A review of options pricing models
From fixed to stochastic volatility

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Abstract

This paper will introduce the basics of options pricing. Starting with the Binomial Tree model and moving on to the Black-Scholes model and the Black-Scholes Greek letters. With a solid theoretical foundation in place, the paper moves on to presenting more advanced options pricing models. The reason being that the Black-Scholes model is pricing options on a fixed volatility. The fixed volatility assumption used by the Black-Scholes model serves well when learning about options. It has, however, been proven empirically that the fixed volatility assumptions is not in line with market behavior. Hence, using the Black-Scholes model for hedging purposes will lead to wrongful hedges.

The fact that the Black-Scholes model is too simple has lead to numerous more sophisticated options pricing models. This paper will review two of these models.

First, the Local Volatility model. After breaking down the model I find that for hedging purposes the model is ill suited. Empirically the model is proven to perform bad when replicating the market behavior. The model implied volatility simply moves in the opposite direction of what is observed in the market. This makes it a bad model for hedging.

With this observation I move on to the SABR model. Contrary to the other models examined in this paper the SABR model applies a stochastic process as part of the volatility measure in the model. This leads to a model that makes good predictions of the market behavior. Empirically the SABR model performs well and the model is widely used by practitioners.

The low interest rate environment has however created a potential problem for the SABR model as it implicitly assumes the interest rates to be strictly positive when $\beta \neq 0$. The paper introduces some adjusted SABR models for the reader without going into detail with the models. However, it is important to know of the existence of these models.

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1 Introduction

In this section I will introduce the paper - starting with the motivation for the subject, moving on to a minor introduction of the theories to come and some of the papers used, to what to expect in the paper.

Most private investors invest their money in stocks, bonds and indexes. This is probably because it is what the media is covering and what is easy accessible when logging in to ones depository at the bank. To some extent it might also be because the complexity of these products are not too high. That way you do not have to be a specialist in order to invest your money in this type of assets.

Investing money is of course embedded with risk. Hence the reward of investing. Most investors are interested in maximizing their return on investment given the amount of risk they add to their portfolio. In theoretical finance the risk/reward is measured in different ways. Common to most investors is that they would like to secure their reward given the risk they take. One way to insure the reward on a portfolio from risk is by buying options. That is, adding options to a portfolio allows the owner of the portfolio to limit the risk on the investment.

Options are a type of financial instrument that can be bought and sold at exchanges the same way a stock or another asset is being traded. The formal definition of an option is

Definition. Owning an option on an underlying asset gives the right, but not the obligation, to buy or sell that asset at a certain price.

To most private investors options are unknown. This might partly be because it is a complex area of investing and partly because it is expensive to trade options. In professional finance, however, options are widely used. One of the reasons are that large banks operate with a division called Markets. This division specializes in market making. That is, they make the market between buyers and sellers. In order to do this, the Market division buys and sells assets. But sometimes they are not able to sell an asset straight away. Or they simply choose to keep that asset because they think they can make more money that way. In order to reduce the risk on their positions, they hedge their positions using options. This is typically done at the end of the day. It is common that each trader hedge their own positions. That way, they know their risk when they leave at night. But the bank or financial institution typically also hedge the overall position of the bank. To do this, the bank typically have a team of people who specializes in options trading. This team is highly specialized within options and handle all customers who need help buying or selling options. The team also handle the banks overall hedging needs.

1.1 The purpose of the paper

In this paper I will investigate the world of options. Options are a large part of professional finance, as also described above. In order to get a better understanding of some of the mechanisms in the financial world, I think it will be beneficial to have an understanding of options. Hence, this is my motivation for writing this paper.

Below I will present some research questions that I will try to answer in this paper. They serve as a guideline for the paper.

Problem. The Black-Scholes model is a well-known options pricing model. Is there a model that handles options pricing for hedging purposes better than the Black-Scholes model?

Problem. What are the arguments for choosing one model over another?

Problem. What are the implications of choosing a wrong or less precise model?

Theoretical finance can be very mathematical. Especially if you choose to explore the more complicated assets classes or areas like for instance options pricing.

I will not pursue the mathematical solution to the models - or try to prove the models and their assumptions mathematically. Instead, I will focus on the general understanding of options and the different options pricing models presented in the paper. Hence, I will accept and apply the mathematically proposed models and their solutions and instead focus on the outcome of the models. Namely the correctness of the pricing from each model and a discussion about the results from an intuitive standpoint. Moreover, I will not go into any trading strategies or anything of the like. I will solely focus on the pricing of options from a hedging perspective.

I will not make any applications of the models in my review. Instead, I will focus on the intuition of the models using figures. The reason being that making an application of each of the models is beyond the scope of this paper.

1.2 Introduction of theory

The Black Scholes model is probably the best known model for options pricing. The model was first introduced by Fischer Black and Myron Scholes in a paper published in 1973 [3]. The same year Robert Merton published an expansion of the model [28], making it capable of pricing options on underlying securities paying out dividends. This model is know as the Black-Scholes-Merton model and will be introduced in this paper.

Today it is clear that the Black-Scholes model has some shortcomings. These are especially relevant when options are used for hedging.

The shortcomings of the Black-Scholes model are the reason for other options pricing models. In this paper I will investigate some of the models in the search for a better or more correct way of pricing options. I will do this by looking at two other models introduced below. The Black-Scholes model is the first model introduced in this paper and it will serve as a sort of baseline model.

The second model the paper is looking at is the Local Volatility Model. This is a model simultaneously developed by Dupire [13] and by Derman and Karni [10]. I will use the approach of Derman and Kani.
The Local Volatility Model is trying to account for the none constant volatility that the Black-Scholes model is failing to address. The way the Local Volatilty Model handles this is by incorporating the volatility smile - the volatility smile will be introduced later in the paper. It turns out that the result of incorporating the volatility smile in the Local Volatility Model might not give the intuitively expected result.

The third model introduced is the SABR model. The model was first proposed by Hagan, Kumar, Lesniewski, and Woodward in 2002 [18]. This model is also trying to handle the non-constant volatility. The way the SABR model solves this is by allowing the volatility to be stochastic.

Applying the SABR model involves calibration of the volatility smile. The calibration is a continuous process as the volatility smile changes over time. The output from the SABR model is applied to the Black-Scholes model to generate new Black-Scholes Greeks for better hedging.

The paper is structured in the following way: Chapter 2 is introducing the basics of options. Starting with the binomial tree structure, moving on to the Black-Scholes-Merton model and on to Itô's formula and ending with the Black-Scholes Greek letters. This should give the reader a solid theoretical understanding of what options are before moving on to more complex models of options pricing.

Chapter 3 is looking at the Local Volatility model. For starters, this chapter introduces the concepts of implied volatility and the volatility smile. Then the Local Volatility model is introduced before it is applied and the model dynamics are discussed.

Chapter 4 is about the Stochastic Alpha Beta Rho model. More commonly known as the SABR model. Before introducing the actual SABR model the Black-Scholes Greek letters are expanded with two additional Greek letters. Then the SABR model is introduced and calibration of the model is explained before the model is discussed.

In chapter 5 the options pricing models are evaluated and compared to each other in order to answer the research questions presented earlier. Chapter 6 is concluding.

2 Introduction to options

In this section I will present some theory needed to understand the basics of options. It is necessary to know the basics of options and options pricing to understand and work with volatility models as this is a particular element in options.

An option is a type of financial derivative. That is, the price of an option is depending on some underlying asset like a stock or a currency. This, in essence, mean that when the underlying asset's price moves in either direction the option price makes an equivalent move. When buying an option it is possible to buy either a put or a call option. The put options value rises as the underlying assets price is declining. A call options value rises as the underlying assets price is rising. This makes it possible for options to be regarded as a form of insurance on a financial asset or an insurance of a portfolio as you can buy options moving in the opposite direction of your assets. It is also possible to sell a call option, meaning that you have a short position in the option. Going short in a call option is like going long in a put option. However, there are some differences in doing so.

There are two types of options. European options and American options. The main difference between them is that the American option has an early exercise element embedded in it. In general, some definitions for working with options are listed below.

Definition 1. An option is a security that gives the right but not an obligation to buy or sell an asset within a specified period of time.

Definition 2. A call option is the kind of option that gives the right to buy a single share of common stock.

Definition 3. An exercise price is the price that is paid for the asset when the option is exercised.

Definition 4. A European option is a type of option that can be exercised only on a specified future date

This section will look at the theory of European options on underlying assets not paying any dividends. The reason for this is that most models are derived on these assumptions. One reason can be that some underlying assets are not paying dividends.
Dividends are often paid by stocks. But options can be traded on a great variety of underlying assets. In fact, the stock market is quite small compared to the market for foreign exchange or interest rates. However, it is possible to expand the models to include underlying assets paying dividends, if needed. Also, the models can be expanded to include American options.

This section is structured so that we start by looking at binomial trees to get the basic understanding of pricing an option. We then move on to look at processes before we dive into the Black Scholes Merton model and Itô's Lemma. All this is gathered by the Greeks before the section is summed up in the conclusion.

2.1 Binomial Tree

The pricing of options usually start with the introduction of binomial trees. Pricing an option this way is done in steps where you start at the final node and then price the option backwards. For the binomial tree model to work some assumptions must be made. The first assumption is the one of no arbitrage opportunities in the model. That is, it should not be possible to buy and sell the option at the same time and then make money of it. Further, we assume that there exist risk less portfolios and risk neutral valuations. This, I will explain further below.

The risk less portfolio is set up under the assumption that it must earn a return equal to the risk free interest rate. That is, as there is no risk to the portfolio there can be no risk premium either. Hence, the investor is requiring the risk free interest rate in return for the investment.

Setting up a portfolio consisting of a long position in some asset denoted as $S_0\Delta$, where S is the price of the asset and Δ is the quantity of the asset, and a short position in an option using the same asset as underlying, the one step binomial tree model can be used to express the risk less portfolio.

$$S_0 u\Delta - f_u = S_0 d\Delta - f_d \tag{2.1}$$

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{2.2}$$

Equation 2.1 is the risk less portfolio. No matter if the underlying assets value goes up or down, the value of the risk less portfolio must equal in order for the portfolio to be risk less. Hence, solving for Δ as in equation 2.2 and using this Δ value ensures that the portfolio is risk less. That is, buying Δ amount of the asset and selling one option ensures that you have a risk less portfolio in the given time period.

Pricing an option using a binomial tree model basically means that you assume that the price can only be one of to possible things in next period given the price in this period. The length of the time period can vary from tree to tree. But when you start to evaluate an option using binomial trees the distance in time between each node must be fixed.

Figure 2.1: Binomial Tree structure

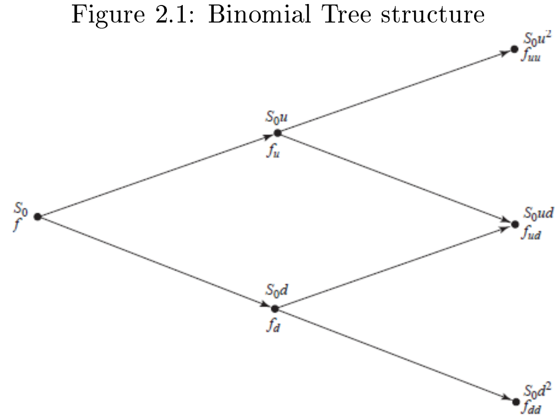

Figure from Hull [22]

Note: The binomial tree structure. Starting in f and moving forward the option price can only take one of two values in each future period. Moving one step forward the option is again facing one of two prices.

Figure 2.1 depicts the binomial tree model structure. Starting at f and moving forward the price of the option in the next period can only be f_u or f_d. The following period the price can be one of the following three - f_{uu}, f_{ud} or f_{dd} - depending on the node in wich you stand in in the middle period of figure 2.1.

When pricing an option using the binomial tree model one starts at the time of expire of the option and work back towards the start of the option. In figure 2.1 this means that one would start by pricing f_{uu}, f_{ud} and f_{dd} using the values of the underlying asset at expiry. The same way, one would price f_u and f_d and in the end one would price f. This is put more formally in the equations below.

$$f = e^{-r\Delta t}\left[pf_u + (1-p)\,f_d\right] \tag{2.3}$$

$$p = \frac{a-d}{u-d} \tag{2.4}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{2.5}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{2.6}$$

$$a = e^{r\Delta t} \tag{2.7}$$

where r is the risk-free interest rate, σ is the volatility and Δt is the change in time.

Equation 2.3 states the calculation of the option value, f. Equation 2.4 is the calculation of p using the input from equation 2.5 to equation 2.7. Equation 2.4 is a generalization of equation 2.2. Calculating the option price this way is called risk neutral pricing. Pricing options this way is based on the assumption that investors are risk neutral. That is, investors do not expect higher return with higher risk. This assumption allows for two simplifying features.The expected return of an investment is equal to the risk free rate. And the discount rate used to calculate the expected payoff from an option is the discount rate.
P is measuring the probability of a node to be reached.

In the limit the Binomial Tree model and the Black Scholes Merton Model is the same model. Later in this section I will present the Black Scholes Merton model, where I will show why the two models are the same.

2.2 Processes

The pricing of a financial asset is following a stochastic process. That is, when a variable is changing value in an uncertain way over time it is said to follow a stochastic process. This stochastic process can be classified either as a discrete or a continuous time process. The discrete time stochastic process is a process where the value of the variable can only change at a certain time. For example once a day or every time you flip a coin. A continuous time stochastic process is one where the value of the variable can change at any time. However, when talking about financial assets the actual change will only take place when the stock exchange is open.

