Star coloring of Hypercubes

Master's Thesis
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Synopsis:

In 2004, G. Fertin, A. Raspaud and B. Reed published their proof of the bounds to the star chromatic number of Hypercubes $\chi_s(Q_n)$. These are two linear functions where the gap between them grows for every dimension n in Q_n. In examples on star coloring of Hypercubes, it was noticed that the lower bound does not hold. For this Master's Thesis, this has been examined and since the bounds are based on the relation made between star coloring and acyclic coloring, this thesis has sought to establish possible criteria to separate them in order to draw the upper and lower bound to the star chromatic number for Hypercubes closer together. This has led to the introduction of the concept of star components and how these with the right criteria will proof that the lower bound is too low for a star coloring of a hypercube to be correct.

Dansk resumé

Dette speciale er skrevet ved Institutet for Matematiske Fag, Aalborg Universitet og behandler begreber indenfor punktfarvning af grafer. Stjernefarvning er en gren af punktfarvning, hvor to nabopunkter skal være farvet forskelligt og hvor en grafisk vej af længde 3 aldrig må blive to-farvet. Som for enhver punktfarvning opstår et optimeringsproblem: Hvad er det mindste mulige antal farver, der findes nødvendig for at kunne punktfarve grafen korrekt? For stjernefarvning kaldes dette det stjerne-kromatiske tal $\chi_s(G)$ for en graf G.

Grafen der behandles i dette speciale er de såkaldte Hypercubes Q_n. Disse grafer er konstrueret ved et kartetisk produkt, hvilket giver for dimension n en n-regulær graf med 2^n punkter og $n2^{n-1}$ kanter.
I 2004, Guillaume Fertin, André Raspaud og Bruce Reed sammen deres bevis for øvre og nedre grænser for the stjernekromatiske tal $\chi_s(Q_n)$.

Hovedvægten for dette speciale lægges på, at G. Fertin, A. Raspaud og B. Reed bygger deres resultater for stjernefarvning på acyklisk farvning og at det acykliske kromatiske tal er mindre end eller lig med det stjerne-kromatiske tal. Med andre ord, alle stjernefarvninger er acykliske, men langt fra alle acykliske farvninger er stjernefarvninger.
Begge grænser for det stjerne-kromatiske tal er linære funktioner afhængige af dimensionen n og når denne vokser, vokser afstanden mellem funktionsværdierne også. Derfor er det af stor interesse i dette speciale, at undersøge og om muligt at indsnævre denne afstand.
Dertil undersøges først udførte stjernefarvninger af Hypercubes for at bestemme givne resultater for disse.
Dernæst undersøges de anvendte argumenter i beviserne for Guillaume Fertin, André Raspaud og Bruce Reeds resultater.

Prolog

This Master of Science Thesis is written by Anette Hyllested Grønhøj in the autumn of 2019 during the last semester of a Master's degree programme in Mathematics at the Department of Mathematical Sciences at Aalborg University.

The primary literature for this paper is "Star coloring of a graph" by G. Fertin, A. Raspaud, and B. Reed, consulted by "Acyclic and k-distance coloring of the grid" by G. Fertin, E. Godard, and A. Raspaud. For basic theory in advance "Graphs and Digraph" by G. Chartrand, L. Lesniak, and P. Zhang has been consulted. A full bibliography is provided on the last page.
The reader is expected to have a certain knowledge of Graph Theory beforehand and to possess the mathematical qualifications corresponding to completion of a bachelor education in Mathematical Sciences as a minimum.

The author wishes to thank her supervisor Oliver Wilhelm Gnilke for his help and supervision, and the staff at Department of Mathematical Sciences at Aalborg University for their support.

Reader's Guide
The first chapter will first introduce general theory on proper vertex coloring in order to define the general k-chromatic number after which the chapter narrows down to define star coloring and its general properties.

The second chapter is devoted to Hypercubes and its star chromatic number $\chi_s(Q_n)$.

The third chapter then treats the results found in the previous chapters and confronts the properties of the lower bound to $\chi_s(Q_n)$.

The final chapter then concludes the thesis with its results and opens up to further investigation.

Sections, figures, mathematical definitions, etc., are numbered according to the chapter, i.e. the first figure in Chapter 2 has number 2.1 etc. References to an equation is on the form (x.y).

Anette Hyllested Grønhøj - Matematik - Aalborg Universitet

Anette Hyllested Grønhøj

Contents

0.1 Introduction

Within the mathematical field of Graph Theory, the perhaps best known and most studied area is inarguably coloring. Within the study of vertex coloring, one is the primary problems is one of optimization. That is, what is the minimum number of colors necessary for the vertex set of a graph such that no two adjacent vertices are assigned the same color? otherwise known as the chromatic number of G, denoted by $\chi(G)$. This number can be difficult to obtain, yet is rather straight forward to verify. Because of this difficulty, boundaries to $\chi(G)$ have constructed various times.

One particular type of vertex coloring is called star coloring. Star coloring consists of two conditions: 1) two adjacent vertices cannot be assigned the same color and 2) any path of length 3 cannot be bicolored.
In 2004, G. Fertin, A. Raspaud and B. Reed published their results to determine boundaries to the star chromatic number for a number of distinct families of graphs. The case of interest to this thesis is the star chromatic number of Hypercubes $\chi_s(Q_n)$. All star color boundaries, in their work, are based on their observed relation between star coloring and acyclic coloring.

The purpose of this thesis is to examine these boundaries to the star chromatic number of Hypercubes $\chi_s(Q_n)$.

0.2 Problem Statement and Thesis

For a Hypercube Q_n of dimension n, the star chromatic number is bounded above and below by

$$\left\lceil \frac{n+3}{2} \right\rceil \leq \chi_s(Q_n) \leq n+1.$$

These bounds are both linear functions according to the variable n. However as n grows, so does the gap between the two bounds.
For this reason, this thesis will examine both bounds both in terms of validity in actual execution as well as the theory precedent to the bounds. The goal of this thesis is to then examine any possibility of improvement to the gap between the two boundaries and present any given results that may emerge during the process.

1 Vertex Colorings

Vertex coloring, in Graph theory, is the assignment of colors to the vertices of G. Unless stated otherwise, this piece of theory is based on [1].
The main focus of this paper is the concept of *star coloring* which is a type of coloring related to the general concept of *proper vertex coloring.* Therefor, an explanation of the latter along with its properties will be given first and then, an elaboration of the main topic will be given.

Definition 1.0.1: Proper Vertex Coloring

A Vertex coloring in which adjacent vertices are assigned different colors is called *proper vertex coloring.*

For simplicity, we choose the positive integers $1,2,\ldots,k$ for some positive integer k, to represent colors. Thus, the proper coloring can be considered as a function

$$c : V(G) \to \mathbb{N}$$

such that $c(u) \neq c(v)$ if u and v are adjacent in G.

In this paper, any coloring is assumed to have been executed through *the Greedy Coloring Algorithm*:

Algorithm 1.0.1: The Greedy Coloring Algorithm

Suppose that the vertices of a graph G are listed in the order $v_1,v_2,\ldots,v_n$.

1. The vertex v_1 is assigned color 1.
2. Once the vertices $v_1,v_2,\ldots,v_j$ have been assigned colors, where $1 \leq j < n$, the vertex v_{j+1} is assigned the smallest color that is not assigned to any neighbours of v_{j+1} belonging to the set $\{v_1,v_2,\ldots,v_j\}$.

If G is colored using k colors, this coloring is referred to as a k-coloring.

Definition 1.0.2: Color class

Assume c is a k-coloring of a graph G.
If $V_i(1 \leq i \leq k)$ is the set of vertices colored i and is nonempty, then V_i is called a *color class.*
The nonempty elements of the set $\{V_1,V_2,\ldots,V_k\}$ gives a partition of $V(G)$.

Due to the proper vertex coloring of G, each of the nonempty color classes $V_i (1 \leq i \leq k)$ is an independent set of vertices of G.

1.1 the k-Chromatic Number

G is called *k-colorable* if there exists a k-coloring of G.
The minimum positive integer k for which G is *k-colorable* is the *chromatic number*, $\chi(G)$ of G.
So, if $\chi(G) = k$, there exists a k-coloring of G, but no $(k-1)$-coloring of G. From this, it is clear that a graph G is k-colorable if and only if $\chi(G) \leq k$.
So, for a graph of order n, which is thus n-colorable, the range for the chromatic number is

$$1 \leq \chi(G) \leq n.$$

This is where $\chi(G) = n$ if and only if $G = K_n$, the complete graph of order n, and $\chi(G) = 1$ if and only if $G = \bar{K_n}$ - the complement of K_n.

Properties 1.1.1: the Calculation of $\chi(G)$

In practice, the chromatic number k is calculated by, first showing that the graph is k-colorable - that is, $\chi(G) \leq k$ - and by showing that every coloring of G requires at least k colors - that is, $\chi(G) \geq k$.

So, in terms of color classes, a k-coloring of a graph G gives a k-partition of G into k color classes.

As there is no general formula for the chromatic number of a given graph, there are bounds given in the following theorems:

Theorem 1.1.1: Upper bound for the chromatic number

For every graph G

$$\chi(G) \leq 1 + \Delta(G), \tag{1.1}$$

where $\Delta(G)$ is the maximum degree of G.

