs 1,	3 vs 2
8)	∞ vs 8,	7 vs 9,	6 vs 1,	5 vs 2,	4 vs 3
9)	∞ vs 9,	8 vs 1,	7 vs 2,	6 vs 3,	5 vs 4

Note that arranging games into rounds using the circle method as described in this section is used equivalently in graph theory by creating a one-factorization of the complete graph of $2n$ vertices. This is the method used by D. de Werra in [11] as well as [12] to study the home-away pattern of a schedule of $2n$ teams for instance, which is the theme of the next section. However, this thesis is restricted to the circle method since it revolves around design theory, so the graph theoretic approach will not be discussed any further in this thesis.

4.3 Venues

The schedules described in the previous section are useful when they are played on neutral ground, for example a group stage of a world cup taking place in a single country. In this section, based on [2, pp. 161-163], unless otherwise specified, the circle method will be applied. It will focus on how to use it such that a league is scheduled in a way where each team plays alternately at home and away as much as possible. The notation is defined such that a match a vs b means that team a plays at home (H) while team b plays away (A).

With this information, the schedule of example 4.3 can be organized in a table containing a *venue sequence* for each team as follows. The table is called a *home-away-pattern* (HAP), inspired by [11].

	R1	R2	R3	R4	R5	R6	R7	R8	R9
∞	H	H	H	H	H	H	H	H	H
1	A	H	H	H	H	A	A	A	A
2	A	A	H	H	H	H	A	A	A
3	A	A	A	H	H	H	H	A	A
4	A	A	A	A	H	H	H	H	A
5	A	A	A	A	A	H	H	H	H
6	H	A	A	A	A	A	H	H	H
7	H	H	A	A	A	A	A	H	H
8	H	H	H	A	A	A	A	A	H
9	H	H	H	H	A	A	A	A	A

Figure 4.1: A HAP for the schedule of example 4.3, where breaks in the schedule are indicated by the red color, and Ri means match-day i.

A repetition of H or A in consecutive positions is called a *break* and as indicated by the red letters of the table above, there are a lot of them by using the circle method, exemplified in the most obvious manner by team ∞ who gets to play all matches at home. Therefore, the circle method needs to be altered slightly so that this is considered, while still maintaining the match-ups given by the circle method. It is constructed such that on match-day 1 the matches are given by the following, organized depending on whether n is even or odd, respectively:

$$\begin{aligned} &\infty \text{ vs } 1,\ \ 2n-1 \text{ vs } 2,\ \ 3 \text{ vs } 2n-2,\ \ 2n-3 \text{ vs } 4, \dots,\ \ n+1 \text{ vs } n, \quad \text{if } n \text{ is even, and} \\ &\infty \text{ vs } 1,\ \ 2n-1 \text{ vs } 2,\ \ 3 \text{ vs } 2n-2,\ \ 2n-3 \text{ vs } 4, \dots,\ \ n \text{ vs } n+1, \quad \text{if } n \text{ is odd} \end{aligned} \tag{4.2}$$

The following match-days are then found by switching ∞ between home and away, while the remaining matches are found by adding 1 and reducing modulo $2n-1$, just as it was the case in the circle method.

Example 4.4
Apply the altered version of the circle method on a league with 10 teams, then the following is obtained:

1)	∞ vs 1,	9 vs 2,	3 vs 8,	7 vs 4,	5 vs 6
2)	2 vs ∞,	1 vs 3,	4 vs 9,	8 vs 5,	6 vs 7
3)	∞ vs 3,	2 vs 4,	5 vs 1,	9 vs 6,	7 vs 8
4)	4 vs ∞,	3 vs 5,	6 vs 2,	1 vs 7,	8 vs 9
5)	∞ vs 5,	4 vs 6,	7 vs 3,	2 vs 8,	9 vs 1
6)	6 vs ∞,	5 vs 7,	8 vs 4,	3 vs 9,	1 vs 2
7)	∞ vs 7,	6 vs 8,	9 vs 5,	4 vs 1,	2 vs 3
8)	8 vs ∞,	7 vs 9,	1 vs 6,	5 vs 2,	3 vs 4
9)	∞ vs 9,	8 vs 1,	2 vs 7,	6 vs 3,	4 vs 5

This fixture list then results in the following HAP, where the red color indicates a break.