The choice between discrete time and continuous time is important when choosing a process. The usage of processes is a way to guess or limit the sample space of a variable. This is a way of managing expectations in a given situation. Hence, for a process to make sense it is important to figure out if it is in discrete time or in continuous time. Once this is settled, you can move on to try to estimate the possible values the variable can have. For instance, if you play heads or tails it makes no sense to use a continuous time process. Flipping a coin is a discrete time maneuver and the fall out can only be one of two things - head or tail. Hence, using the correct process enables you to limit the sample space for this particular game. That way you can easier make predictions about the fall out of the game.

Given that flipping a coin is a fairly simple game the gain from using a process might seem a bit small since you can easily see the outcome. When working with more complicated stuff like estimating tomorrows price on a stock it might be beneficial using processes.

The Wiener process is a particular type of Markov Stochastic Process with a mean of zero and a variance of one. A variable z follows a Wiener process if the change in z for a small period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t} \tag{2.8}$$

where $\Delta z = z_t - z_{t-1}$ and $\Delta t = t_t - t_{t-1}$ and ϵ has a standard normal distribution $\phi(0,1)$. The values of Δz must be independent for any two different short time intervals. It follows from equation 2.8 that Δz follows a normal distribution with a mean of zero, variance of Δt and a standard deviation of $\sqrt{\Delta t}$. Evaluating the change in z on a relatively long time period T, the change in z can be written as $z(T) - z(0)$. Thinking of this as the sum of changes in z in N small intervals with a length of Δt, we get

$$N = \frac{T}{\Delta t}$$

From this it follows that

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{2.9}$$

From equation 2.9 it follows that $z(T) - z(0)$ is normally distributed with mean zero, variance T and standard deviation of $\sqrt{T}$.

It follows from the fact that Δz must be independent form any two short time intervals that z follows a Markov Process. We can evaluate z in the limit as $\Delta t \to 0$. That is, the basic Wiener Process can be expressed as dz, meaning that it has the properties of Δz as $\Delta t \to 0$. This allows for the Generalized Wiener Process.

$$dx = a\ dt + b\ dz \tag{2.10}$$

where a and b are constants.

The General Wiener Process, equation 2.10, has a drift rate of zero and a variance of one - $a\ dt$ being the drift element and $b\ dz$ being the variability or noise element of x. The drift rate of zero means that the expected value of z at any given future time equals the current value. The variance rate of one means that the variance of change in z is equal to the change in time. In other words, the expectation for next periods value is equal to this periods value. The lack of a drift means that the change in value from period to period can be explained with the change in time.

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t} \tag{2.11}$$

Equation 2.11 expresses a change in x given a small change in time t. Like in equation 2.8 ϵ has a standard normal distribution. Thus Δx has a standard normal distribution with mean $a\Delta t$, variance of $b^2\Delta t$ and standard deviation of $b\sqrt{\Delta t}$. It follows from previously that the change in value of x in any time interval T is normally distributed with a mean of aT, variance of b^2T and standard deviation of $b\sqrt{T}$.

The Itô Process is a stochastic process much like the Generalized Wiener Process. The main difference being that the Itô Process uses functions for the parameters a and b so that they are functions of the underlying variable x and of time t.

$$dx = a\left(x,t\right)\Delta t + b\left(x,t\right)\epsilon\sqrt{\Delta t} \tag{2.12}$$

Equation 2.12 expresses an Itô Process. Making the parameters a and b functions of x and t allow for change in the drift rate and in the variance rate. In a small time interval between t and $t+\Delta t$ the variable x changes to $x+\Delta x$ where Δx is expressed in equation 2.13.

$$\Delta x = a\left(x,t\right)dt + b\left(x,t\right)dz \tag{2.13}$$

To use equation 2.13 involves the approximation that the drift rate and variance rate remain constant at their value at time t during the time interval between t and $t+\Delta t$. The Itô Process, equation 2.12, is a Markov because the change in x at time t only depends on the value of x at time t.

2.3 The Black Scholes Merton Model

In subsection 2.1 I mention that the Binomial Tree model and the Black Scholes Merton model is the same, when evaluating the Binomial Tree model in the limit. Deriving the Black Scholes Merton model from the Binomial Tree model is done by letting the time steps in the Binomial Tree model approach infinity. I will derive the Black Sholes Merton below.

A tree with n time steps is used to value a European call option with strike price K and life T. Each step in the binomial tree is of length T/n. j is the number of upward movements and $n-j$ is the number of downward movements on the tree. The final stock price is then $S_0u^jd^{n-j}$ where u is upward movement and d is downward movement. S_0 is the initial stock price. The payoff from a European call option is then

$$max\left(S_0u^jd^{n-j}-K,0\right)$$

The probability of exactly j upward and $n-j$ downward movements in the binomial distribution is given by the equation

$$\frac{n!}{(n-j)!j!}p^j\left(1-p\right)^{n-j}$$

Following the above mentioned argument the expected payoff from the call option is then given as

$$\sum_{j=0}^{n}\frac{n!}{(n-j)!j!}p^j\left(1-p\right)^{n-j} max\left(S_0u^jd^{n-j},0\right)$$

The binomial tree represents movements in a risk-neutral world and can be discounted using the risk-free rate r. This gives the option price

$$c=e^{-rT}\sum_{j=0}^{n}\frac{n!}{(n-j)!j!}p^j\left(1-p\right)^{n-j} max\left(S_0u^jd^{n-j},0\right) \tag{2.14}$$

As equation 2.14 is for a call option the value of the option is only of interest when the stock price is greater than the strike price as the option price is zero otherwise. Hence, the following can be implemented.

$$S_0u^jd^{n-j}>K$$

or

$$ln\left(S_0/K\right)>-jln\left(u\right)-\left(n-j\right)ln\left(d\right)$$

as $u=e^{\sigma\sqrt{T/n}}$ and $d=e^{-\sigma\sqrt{T/n}}$ the above condition becomes

$$ln\left(S_0/K\right)>n\sigma\sqrt{T/n}-2j\sigma\sqrt{T/n}$$

or

$$j>\frac{n}{2}-\frac{ln\left(S_0/K\right)}{2\sigma\sqrt{T/n}}$$

With this, equation 2.14 is written as

$$c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \left(S_0 u^j d^{n-j} - K\right) \tag{2.15}$$

Where

$$\alpha = \frac{n}{2} - \frac{ln\,(S_0/K)}{2\sigma\sqrt{T/n}} \tag{2.16}$$

I define the following

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} u^j d^{n-j} \tag{2.17}$$

and

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \tag{2.18}$$

which gives equation 2.19 below.

$$c = e^{-rT} (S_0 U_1 - K U_2) \tag{2.19}$$

If we start by evaluating U_2. As the number of steps in the binomial distribution approaches infinity the distribution approaches a normal distribution. If the number of steps is denoted n and the probability of success is denoted p, the probability distribution of the number of successes is approximately normal with mean np and standard deviation $\sqrt{np\,(1-p)}$. In equation 2.19U_2 is the probability of the number of successes being more than α. From the properties of the normal distribution, it follows that, for large n's approaching infinity,

$$U_2 = N\left(\frac{np - \alpha}{\sqrt{np\,(1-p)}}\right) \tag{2.20}$$

where N is the cumulative normal distribution function. Substituting for α in equation 2.20, U_2 becomes

$$U_2 = N\left(\frac{ln\,(S_0/K)}{2\sigma\sqrt{T}\sqrt{p\,(1-p)}} + \frac{\sqrt{n}\left(p - \frac{1}{2}\right)}{\sqrt{p\,(1-p)}}\right) \tag{2.21}$$

From equation 2.4 to equation 2.7 in subsection 2.1 I have

$$p = \frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

When n tends to infinity $p\,(1-p)$ tends to $\frac{1}{4}$ and $\sqrt{n}\left(p-\frac{1}{2}\right)$ tends to

$$\frac{\left(r-\sigma^2/2\right)\sqrt{T}}{2\sigma}$$

Evaluating U_2 in the limit as n goes to infinity equation 2.21 becomes

$$U_2 = N\left(\frac{ln\,(S_0/K) + \left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right) \tag{2.22}$$

Moving on from U_2 to evaluating U_1. From equation 2.17 I have the below expression. I have just rearranged it for a cleaner expression.

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!}\,(pu)^j\,[(1-p)\,d]^{n-j} \tag{2.23}$$

Defining

$$p^* = \frac{pu}{pu + (1-p)\,d} \tag{2.24}$$

It follows that

$$1 - p^* = \frac{(1-p)\,d}{pu + (1-p)\,d}$$

Using this, I can rewrite equation 2.23 as

$$U_1 = [pu + (1-p)\,d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!j!}\,(p^*)^j\,(1-p^*)^{n-j}$$

The expected return in a risk neutral model is equal to the risk-free interest rate, r. From this it follows that $pu + (1-p)\,d = e^{rT/n}$ and then it follows that

$$U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!}\,(p^*)^j\,(1-p^*)^{n-j}$$

Using the same argumentation as with U_2 the above expression shows that U_1 involves a binomial distribution with the probability of an up movement is p^*

instead of p. Again, the binomial distribution goes towards a normal distribution as n goes to infinity which gives

$$U_1 = e^{rT} N \left(\frac{np^* - \alpha}{\sqrt{np^* (1 - p^*)}} \right)$$

Again, substitution for α gives

$$U_2 = e^{rT} N \left(\frac{ln\,(S_0/K)}{2\sigma\sqrt{T}\sqrt{p^*(1-p^*)}} + \frac{\sqrt{n}\left(p^* - \frac{1}{2}\right)}{\sqrt{p^*(1-p^*)}} \right)$$

And substituting for u and d gives

$$p^* = \left(\frac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}} \right) \left(\frac{e^{\sigma\sqrt{T/n}}}{e^{rT/n}} \right)$$

By the same argument as with U_2, expanding the exponential function and letting n go to infinity $p^*(1-p^*)$ goes towards $\frac{1}{4}$ and $\sqrt{n}\left(p^* - \frac{1}{2}\right)$ goes towards

$$\frac{\left(r + \sigma^2/2\right)\sqrt{T}}{2\sigma}$$

resulting in

$$U_1 = e^{rT} N \left(\frac{ln\,(S_0/K) + \left(r + \sigma^2/2\right) T}{\sigma\sqrt{T}} \right) \tag{2.25}$$

The final model is then expressed using equations 2.19, 2.22 and 2.25.

$$c = S_0 N\,(d_1) - K e^{-rT} N\,(d_2) \tag{2.26}$$

$$p = K e^{-rT} N\,(-d_2) - S_0\,(-d_1) \tag{2.27}$$

where

$$d_1 = \frac{ln\,(S_0/K) + \left(r + \sigma^2/2\right) T}{\sigma\sqrt{T}} \tag{2.28}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{ln\,(S_0/K) + \left(r - \sigma^2/2\right) T}{\sigma\sqrt{T}} \tag{2.29}$$

With this, we are able to price an option. However, the more interesting question is often how the future price of an option is evolving in relation to building a portfolio. To this, the Black Scholes differential equation is usable. We shall look more in to this equation later. For now I only state it below.

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \tag{2.30}$$

In this subsection I have derived the price of a call option . By the same argumentation one can derive the price of a put option. However, I only state the put option equation, equation 2.27, in this subsection. What should be quite clear from equation 2.26 and equation 2.27 is that the main difference between calculating the call and the put price is the difference between stock price and strike price. With a call option the option has a positive value when the strike price is below the stock price, whereas the put option has a positive value when the strike price is above the stock price. Both equation 2.26 and equation 2.27 make use of equation 2.28 and equation 2.29, but with opposite sign.

2.4 Itô's formula

In this subsection I will present some theory on Ito's lemma. However, I will not make a full derivation of Ito's formula as it is not the scope of the paper. Instead, I will introduce Ito's lemma and show the relation to the Black Scholes Merton model showed in subsection 2.3.

The price of an option is essentially a function two things. The underlying asset and time. The underlying asset can be regarded as a stochastic variable as its price is evolving in a stochastic way. Hence, an option is a function of some stochastic variable and time. In subsection 2.2 I introduced the Itô process as

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

where a and b are functions of x, t and dz, where dz is a Wiener process as introduced in subsection 2.2. The variable x in the above equation has a drift rate of a and a variance rate of b^2. Substituting x for S, as I will look at a particular situation with a spot price, S, I restate the process as

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dX(t) \tag{2.31}$$

where μ and σ are constants. Using Itô's Lemma on a function $f = f(S,t)$as a function of S and t gives

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}(S,t)\,dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(S,t)\,dS^2 \tag{2.32}$$

if the function f in equation 2.32 follows the Itô process in equation 2.31, equation 2.31 can be inserted in equation 2.32 and gives

$$df = \left(\frac{\partial f}{\partial t}(S,t) + \mu S\frac{\partial f}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}(S,t)\right)dt + \sigma S\frac{\partial f}{\partial S}(S,t)\,dX \tag{2.33}$$

Using the argument of risk neutrality I can use the risk free interest rate. That is, if a portfolio is eliminated for risk an investor will expect the portfolio to grow with the risk free rate. In this case the portfolio growth as time passes is of interest. In subsection 2.5 I will go through this more thoroughly. For now, let us just accept that delta, Δ, is a measure of change in time for some quantity of asset.

$$d\left(f+\Delta S\right)=\left(\frac{\partial f}{\partial t}\left(S,t\right)+\mu S\frac{\partial f}{\partial S}\left(S,t\right)+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}\left(S,t\right)+\Delta\mu S\right)dt+\Delta S\left(\frac{\partial f}{\partial S}+\Delta\right)dX \tag{2.34}$$

where $\Delta=-\frac{\partial f}{\partial S}\left(S,t\right)$. Using this equation 2.34 reduces to

$$d\left(f+\Delta S\right)=\left(\frac{\partial f}{\partial t}\left(S,t\right)+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}\left(S,t\right)\right)dt \tag{2.35}$$

This reduction has eliminated randomness and the expected growth rate is now the risk free interest rate. Hence, we can state that

$$\frac{\partial f}{\partial t}\left(S,t\right)+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}\left(S,t\right)=r\left(f-S\frac{\partial f}{\partial S}\right) \tag{2.36}$$

Rearranging equation 2.36 and dropping the notation (S,t) gives

$$rf=\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} \tag{2.37}$$

Equation 2.37 is equivalent with the Black Scholes differential equation introduced in subsection 2.3. We shall look more into the equation in subsection 2.5.