Proof. Assume that the greedy coloring algorithm 1 is applied and that the vertices of G are listed as done in the algorithm.
Then v_1 is assigned color 1 and for $2 \leq i \leq n$, the vertex v_i is either assigned color 1 or the color $k+1$, where k is the largest integer where all of the colors 1,2,...,k are used to color the neighbours of v_i in the set $S = \{v_1, v_2, \ldots, v_{i-1}\}$.
Since at most $\deg(v_i)$ neighbours of v_i belongs to S, the largest value of k is $\deg(v_i)$.
Therefore, the color assigned to v_i is at most $1 + \deg(v_i)$ and thereby

$$\chi(G) \leq \max_{1 \leq i \leq n} \{1 + \deg(v_i)\} = 1 + \Delta(G),$$

which is the desired result. ∎

In order to determine a lower bound for the chromatic number of G, the chromatic number of its subgraph H is introduced:

Theorem 1.1.2

If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.

Proof. Assume that $\chi(G) = k$.
Then there exists a k-coloring c of G.
Since c assigns distinct colors to every two adjacent vertices in G, the coloring c does the same in H.
Therefore, H is k-colorable and thus, $\chi(H) \leq k = \chi(G)$. ∎

So, in view of theorem 1.1, if a graph contains a complete subgraph, also known as a *clique*, K_k, then $\chi(G) \geq k$. The *clique number* $\omega(G)$ of a graph G denotes the largest clique of G and a clique of order k is called a k-clique.
Thus an immediate result as a consequence of theorem 1.1 is given:

Corollar 1.1.1: a Lower Bound for the Chromatic Number of G

For every graph G, $\chi(G) \geq \omega(G)$.

The theory about bounds to $\chi(G)$ is the main concern within this paper about *star coloring* and the *star chromatic number* of a graph G.

1.2 Star Coloring of a graph G

In this part of proper vertex coloring, it may be useful that in [1], it is noted that a *star* S_n, in graph theory, is the complete biparte graph $K_{1,t}$, for an integer tm of order $n = 1 + t$.
This section is otherwise based on [2].
In figure 1.1, examples of proper vertex colored star graphs are given. The original picture is acquired from [3]

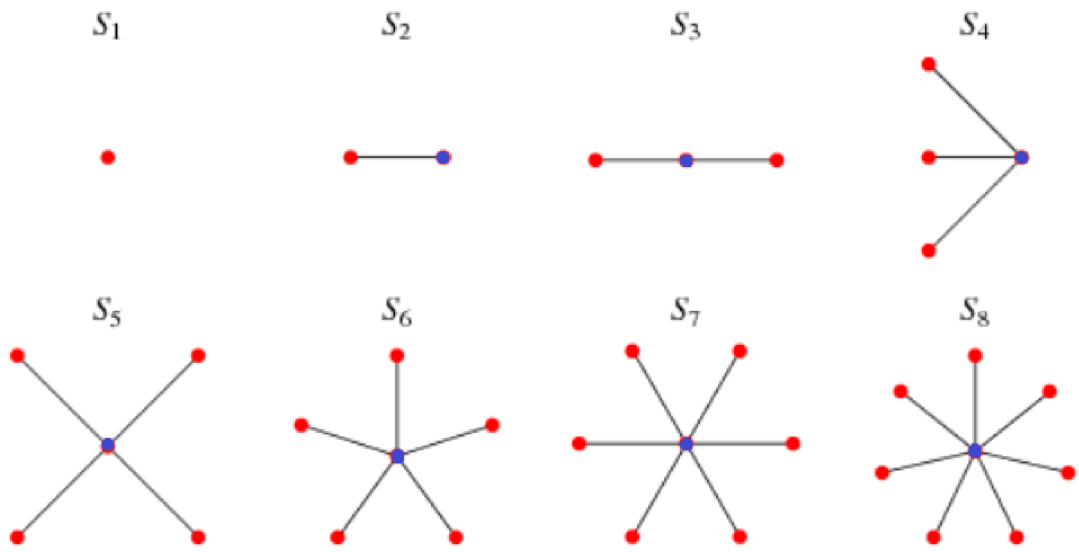


Figure 1.1. Proper vertex colored Star graphs

This understanding of what a star graph is, will aid in the understanding of what a star coloring is.

Definition 1.2.1: Star Coloring

A star coloring of a graph G is a proper vertex coloring of G such that no path of length 3 in G is bicolored.

The star graphs shown in figure 1.1 show that the maximum path length of a star is 2. So, in terms of vertex coloring, this means that, in order to maintain the star formed by the relationship between two colors, the second a path of a graph reaches length 3, a third colors needs to be added. An example of star coloring of a graph is provided by [4] and is given in figure 1.2. Comparing this to figure 1.1 visualizes the definition of star coloring.

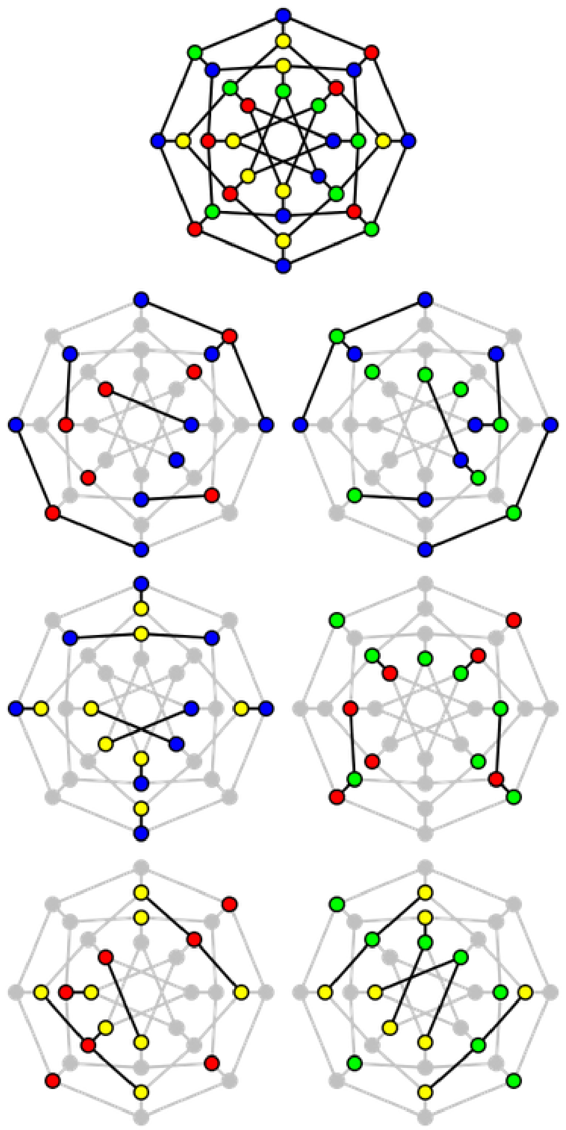


Figure 1.2. An example of star coloring of a graphs

Along with this definition, the *star chromatic number*, denoted by $\chi_s(G)$, is the minimum number of colors needed to star color a given graph G.
Thus, in terms of a star graph S_n, $\chi_s(S_1) = 1$ and $\chi_s(S_n) = 2$ for any $n > 1$. In figure 1.2, $\chi_s(G) = 4$.

1.3 The Star Problem

The process to achieve the chromatic number (star or k) does not have a definitive algorithm, and depends on repetitive use of the greedy coloring algorithm 1 until a minimum number of colors for the given type of coloring is achieved.
A current method in order to realize any type of shortcut to this Brute Force approach - where you simply try any possible approach until you obtain the wanted end result - is, as done for any graph in theorems 1.1 and 1.1, by establishing boundaries to the chromatic

number depending on the structure of the graph in question.

In chapter 2.1, this paper turns its face to the bounds for the star chromatic number for Hypercubes presented in theorem 2.1.1. However, in order to successfully prove the bounds of this theorem, results from [2] and [5] need to be noted.

Firstly, [2] draws a connection between star coloring and acyclic coloring:

Definition 1.3.1: Acyclic Coloring, $a(G)$

An acyclic coloring of a graph G is a proper vertex coloring of G such that no cycle in G is bicolored.
Here, the acyclic chromatic number is denoted $a(G)$.

An acyclig graph is also known as a forest, as an acyclic graph contains no cycles and thus, only consists of trees. An understanding of this is useful for the general understanding of the proof of theorem 1.3.

[2] notes that for any graph G, any star coloring of G is also an acyclic coloring of G:

"indeed, a cycle in G can be bicolored if and only if it is of even length, that is of length greater than or equal to 4. However, by definition of a star coloring, no path of length 3 in G can be bicolored."

So, for a cycle of length 3, a triangle, star coloring and acyclic coloring, by definition, are the same, as star coloring never moves beyond a path length of 3.
This gives the observation, that

$$a(G) \leq \chi_s(G). \tag{1.2}$$

A result that is necessary in order to prove theorem 2.1.1, is proven via use of the acyclic chromatic number and observation (1.2):

Theorem 1.3.1

For any graph $G = (V,E)$, let $n = |V|$, $m = |E|$ and $\gamma = 4n(n-1) - 8m + 1$.
Then, we have:

$$\chi_s(G) \geq a(G) \geq \frac{2n + 1 - \sqrt{\gamma}}{2} \tag{1.3}$$

Proof. Let $a(G) = p$.
let V_i, $1 \leq i \leq p$, be color class i in an acyclic coloring of G using p colors.
By definition, the subgraph of G induced by any $V_i \cup V_j$, $1 \leq i < j \leq p$, is a forest.
Let $e_{i,j}$ be the set of edges covered by this forest.
Then, for any two distinct pairs of vertices (i_1,j_1) and (i_2,j_2) with $1 \leq i_1 < j_1 \leq p$ and $1 \leq i_2 < j_2 \leq p$, we have $e_{i_1,j_1} \cap e_{i_2,j_2} = \emptyset$.