	R1	R2	R3	R4	R5	R6	R7	R8	R9
∞	H	A	H	A	H	A	H	A	H
1	A	H	A	H	A	H	A	H	A
2	A	H	H	A	H	A	H	A	H
3	H	A	A	H	A	H	A	H	A
4	A	H	A	H	H	A	H	A	H
5	H	A	H	A	A	H	A	H	A
6	A	H	A	H	A	H	H	A	H
7	H	A	H	A	H	A	A	H	A
8	A	H	A	H	A	H	A	H	H
9	H	A	H	A	H	A	H	A	A

Figure 4.2: A HAP for the schedule above, where breaks in the schedule are indicated by the red color, and Ri means match-day i.

Notice that in this example, ∞ and 1 have no breaks, while every other team has one. This is in fact the best possible scenario and holds in general since there are exactly two teams that can have a perfectly alternating HAP, and since every team has to meet once, no team can have the same HAP. Therefore, the minimum number of breaks in a schedule of $2n$ teams, that meet each other once, is given by $2n-2$. Furthermore, notice that teams ∞ and 1 have *complementary* venue sequences, that is, one team plays at home when the other is away. The same goes for teams 2 and 3, 4 and 5, 6 and 7, 8 and 9. This is a useful property when there are two teams sharing the same venue for instance, thereby avoiding any conflicts regarding who gets to use the venue.

Additionally, if a schedule for an odd number of teams is required, simply remove all the games involving team ∞. Then all teams will still meet each other once, however, on match-day i, team i will not play, but it can be seen from example 4.4 that doing this will result in a perfectly alternating HAP. It is also possible to instead minimize the number of days with breaks, instead of the number of breaks, as described by D. de Werra in [11]. However, returning to the case of $2n$ teams, suppose now that each team are to meet twice during a season. The simplest way to construct this schedule is to make the second half of the tournament the exact same match-ups but with venues interchanged. However, some slight alterations have to be made such that no teams have consecutive breaks.

Theorem 4.7:

A league schedule can be constructed for $2n$ teams, with $4n-2$ rounds and where each team meets twice, such that:

1) the second half of the schedule is exactly the same as the first half but with venues interchanged,
2) there are $6n-6$ breaks in total, and
3) no team has consecutive breaks

Proof
Let the first half of the schedule be constructed using the altered circle method and let the second half of of the season be exactly the same as the first half but with the venues interchanged. Assume that team i has x_i breaks in the first half. Then if x_i is odd, team i will have a break between halves, since there are an odd number of match-days in each half.

Next, assume that there are a total of s teams for which x_i is odd, then there are $2n-s$ teams for which x_i is even, and thus the total number of breaks is given by $2\sum x_i + s$. However, as previously described, using the altered circle method results in exactly two perfectly alternating venue sequences, while the remaining $2n-2$ teams will have x_i odd, so $s = 2n-2$. But the two perfectly alternating teams will naturally have $x_i = 0$, and therefore $\sum x_i = 2n-2$, and hence the total number of breaks using the circle method is given by:

$$2\sum x_i + s = 2(2n-2) + (2n-2) = 6n-6$$

Finally, it needs to be shown that no team has any consecutive venue breaks. There are no consecutive breaks in the first half using the altered circle method, and the only teams who might have a consecutive break are teams $2n-2$ and $2n-1$ since they have a break on match-day $2n-1$ according to the altered circle method. Furthermore, according to the altered circle method from equation (4.2), team ∞ plays at home vs odd teams and away vs even teams. Moreover, using equations (4.2) once more, it follows that the second fixture is not altered, and hence the following fixtures for teams $2n-2$ and $2n-1$ are obtained using the altered circle method when fixtures of the second half are reversed:

- Day $2n-2$: $2n-2$ vs ∞, $2n-3$ vs $2n-1$, ...
- Day $2n-1$: ∞ vs $2n-1$, $2n-2$ vs 1, ...
- Day $2n$: 1 vs ∞, 2 vs $2n-1$, $2n-2$ vs 3

This gives team $2n-2$ three home games and team $2n-1$ three away games in a row. Thus, the circle method needs to be slightly altered once more. The issue can be fixed quite simply by interchanging the venues of the fixtures $2n-2$ vs ∞ and ∞ vs $2n-1$. As it follows from the fixtures above, this gives no consecutive venue breaks for any teams. However, it needs to be shown that making this alteration does not change the minimum of $6n-6$ breaks proven above.