In this subsection I derived the relationship between the Black Scholes differential equation and Itô's Lemma. With this, we are ready to look at the Greek letters before round this section off.

2.5 The Greeks

This subsection looks into the Greeks. The Greeks are a common term used in explaining the different parts that are used when hedging your portfolio against different changes like a change in price, in volatility or in time. In total, the Greeks consist of five different measures. Delta, the measure of change in price in the underlying asset. Theta, the passage of time. Gamma, the speed of change in the price of the underlying asset. Vega, the change in value with respect to volatility. Rho, the change in value with respect to the interest rate.

Below I will introduce each of the Greek letters more in depth. The Greeks, as they are called in 'jargon', are the language used when traders are talking about hedging a portfolio. Hence, the Greeks are an important part of understanding options. I will return to the Greek letters and hedging later in this paper.

Delta,Δ, is defined as the rate of change of the option price with respect to the price of the underlying asset. From a graphical perspective it is the slope of the curve relating the option price with the underlying assets price. The delta value is a reflection of the change in the option price as the underlying asset changes value. Hence, a delta value of 0.75 means that when the underlying assets price changes with one unit, the option price changes with 0.75 unit. This relationship can generally be shown as

$$\Delta = \frac{\partial c}{\partial S}$$

where c is the price of a call option and S is the price of the underlying asset.

A European call option on non-dividend-paying underlying asset can be set up as equation 2.38.

$$\Delta\left(call\right) = N\left(d_1\right) \tag{2.38}$$

In equation 2.38, d_1 corresponds to equation 2.28 in subsection 2.3 above. $N\left(x\right)$ is the cumulative distribution function for a standard normal distribution - calculated as $N\left(x\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$. Equation 2.38 is the formula for the delta of a long position in a call option. The delta of a short position in a call option is denoted $-N\left(d_1\right)$.

Like with the European call option, a put option can also be set up. This is done in equation 2.39.

$$\Delta\left(put\right) = N\left(d_1\right) - 1 \tag{2.39}$$

Whereas a long position in a call option has a positive delta, a long position in a put option has a negative delta. This mean that to use delta hedging for a long position in a call option one must maintain a short position of $N\left(d_1\right)$of the underlying asset for each call option purchased. For a long position in a put option delta is negative. Hence, to hedge this one must hold a long position in the underlying asset. '

The calculation of delta, as done above, is just as easily done for a portfolio. In a portfolio delta is dependent on a single assets price, S.

$$\frac{\partial \Pi}{\partial S}$$

where Π is the value of the portfolio.

The delta of a portfolio is calculated as the sum of the individual assets deltas. If a portfolio is constructed with w_i as the individual assets portfolio weight and Δ_i as the individual assets delta, the portfolio delta is given as equation 2.40.

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i \tag{2.40}$$

Theta, Θ, of a portfolio is a measure of the rate of change of the value of the portfolio with respect to the passage of time, all else being equal. That is, how is the value of a portfolio changing if the only other thing that changes is time. For a European call option this can be shown as in equation 2.41.

$$\Theta\left(call\right) = -\frac{S_0 N'\left(d_1\right)\sigma}{2\sqrt{T}} - rKe^{-rT}N\left(d_2\right) \tag{2.41}$$

where the probability density function for a standard normal distribution is given as

$$N'\left(x\right) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

The theta for a put option is given as

$$\Theta\left(put\right) = -\frac{S_0 N'\left(d_1\right)\sigma}{2\sqrt{T}} + rKe^{-rT}N\left(-d_2\right) \tag{2.42}$$

The theta of a put option exceeds the theta of a call option by rKe^{-rT} because $N\left(-d_2\right) = 1 - N\left(d_2\right)$ in equation 2.42. Hence, the passage of time on a put option has a larger impact than the passage of time on a call option. When calculating theta as in equation 2.41 and equation 2.42, theta is calculated on a yearly basis. Usually one is interested in knowing theta on a daily basis or on a fixed period of time. Hence, dividing the theta value by 365 you have the value on a daily basis.

Theta is usually negative. This is intuitive as the value of an option usually declines as time passes with all else being equal. Usually, theta is not a parameter used when hedging ones portfolio. The reason is that it is expected that time will pass. Hence, it make no sense trying to hedge this. Further, all else is not equal as time passes. The price of the underlying asset will change with time.

Gamma, Γ, of a portfolio of options on some underlying asset is a measure of the rate of change in the delta of the portfolio with respect to the price of the underlying asset. That is, gamma is a measure of how fast the delta is changing. The gamma of a portfolio is found as the second partial derivative of the portfolio with respect to price.

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} \tag{2.43}$$

The value of gamma is a reflection of the speed of change in delta. Hence, a small gamma value means that delta changes slowly, whereas a large gamma value means that delta changes faster. The speed of change in delta plays a role when hedging ones portfolio. If the speed of change in delta i slow one can less frequently adjust the delta hedge of the portfolio in order to keep the portfolio hedged delta neutral. If gamma is highly negative or highly positive, delta is very sensitive to the price of the underlying asset. This means that if the objective is to keep a portfolio hedged delta neutral one has to pay a lot of attention to the hedge as small changes in the price of the underlying asset makes a big impact on the portfolio.

Figure 2.2: Hedging error using delta hedging

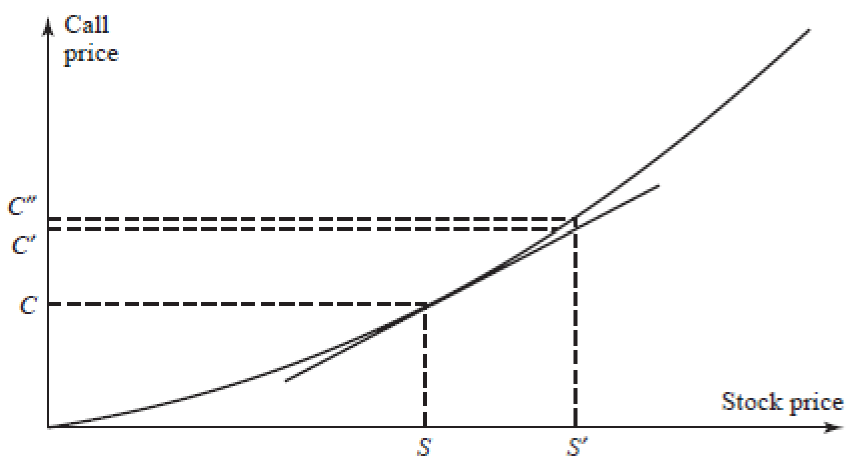

Figure from Hull [22].
Note: Consider a price change for S to S'. If delta hedging a portfolio one would assume that the call price of a option moves from C to C'. In fact the price moves from C to C'' which is not captured when using delta hedging as delta do not account for the curvature in the relationship between the price of the option and the price of the underlying asset. The difference between C' and C'' leads to a hedging error making the portfolio non delta neutral. The curvature of the relationship between the option price and the price of the underlying asset is captured by gamma. Hence, using gamma when hedging ensures that the portfolio is correctly hedged for changes in the price of the underlying asset.

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For a European call or put option on an underlying asset without dividend payouts gamma is calculated as

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \tag{2.44}$$

Where the input parameters are as defined earlier in this section. Gamma of a long position in an option is always positive and varies with S_0 as in figure 2.3.

Figure 2.3: The variation in gamma with the underlying asset

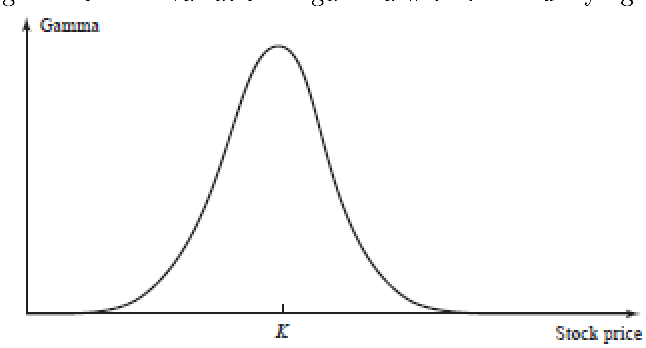


Figure from Hull [22].
Note: Gamma is normally distributed around K for a long position in an option.

To make a portfolio gamma neutral one must add a position in an instrument that is not linearly dependent on the underlying asset of the option in ones portfolio. This is because the underlying asset of the option does not have any gamma. If a delta neutral portfolio has a gamma equal to Γ and a traded option has a gamma equal to Γ_T and the number of traded options added to the portfolio is equal to w_T, the gamma of that portfolio is equal to

$$w_T \Gamma_T + \Gamma$$

To make this portfolio gamma neutral the position in the traded option must be$-\Gamma/\Gamma_T$. When including the traded option position in ones portfolio it is likely that the delta hedging of the portfolio changes. Hence, the position in the underlying asset in ones portfolio has to change as well to secure that the portfolio maintain delta neutrality.

Vega of a portfolio of options on some underlying assets is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset. That is, how is the value of a portfolio changing with fluctuations in the volatility in the underlying assets.

$$Vega = \frac{\partial \Pi}{\partial \sigma}$$

If vega is either highly negative or highly positive, the portfolio is very sensitive to small changes in the volatility of the underlying assets. That is, the portfolio value will be very volatile if vega is taking on large positive or negative values. If vega is close to zero the portfolio fluctuations will be small with fluctuations in volatility. Like with gamma, the underlying asset has zero vega. The vega of a portfolio can, like with gamma, be changed by adding a position in a traded option. Let $Vega$ be the vega of a portfolio and $Vega_T$ the vega of a traded option. A position of $-Vega/Vega_T$ in a traded option makes the portfolio vega neutral. To make a portfolio both vega and gamma neutral two traded derivatives dependent on the underlying asset must usually be added.

A European call or put option on an underlying asset not paying dividends is given by

$$\nu = S_0\sqrt{T}N'(d_1)$$

Where the inputs are given as previously described. The vega of a long position in an option is always positive.

Rho, ρ, of a portfolio of options is a measure of the change in value of the portfolio with respect to the interest rate. That is, how does the portfolio value change when the interest rate changes, all else equal.

$$\rho = \frac{\partial \Pi}{\partial r}$$

For a European call option on an underlying asset not paying any dividends rho is

$$rho\,(call) = KTe^{-rT}N\,(d_2)$$

and a put options is

$$rho\,(put) = -KTe^{-rT}N\,(-d_2)$$

To sum up, the relationship between the greeks and the Black Scholes Merton model can be illustrated as below

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \tag{2.45}$$

Equation 2.45 is known as the Black Scholes Merton Differential Equation. Substituting f with a portfolio Π gives

$$r\Pi = \frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2}$$

As shown previously in this section

$$\Theta = \frac{\partial \Pi}{\partial t}, \quad \Delta = \frac{\partial \Pi}{\partial S}, \quad \Gamma = \frac{\partial \Pi}{\partial S^2}$$

Inserting this gives

$$r\Pi = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma \tag{2.46}$$

Equation 2.46 illustrates the use of the greeks. It might be easier to interpret equation 2.46 using greek letters as notation compared to equation 2.45. Further, the interpretation of a delta neutral portfolio is probably easier using the greek letter notation as it is easier to spot the impact in the equation.

2.6 Conclusion

In section 2 I have presented the basic theory needed to understand the complexity of options. The pricing of an option usually starts with binomial trees. In subsection 2.1 I introduced the theory of binomial trees. In essence, the pricing of options using binomial trees are done going backwards in the tree illustrated by figure 2.1. This can be formalized by equation 2.3 restated below.

$$f = e^{-r\Delta t}\left[pf_u + (1-p)\,f_d\right]$$

Equation 2.4 to equation 2.7 can be used to expand equation 2.3. The binomial tree model builds on the assumption that investors are risk neutral. This allows for risk neutral pricing. However, investors are not risk neutral in practice. If investors were risk-neutral they had no incentive to invest in options as they would be satisfied with the risk-free interest as the return on investment. But the assumption makes it possible to build the model.

In subsection 2.2 I looked at processes. In essence, processes are about classification. The pricing of options can be classified as a stochastic process as the price is evolving in an unknown manner. But by characterizing it as a stochastic process some properties are following. For instance, characterizing something as a Wiener process means that it has a mean of zero and a variance of one, which can be helpful going forward.