It can be seen that the number of pairwise distinct pairs of colors is equal to $\frac{p(p-1)}{2}$, and that over these $\frac{p(p-1)}{2}$ pairs of colors, each color $1 \leq k \leq p$ appears $p-1$ times. Moreover, for each pair (i,j), with $1 \leq i < j \leq p$, we have $|e_{i,j}| \leq |V_i| + |V_j| - 1$. Thus, it additionally holds that $\sum_{(i,j)} |e_{i,j}| = m$.

Combining these two results, we obtain that

$$m \leq \sum_{(i,j)} |V_i| + |V_j| - 1, \tag{1.4}$$

that is

$$m \leq (p-1)(\sum_{k=1}^{p} |V_k|) - \frac{p(p-1)}{2}. \tag{1.5}$$

Since $\sum_{k=1}^{p} |V_k| = n$, we get that $m \leq n(p-1) - \frac{p(p-1)}{2}$, which by regular Algebra gives

$$p^2 - (2n+1)p + 2(m+n) \leq 0. \tag{1.6}$$

Let $\gamma = 4n(n-1) - 8m + 1$.
Since $m \leq \frac{n(n-1)}{2}$, we have $\gamma \geq 1$ in all cases. Thus, we can conclude that

$$\frac{2n+1-\sqrt{\gamma}}{2} \leq p \leq \frac{2n+1+\sqrt{\gamma}}{2}.$$

However, we can see that the upper bound is not relevant, as we always have $m \leq \frac{n(n-1)}{2}$, that is $\gamma \geq 1$; hence the least value for $\frac{2n+1+\sqrt{\gamma}}{2}$ of $n+1$.
However, it is obvious that $p \leq n$ in all cases.
Thus, along with the observed relation between $a(G)$ and $\chi_s(G)$ the result in the theorem is given. ∎

In [5], it is stated that this general lower bound is optimal for several families of graphs. Examples mentioned are graphs such as trees (where $\gamma = (2n-3)^2$ and thus $a(G) \geq 2$), cycles (where $2n - 4 < \sqrt{\gamma} < 2n - 3$ and thus, $a(G) \geq 3$), and complete graphs (in which case, $\gamma = 1$ and thus, $a(G) \geq n$).
[2] additionally states that a star coloring can be noted as an acyclic coloring such that if we have two color classes, then the induced subgraph is a forest composed only of stars.

the Hypercube Q_n 2

In this chapter, Hypercubes are formally defined after which the the star chromatic number for Hypercubes are proven based on [5] and [2]. This is then followed by examples of star colorings.
The introductory definition of a Hypercube and its general properties are based on [1].
A *Hypercube*, also known as an n-cube, is denoted Q_n.

Definition 2.0.1: the n-cube Q_n

If $n = 1$, $Q_n = K_2$.
For $n \geq 2$, Q_n is recursively defined at the cartesian product

$$Q_{n-1} \times K_2$$

of Q_{n-1} and K_2.

This is illustrated in a Graphic Interchange Format (a GIF) in [6] which illustrates the cartesian product as the transition from Q_n to Q_{n+1} from a single vertex to a line, Q_1, to a square, Q_2, to a cube, Q_3, and then into a tesseract, Q_4.
The graph Q_n is thus an n-regular graph of order 2^n.
The n-cubes for n = 1,2 and 3 are illustrated in figure 2.1

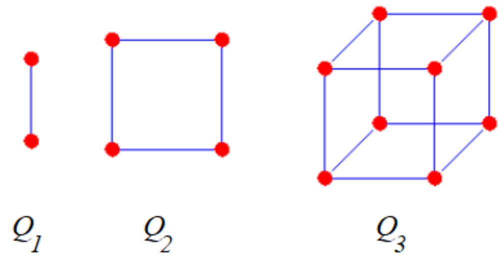


Figure 2.1. the n-cubes, Q_1,Q_2 and Q_3

The vertex set $V(Q_n)$ of a given n-cube can be represented in binary code, using ordered n-tuples $(a_1,a_2,\ldots,a_n)$ or $a_1a_2\cdots a_n$ where $a_i \in \{0,1\}$ for $1 \leq i \leq n$. This is used such that two vertices are adjacent if and only if the corresponding n-tuples differ at precisely one coordinate. For the hypercubes in figure 2.1 the binary presentation is illustrated in figure 2.2.

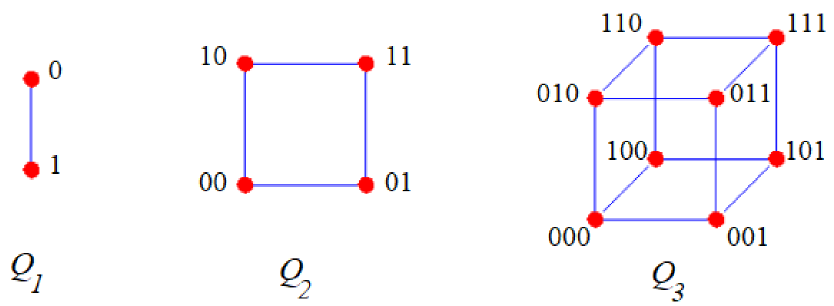


Figure 2.2. the n-cubes from 2.1 with binary representations of the vertex set attached.

This use of bitstrings to identify the elements in $V(G)$ of a hypercube Q_n is a tool often used within coding theory.

In [7], Hamming distance is often used to quantify the extent to which two bitstrings of the same dimension differ. A formal definition is given by:

Definition 2.0.2

the Hamming distance between two bitstrings, x and y is defined as

$$d(x,y) = |\{i : x_i \neq y_i\}|$$

For instance, between bitstrings 111 and 000 in figure 2.2, the Hamming distance is 3.
An early application was in the theory of error-correcting codes where the Hamming distance measured the error introduced by noise over a channel when a message, typically a sequence of bits, is sent between its source and destination.

The binary representation is here used to prove the upper bound to the star chromatic number for Hypercubes.

2.1 the Star Coloring of Hypercubes

In chapter 1.1, it was proven that although there were no direct computation of the k-chromatic number, there were both upper and lower bounds to it. In fact, [2] gives an estimate for upper and lower bounds to the star chromatic number $\chi_s(Q_n)$ in theorem 2.1.1.

2.1.1 Acyclic coloring of Hypercubes and Grids

In [5], Hypercubes are explained via the construction of graphs called Grids whose elements in $V(G)$ also are represented by coordinates presented in tuples according to dimension.

Definition 2.1.1: the 2-dimensional Grid

the 2-dimensional Grid graph $G(m,n)$ is the Cartesian product $P_m \times P_n$ of path graphs on m and n vertices, respectively.
This graph has mn vertices
and $(m-1)n + (n-1)m = 2mn - m - n$ edges.
A vertex in $G(m,n)$ is identified with tuples (m_i,n_i)

Expanding this, the d-dimensional Grid is defined as

Definition 2.1.2: the d-dimensional Grid

Let $d \in \mathbb{N}$ and $(n_1,\dots,n_d) \in \mathbb{N}^d$, with $n_i \geq 2$ for any $1 \leq i \leq d$. The d-dimensional grid of lengths $n_1,\dots,n_d$, denoted by $G_d(n_1,\dots,n_d)$, is the following graph:

$$\begin{aligned} V(G_d(n_1,\dots,n_d)) &= [1,n_1] \times [1,n_2] \times \dots \times [1,n_d] \\ E(G_d(n_1,\dots,n_d)) &= \{\{u,v\} | u = (u_1,\dots,u_d), v = (v_1,\dots,v_d), \\ &\quad \text{and there exists } i_0 \text{ such that } \forall i \neq i_0, u_i = v_i, \\ &\quad \text{and } |u_{i_0} - v_{i_0}| = 1\} \end{aligned}$$

Additionally for any d-dimensional Grid $G_d(n_1,\dots,n_d)$, we have:

$$\begin{aligned} |V(G_d(n_1,\dots,n_d))| &= n_1 \times \dots \times n_d \\ |E(G_d(n_1,\dots,n_d))| &= n_1 \times \dots \times n_d \times (d - \sum_{i=1}^{d} \frac{1}{n_i}) \end{aligned}$$

The attentive reader may have noticed that for $n_i = 2$, $1 \leq i \leq d$, the d-dimensional grids are in fact hypercubes Q_n of dimension $n = d$.

In theorem 2.1.1 the proof of the upper bound is based on the proof of the upper bound for acyclic coloring of d-dimensional Grids.

Theorem 2.1.1

Let $n_1,\dots,n_d \in \mathbb{N}$ with $n_i \geq 2$ for any $1 \leq i \leq d$.
For any grid $G_d(n_1,\dots,n_d)$ of dimension d,

$$a(G_d(n_1,\dots,n_d)) \leq d + 1$$

Proof. Each vertex u of $G_d(n_1,\dots,n_d)$ is defined by its coordinates, ie. $u = (x_1,x_2,\dots,x_d)$, where $0 \leq x_i \leq n_i - 1$.
Let us define the following coloring:
Each vertex $u = (x_1,x_2,\dots,x_d)$ is assigned color

$$c(u) = (\sum_{i=1}^{d} i \cdot x_i) \bmod d + 1 \tag{2.1}$$

which uses no more than $d + 1$ colors.