As seen from the fixtures above, the break for team $2n-1$ is removed and it receives a perfectly alternating venue sequence. Furthermore, team ∞ will now have a break between halves using this construction instead of team $2n-1$. Meanwhile team $2n-2$ no longer has a break on day $2n-1$, but instead on day $2n-2$ since they have an away game on day $2n-3$ as well. Finally, the break between halves for team $2n-2$ remains unchanged. All of these fixtures for teams ∞, $2n-1$ and $2n-2$ are summarized below, and hence the minimum of $6n-6$ breaks remain using this modification of the altered circle method.

- Day $2n-3$: $2n-3$ vs ∞, $2n-4$ vs $2n-2$, $2n-1$ vs $2n-5$
- Day $2n-2$: ∞ vs $2n-2$, $2n-3$ vs $2n-1$, ...
- Day $2n-1$: $2n-1$ vs ∞, $2n-2$ vs 1, ...
- Day $2n$: 1 vs ∞, 2 vs $2n-1$, $2n-2$ vs 3

■

4.4 Carry-over Effects

In the previous section, a tournament consisting of $2n$ teams was constructed such that there was a minimum number of breaks in the venue sequence. However, this method has one disadvantage which will be presented in this section, based on [2, pp. 164-165], unless otherwise specified.

Consider example 4.4 from the previous section, where a schedule has been constructed using the altered circle method. Here, the teams 2,3,4,5,6,7, and 9 all meet team 1 immediately after team 8. Thus, if team 8 were a weak team, then team 1 could potentially complain that their opponent could be high with confidence, boosted by playing the weaker team 8. On the other hand, if team 6 is a strong physical team, then team 1 would be at an advantage since their opponents could potentially be suffering physically from their previous encounter. These examples apply to teams 1,3,4,5,6, and 8 as well since they all play team 9 immediately after team 7. Therefore teams 3,4,5, and 6 all have these advantages/disadvantages from previous matches, and thus these teams have justifiable complaints if it is a disadvantage, while the remaining teams have so if it is an advantage. The opponents of each team in order are summarized below, where the aforementioned consecutive opponents are marked by red.

	R1	R2	R3	R4	R5	R6	R7	R8	R9
∞	1	2	3	4	5	6	7	8	9
1	∞	3	5	7	9	2	4	6	8
2	9	∞	4	6	8	1	3	5	7
3	8	1	∞	5	7	9	2	4	6
4	7	9	2	∞	6	8	1	3	5
5	6	8	1	3	∞	7	9	2	4
6	5	7	9	2	4	∞	8	1	3
7	4	6	8	1	3	5	∞	9	2
8	3	5	7	9	2	4	6	∞	2
9	2	4	6	8	1	3	5	7	∞

Hence it is optimal to find a way to balance out these so-called carry-over effects. A construction of such a schedule is possible for $2n$ teams if n is a power of 2, but before this can be proven the following definition is required.

Definition 4.8: Primitive Element [2, p.32]

A non-zero element $\theta \in \mathbb{F}_q$ is called a primitive element if $\theta, \theta^2, \theta^3, \ldots, \theta^{q-1} = 1$ are exactly all the non-zero elements of $\mathbb{F}_q$.

Example 4.5
Consider $\mathbb{F}_{2^3}$ with polynomial $f(x) = x^3 + x + 1$. Then $\mathbb{F}_{2^3}$ consists of 7 elements:

$$x^2 + x + 1, x^2 + x, x^2 + 1, x^2, x + 1, x, 1$$

The element x is a primitive element in $\mathbb{F}_{2^3}$ since:

- $x^1 = x$
- $x^2 = x^2$
- $x^3 = x + 1$
- $x^4 = x \cdot x^3 = x(x+1) = x^2 + x$
- $x^5 = x^2 \cdot x^3 = x^2(x+1) = x^3 + x^2 = x^2 + x + 1$
- $x^6 = x^3 \cdot x^3 = (x+1)(x+1) = x^2 + 1$
- $x^7 = x \cdot x^6 = x \cdot (x^2+1) = x^3 + x = x + 1 + x = 1$

With this, it is possible to present a balanced construction of a schedule for $2n$ teams, however, it only works when n is a power of 2.