In subsection 2.3 I introduced the Black Scholes Merton model. I showed how to derive the model from the binomial tree model, which lead to equation 2.26 and equation 2.27 with equation 2.28 and equation 2.29 as input. I also introduced the Black Scholes differential equation, equation 2.30, restated below.

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

The Black Scholes differential equation was used in subsection 2.4 to show the relationship between Itô's Lemma and the Black Scholes model.

In subsection 2.5 I introduced the Greek letters and the calculation of those. Using the the Black Scholes differential equation and substituting the price of an option, f, with a portfolio, Π, I got

$$r\Pi = \frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2}$$

Inserting the greek letters I got

$$r\Pi = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma$$

where

$$\Theta = \frac{\partial \Pi}{\partial t}, \quad \Delta = \frac{\partial \Pi}{\partial S}, \quad \Gamma = \frac{\partial \Pi}{\partial S^2}$$

Here, delta is the change in the portfolio value as the underlying assets are changing in value. Theta is the change in portfolio value with the passage of time. And Gamma is the speed of change in delta.

Until now the focus has been on understanding what an option is and how to price it. The models used so far are build on some assumptions. The Black Scholes Merton model assumes that there is constant volatility. This is obviously not the case in real life. Hence, other models might offer better insights if they include the volatility element? The same way it is assumed that investors are risk neutral. That means that investors settle with a return equal to the risk free interest rate. This is obviously not true either. Investors have an expectation about a payoff. This expectation varies from investor to investor. But most expect something higher than the risk free interest rate. This is to be examined going forward.

3 Local Volatility Model

In this section I will introduce the local volatility model. The Local Volatility Model tries to solve the issue of constant volatility from the Black Scholes model. This is done by making the volatility a function of time and of the price of the underlying asset.

In section 2 I introduced the basic theory needed to understand options and options pricing. One of the things introduced was the Black Scholes Merton model in subsection 2.3. The Black Scholes model is one of the most famous models for pricing options. A continuing problem with the model though, is that it treats volatility as a fixed component of the options price. Keeping the volatility fixed is a mistake and it will lead to wrong and unstable hedges of portfolios. This is undesirable for investors as it is costly to rehedge. But more importantly, if the market makes big moves that are unfavorable to the investors position, the investor could loose a lot of money on the investment. The Local Volatility Model tries to cope with this by making the volatility dependent of time and of price. That way, the model should be more correct and help investors to hedge their portfolios right.

$$r\Pi = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma$$

In subsection 2.5 I accounted for the Black Scholes greek letter. Equation 2.46 from subsection 2.5 is restated above. Remembering that equation 2.46 is the equivalent of equation 2.37, it is easy to see that the volatility is a constant. In this section I will dive more into the volatility component, σ. The Black Scholes model assumes that the volatility is constant throughout time and with different prices.
Empirically it has been proven that the volatility is not constant and the Black Scholes assumption turns out not to be true. Hence, other models for volatility has emerged. The Local Volatility Model being one of them. The model is originally presented by Derman and Kani [10] and by Dupire [13] [14].

Before the Local Volatility Model is presented, subsection 3.1 will introduce the concept of implied volatility and the volatility smile. Those are critical concepts to understand when moving from the Black Scholes Model and into volatility models. Subsection 3.2 will introduce the Local Volatility Model. In subsection 3.3 I will explain the concepts for setting up the model. In subsection 3.4 I will make calculations using the model setup. Subsection 3.5 touches upon the dynamics of the model before subsection 3.6 concludes.

3.1 Implied volatility and the volatility smile

Before moving on to setting up the Local Volatility Model, touching upon a few things are necessary. Local volatility models often try to price options by taking account of the volatility smile of options with a particular strike price and a particular time to maturity. To understand what a volatility smile is, we must first understand what implied volatility is.

In the Black Scholes model the volatility is kept constant. Black and Scholes simply assume that the volatility is fixed and not a function of for instance time or strike price. This means that the volatility curve from the Black Scholes models point of view is flat. Calculating the volatility in the Black Scholes setup is done iterative. Hence, when you have the price of an options, you make an iterative estimate of the volatility until the correct value is reached.

The move from volatility to implied volatility is basically a switch from theoretical options pricing to options pricing in the market. That is, the implied volatility is the calculated volatility using the market price of a given option. The market price of an option is not necessarily the same as the theoretical price of the option. Hence, the implied volatility might also be different. The implied volatility is often referred to as the estimated future volatility of an option because it uses market prices.

Plotting options with the same time to maturity in a plot with the strike price on the X axis and the implied volatility on the Y axis gives the volatility smile

Figure 3.1: The volatility smile

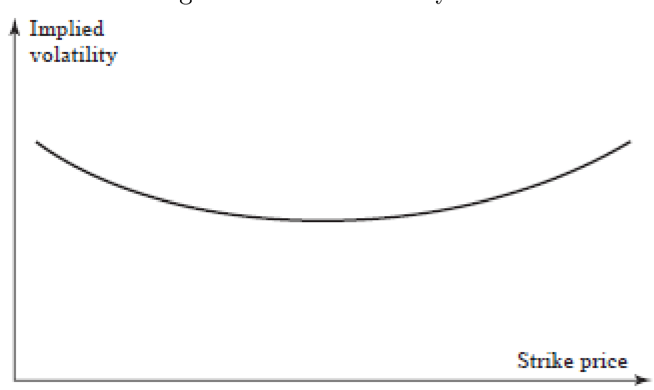


Figure from Hull [22]
Note: The figure depicts a volatility smile of an option with a given time to maturity. The implied volatility is increasing as the option moves away from the money in either direction. Hence, if the option is deep in or deep out of the money, the implied volatility is greater than if the option is at the money.

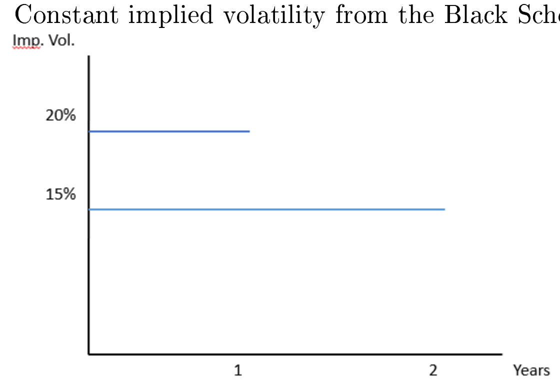
Figure 3.1 depicts the volatility smile. The implied volatility is increasing as the option is moving away from the money in either direction. Hence, if the option is deep in or deep out of the money, the implied volatility is greater than if the option is at the money.

The put-call parity secures that the volatility smile for a call option is equal to the volatility smile of a put option with the same strike price and time to maturity.

3.2 The Model

One of the problems with the Black Scholes model holding volatility constant throughout time is that the volatility can be different for overlapping time periods. For instance, the model allow for the volatility to be some value - say 20% - with a maturity of one year. At the same time, the volatility can be some other lower value - say 15% - for a maturity time of two years. Hence, the model suggests that the volatility is constant around a higher level for a year, while it at the same time suggests that the volatility is at another lower level in the same period.

Figure 3.2: Constant implied volatility from the Black Scholes model



Note: The figure depicts the implied volatility from the Black Scholes model. The X axis is time and the Y axis is the implied volatility. It is easy to see that the implied volatility is constant through time.

Figure 3.2 depicts this problem. Here the above two mentioned volatilities are plotted. As the options in question have the same starting time but different maturities there is an overlap in the time period. This basically gives the problem of the volatilities being different within the same period of time. Plotting the volatilities clearly stresses the issue of different volatilities with overlapping time. This issue is however fixed by letting the volatility be time dependent as Merton [28] did.

$$\frac{dS}{S} = \mu(t)\,dt + \sigma(t)\,dW \tag{3.1}$$

Equation 3.1 is an Itô process as described in subsection 2.2. In equation 3.1 $\mu(t)$ is the risk-neutral drift dependent on time and $\sigma(t)$ is the local volatility dependent on time. This corresponds to what Merton did.

Taking account of time is only one part of the problem with the implied volatility calculated from the Black Scholes model. The other part is taking account of the price of the underlying asset. The dependence of implied volatility on the strike price for a given maturity must in some way be handled. This is the volatility smile, introduced in subsection 3.1, doing.

Incorporating the volatility smile in a model is the challenge that must be solved. One of the big challenges when incorporating a volatility smile in a given model is that the model can become very complex and lose its completeness.

Dupire [13] [14] and Derman and Kani [10] tries to solve the issue of incorporating a volatility smile in a model.

$$\frac{dS}{S} = r\left(t\right)dt + \sigma\left(S,t\right)dW \tag{3.2}$$

Equation 3.2 is very similar to equation 3.1. The main difference is that the volatility in equation 3.2 includes the spot price of the underlying asset. The volatility in equation 3.2 is a deterministic function of the spot price and of time. A deterministic function is a function that returns the same result every time the function is called, given that the input is the same. In other words, the volatility component of equation 3.2 is unchanged, if the input spot price and the input time is unchanged.

Equation 3.2 is the equation the Local Volatility Model tries to answer. Dupires approach to solving equation 3.2 is different from the approach of Derman and Kani. I will use the approach of Derman and Kani.

3.3 Setting up the model

In this subsection I will do the formal setup of the model. For an easier overview I will structure the setup in the following way. I start by introducing the notation of the model. Then I show some figures before introducing equations. Lastly, I explaining the intuition.

S_0 The spot price of the underlying asset we are calculating the options price for. This price is known when $t = 0$. Hence, this is the initial price of the underlying asset.

s_i This is the known stock price at node (n, i) at level n node i. This is also the strike value for options expiring at level $n + 1$.

S_{i+1} The unknown state value reached after an upward move in the tree. Think of it as the new spot price after an upward move.

S_i The unknown state value reached after a downward move in the tree. Think of it as the new spot price after a downward move.

p_i The risk neutral probability of moving from s_i to S_{i+1}. Equivalently, the probability of moving from s_i to S_i is $1 - p_i$.

F_i The known forward price one period forward from the know price s_i.

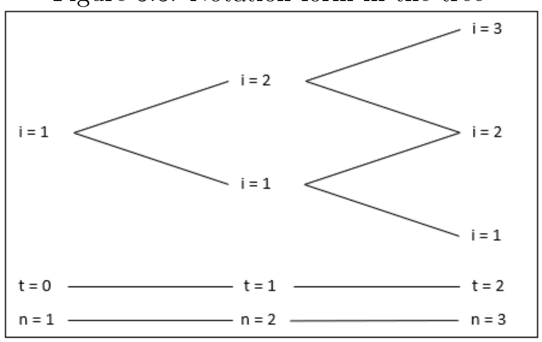
λ_i The Arrow-Debreu price calculated using forward induction.

t Time in years.

n node.

Now that the notation is in place, the figure below depicts the way in which the subscripts are used in the binomial tree.

Figure 3.3: Notation form in the tree



Note: t is the time measure. n is the node group. i is the specific node we are looking at. The specific node, i, is always starting from the bottom with value 1 as show in the figure.

We are now almost ready to look into the mathematics of the model. But before doing that, there are a few things to note. This model is trying to calculate transition probabilities, p_i, and future unknown state values, S_i. To do so, the model maintains the spot price, S_0, as a central node throughout the tree - figure 3.4a shows this as the central node maintains the spot price of 100 - known as the centering constraint. This is done to ensure uniquely determined parameters. Now, the market prices can be calculated using the observed volatility smile.

At a given time $n > 0$ there are $2n + 1$ parameters describing the transition from time n to time $n + 1$ - the new stock price S_i and the n transition probabilities p_i at time $n + 1$. The information at hand at time n is n forward prices and n option prices. Hence, to determine $2n + 1$ parameters using $2n$ equations something must be done. Adding the centering constraint solves this issue and makes it possible to uniquely determine all parameters within the time step.

The first step in setting up the model is to acknowledge that the forward price corresponding to S_i is $F_i = e^{r\Delta t} S_i$. With this, the following identity equation must hold.

$$F_i = p_i S_{i+1} + (1 - p_i) S_i \tag{3.3}$$

Equation 3.3 is the known forward price. Reminding our self that S_{i+1} denotes an upward move, S_i a downward move and that p_i is the probability.

The value of a call option is now represented by $C(K, t_{n+1})$ with a strike K and maturity t_{n+1}.

$$C(K, t_{n+1}) = e^{-r\Delta t} \sum_{j+1}^{n} \{\lambda_j p_j + \lambda_{j+1}(1 - p_{j+1})\} max(S_{j+1} - K, 0) \tag{3.4}$$

Using the Arrow-Debreu prices, λ_i, equation 3.4 states the value of a call option. $e^{-r\Delta t}$ indicates that we discount back to time 0. Setting $K = s_i$ allows for equation 3.4 to be simplified.

$$C(s_i, t_{n+1}) = e^{-r\Delta t} \left[\lambda_i p_i (S_{i+1} - s_i) + \sum_{j=i+1}^{n} \lambda_j (F_j - s_i) \right] \tag{3.5}$$

Equation 3.5 is the simplified expression. From this, only up moves in the tree will have a positive impact, whereas down moves will have zero impact. The forward statement in equation 3.3 is applied to equation 3.5 as well. Equation 3.5 can now be evaluated in two steps. The first part of the bracket is only adding value with some probability, p_i, whereas the last part of the bracket for sure is adding value. This value is the sum of all differences between the forward price and the stock price. The unsure value is only added if there is an upward move in the tree.