First we show that this is a proper coloring:
Assume that two adjacent vertices u and u' are assigned the same color c.
Assume also that the coordinates of u and u' differ in the j-th dimension. That is,

$$u = (x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_d) \text{ and}$$
$$u' = (x_1,x_2,\dots,x_{j-1},x_j \pm 1,x_{j+1},\dots,x_d)$$

and by definition of $c(u)$ and $c(u')$ we have

$$j \cdot x_j + \sum_{i=1,i\neq j}^{d} \equiv (j \pm 1) \cdot x_j + \sum_{i=1,i\neq j}^{d} i \cdot x_i \bmod d+1.$$

This gives $\pm j \equiv 0 \bmod d+1$, but $j \in [1;d]$, so this is impossible and thus, two adjacent vertices cannot have the same color by this coloring.

Now we prove that this is an acyclic coloring:
Take any two distinct colors c_1 and c_2 from coloring (2.1). Assume two adjacent vertices u_1 and u_2 in $G_d(n_1,\dots,n_d)$ are assigned the colors c_1 and c_2, respectively.
Suppose that the coordinates of u_1 and u_2 differ on dimension j.
By definition of the coloring (2.1), we get the following equality:

$$c_1 - c_2 \equiv \pm j \bmod d+1.$$

W.l.o.g., suppose that $c_1 > c_2$.
Thus, there exists only one neighbor u_2' of u_1 other than u_2 for which $c(u_2') = c(u_2) = c_2$.
This neighbor u_2' differs in the $(d+1-j)$-th coordinate from u_2.
Now, to find a neighbor for u_2', other than u_1, which is assigned color c_1, using the same argument as for finding u_2', it is notable that it must differ from u_1 in the j-th coordinate.
By induction it comes to light that for any distinct pair of colors c_1 and c_2 with $c_1 > c_2$, the vertices assigned either c_1 or c_2 form a cycle if and only if they lie in the same 2-dimensional subgrid of $G_d(n_1,\dots,n_d)$. Here, this subgrid $G_2(c_1,c_2)$ is induced by dimension $j = c_1 - c_2$ and $j' = d+1-j$ - cf. Figure 2.3.

No bicolored cycle can exist in that case, because in $G_2(c_1,c_2)$, the only authorized moves between a given pair of colors will be either N and W; N and E; S and W; or S and E.
Therefore, the path will take on a stair-like shape and no cycle can be created. ∎

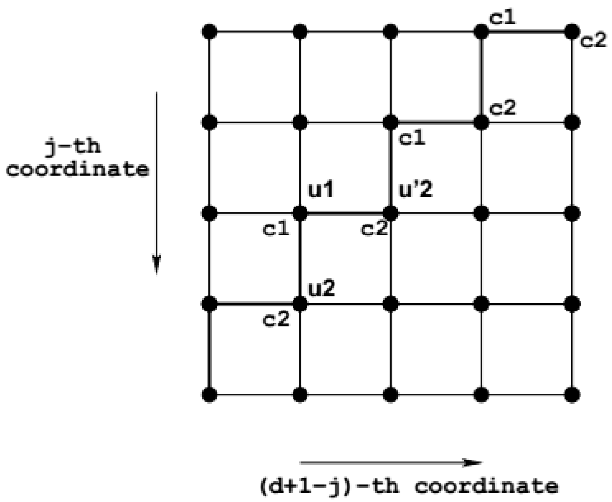


Figure 2.3. From [5], $G_2(c_1{,}c_2)$, the subgrid of $G_d(n_1,\dots{,}n_d)$ induced by dimensions $j = c_1 - c_2$ and $j' = d + 1 - j$

The coloring defined in the proof of theorem 2.1.1 is used in the proof of theorem 2.1.1:

Theorem 2.1.2: (Star-Coloring of Hypercube of dimension n, Q_n

For any n-dimentional hypercube, Q_n,

$$\left\lceil \frac{n+3}{2} \right\rceil \leq \chi_s(Q_n) \leq n+1 \tag{2.2}$$

Proof. The proof will begin with a proof of the lower bound and then prove the upper bound.
The lower bound is a direct consequence of theorem 1.3, where $|V| = 2^n$ and size $m = n \cdot 2^{n-1}$, which means we have

$$\chi_s(Q_n) \geq \frac{2|V| + 1 - \sqrt{\gamma}}{2},$$

where $\gamma = 4|V|(|V| - 1) - 8m + 1$.
Now we prove that $\chi_s(Q_n) > \frac{n+2}{2}$.
For this, we now show that $f(|V|{,}n) = \frac{2|V|+1-\sqrt{\gamma}}{2} - \frac{n+2}{2} > 0$.
Note that

$$\begin{aligned} f(|V|{,}n) &= \frac{(2|V| - 1 - n) - \sqrt{\gamma}}{2} \\ &= \frac{(2|V| - 1 - n) - \sqrt{\gamma}}{2} \cdot \frac{(2|V| - 1 - n) + \sqrt{\gamma}}{(2|V| - 1 - n) + \sqrt{\gamma}}. \end{aligned}$$

That is, $f(|V|{,}n) = \frac{(2|V|-1-n)^2-\gamma}{2((2|V|-1-n)\sqrt{\gamma})}$.

However, the denominator, $D(|V|{,}n) = 2((2|V| - 1 - n) + \sqrt{\gamma})$, is positive for any $n \geq 1$, as $\sqrt{\gamma} \geq 1$ in all circumstances and $|V| = 2^n$.

Therefore, it is enough to show that the nominator $f'(|V|,n) = (2|V| - 1 - n)^2 - \sqrt{\gamma}$ is positive in order to prove the lower bound.

$$f'(|V|,n) = (2|V| - 1 - n)^2 - (4|V|^2 - 4|V| - 8m + 1),$$

thus, $f'(|V|,n) = n^2 - 4|V|n + 2n - 8m$.
Since $m = \frac{|V|n}{2}$, it is concluded that $f'(|V|,n) = n(n+2) > 0$ for any $n \geq 1$.Therefore, $\chi_s(Q_n) > \frac{n+2}{2}$, that is $\chi_s(Q_n) \geq \left\lceil \frac{n+3}{2} \right\rceil$.

In order to prove the upper bound we give the following coloring:
Suppose the vertices of Q_n are labeled according to their binary presentation; that is, every vertex $u \in V(Q_n)$ is labeled as such: $u = b_1 b_2 \dots b_n$, with every $b_i \in \{0,1\}$ for $1 \leq i \leq n$.
We then assign a color $c(u)$ to u according to the following equation:

$$c(u) = \sum_{i=1}^{n} i \cdot b_i \bmod n+1.$$

From the proof of theorem 2.1.1, we have that this coloring is acyclic and that any bicolored path in Q_n can only appear in a copy of a 2-dimensional square Q_2. So, since this coloring is acyclic, we can conclude that no bicolored path of length strictly greater than 2 can appear, and thus it is a star coloring. ∎

These bounds are given in the shape of two functions of one independent variable n. These are plotted with the aid of [8], and displayed in Figure 2.4:

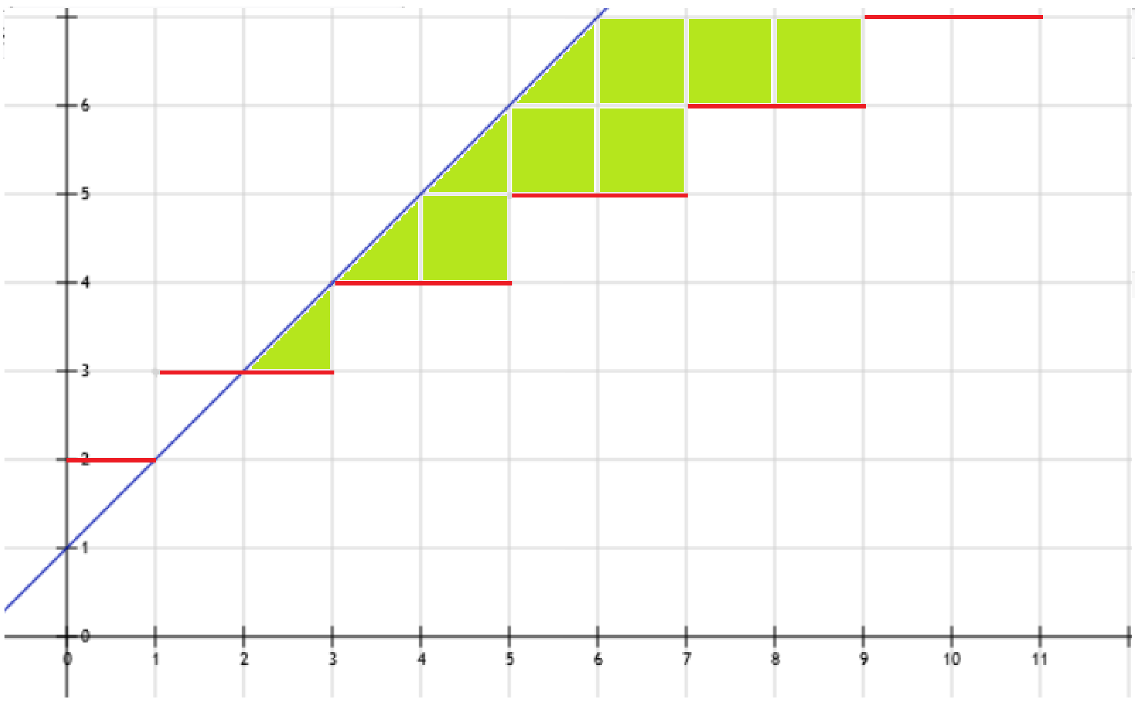


Figure 2.4. Graphic presentation of the upper (blue) and lower (red) boundaries to $\chi_s(Q_n)$ where $x = n$

Notice that the red lines cover the intervals [0,1],]1,3],]3,4],]4,5] . . . of the $x = n$ on the first hand axis.
The values of both boundaries for $1 \geq n \geq 7$ are given Table 1 below:

Table 1

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$\left\lceil \frac{n+3}{2} \right\rceil$	2	3	3	4	4	5	5
$n+1$	2	3	4	5	6	7	8

From both figure 2.4 and the table, it is visible that they both take on a linear behaviour but are different in scopes. Because of this, the distance between the extrema grows with every dimension n. This means that for a hypercube of higher dimension n, the more cases of star colorings of that hypercube can be accepted as a minimum star coloring.