Theorem 4.9:

If n is a power of 2, a cyclic league schedule for $2n$ teams can be constructed in a way where the following holds:

$$\text{There are no choices of } x, y, z \text{ and } w \text{ such that teams } x \text{ and } y \text{ both play team } z \text{ immediately after playing team } w \tag{4.3}$$

Proof
Let $2n = 2^r = q$ and let θ be a primitive element of $\mathbb{F}_q$. For $1 \le j \le q-1$, define match-day j to consist of the matches

$$\theta^i \text{ vs } (\theta^i + \theta^j) \tag{4.4}$$

Suppose that condition (4.3) does not hold. Then for j and J:

$$x = w + \theta^{j-1}, \quad x = z + \theta^j, \quad y = w + \theta^{J-1}, \quad y = z + \theta^J \tag{4.5}$$

Since the calculations are made in $\mathbb{F}_{2^r}$, the coefficients are calculated modulo 2, and therefore -1=1. Hence it follows that:

$$\theta^i + \theta^j = \theta^k \Leftrightarrow \theta^i = \theta^k - \theta^j = \theta^k + \theta^j$$

Using this fact in construction (4.4) with $\theta^i = x$ and $\theta^k = z$ it follows that $x + \theta^j = z \Leftrightarrow x = z + \theta^j$, and therefore by writing $x = z + \theta^j$ it means that team x plays team z on match-day θ^j, and similarly for the remaining elements of equation (4.5). However, if the conditions of equation (4.5) holds, then;

$$y - x = w + \theta^{J-1} - (w + \theta^{j-1}) = z + \theta^J - (z + \theta^j) \Leftrightarrow \theta^{J-1} - \theta^{j-1} = \theta^J - \theta^j$$

Therefore:

$$\theta^{J-1} - \theta^{j-1} = \theta^J - \theta^j \Leftrightarrow \theta^J - \theta^{J-1} = \theta^j - \theta^{j-1} \Leftrightarrow \theta^{J-1}(\theta - 1) = \theta^{j-1}(\theta - 1) \Leftrightarrow \theta^{J-1} = \theta^{j-1}$$

However, then it follows from construction (4.5) that $x = y$, and therefore condition (4.3) is satisfied.

Finally, if θ^k plays θ^l on round j, then $\theta^k = \theta^l + \theta^j$ by construction (4.5) and so $\theta^{k+1} = \theta^{l+1} + \theta^{j+1}$, that is, team θ^{k+1} plays θ^{l+1} on round $j+1$. Therefore, by replacing θ^i and 0 by i and ∞, respectively, a cyclic schedule is obtained. ■

Example 4.6 (Balanced schedule for n=4)
Take the primitive element of $\mathbb{F}_8$ that satisfies $\theta^3 = \theta + 1$. Then by using example 4.5 the following elements are given by:

$$\theta^4 = \theta^2 + \theta, \quad \theta^5 = \theta^2 + \theta + 1, \quad \theta^6 = \theta^2 + 1, \quad \theta^7 = 1$$

The first round of matches are constructed by (4.4) to be on the form θ^i vs $(\theta^i + \theta)$, that is

$$0 \text{ vs } \theta, \quad \theta^2 \text{ vs } \theta^4, \quad \theta^3 \text{ vs } \theta^7, \quad \theta^5 \text{ vs } \theta^6$$

This results from using example 4.5 once more to see that $\theta^2 + \theta = \theta^4$, $\theta^3 + \theta = \theta + 1 + \theta = 1 = \theta^7$, and $\theta^5 + \theta = \theta^2 + \theta + 1 + \theta = \theta^2 + 1 = \theta^6$. Therefore, the following cyclic schedule is obtained:

1) ∞ vs 1, 2 vs 4, 3 vs 7, 5 vs 6

2) ∞ vs 2, 3 vs 5, 4 vs 1, 6 vs 7

3) ∞ vs 3, 4 vs 6, 5 vs 2, 7 vs 1

4) ∞ vs 4, 5 vs 7, 6 vs 3, 1 vs 2

5) ∞ vs 5, 6 vs 1, 7 vs 4, 2 vs 3

6) ∞ vs 6, 7 vs 2, 1 vs 5, 3 vs 4

7) ∞ vs 7, 1 vs 3, 2 vs 6, 4 vs 5

Listing each team's opponents in order clarifies that condition (4.3) is satisfied:

	R1	R2	R3	R4	R5	R6	R7
∞	1	2	3	4	5	6	7
1	∞	4	7	2	6	5	3
2	4	∞	5	1	3	7	6
3	7	5	∞	6	2	4	1
4	2	1	6	∞	7	3	5
5	6	3	2	7	∞	1	4
6	5	7	4	3	1	∞	2
7	3	6	1	5	4	2	∞