As the forward prices are known and a volatility smile is given, only the transition probability, p_i, and the underlying asset after an upward move, S_{i+1}, are unknown. Combining equation 3.3 and equation 3.5 can solve this. However, combining the two equations add an unknown, S_i. Making the centering around S_0 makes it possible to start at the central node and work upwards in the tree. Solving equation 3.3 and equation 3.5 simultaneously yields

$$S_{i+1} = \frac{S_i \left[e^{r\Delta t} C(s_i, t_{n+1}) - \sum \right] - \lambda_i s_i (F_i - S_i)}{\left[e^{r\Delta t} C(s_i, t_{n+1}) - \sum \right] - \lambda_i (F_i - S_i)} \tag{3.6}$$

$$p_i = \frac{F_i - S_i}{S_{j+1} - S_i} \tag{3.7}$$

where $\sum = \sum_{j=1}^{i-1} \lambda_j (s_i - F_j)$ from equation 3.5.

A key element in calculating state values and transition probabilities this way is knowing the value of the central node. When not knowing this, a different approach must be used. This is for instance when moving from $n = 1$ to $n = 2$. Derman and Kani uses the natural logarithm to solve this issue.

$$Log\left(S_0\right) = \frac{log\left(S_{i+1}\right) + log\left(S_i\right)}{2}$$

which is equivalent to

$$S_i = \frac{S_0^2}{S_{i+1}} \tag{3.8}$$

Substituting equation 3.8 in to equation 3.6 and rearranging yields

$$S_{i+1} = \frac{S_0\left[e^{r\Delta t}C\left(S, t_{n+1}\right) + \lambda_i S_0 - \sum\right]}{\lambda_i F_i - e^{r\Delta t}C\left(S_0, t_{n+1}\right) + \sum} \tag{3.9}$$

The implied volatility can now be calculated for each node in the tree based on the possible state values and the transition probability.

$$\sigma_i = \sqrt{p_i\left(1 - p_i\right)}log\left(\frac{S_{i+1}}{S_i}\right) \tag{3.10}$$

3.4 Applying the model

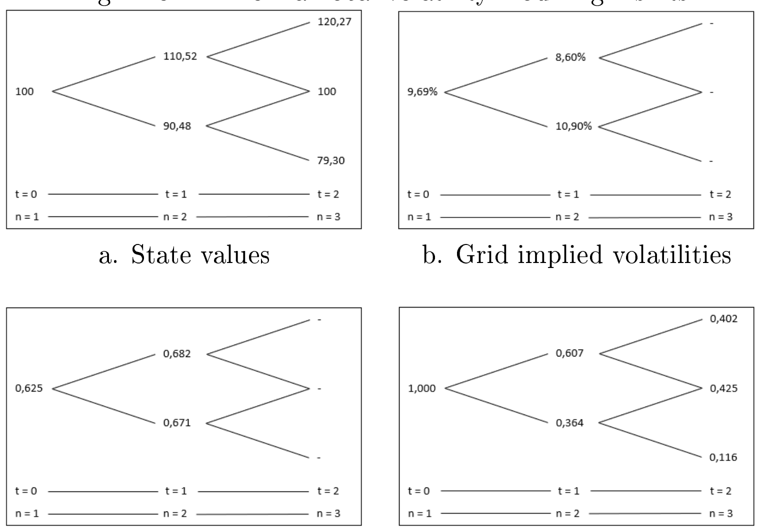
In this subsection I will apply the model on a sample calculation. That way, it should be easier to see how the model works.

For this calculation, the setting is as follows.

- $t \in \{0, 1, 2\}$ and $n \in \{1, 2, 3\}$.
- The spot price of the underlying asset is $S_0 = 100$.
- The risk free interest rate is 3%.
- The volatility smile has an at the money volatility of 10% and a 0,5 percentage point change for every 10 unit change in the strike price starting at $K = S_0$. This is calculated as

$$\sigma_{imp}\left(K\right) = 10\% - \frac{0,5\% \times \left(K - S_0\right)}{10} \tag{3.11}$$

Figure 3.4: Binomial local volatility modeling results



a. State values b. Grid implied volatilities

c. Transition probabilities d. Arrow-Debreu prices

Note: Figure a shows the results of the state values - $s_{(n,i)}$- when solving the local volatility model. Figure b is the grid implied volatilities - $\sigma_{(n,i)}$. Figure c is the transition probabilities - $p_{(n,i)}$.Finally, figure d is the Arrow-Debreu pricses for the given option - $\lambda_{(n,i)}$.

Figure 3.4 shows the results from solving the Local Volatility Model with the characteristics listed in the beginning of this subsection. Below is the calculations corresponding the results in figure 3.4.

The objective is to essentially calculate all state values in the tree - that is, solving the tree of figure 3.4a. To do that, the other trees must be solved as well. This is a step wise procedure where one value at the time is found. To calculate the state value of $(n,i) = (2,2)$, the Arrow-Debreu price of $(n,i) = (1,1)$ must be know. As this is the spot price note, the Arrow-Debreu price is simply equal to one. Solving $S_{(2,2)}$, we must first solve $C(100,1)$, as there is no central node in node group two. That is, we can only move up or down in the node group making this a special case. Hence, we must use equation 3.9 to calculate the price. To calculate $C(100,1)$ I use the implied volatility smile. As the option has a strike price of 100, the implied volatility is $\sigma_{imp}(100) = 10\%$ cf. equation 3.11. Using this I get $\sigma = \sigma_{imp} = 10\%$ and $C(100,1) = 6,38$. Inserting this information and the fact that node $(2,2)$ is the highest node in the group, $S_{(2,2)}$ is calculated below.

$$S_{(2,2)} = \frac{100\left[e^{3\%} \times 6,38 + 1 \times 100 - 0\right]}{1 \times 100\,(1+3\%) - e^{3\%} \times 6,38 + 0} = 110,52$$

As mentioned, node $(2,2)$ is the highest in the group, meaning that the $\Sigma-term$ is equal to zero. The result of $S_{(2,2)}$ is now used to calculate $S_{(2,1)}$ using equation 3.8.

$$S_{(2,1)} = \frac{100^2}{110,52} = 90,48$$

With the prices at node two in place it is now possible to calculate the transition probability $p_{(1,1)}$ using equation 3.7.

$$p_{(1,1)} = \frac{100 \times 1,03 - 90,48}{110,52 - 90,48} = 0,625$$

Using the transition probability and the state values calculated above, the grid implied volatility is calculated as

$$\sigma_{(1,1)} = \sqrt{0,625 \times (1 - 0,625)} log \left(\frac{110,52}{90,48} \right) = 9,69\%$$

Using equation 3.11 to calculate the implied volatility yields

$$\sigma_{imp} (110,52) = 10\% - \frac{0,5\% \times (110,52 - 100)}{10} = 9,47\%$$

The same way as above, the next steps in the tree is calculated. This iterative process is carried out until the whole tree is calculated. I will not calculate more steps, as it offers no more information. Instead, I will discuss the model in the next subsection.

3.5 Model Dynamics

In subsection 3.4 I showed how to calibrate the model to a volatility smile in a numerical example. Here, the volatility smile was given by equation 3.11. With the knowledge of how to solve the model I will now move on to the dynamics of the model. In other words, how is this a good model from an intuitive standpoint.

To better evaluate the dynamics of the model, I start by simplifying the initial setup by removing the time dimension from equation 3.2.

$$dS = \sigma_{loc} (S) \, SdW \tag{3.12}$$

Evaluating the model in this particular setup is the focus of a paper by Hagan and Woodward [19]. In this paper, Hagan and Woodward give an approximation of the implied volatility from the Black-Scholes model. Equation 3.12 is stated on spot prices, whereas Hagan and Woodward are investigating on forward prices as stated below - under the forward measure previously introduced.

$$dF = \sigma_B F dW \tag{3.13}$$

With this, the Black-Scholes volatility for a given strike price, K, and a give forward price, F, is

$$\sigma_B(K,F) = \sigma_{loc}\left(\frac{F+K}{2}\right)\left[1 + \frac{1}{24}\frac{\frac{d^2\sigma_{loc}}{df^2}\left(\frac{F+K}{2}\right)}{\sigma_{loc}\left(\frac{F+K}{2}\right)}(F+K)^2 + ...\right] \tag{3.14}$$

Intuitively, from looking at equation 3.14, it is clear that the contribution from the first term - $\sigma_{loc}\left(\frac{F+K}{2}\right)(\times 1)$ - is greater than from the second term - $\sigma_{loc}\left(\frac{F+K}{2}\right)\left(\times\frac{1}{24}\right)$. Hagan et al. [18] states that the contribution from the second term is usually less than 1%. From a pricing perspective the second term is important. But from an analytical point it makes sense to omit the second term leaving us with

$$\sigma_B(K,F) = \sigma_{loc}\left(\frac{F+K}{2}\right) \tag{3.15}$$

With this reduced equation in place I now turn to the analytical evaluation of the model. Suppose a forward price observed today denoted as F_0 and a strike price K. Together they form a volatility smile denoted as $\sigma_B^0(K)$. With $\sigma_B^0(K)$ and equation 3.15, for the model to be calibrated it must hold that $\sigma_{loc}(F) = \sigma_B^0(2F - F_0)$ since

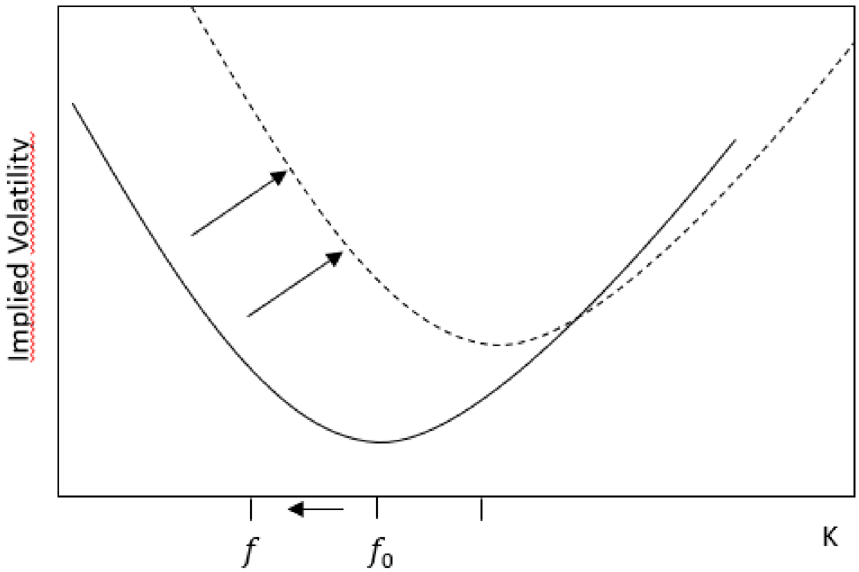
$$\sigma_B^0(2F - F_0) = \sigma_{loc}\left(\frac{F_0 + [2F - F_0]}{2}\right) = \sigma_{loc}(F) \tag{3.16}$$

If the current forward price, F_0, changes to some other forward value, F, equation 3.15 together with equation 3.16 imply that

$$\begin{aligned}
\sigma_B(K,F) &= \sigma_{loc}\left(\frac{F+K}{2}\right) \\
&= \sigma_B^0\left(2\left[\frac{F+K}{2}\right] - F_0\right) \\
&= \sigma_B^0(K + [F - F_0])
\end{aligned}$$

This is the new implied volatility for a an option with a strike price, K, an initial forward price of F_0 and a new forward price of F. Figure 3.5 depicts the situation where the forward price decreases from F_0 to F.

Figure 3.5: Implied volatility if the forward price decreases



Note: Plot of a volatility smile moving as the forward price is changing. The interesting point is that the volatility smile is moving in the opposite direction of the forward price change.

Assuming an implied volatility of $\sigma_B^0(K, F_0) = \sigma_B^0(K + 0)$ is observed. Letting the forward level decrease so that $F < F_0$ leaving us with a new implied volatility being $\sigma_B(K, F) = \sigma_B^0(K + [F - F_0])$. When the forward price decreases from F_0 to F the volatility smile moves up and to the right. This is a move in the opposite direction of the underlying asset. Hagan et al. [18] find that this move is the opposite of the typical market behavior. The typical market behavior being that volatility smiles move in the same direction as the underlying asset.

This opposite movement of the implied volatility in the model can lead to potential wrongful hedging. Letting $C(K, F, \sigma_B(K, F))$ denote the call value of an option with strike price K, forward price F and volatility calculated from the local volatility model being $\sigma_B(K, F)$. Calculating the delta risk for this option as

$$\Delta_C = \frac{\partial C(K, F, \sigma_B(K, F))}{\partial F} + \frac{\partial C(K, F, \sigma_B(K, F))}{\partial \sigma_B} \frac{\partial \sigma_B(K, F)}{\partial F} \tag{3.17}$$

The first term of equation 3.17 is the standard delta risk from the Black-Scholes model using constant volatility. The second part of equation 3.17 is the correction term. This is a result of the volatility being a function of the forward price. As a result of the Local Volatility model handling the implied volatility wrongfully, as depicted in figure 3.5, the delta risk is wrongfully hedged. That is, using the Local Volatility model one might under hedge when the forward price rises and over hedge when the forward price decreases.

3.6 Conclusion

In this chapter I started by introducing implied volatility and the volatility smile. The implied volatility is the market generated volatility - volatility based on market prices. The implied volatility is sometimes referred to as the estimated future volatility of an option based on the market price.