Star colorings of Hypercubes of three dimensions are displayed in chapter 2.2:

2.2 Examples of star colorings

In these examples the vertices are both given an actual color as well as a corresponding number $x \in \mathbb{N}$.
Since Q_1 only consists of two adjacent vertices, this case is trivial and we move on to the next dimension.

2.2.1 Hypercube of 2nd dimension (a Square):

For this dimension, only one star chromatic number, 3, was given in the table.
By use of the greedy coloring algorithm approach, by assigning the first color, 1/red, it was easily possible to assign the two adjacent vertices with the color 2/yellow and then in order to meet the criteria for a star coloring, the final vertex was assigned the color 3/green. This is illustrated in figure 2.5.

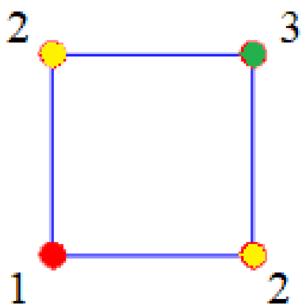


Figure 2.5.

Although the above explanation of the coloring process is sufficient, the greedy coloring is additionally illustrated in figure 2.6 with starting point being the vertex with the binary code 00.

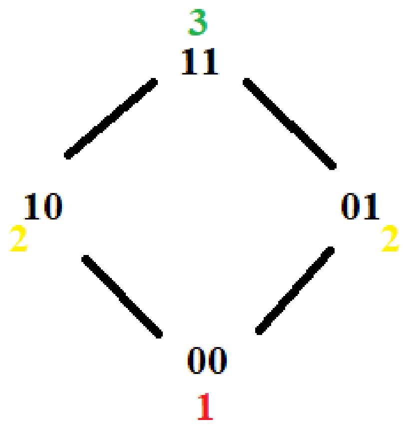


Figure 2.6.

2.2.2 Hypercube of 3rd dimension (a Cube):

For dimension 3, the star chromatic number is limited to 3 and 4.
In figure 2.7, a star coloring by use of 4 colors is shown. It is notable that it is even symmetric across K_2 of the cartesian product $Q_2 \times K_2$ with an oscillation of 180 degrees. In other words, the two vertices with the same color are placed in the corners that are the farthest away from each other - they have a minimum path of length 3 apart.

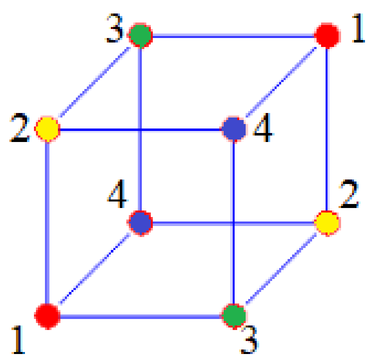


Figure 2.7.

This coloring by use of binary coding, starting at vertex 000 and moving upward, is shown in figure 2.8.

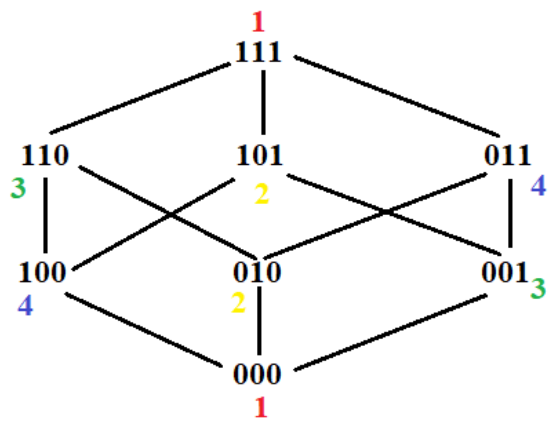


Figure 2.8.

Since the cube in dimension 3 is relatively easy to comprehend visually, its coloring was achieved within a relatively short time frame. For the sake of curiosity, let us see if a symmetric star coloring is also possible for the Tesseract (Q_4).

2.2.3 Hypercube of 4th dimension (a Tesseract):

Although this one was no easy task, it was possible to achieve some sort of symmetry to the Tesseract, while it was drawn up in a 2 dimensional environment as shown in figure 2.9.

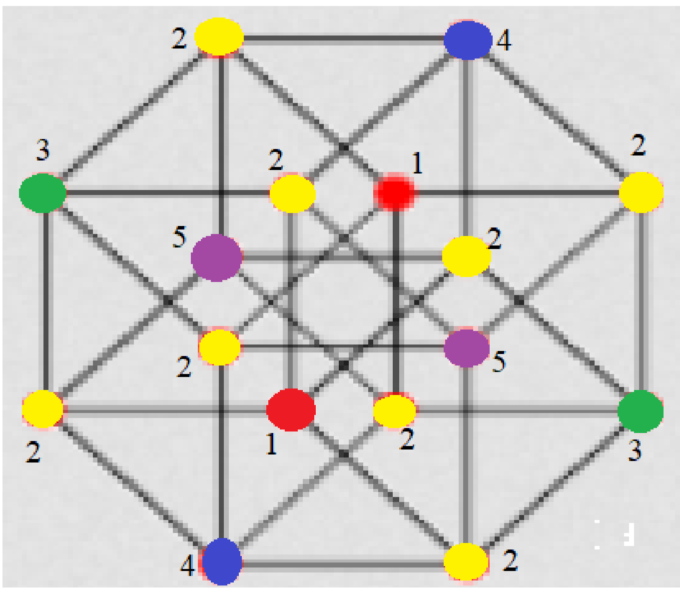


Figure 2.9.

This coloring with use of binary code, as in the two previous examples, is shown in figure 2.10

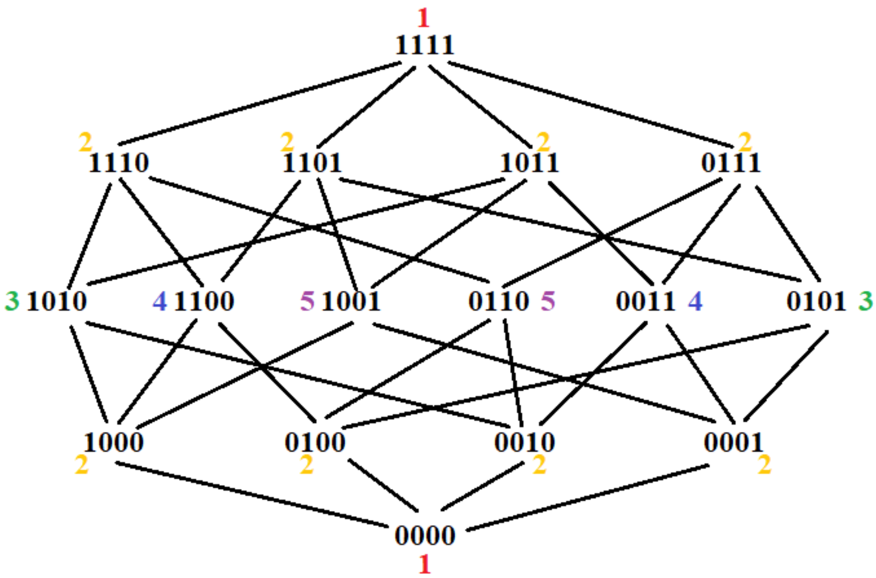


Figure 2.10.

Again, this was only possible with the upper bound of 5 colors, although attempts on fewer were made. Consider the tesseract as its cartesian product $Q_3 \times K_2$, illustrated in figure 2.11 where the red and blue color illustrates the two Q_3s and the yellow color illustrates K_2.
This makes it visible that, like in example 2.2.2 about Q_3, the two vertices with the same color are placed in the corners that are the farthest away from each other - they have a minimum path of length 4 apart. This may not be so clear due to the many uses of color 2/yellow. So, in viewing figure 2.9 and figure 2.11 together and, as an example consider the positions of the vertices with color 5, the explanation may become clearer.

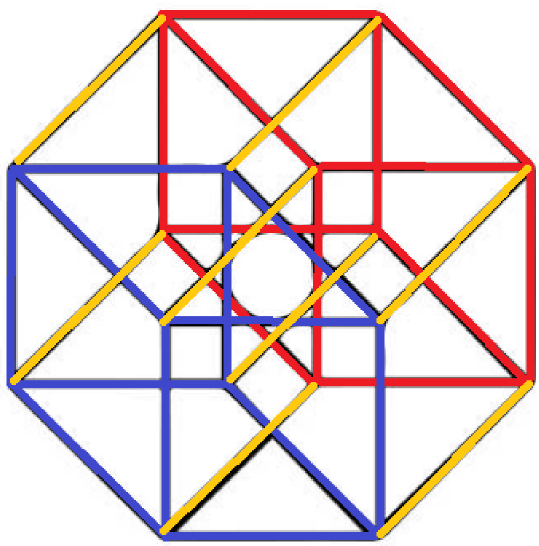


Figure 2.11.

This additionally shows that the two subgraphs Q_3 are also star colored and indeed in the exact same way only that they have been oscillated 180 degrees around the first axis of the 3 dimensional coordinate system and then by 90 degrees around the second axis. This is also similar to the symmetric star coloring of the cube Q_3 as mentioned in example 2.2.2.