Therefore, a balanced schedule for 8 teams has been found since there are no repeated sequences of numbers. This construction does not take into account whether a team plays at home or away, unlike how it was the case in section 4.3. However, it happens to be possible to construct a HAP with 8 breaks, using only the match-ups from this balanced construction as follows:

1) ∞ vs 1, 4 vs 2, 7 vs 3, 5 vs 6

2) 2 vs ∞, 3 vs 5, 1 vs 4, 6 vs 7

3) ∞ vs 3, 4 vs 6, 5 vs 2, 7 vs 1

4) 4 vs ∞, 5 vs 7, 3 vs 6, 1 vs 2

5) ∞ vs 5, 6 vs 1, 7 vs 4, 2 vs 3

6) 6 vs ∞, 7 vs 2, 5 vs 1, 3 vs 4

7) ∞ vs 7, 1 vs 3, 2 vs 6, 4 vs 5

This results in the following HAP, where the breaks are indicated with red.

	R1	R2	R3	R4	R5	R6	R7
∞	H	A	H	A	H	A	H
1	A	H	A	H	A	A	H
2	A	H	A	A	H	A	H
3	A	H	A	H	A	H	A
4	H	A	H	H	A	A	H
5	H	A	H	H	A	H	A
6	A	H	A	A	H	H	A
7	H	A	H	A	H	H	A

Note that this is not a general construction of a HAP, but only works for this specific example. For example, a schedule of 4 teams would result in the minimum of 2 breaks. However, it does have some advantages to discuss. With a total of 8 breaks, this is not the minimum of 6 breaks presented in section 4.3, which in this instance goes against teams 4 and 6, who could then complain that they have two breaks while teams ∞ and 3 have none. However, unlike the altered circle method there are no carry-over effects.

Additionally, according to [11], a schedule of $2n$ teams has a minimum of $\lceil \log_2(n) \rceil$ days with breaks, and with at most $2\lfloor \frac{1}{2}n \rfloor \lceil \log_2(n) \rceil$ breaks. Since $n = 4$ in the example above, this schedule is constructed exactly like this. Furthermore, in this schedule teams ∞ and 3 have complementary venue sequences and so does teams 4 and 6, 1 and 7, and 2 and 5, and therefore maintains the advantages from this as described in section 4.3. Finally, should a second half be necessary, it will not produce the minimum of 18 breaks, described by theorem 4.7, but instead a total of 20 breaks, where teams 1,2,5, and 7 have breaks between halves, however, the advantage of no carry-ovver effects remain.

Chapter 5

Conclusion

In this thesis various results on combinatorial design theory has been presented, most importantly the Bryck-Ryser-Chowla-theorem, which excludes the existence of designs with parameters that does not satisfy certain properties. Furthermore, the theorem was related to projective planes of order n, giving an easier to apply corollary of the Bryck-Ryser-Chowla theorem.

Resolvable designs were presented in order to describe a resolvable $(2n, 2, 1)$ balanced incomplete block design with the $i'th$ parallel class consisting of the pairs of teams playing each other on match-day i. A construction was presented where the first round is given by the matches

$$\infty \text{ vs } 1, \;\; 2 \text{ vs } 2n-1, \;\; 3 \text{ vs } 2n-2, \;\; \dots, \;\; n \text{ vs } n+1$$

and the following matches are found by adding 1 and reducing modulo $2n-1$ to each team until a schedule of $2n-1$ match-days are found. This works because the pairs $\{2, 2n-1\}$, $\{3, 2n-2\}$, $\dots$, $\{n, n+1\}$ form a difference system in $\mathbb{Z}/(2n-1)\mathbb{Z}$. This schedule is then altered to consider whether a team plays at home or away such that a minimum of $2n-2$ venue breaks are in the schedule. This schedule is then expanded such that all teams meet twice, where the second half is the same as the first, but with venues interchanged and with a total of $6n-6$ breaks, none of which are consecutive.

This schedule, however, has one flaw called the carry-over effect, since the construction has repeated sequences of numbers, which means that certain teams meet the same teams in the same order. A new construction is therefore introduced for a schedule of $2n$ teams where n is a power of 2 and where match-day j consists of the matches

$$\theta^i \text{ vs } (\theta^i + \theta^j)$$

for $1 \leq j \leq q-1 = 2n-1$ and where θ is the primitive element of $\mathbb{F}_{2n}$. However, although this is a balanced schedule in terms of the carry-over effect, it no longer has the minimum of $2n-2$ breaks if the venue is considered.

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