Figure 3.1 plotted a volatility smile. The volatility smile is a way of showing how the implied volatility changes when the strike price changes. As the price moves further away from the strike price in either direction, the implied volatility changes. Remember how the implied volatility is higher when the option is either far out of the money or far in the money and how the implied volatility gradually decreases as the strike price moves closer at the money. This forms a curve that resembles just like a smile - hence the name.

The implied volatility and the volatility smile are both important concepts to understand as they play a big role in the pricing of options. In particular when looking at alternative options pricing models to the Black-Scholes model.

In section 2 I introduced the Itô formula. In this, the volatility was kept constant. This formula looked something like the one below, where $r(t) = \mu$.

$$\frac{dS}{S} = r(t)\,dt + \sigma dW$$

In this section I found that the question the Local Volatility model tries to answer is equation 3.2 restated below.

$$\frac{dS}{S} = r(t)\,dt + \sigma(S,t)\,dW$$

The main difference between the two above stated equations are the addition of time and price. But the addition of time and price is exactly what the Black-Scholes model is missing. Hence, answering this question might be the answer to doing options pricing. This is at least the approach of Derman and Kani. They came up with a model to solve the issue of assumed constant volatility. That is, the model should solve equation 3.2 to handle the problem of assumed constant volatility. To solve this, the model proposes a variety of equations. The reason being that they want to keep the mathematics at a level not too complex. Below is the restated model highlighting the most important equations in solving this model.

$$F_i = p_i S_{i+1} + (1 - p_i)\,S_i$$

The first assumption of the model is that of the forward price. It states that the forward price is equal to the discounted spot price. This assumption leads to the relationship stated above, which is that the forward price is equal to the probability of the spot price moving up or down in the pricing tree. This leads to the pricing of a call option stated below.

$$C(s_i, t_{n+1}) = e^{-r\Delta t}\left[\lambda_i p_i (S_{i+1} - s_i) + \sum_{j=i+1}^{n} \lambda_j (F_j - s_i)\right]$$

Solving the above two stated equations simultaneously leads to the pricing of an upward move in the tree.

$$S_{i+1} = \frac{S_i\left[e^{r\Delta t}C(s_i, t_{n+1}) - \sum\right] - \lambda_i s_i (F_i - S_i)}{\left[e^{r\Delta t}C(s_i, t_{n+1}) - \sum\right] - \lambda_i (F_i - S_i)}$$

And calculating the probability as

$$p_i = \frac{F_i - S_i}{S_{j+1} - S_i}$$

The pricing of a downward move in the tree can be calculated like

$$S_i = \frac{S_0^2}{S_{i+1}}$$

Substituting S_i into equation3.6 and rearranging and you get

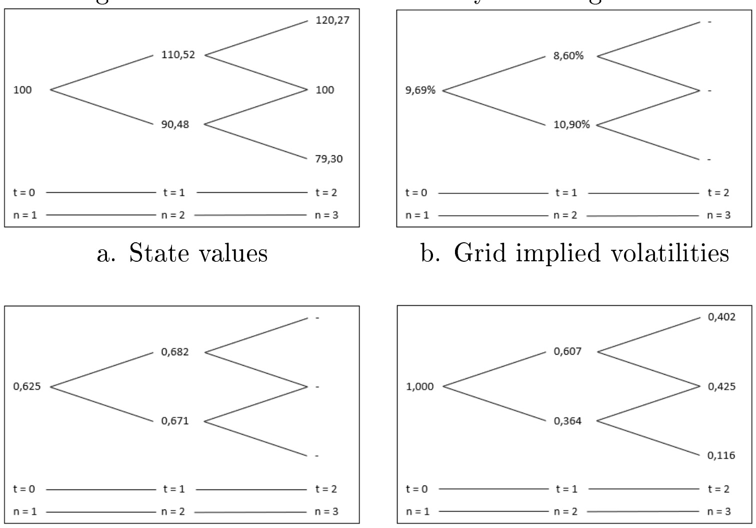
$$S_{i+1} = \frac{S_0\left[e^{r\Delta t}C(S, t_{n+1}) + \lambda_i S_0 - \sum\right]}{\lambda_i F_i - e^{r\Delta t}C(S_0, t_{n+1}) + \sum}$$

This gives the implied volatility of the model as

$$\sigma_i = \sqrt{p_i (1 - p_i)}log\left(\frac{S_{i+1}}{S_i}\right)$$

Applying the model like I did in subsection 3.4 and writing the results from the calculations in a binomial tree structure like in figure 3.4 restated below.

Figure 3.6: Binomial local volatility modeling results



a. State values

b. Grid implied volatilities

c. Transition probabilities

d. Arrow-Debreu prices

Note: Figure a shows the results of the state values - $s_{(n,i)}$- when solving the local volatility model. Figure b is the grid implied volatilities - $\sigma_{(n,i)}$. Figure c is the transition probabilities - $p_{(n,i)}$.Finally, figure d is the Arrow-Debreu pricses for the given option - $\lambda_{(n,i)}$.

With the calibration of the model to a volatility smile in place, the dynamics of the model was examined. The examination of the dynamics ended up with an equation for the implied volatility restated below.

$$\sigma_B\left(K,F\right) = \sigma_B^0\left(K + \left[F - F_0\right]\right)$$

This is the Black Scholes volatility as a function of strike price and forward price. Using the the Black Scholes implied volatility I found that the implied volatility from the Local Volatility Model moves in the opposite direction of the price of the underlying asset. This is intuitively difficult to understand as it is the opposite behavior of what is the typically observed market behavior. The movement is depicted in figure 3.5.

Calculating the delta risk of a call option using the volatility from the Local Volatility Model leads to the below stated equation.

$$\Delta_C = \frac{\partial C\left(K,F,\sigma_B\left(K,F\right)\right)}{\partial F} + \frac{\partial C\left(K,F,\sigma_B\left(K,F\right)\right)}{\partial \sigma_B}\frac{\partial \sigma_B\left(K,F\right)}{\partial F}$$

Since the volatility from the Local Volatility Model moves in the opposite direction of what is empirically documented, the second part of the delta-equation above is incorrect. As this is incorrect the delta-hedging becomes incorrect.

Hence, the Local Volatility Model does not deliver the desired properties for hedging purposes.

4 The SABR Model

This section is introducing the SABR Model. The SABR model is an options pricing model allowing stochastic parameters in the model. That way, it might be a better suited model for handling volatility.

4.1 Vanna and Volga

Before delving in to the SABR model, a few things must be touched upon. In section 2 I gave an introduction to the basics of options. In this introduction I introduced the Greek letters for options pricing. The theory presented in subsection 2.5 about the Greek letters must be expanded with a few more letters. Namely the Greek letters Vanna and Volga.

Vanna is defined as

$$Vanna = \frac{\partial^2 V_{call}}{\partial S \partial \sigma} \tag{4.1}$$

Vanna is the second order derivative of the value of the option with respect to the price of the underlying asset and the volatility. This is the same as being the sensitivity in delta with respect to volatility.
Why is this of interest? It is of interest because the delta hedging of a portfolio changes with changing volatility. If accounting for this, the portfolio might be better and perhaps cheaper hedged.

Volga is defined af

$$Volga = \frac{\partial V_{call}}{\partial \sigma^2} \tag{4.2}$$

Volga is the second order derivative of the value of the option with regards to volatility. Being the second order, it is the same as being the volatility of the volatility. Why is this of interest? It is of interest because the volatility of the volatility reveals something about the market sentiment. That is, should I hedge my portfolio for rising volatility or should I hedge my portfolio for a more calm environment?

I will return to the Greek letters later in this chapter as the findings of the SABR model will be applied to the Black-Scholes Greeks.

4.2 Introduction to the SABR model

The Stochastic Alpha Beta Rho model, also known as the SABR model, is a model that tries to handle the stochastic volatility of options pricing. One can think of the SABR model as a sort of add on to the Black-Scholes model.

The Black-Scholes model is a model that prices options. As mentioned previously in this paper, the Black-Scholes model prices options on fixed volatilities. To get a more correct pricing, expanding the model with a non-constant volatility is a possibility. This is where the SABR model becomes handy. Where the Black-Scholes model handles volatility by keeping it constant and the Local Volatility Model presented in section 3 handles it by letting the volatility be locally constant, the SABR model is handling the volatility by allowing it to be a function of time, strike price and current forward price.

The SABR model was originally published by Hagan et. al. in 2002 [18]. This paper is taking the same approach as of Hagan et. al. One of the attractive things of the Hagan et. al. approach is that they try to keep the model as simple as possible. Other papers are evaluating the SABR model using more advanced mathematical methods that are outside the scope of this paper.

The SABR model is given by three equations. One equation for a forward price process, one equation for a volatility process and one equation with a correlation coefficient for the two Brownian motions included in the first two equations. The three equations are stated below after a definition of the variables used.

α_t is the stochastic volatility. α is the variable reflecting the level of volatility smile curve.

f_t is the forward price at time t.

β is the exponent of the forward rate.

v is the volatility of the volatility. This is the variable that controls the curvature of the volatility smile curve.

ρ is the correlation between the Brownian Motions.

σ_{ATM} At the money volatility.

$\sigma_B(K, f)$ Implied volatility from the Black-Scholes model.

With the notation in place, the model looks like,

$$df_t = \alpha_t f_t^{\beta} dW_t^1 \tag{4.3}$$

$$d\alpha_t = \nu \alpha_t dW_t^2 \tag{4.4}$$

$$dW_t^1 dW_t^2 = \rho dt \tag{4.5}$$

Equation 4.3 is the forward price process. The forward price process is given by volatility, forward price to the power of β and a Brownian motion. Changes to the forward price is given by volatility and the Brownian motion.

Equation 4.4 is the volatility process. This is given by the volatility itself, the volatility of volatility and a Brownian motion. Changes to the volatility is then given by the volatility of volatility and a Brownian motion.

Equation 4.5 is the correlation between the two Brownian motions. This is measured with the correlation coefficient ρ.

The model can be solved using Monte Carlo but this is a tedious process. Instead, Hagen et. al. solves the model using singular perturbation techniques. The results from using this technique is presented in the next section.

4.3 Solving the SABR model

In subsection 2.3 I introduced the Black-Scholes-Merton model for pricing options. More specifically, I ended up with the equations 2.26 to 2.29 for the final model. Using the same approach for valuing options, the SABR model can be solved by singular perturbation. Approximating the implied volatility this way, $\sigma_B(K, f)$, gives the following result

$$\sigma_B(K,f) = \frac{\alpha \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}v^2\right] t_{ex}\right\}}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24} log^2\left(\frac{f}{K}\right) + \frac{(1+\beta)^4}{1920} log^4\left(\frac{f}{K}\right)\right\}} \times \left(\frac{z}{x(z)}\right) \tag{4.6}$$

where

$$z = \frac{v}{\alpha}(fK)^{(1-\beta)/2} log\left(\frac{f}{K}\right) \tag{4.7}$$

and

$$x(z) = log\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right) \tag{4.8}$$

Using the same technique for at-the-money options, the case where $K = f$, the formula simplifies to

$$\sigma_{ATM} = \sigma(f, f) = \frac{\alpha}{f^{(1-\beta)}} \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{(2-2\beta)}} + \frac{\rho \beta v \alpha}{4 f^{(1-\beta)}} + \frac{2-3\rho^2}{24} v^2 \right] t_{ex} \right\} \tag{4.9}$$

Equations 4.6 and 4.9 are the main results of the 2002 paper by Hagen et. al. [18]. This result is a replication of the volatility given a certain volatility smile. This way, the model is able to give a value for the volatility given different input factors.

With this result in place it is now time to look at the calibration of the SABR model in order to make consistent estimates.

4.4 Calibrating the model

The parameters in the SABR model must be calibrated before the model is functioning as intended. That is, before the model is calibrated it will not be able to fit the volatility smile properly. This means that any volatility predictions are wrong and will lead to mistaken hedging. As most traders are applying the SABR model to help making hedging correct, wrongful calibration is not desirable.

4.4.1 The beta value

When calibrating the SABR model, the β value is the first parameter to be fixed. This is done manually by the trader. The β value is fixed based on the traders own belief given the market conditions - in other words, the traders intuition about the market he is participating in.

In the SABR model the β value is limited to be between zero and one. Usually the β value is set high when the interest rates are high. Similarly the β value is set low when the interest rates are low. In the following I will show the two corner solution values of β. The β value is limited to be between 0 and 1 and hence the corner solutions are a β value of 0 and a β value of 1.

Setting $\beta = 0$ makes the forward process normally distributed. By Setting $\beta = 0$ the forward process is reduced to the expression in equation 4.10.

$$df_t = \alpha_t dW_t \tag{4.10}$$

The reduced forward process means that the forward price increments are stochastic normally distributed in within the model. More specifically, the forward price movements are normally distributed with a mean of zero and a log-normal distributed stochastic standard deviation.