2.3 Results

The above examples have shed the following light on what properties the theorem 2.2 provide for a hypergraph:

In third and fourth dimension, it was proved that the cartesian product which constitutes all hypergraphs of dimension n constitutes of two identically star colored hyper-subgraphs of dimension $n-1$ connected via K_2. The last made possible by oscillation(s) in the three dimentional space.
It is additionally notable that a symmetric star coloring was only possible by use of the upper bound to the star chromatic number in theorem 2.1.1.
Since these results have such strong similarities in both third and fourth dimension, they provide an indication that the same results may very well be possible for dimension 5 as well.

From this chapter, it is clear that the lower bound in theorem 2.1.1 is off. In the next chapter, criteria that may lie as base for the proof of this will be introduced

the Acyclic Lower Bound 3

As observed in the proof of theorem 2.1.1 the lower bound is based on the acyclic lower bound in theorem 1.3. We know that star colorings are acyclic colorings, however, not all acyclic colorings are star colorings.
Therefore, the proof for the acyclic lower bound 1.3 is examined to and in the sum (1.4):

$$m \leq \sum_{(i,j)} |V_i| + |V_j| - 1$$

it is noticed that "-1" is the subtraction of one component.
Now, this is for acyclic coloring, so, for star coloring, assume

$$m = \sum_{(i,j)} |V_i| + |V_j| - c_{ij}. \tag{3.1}$$

Here, let c_{ij} be the number of star components between the vertices assigned the colors i and j.

Definition 3.0.1

the number of star components between two color classes V_i and V_j is defined as

$$c_{ij} = \{\forall u \in V_i \text{ and } \forall v \in V_j \text{ where } uv \in E(G) | \# K_{1,t}\}$$

This definition does not account for isolated vertices that may occur in the subgraph constituted by the color classes V_i and V_j. This is the topic of the following theorems 3, 3 and the subsequent corollary.

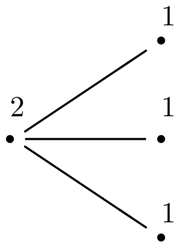
Theorem 3.0.1

Consider a star colored Hypercube Q_n. Then, for adjacent vertices of two color classes $|V_i| \leq |V_j|$, where $u \in V_i$ and $v \in V_j$. The star components are given by:

$$c_{ij} = \min\{|V_i|,|V_j|\}.$$

Proof. Let, $k \leq n$
Consider two color classes where k vertices belong to color class V_i and n vertices to V_j. Assume no isolated vertices.
Since this is a star coloring, the resulting subgraph by V_1, V_2 and the edges connecting them is a forest of $K_{1,t}$ trees for $t \in \mathbb{N}$.
For $|V_i| < |V_j|$, Consider graph A below, where $|V_2| < |V_1|$ and the two subsets equals a

forest of stars, then $c_{12} = |V_2|$.
Consider Graph A as two color classes of k in $|V_i|$ and n in $|V_j|$. This expands the above to, that for two adjacent color classes $|V_i| < |V_j|$ in a star coloring of a Hypercube, then $c_{i,j} = |V_i|$.



Graph A

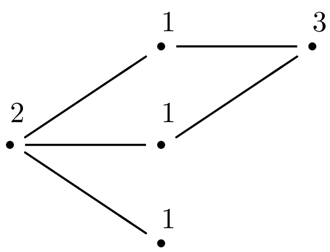
For k and n such that $|V_i| = |V_j|$, the subgraph of the connected graph G, is at least a forest of K_2 trees, and in that case, $|V_i| = c_{ij} = |V_j|$.
If the subgraph constitutes of a mix of different $K_{1,t}$ trees other then K_2, the number of star components is the number of roots to every tree, that is the number of vertices of degree $deg(v) > 1$.
So, for vertices of color classes $|V_i| \leq |V_j|$ where $u \in V_i$ and $v \in V_j$ and $uv \in E(G)$, for the general graph G, the result holds. ∎

For a star colored hypercube of dimension $n \geq 2$, a subgraph of two adjacent color classes would result in isolated vertices that used to be connected to at least a third color class in order to avoid bicolored cycles or paths of length 3 or above. Theorem 3 did not concern with these, at they are paid attention to in 3.

Theorem 3.0.2

For two non-adjacent color classes $|V_i| < |V_j|$ separated by a third color class V_k c_{ij} is $|V_i|$ times the path length between color class V_i and V_j

Proof. For non-adjacent color classes, c_{ij} is the number of aligned star components between color class V_i and V_j.
Considering the proof of theorem 3, it becomes apparent that, as the minimum requirement of a star component is that it has the shape of K_2 between two adjacent color classes, the number of star components from color class V_i to V_j is equal to $|V_i|$ times the number of edges between the color classes. This holds, due to the symmetry of the star coloring of Hypercubes. Thus, in Graph B, $c_{12} = 1$, $c_{13} = 1$ and $c_{23} = 1 + 1 = 2$.



Graph B

∎

These results of theorem 3 and 3 lead to the following collorary:

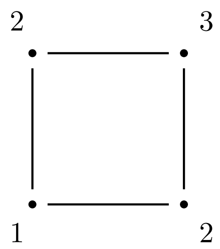
Corollar 3.0.1

For an n-colorable star colored graph with color classes $V_1, \dots, V_n$, where $|V_1| \leq \dots \leq |V_n|$ and V_1 and V_n are separated by $n-2$ color classes:

$$c_{1n} = |V_1| \cdot (n-1)$$

Remark: These theorems 3 and 3 and corollary assume symmetry such that all paths between color i and j are of equal length. Therefore, this is a concept open for further examination.

As an example, consider the star colored square Q_2 from chapter 1.



Here, the vertex assigned color 2 along with the two adjacent vertices assigned the color 1 constitutes one star component. So, $c_{12} = 1$.
This is the exact same case with the vertex colored 3 with the adjacent vertices colored 2 - $c_{23} = 1$. But c_{13} constitutes of both previous two star components, so $c_{23} = 2$.

Therefore, for the star colored square Q_2, (3.1) gives:

$$1{,}2 : 2 + 1 - 1 = 2$$
$$1{,}3 : 1 + 1 - 2 = 0$$
$$2{,}3 : 2 + 1 - 1 = 2$$

and the sum of this is 4 - the size of Q_2. The sum in (1.4) would have given the number 5 - larger than the size of Q_2.

Returning to the proof of theorem 1.3, we know that the number of pairwise distinct pairs of colors is equal to

$$\frac{p(p-1)}{2}$$

for p colors. This is precisely $\sum_{(i,j)} c_{ij}$. So, from (1.5) and (3.1), we obtain for any graph G of order $V(G)$

$$m = (p-1)V(G) - \sum c_{ij} \tag{3.2}$$

and by isolating the number of colors p, we get

$$\frac{m + \sum c_{ij}}{V(G)} + 1 = p \tag{3.3}$$

Thus, for Hypercubes Q_n, this is:

$$\frac{n2^{n-1} + \sum c_{i,j}}{2^n} + 1 = p. \tag{3.4}$$

3.1 Star components in star colored Hypercubes

In chapter 1, it was presumed that all star colorings are executed via the greedy coloring algorithm 1. However, in this chapter algorithms fitted to star color hypercubes to obtain $\chi_s(Q_n)$ are drawn up. In chapter 2.1, Consider the star coloring in figures 2.6 and 2.10. From these a special type of coloring algorithm appears possible.
By assigning the same color, A, to vertices of either even or odd weighted bitcode, c_{ij} can be evaluated as $c_{Aj} = |V_j|$. This is because $|V_A|$ in a star coloring of Hypercubes automatically becomes strictly greater than the cardinality of any other color class.

Algorithm 1: Even or Odd Algorithm

```
Result: Star coloring of Hypercube Q_n
All vertices with bitcode of odd weight are assigned color A;
while bitcode is of even weight do
    assign a unique color i other than A if the even bitcode differs in precisely 2 bits
     then
        assign a new color;
    else
        assign color i;
    end
end
```

In figures 2.10 and 2.6, it shows that the algorithm works in 2nd and 4th dimensional Hypercubes.

Remark: The importance here, is that you can decide to assign the color A to vertices of even weight instead as long as consistency is maintained.

In figure 3.1, the Hypercube of dimension 5 Q_5 is star colored using the Even or Odd Algorithm 1. This time, the color A is assigned to every vertex of even weighted bitcode. The vertices of bit weight $w(v) = 1$ are then assigned a unique color each. This creates a base for the coloring of vertices of weight $w(v) = 3$.
Thus, vertex 01110 cannot be assigned the same color as 01000,00100 and 00010. This leaves the choice between two of the remaining base colors from 10000 or 00001.

Remark: It becomes apparent, in figure 3.1, that this method does not hold for the maximum of the star chromatic number of Q_5 in theorem 2.1.1 - 6 colors.

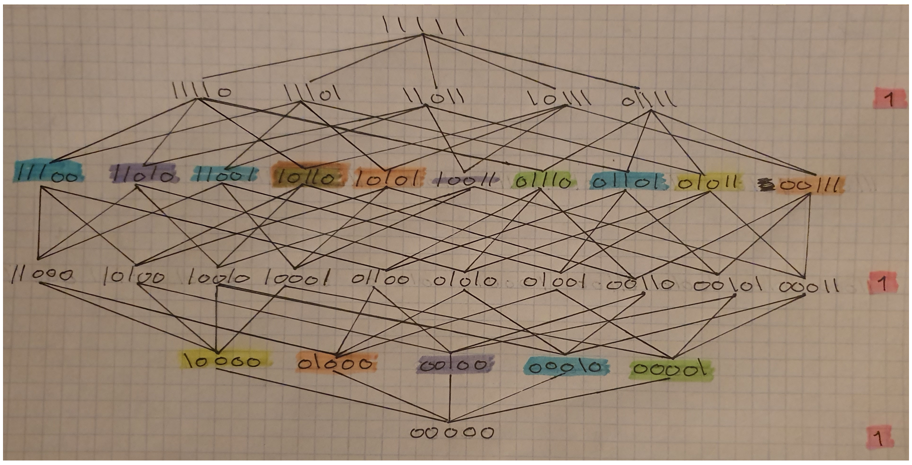


Figure 3.1.

This is also the case for Q_3 as shown in figure 3.2

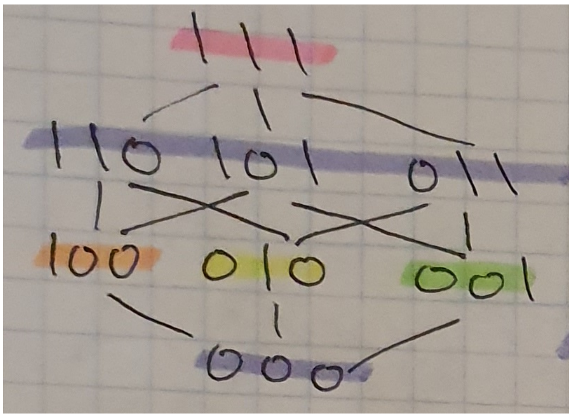


Figure 3.2.

In both cases, Q_3 and Q_5, $n+2$ colors are needed at least for this method to hold and not $n+1$ - the upper bound for the star chromatic number.

The Even or Odd Algorithm 1, does, however, fit the star chromatic number for Q_2 and Q_4 as shown in figures 2.6 and 2.10, where all vertices of odd bit weight are assigned color $A = 2$.

Thus, for hypercubes of even dimension, this method of coloring excludes any bicolored cycles or paths of length 3 or above and works by the upper bound of theorem 2.1.1.
So, considering hypercubes Q_n of even dimension n, the color classes from the Even or Odd coloring form a complete bipartition $K_{1,n}$ as illustrated in figures 3.3 and 3.4.

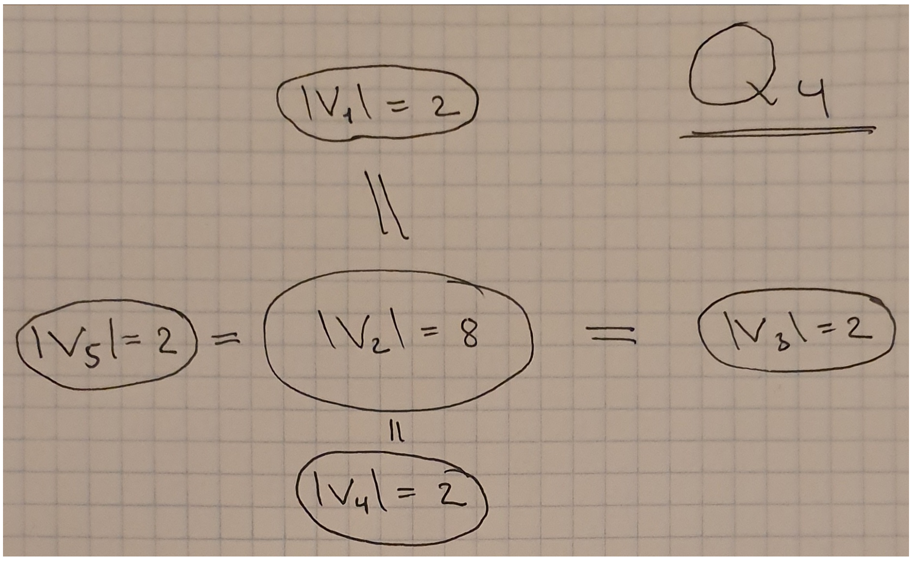


Figure 3.3.

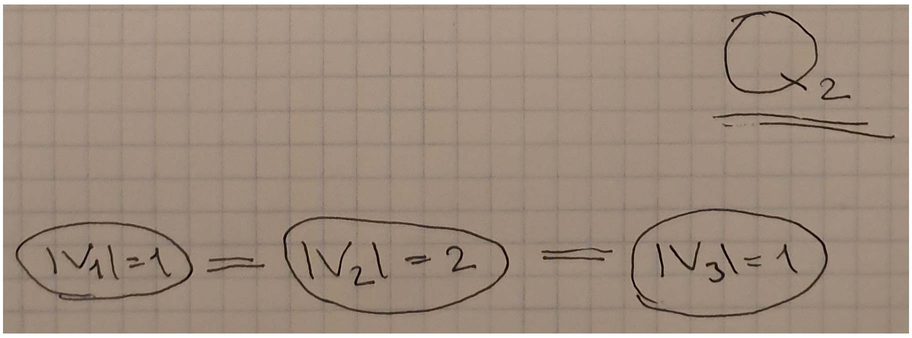


Figure 3.4.

This complete bipartite structure ensures that the number of star components c_{ij} of colors i and j is either $|V_j|$ for adjacent color classes $|V_i| \geq |V_j|$. For non-adjacent color classes, $c_{ij} = 2|V_j|$ if $|V_i| \geq |V_j|$.

3.1.1 the Sum of Star Components

In chapter 2.1, the lower bound for the star chromatic number of Hypercubes was given and it was calculated for $1 \leq n \leq 7$. It was only possible of achieve the lower bound for dimensions 1 and 2, but for these the upper and lower bound were identical. In dimension 3 and 4, a star coloring was only possible with the upper bound.

In Table 2, calculated $\chi_s(Q_n)$ from chapter 2.1 and figure 3.5 are added to Table 1

Table 2

	n=1	n=2	n=3	n=4	n=5	n=6	n=7
$\left\lceil \frac{n+3}{2} \right\rceil$	2	3	3	4	4	5	5
$n+1$	2	3	4	5	6	7	8
actual calculations	2	3	4	5	6		

In this chapter, we investigate the inequality:

Theorem 3.1.1

For a Hypercube Q_n

$$\frac{n2^{n-1} + \sum c_{ij}}{2^n} > \left\lceil \frac{n+3}{2} \right\rceil \tag{3.5}$$

Proof. the validity of equation (3.5) lies with the sum of star components.
For Q_3, the claim of theorem 2.1.1, by equation (3.4) gives that $\sum c_{ij} = 4$ - that there will in total be four stars in a cube Q_3 by use of $p = 3$ colors.
This is impossible for a star coloring.

For Q_4, we get

$$\frac{4 \cdot 2^3 + \sum C_{ij}}{2^4} + 1 = 4, \tag{3.6}$$

which gives $\sum c_{ij} = 16$ by use of $p = 4$ colors.

In a hypercube of any dimension n, the smallest cycle is a square of path length 4 because of the Cartesian product $Q_{n-1} \times K_2$. such a cycle requires at least 3 colors to be both acyclic and star colored. However, when the regularity of the hypercube grows by one every cycle of path length 4 becomes interconnected into several 4-cycles and thus, 3 colors cannot be enough for a cube and by the same argument, 4 colors cannot be enough for a tesseract Q_4. ∎

This proof is incomplete as it solely depends on $\sum C_{ij}$ and that C_{ij} is properly defined for Hypercubes. That is, the key to correct the lower bound to the star chromatic number for Hypercubes follows a clear lower bound to the total sum of star components. That is,

$$16 = \frac{3 \cdot 32}{6} \leq \sum c_{ij}$$

for Q_4.

To give an indication of this, The actual results from Table 2 is illustrated in Equation 3.5 for Q_3 and Q_4:

for Q_3 this would be:

$$\frac{12 + \sum c_{ij}}{8} + 1 > 3$$

for this to hold, $\sum c_{ij} > 4$ is necessary.
For the symmetric star coloring of Q_3 in figure 2.8, we obtain the c_{ij} values:

$$
\begin{aligned}
c_{12} &= 2\\
c_{13} &= 2\\
c_{14} &= 2\\
c_{23} &= 2\\
c_{24} &= 2\\
c_{34} &= 2
\end{aligned}
$$

and thus, $\sum c_{ij} = 12 > 4$.

<u>**For Q_4 the equation is:**</u>

$$\frac{32 + \sum c_{ij}}{16} + 1 > 4$$

and for this to hold, we need $\sum c_{ij} > 16$ and from figure 2.10, we get:

$$
\begin{aligned}
c_{12} &= 2\\
c_{13} &= 4\\
c_{14} &= 4\\
c_{15} &= 4\\
c_{23} &= 2\\
c_{24} &= 2\\
c_{25} &= 2\\
c_{34} &= 4\\
c_{35} &= 4\\
c_{45} &= 4
\end{aligned}
$$

and $\sum c_{ij} = 32 > 16$

From these two examples, another result emerges that:

Theorem 3.1.2

For a Hypercube Q_n

$$\sum c_{ij} = m \tag{3.7}$$

Proof. Hypercubes Q_n, because of the Cartesian product, derives from n cartesian products of K_2 to a vertex as

$$
\begin{aligned}
Q_4 &= Q_3 \times K_2\\
&= (Q_2 \times K_2) \times K_2\\
&= ((Q_1 \times K_2) \times K_2) \times K_2\\
&= (((Q_0 \times K_2) \times K_2) \times K_2) \times K_2,
\end{aligned}
$$

Where Q_0 is but an isolated vertex.
$K_2 \equiv S_2$ and for a star colored Hypercube Q_n, that may be solely containing S_2 stars, like Q_3 in figure 2.7, or be a complete bipartite set $K_{1,t}$ of color classes, as shown in figures 2.9, 3.3 and 3.4. Thus, the total number of stars, $\sum c_{ij}$, equals the total number edges in the graph. ∎

In this chapter many exciting results have arisen:

Above all that improvement to the lower bound in theorem 2.1.1 is possible with the introduction of the total sum of star components. Although this is still a concept that needs improvement in itself, this separates star coloring from acyclic coloring and could complete the proof of theorem 3.1.1 and thus, that the actually calculated $\chi_s(Q_n)$ are in fact the smallest number k for which Q_n is star colorable. That is, $\left\lceil \frac{n+3}{2} \right\rceil$ is strictly too low for dimensions $n > 2$.