With this reduced forward process the implied normal volatility is given by equation 4.11.

$$\sigma_N(K) = \epsilon\alpha\left\{1 + \frac{2-3\rho^2}{24}\epsilon^2 v^2 \tau_{ex} + ...\right\} \tag{4.11}$$

The implied normal volatility from the Black-Scholes model is given by equation 4.12.

$$\sigma_B(K) = \epsilon\alpha\frac{log(f/K)}{f-K} \times \left(\frac{\varsigma}{\hat{x}(\varsigma)}\right) \times \left\{1 + \left[\frac{\alpha^2}{24fK} + \frac{2-3\rho^2}{24}v^2\right]\epsilon^2\tau_{ex} + ...\right\} \tag{4.12}$$

where ς and $\hat{x}$ are given by

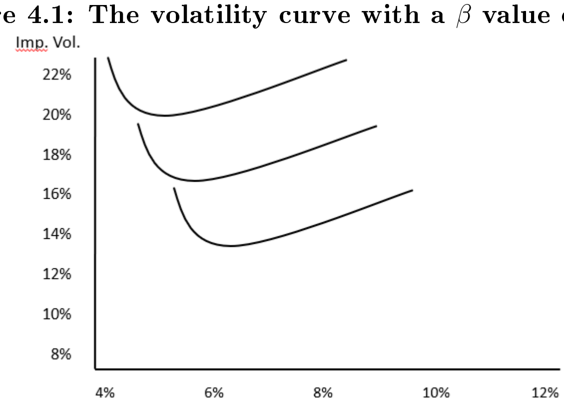
$$\varsigma = \frac{v}{\alpha}\sqrt{fK} \times log\left(\frac{f}{K}\right)$$

and

$$\hat{x}(\varsigma) = log\left(\frac{\sqrt{1-2\rho\varsigma+\varsigma^2}-\rho+\varsigma}{1-\rho}\right)$$

Setting the β value to zero further means that the volatility curve will behave as depicted in figure 4.1.

Figure 4.1: The volatility curve with a β value of zero



Note: On the x-axis is given the underlying forward rate and on the y-axis is given the model implied volatility. Starting from the top left corner a volatility smile is given. As the underlying forward rate rises the volatility smile moves down to the right. The model implied volatility is decreasing as the the underlying forward rate rises.

Starting from the top left corner a volatility smile is given. As the underlying forward rate rises the volatility smile moves down and to the right. Hence, the model implied volatility is decreasing as the the underlying forward rate rises.

Setting $\beta = 1$ makes the forward process a log normal process. The forward process is reduced to the expression in equation 4.13.

$$df_t = \alpha_t f_t dW_t \tag{4.13}$$

The forward process is now very similar to the Black-Scholes model. In fact, if $v = 0$ the SABR model reduces to the Black-Scholes model. The forward rate in equation 4.13 follows a log-normal distribution meaning that it has a non-negative property.

With this reduced forward process the implied normal volatility is given by equation 4.14

$$\sigma_N(K) = \epsilon\alpha \frac{log(f/K)}{f-K} \times \left(\frac{\varsigma}{\hat{x}(\varsigma)}\right) \times \left\{1 + \left[-\frac{1}{24}\alpha^2 + \frac{1}{4}\rho\alpha v + \frac{1}{24}\left(2-3\rho^2\right)v^2\right]\epsilon^2\tau_{ex} + ...\right\} \tag{4.14}$$

The implied normal volatility from the Black-Scholes model is given by equation 4.15

$$\sigma_B(K) = \epsilon\alpha \times \left(\frac{\varsigma}{\hat{x}(\varsigma)}\right) \times \left\{1 + \left[\frac{1}{4}\rho\alpha v + \frac{1}{24}\left(2-3\rho^2\right)v^2\right]\epsilon^2\tau_{ex} + ...\right\} \tag{4.15}$$

where ς and $\hat{x}$ are given by

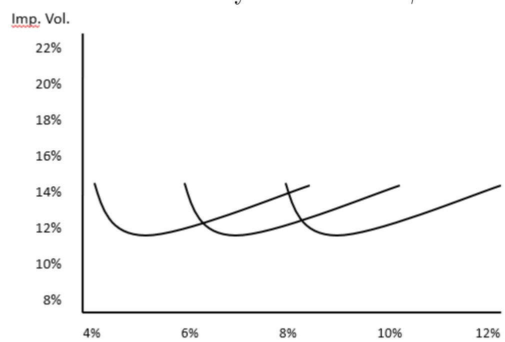
$$\varsigma = \frac{v}{\alpha} log\left(\frac{f}{K}\right)$$

and

$$\hat{x}(\varsigma) = log\left(\frac{\sqrt{1-2\rho\varsigma+\varsigma^2}-\rho+\varsigma}{1-\rho}\right)$$

Setting the β value to one further means that the volatility curve will behave as depicted in figure 4.2.

Figure 4.2: The volatility curve with a β value of one



Note: On the x-axis is given the underlying forward rate and on the y-axis is given the model implied volatility. With $\beta = 1$ the volatility smile is moving horizontally as the underlying forward rate is rising.

With a β value of one the volatility smile is moving horizontally. Hence, the model implied volatility is limited to being the smile curve. With a β value of zero the model implied volatility is changing with changing values in the underlying forward rate. Instead of a limited model implied volatility like in the case of β equals one, the model implied volatility is now responding to the underlying forward rate and at the same time also moving along the smile curve.

If the underlying forward rate is high the model implied volatility is low when the β value is set to zero. This is in accordance with what I wrote earlier - that a β value of zero is applied when the interest rates are low whereas a β value of one is applied when the interest rates are high. Figure 4.1 and figure 4.2 displays the outcome of this. But the two figures also display something about how often one might have to re-calibrate the model.

One thing to remember about the re-calibration of the model is that the interest rates usually do not make large moves. However, when large movements are happening it is usually due to some sort of crisis in the markets. When this happens chances are the model must be re-calibrated often in order to reflect the market conditions.

4.4.2 Estimating α, v and ρ

With the β value fixed the remaining parameters can be estimated. The remaining parameters being α, v and ρ. One way of estimating the parameters is by assigning values to ρ and v. From there the α parameter can be estimated using the σ_{ATM} - at the money implied volatility - information found in equation 4.9 restated below.

$$\sigma_{ATM} = \sigma(f,f) = \frac{\alpha}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{(2-2\beta)}} + \frac{\rho\beta v\alpha}{4f^{(1-\beta)}} + \frac{2-3\rho^2}{24}v^2\right]t_{ex}\right\} \tag{4.16}$$

Rearranging equation 4.16 I find

$$A\alpha_0^3 + B\alpha_0^2 + C\alpha_0 - \sigma_{ATM}f^{(1-\beta)} = 0 \tag{4.17}$$

where $A = \left[\frac{(1-\beta)^2 T}{24f^{(2-2\beta)}}\right]$, $B = \left[\frac{\rho\beta v T}{4f^{(1-\beta)}}\right]$ and $C = \left[1 + \frac{2-3\rho^2}{24}v^2 T\right]$. This is a cubic with up to three real roots. Typically there will only be one real root. Should there be more than one real root the smallest possible real root should be chosen [32].

With this, the model is essentially calibrated.

When applying the model to trading data some optimization is needed. It is in reality a question about minimizing the sum of squared errors of v and ρ leaving the optimization problem to be

$$\underset{v,\rho}{min}\sum_i \left(\overline{\sigma}_i - \sigma_B\left(v,\rho,\alpha\left(v,\rho,\sigma_{ATM}\right),K_i,f,\beta\right)\right)^2 \tag{4.18}$$

where $\overline{\sigma}_i$ is the market implied volatility and σ_B is the Black-Scholes implied volatility using the SABR implied volatility. With all parameters estimated the SABR model is properly calibrated.

4.5 The properties of the model

When the model is properly calibrated it is a single self-consistent model for all strikes K which means that the risks calculated at one strike is consistent with strikes calculated at other strikes. With the risks calculated being consistent across strike prices the risks of options on the same underlying asset can be added together. This is a model specific property that means that instead of hedging the risk of every position one by one they can be grouped and one can hedge the residual risk of all the positions instead.

Lets look at the the Greek letters coming from the SABR model. To start, the value of a call option is given by

$$V_{call} = BS\left(f,K,\sigma_B\left(K,f\right),t_{ex}\right) \tag{4.19}$$

This is the value by the Black-Scholes model where the volatility component $\sigma_B\left(K,f\right) \equiv \sigma_B\left(K,f;\alpha,\beta,\rho,v\right)$ is given by equation 4.9 to equation 4.8.

The Vega risk is found by differentiating the value of the call option with respect to the volatility - in the SABR model it is α. It can also be σ_{ATM} as it was used to fit the model with.

$$\begin{aligned}
Vega &\equiv \frac{\partial V_{call}}{\partial \alpha} = \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B \left(K, f, \alpha, \beta, \rho, v\right)}{\partial \alpha} \\
&= \frac{\partial BS}{\partial \sigma_B} \times \frac{\frac{\partial \sigma_B (K,f,\alpha,\beta,\rho,v)}{\partial \alpha}}{\frac{\partial \sigma_{ATM} (f,\alpha,\beta,\rho,v)}{\partial \alpha}} \\
&\approx \frac{\partial BS}{\partial \sigma_B} \times \frac{\sigma_B \left(K, f\right)}{\sigma_{ATM} \left(f\right)} \\
&\approx \frac{\partial BS}{\partial \sigma_B} \times \frac{\sigma_B \left(K, f\right)}{\sigma_B \left(f, f\right)}
\end{aligned}$$

Next is Vanna and Volga as introduced in the beginning of this section. Vanna is given as

$$Vanna = \frac{\partial V_{call}}{\partial \rho} = \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B \left(K, f; \alpha, \beta, \rho, v\right)}{\partial \rho} \tag{4.20}$$

Vanna is the partial derivative of the value of a call option with respect to ρ - the Δ sensitivity with respect to volatility.

Volga is given as

$$Volga = \frac{\partial V_{call}}{\partial v} = \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B \left(K, f; \alpha, \beta, \rho, v\right)}{\partial v} \tag{4.21}$$

Volga is the volatility of volatility. In the SABR model it is found as the partial derivative with respect to v.

Now, lets look at the delta value from the SABR model.

$$\Delta = \frac{\partial V_{call}}{\partial f} = \frac{\partial BS}{\partial f} + \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B \left(K, f; \alpha, \beta, \rho, v\right)}{\partial f}$$

The Δ is the partial derivative with respect to the forward price plus a correction term with respect to the volatility from the SABR model. Hence it is the stochastic volatility instead of a fixed predetermined volatility.

4.6 Conclusion

In this section I started by expanding the greek letters introduced in section 2. The expansion was necessary in order to evaluate the SABR model that was introduced in subsection 4.2.

After defining all variables in the model, the SABR model was shown as

$$df_t = \alpha_t f_t^{\beta} dW_t^1$$

$$d\alpha_t = \nu \alpha_t dW_t^2$$

$$dW_t^1 dW_t^2 = \rho dt$$

In subsection 4.3 I solved the model finding equations 4.6 and 4.9 to be the main results of the 2002 paper by Hagen et. al. [18].

With the main result in place and a general understanding of the model I moved on to calibrating the model in subsection 4.4.

In order to get consistent results from the model, the model estimates must be fitted to observed volatility curves. To do that the models β value is fixed at some value estimated by the trader. Depending on the chosen β value different scenarios play out. Figure 4.1 and figure 4.2 show the volatility curve movement with a β value of zero and a β value of one. These values are the corner solutions for the model.

With a chosen β value the parameters α, v and ρ can be estimated. This is an optimization problem where v and ρ should be minimized. To fit these values α can be set equal to σ_{ATM}, which is the at-the-money volatility.

With the model fitted I moved on to looking at the properties of the model in subsection 4.5. Here I evaluated the value of a call option using the SABR model.

$$V_{call} = BS\left(f, K, \sigma_B\left(K, f\right), t_{ex}\right)$$

The call is defined as the Black-Scholes model using the SABR volatility. With this setup the Δ value is found as

$$\Delta = \frac{\partial V_{call}}{\partial f} = \frac{\partial BS}{\partial f} + \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B\left(K, f; \alpha, \beta, \rho, v\right)}{\partial f}$$

Here, the Δ is the partial derivative of the Black-Scholes model with respect to the forward price plus a correction term with respect to the volatility from the SABR model.

The Vega value is found similarly, but instead of the partial derivative with respect to the forward price it is now the partial derivative with respect to α - the model volatility.

$$Vega \equiv \frac{\partial V_{call}}{\partial \alpha} = \frac{\partial BS}{\partial \sigma_B} \times \frac{\partial \sigma_B\left(K, f, \alpha, \beta, \rho, v\right)}{\partial \alpha}$$

Remembering from subsection 2.5 that Vega is representing the change in value with respect to volatility.

With Vega in place it is now time to focus on Vanna and Volga. Initially I described the intuition in the beginning of this section in subsection 4.1. Equation 4.20 and equation 4.21 from subsection 4.5 states the Vanna and Volga values from the SABR model. Remembering that Vanna is the Delta sensitivity to volatility changes represented by ρ in the SABR model and that Volga is the volatility of volatility represented by v. The delta sensitivity to volatility changes is essentially how much the Delta position is changing with changes in volatility. The volatility of volatility is a measure of how much the volatility is changing when it is changing.

5 Analyzing the Models

In this section I will compare the different models presented in the previous sections. The Black-Scholes model is the benchmark model for this discussion as this model is used to price options in general.

5.1 Comparing the models

Until now this paper has reviewed numerous options pricing models. By now the reader should have a clear idea about what options are and how to price them. Now, it is time to compare the three main models of this paper and to understand the differences between them. After reading this subsection the reader should have a clear understanding of why a particular model is used and what that models key focus is.

5.1.1 Black Scholes

The Black-Scholes model is in many ways the go-to model for options pricing. Especially when you are first introduced to options. If you attend a finance class at university chances are that you have heard about the Black-Scholes model.