For even dimensions n, the color classes from a star coloring of Hypercubes produce a complete bipartite graphs $K_{1,n}$.
Bipartite graphs $K_{x,y}$ are extensively used in modern coding theory. In [9], The function of bipartite graphs in error correcting codes is explained in short. This is where the vertices in the x part of the graph are bits of information that need to be preserved and corrected if corrupted. The vertices on the y part of the graph are parity checks. By use of the parity checks, errors can be corrected if some of the bits are corrupted. Low density parity check (LDPC) codes are used in satellite TV transmission, the relatively new 10G Ethernet standard, and part of the WiFi 802.11 standard.
This, along with the binary coding already present in the presentation of vertices in Hypercubes, promotes some of the uses in modern coding that can be provided by star coloring.

3.2 the Vector Space of Star Coloring Hypercubes

Achieving a symmetric star coloring of, for instance, Q_5, two vertices with distance 4 can be assigned the same color $c(00100) = c(11011)$. With linear coding this can be equivalent to a map

$$\mathbb{F}_2^n \Rightarrow \mathbb{F}_2^n.$$

In order to validify this approach, the space needs to to be split into cosets. Denote this measure of cosets C.
First, C, must contain the code 00000 - the dimension 0 vector space.
To then get a vector space of dimension 1, add 11100.
If 00111 is also added, then the sum of the two must be added to maintain the vector space - $11100 + 00111 = 11011$. This gives 4 elements in a 2 dimensional vector space.

$$C = \{00000, 11100, 00111, 11011\}$$

For the star coloring, we then color two vertices x and y with the same color whenever $x - y$ is in C. These two vertices must be of path length 3 apart, so their bit code have

Hamming distance of at least 3.
Observing the result of this algorithm, in figure 3.5, it is apparent that too many colors are in use considering the upper bound in theorem 2.1.1. However, what is also apparent is that pairs of distinct colors/cosets could be united. The criteria for this union to be valid is that the resulting new linear space is with minimum Hamming distance 2. In figure 3.5 such a union has been constructed of cosets [00000] and [01110] under the color green.
This union is a new code/linear space with minimum Hamming distance 2. Hence, the coloring is still valid.

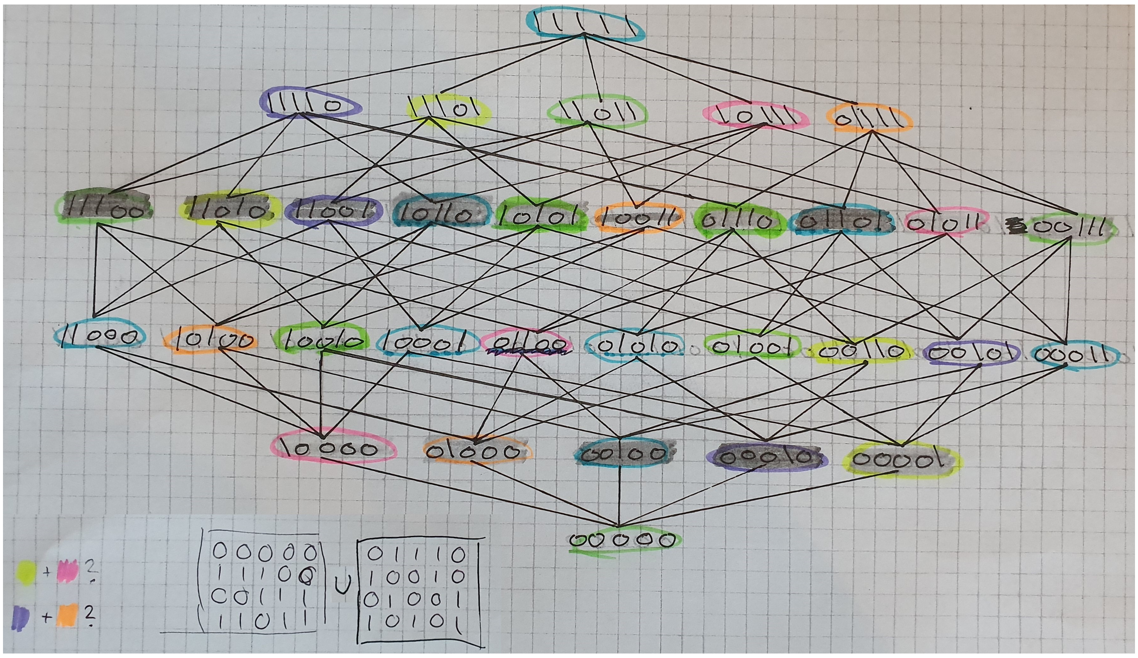


Figure 3.5.

After this union it can be asked whether any other two color classes can be united. In the same figure, it is visible that any additional union will result in either bicolored cycles; bicolored paths of length 3 or above; and/or even adjacent vertices assigned the same color.

This algorithm is formally defined in Algorithm 2:

Algorithm 2: Coset Algorithm

```
Result: Star coloring of Hypercube Q_n
Take a measure of cosets C ∈ F_2^n and assign this measure a unique color.;
while given two vertices x and y do
    consider x − y;
    if x − y ∈ C then
        assign both the same color;
    else
        assign each a different color;
    end
end
while given two resulting color classes V_i and V_j do
    consider V_i ∪ V_j;
    if ∀x,y ∈ V_i ∪ V_j has min{d(x,y)} ≥ 2 then
        unite the color classes under the same color;
    else
        consider another union;
    end
end
```

For Q_5 in view of Algorithm 2, the resulting color classes do not form a bipartition.

This approach, however, does not work for Q_4 since there is no code of length 4. For Q_5, a 2-dimensional code of minimum weight 3 was possible and thus, every coset had had four elements.
But for length 4 the only codes of minimum weight are 1-dimensional and thus the cosets would only contain 2 elements.

In fact, this algorithm holds for dimensions $n \in \{2^n - 1, 2^n - n, 3\}$, this is because the codes of these dimensions are have lengths that enable a minimum weight 3. For code length 3 - Q_3 - the coset is $\{000,111\}$ which by algorithm 2 gives exactly the coloring of Q_3 in figure 2.8.
With this logic, the coset to the algorithm 2 coloring of Q_6 will be of dimension 3 and hold as much as 8 elements.

This gives two algorithms that give the upper bound in theorem 2.1.1.

Conclusion 4

The main issue of this paper has been the lower bound to the star chromatic number of Hypercubes Q_n

$$\left\lceil \frac{n+3}{2} \right\rceil \leq \chi_s(Q_n) \leq n+1.$$

This lower bound is based on the general lower bound

$$\chi_s(G) \geq a(G) \geq \frac{2n+1-\sqrt{\gamma}}{2},$$

and from its proof, it was obtained that for acyclic coloring, $m \leq \sum\limits_{(i,j)} |V_i| + |V_j| - 1$. This is where the theory about that in star coloring, as opposed to acyclic coloring, there could possibly be more components to subtract than merely 1 as $\chi_s(G) \geq a(G)$.
This thesis has yet to fully prove that for $|V_i| \leq |V_j|$, $c_ij = |V_ij|$ in every case as the cases used still rely heavily on symmetry and equal path length.
In this thesis, this concept of star components c_{ij} has been introduced, but must no less be studied further to be complete, and to prove that for $\left\lceil \frac{n+3}{2} \right\rceil = p$ the equation

$$\frac{n2^{n-1} + \sum c_{i,j}}{2^n} + 1 = p$$

gives an incorrect sum of components. Ultimately, a lower bound to this sum of star component would provide solid criteria that would have lasting consequences for the estimated $\chi_s(Q_n)$.

For this verification, this thesis has additionally drawn up two algorithms that result in star colorings that use the upper bound of the star chromatic number.
In chapter 3, the Even or Odd Algorithm 1 was defined. For Q_n of even dimension n, this algorithm gives a complete bipartition of the color classes $K_{1,t}$ for $1 + t = \chi_s(Q_n)$.
The Coset Algorithm 2 utilizes cosets of vector spaces to insure path lengths 3 between vertices of the same color. This algorithm holds for dimensions $n \in \{2^n - 1, 2^n - n, 3\}$.
These may aid in any further definition of star components.

Bipartition is a concept used in error correction programs in detecting and correcting corruption in the information in the 1 part of $K_{1,t}$. The vertex set is presented by binary coded bitstrings which is also a concept used within error correction programs in the detection of Hamming distance between compared bitstrings. Therefore, star coloring of Hypercubes provides a further addition to the tools within the subject of error correction.

Therefore, due to the incompletion to the concept of star components, the improvement these still needs further attention in the future in order to improve the boundaries of $\chi_s(Q_n)$ via this method.

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