However, if you enter into professional finance you will soon realize that the Black-Scholes model is not the best model in practice. The big issue with the Black-Scholes model is that it assumes that the volatility is constant. Hence, the model volatility is constant whereas the market volatility is changing when the price of the underlying asset is changing or when the time to maturity of the option is changing.

The implication of the Black-Scholes model using a fixed model volatility is that the model is not well suited for hedging purposes. The reason being that the hedging will be off in the exact moment of time the hedging is made.

Even though the Black-Scholes model is not well suited for hedging purposes the model is still used. For one, because it is intuitive when learning about options. It takes the student into the options universe without making it too complex. But most importantly, the Black-Scholes model is used in professional finance when quoting prices. That is, when traders are quoting prices to other traders they use the Black-Scholes price of an option. That might seem a bit strange as everyone knows that the prices are quoted on fixed volatilities. But that is the market standard.

In his 2008 letter to investors Warren Buffett reviewed the Black Scholes model. Here, Warren Buffett acknowledges the importance of the Black-Scholes model by writing that the model has approached the status of holy writ. He then gives an example of how the model is wrong if applied to longer maturities. This leads him to write

> *"If the formula is applied to extended time periods, however, it can produce absurd results. In fairness, Black and Scholes almost certainly understood this point well."*

Warren Buffett, (Annual letter to shareholders 2008, page 20 [4])

With this, Buffet touches upon one of the other issues of the Black-Scholes model. In Buffett's example he uses 100 years as the time frame to prove his point. His point is however equally valid on shorter time horizons - that when trying to estimate the Greek's it is only possible to do on shorter time horizons. In other words - the further out in time one wishes to hedge the more insecure. When thinking about it, this makes perfect sense. It is easier to estimate what happens tomorrow - all things equal - than it is estimating what happens in two years time.

One thing that is in favor of the Black Scholes model is that hedging positions usually run for a fairly short period of time. Since hedging essentially is an insurance contract on an underlying asset that might appreciate or depreciate in value over time there is a natural life span on a hedging position. This is intuitive. When a portfolio is hedged, it is done at a point in time where the owner of the portfolio has a certain view on the market. This view is in part reflected in the pricing of the assets in that market. When the market moves, the hedge either kicks in at secures the value of the portfolio or the hedge becomes irrelevant because the market moves in the opposite direction. Hence, the hedging position must be re-hedged in order to become relevant again.

5.1.2 Local Vol

The Local Volatility Model was introduced in section 3. Here I found the implied model volatility to move in the opposite direction of what is intuitively expected. This is depicted in figure 3.5. Here, the volatility curve moves up and to the right when the forward price is falling compared to the initial forward price.

What seems to be intuitively wrong is at the same time proven to be empirically wrong. That is, the observed market behavior is opposite of what the model predicts. To this, Hagan et. al. says

> *"... Due to this contradiction between model and market, delta and vega hedges derived from the model can be unstable and may perform worse than naive Black-Scholes' hedges."*
>
> Hagan. et. al., (Hagan et. al 2002, page 1 [18])

Hagan et. al. argument about the Local Volatility Model as a hedging instrument seems rather convincing. However, as Jim Gatheral points out

> *"It is unlikely that Dupire, Derman and Kani ever thought of local volatility as representing a model of how volatilities actually evolve (...) the idea is more to make a simplifying assumption that allows practitioners to price exotic options consistently with the known prices of vanilla options."*
>
> Jim Gatheral, (Gatheral 2002, page 6 [16])

According to the citation of Gatheral above, the intention of the Local Volatility model was never to be a hedging instrument. Instead, the intent of the model was to provide an instrument that could help price exotic options using a volatility smile of vanilla options. That is, to be able to price more complex options using a volatility smile from the most basic options available.

As the volatility smile in the Local Volatility Model moves in the opposite direction of what is observed in the market it is clear that the model is ill suited for hedging purposes. This is clear as the model will lead to wrongful hedging. With this knowledge it is no surprise that other models has come forward. However, with the statement of Jim Gatheral it is likely that the model was never intended for hedging.

5.1.3 SABR

In section 4 I reviewed the SABR model. After solving the model and showing how to calibrate it to a volatility smile I found that the model makes consistent estimates. Hagan et. al. puts it this way

> *"The SABR model also predicts that whenever the forward price f changes, the implied volatility curve shifts in the same direction and by the same amount as the price f. This predicted dynamics of the smile matches market experience."*
>
> Hagan. et. al., (Hagan et. al 2002, page 16 [18])

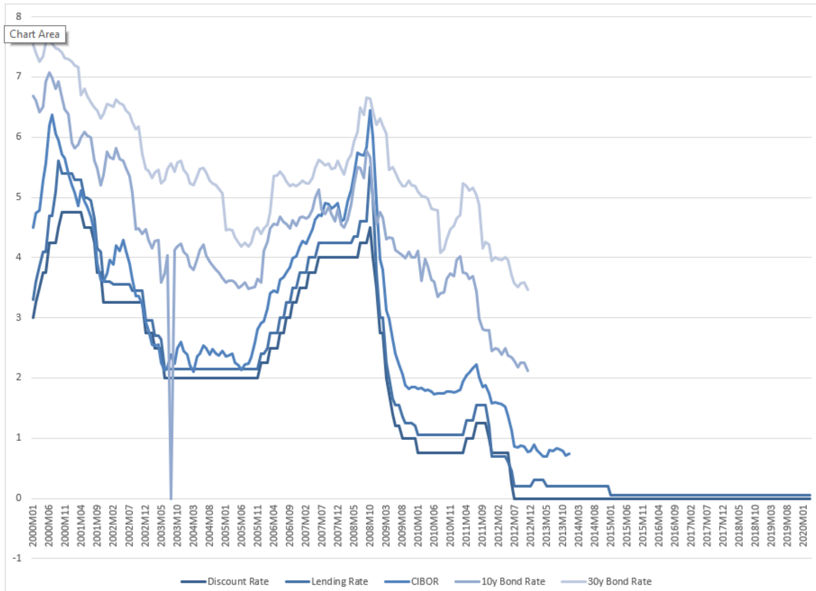
With this it is reasonable to believe that the model is well suited to pricing options and applicable for hedging. From my previous job in Nordea Markets I know that the SABR model is the market standard within options pricing models.

With this knowledge the next logical question to ask is why the SABR model is market standard? In my opinion it comes down to a few things. Most importantly because it integrates with the Black-Scholes model by fixing the issue with constant volatility from the Black-Scholes model.

Further, as Hagan et. al. states, and I restate above, the SABR model predictions are close to what is observed empirically in the financial markets. As this is the case a lot of traders are using the SABR model. This is a further strength of the model. Namely that a lot of people are using the same model for pricing. The strength in this is that people have trust in the prices given by other traders. Put differently. Either everyone is correct or everyone is wrong in their pricing of options.

The SABR model was first proposed in the paper by Hagan et. al. in 2002. At that time the interest rates were well above zero percent.

Figure 5.1: Plot of Danish interest rates from 2000 to 2019



Note: Plot of Danish interest rates month by month from 2000 to 2019. The data shows a clear downward sloping trend for the entire period. The Cibor rate, 10y bond rate and 30y bond rate data expire earlier than the discount rate and the lending rate. [9]

Figure 5.1 shows a sample of Danish interest rates. The interest rates are showing a clear downward sloping trend over time. The interest rates are especially dropping around the last part of 2008. The data points for the Cibor rate, 10 year bond rate, and 30 year bond rate are not available past 2013. However, both the Cibor rate and the long term bond rates have been dropping in the time period.

The low interest rate market raised some issues with the SABR model. The SABR model with $\beta \neq 0$ is implicitly assuming the interest rates to be strictly positive. The new low interest market violates this assumption. Hence, the SABR model has been reviewed and new extensions to the SABR model appeared.

5.2 SABR Models for low interest rate markets

With the low interest rate environment the extensions to the SABR model are important. For that reason I will briefly mention some of the extensions below. Breaking down the models, however, is beyond the scope of this paper.

The shifted SABR model [25] is one of the simplest models trying to coupe with the low interest rate environment. The shifted SABR model using the forward rate is setup as

$$\begin{aligned} dF_t &= \sigma_t (F_t + s)^{\beta} dW_t^1,\ F(0) = f \\ d\sigma_t &= v\sigma_t dW_t^2,\ \sigma(0) = \alpha \end{aligned}$$

$dW_t^1 dW_t^2 = \rho dt$ and s is a positive deterministic shift.

Using the initial values $F(0) = f$ and $\sigma(0) = \alpha$ makes the model quite similar to the SABR model derived in section 4. The only real difference being the addition of the shift, s. The shift, however, changes the lower boundary from 0 to -s allowing the F_t to reach negative levels.

Another extension to the SABR model is the Free Boundary SABR model [1]. The forward rate is assumed to have the following dynamics

$$\begin{aligned} dF_t &= v_t |F_t|^{\beta} dW_t^1,\ F(0) = F_0 \\ d\sigma_t &= \gamma v_t dW_t^2,\ v(0) = v_0 \end{aligned}$$

$dW_t^1 dW_t^2 = \rho dt$, and with $0 \leq \beta < \frac{1}{2}$, and a free boundary.

The free boundary model is not bounding how negative the interest rate can become. This makes the Free Boundary SABR model quite flexible when applying it to market data.

5.3 Conclusion

In this subsection I have compared the three models presented in this paper. The Black-Scholes model, the Local Volatility model and the SABR model.

I found that the Black-Scholes performance over long time periods becomes absurd. But also that Black and Scholes probably knew this. None the less the Black-Scholes model is still widely used for a lot of reasons. And nothing suggests that this will change.

The Local Volatility model was found to move in the opposite direction of the observed market data. I this subsection it is stated that the model was probably not made to try to replicate the volatility behavior. Instead it was made as a tool for easily pricing exotic options. Hence, the intent of the model was never to use it for hedging of portfolios.

Lastly, I find that the SABR model is handling the volatility smile in a desirable way. The SABR model output is in line with what is observed in the market. For that reason the SABR model is well suited for hedging purposes. One issue with the model, however, is that it relies on an implicit assumption of the interest rates being strictly positive. In the current low interest rate environment this assumption might be violated. For that reason I finish this subsection with introducing some adjusted SABR models.

6 Conclusion

In this paper I have taken the reader through some of the theory regarding options pricing. In section 2 I introduced the reader for options in general - starting with the binomial tree model and ending with the Black-Scholes model and the Black-Scholes Greek letters [22].

In section 3 I reviewed the Local Volatility model. This is a model simultaneously presented by Dupire [14] and by Derman and Kani [10]. The objective of the Local volatility model is to incorporate the volatility smile in the pricing of options.

Form the Local Volatility model I moved on to reviewing the SABR model in section 4. The SABR model is presented by Hagan et. al. [18]. The strength of the SABR model is that it allows the volatility to be stochastic which none of the previous reviewed models do.

After reviewing the SABR model I analyzed the different models before introducing the reader to some adjusted SABR models.

Common to most of the theory reviewed in this paper is that it builds on the Black-Scholes model in some sense. This makes sense since the Black-Scholes model is still the model used when quoting options prices.

In subsection 1.1 I wrote the purpose of this paper - including some problems that I wanted to answer with this paper. In total, I wrote down three questions. They are restated below.

Problem. The Black-Scholes model is a well-known options pricing model. Is there a model that handles options pricing for hedging purposes better than the Black-Scholes model?

Problem. What are the arguments for choosing one model over another?

Problem. What are the implications of choosing a wrong or less precise model?

The question to answer first is if there is a model to price options for hedging purposes that are better than the Black-Scholes model? In the paper I have reviewed a lot of theory about the different models. It is clear that the Black-Scholes model is still used for a variety of things. But it is also clear that the Black-Scholes model is not the best model for hedging purposes. So the short answer to the question is: Yes, there are models better suited to price options for hedging purposes. The logical next question then is - which models?

I reviewed the Local Volatility model as an alternative to the Black-Scholes model. But it turns out that this model is not ideal for hedging purposes. As Gatheral stated.

> *"It is unlikely that Dupire, Derman and Kani ever thought of local volatility as representing a model of how volatilities actually evolve*

(...) the idea is more to make a simplifying assumption that allows practitioners to price exotic options consistently with the known prices of vanilla options."

Jim Gatheral, (Gatheral 2002, page 6 [16])

Gatheral's statement is a statement about the model dynamics. It turns out that the models implied volatility moves in the opposite direction of that observed in the market. This is depicted in figure 3.5. The implication of this is that the delta hedging of the option is wrong.

Disregarding the Local Volatility model as a model of choice when hedging - as it produces wrongful hedges and never was intended for risk management, according to Gatheral - I moved on to the SABR model.

The SABR model turns out to be a better solution. One of the main arguments for a model like the SABR model is that it allows for stochastic volatility. According to Gatheral [16], most people engaged in the stock markets agree that the prices are evolving stochastically. For that reason it makes sense to let the volatility evolve stochastically.

When comparing the SABR model results to that observed in the markets the SABR model performs well. Hence, the SABR model handles option pricing for hedging purposes better than the Black-Scholes model - this is probably why the SABR model is widely used in professional finance today.

The arguments for choosing one model over another should be quite clear after reading this paper. As some models are performing really bad you might be worse off using those models than using the simple Black-Scholes model - or not hedging at all. The implications of choosing a wrong or less precise model is at best more expensive hedging. At worst it is wrongful hedging meaning that you are not covered the way you thought. Using a model like the Local Volatility model when hedging could end up meaning that you are way more exposed than you thought.

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