Results on Magnetic Pseudodifferential Operators and Acoustic Black Hole Optimization

Benjamin Støttrup

Department of Mathematical Sciences

August 2019

Master's Thesis - Qualification Exam

STUDENT REPORT

AALBORG UNIVERSITY

AALBORG UNIVERSITY
STUDENT REPORT

Department of Mathematical Sciences
Aalborg University
http://www.aau.dk

Title:
Results on Magnetic Pseudodifferential Operators and Acoustic Black Hole Optimization

Theme:
Applied Mathematical Analysis

Project Period:
Spring Semester 2019

Student:
Benjamin Støttrup

Supervisor:
Horia Cornean

Page Numbers: 29

Date of Completion:
August 9, 2019

Abstract:

This thesis contains a summary of the research I have done as part of the first two years of my PhD study and a plan for the research I intend to conduct as part of the remaining two years of my 4+4 PhD. It contains detailed summaries of two papers. The first paper is published and treats magnetic pseudodifferential operators and their spectral properties. The second paper is unpublished and treats the acoustic phenomenon known as acoustic black holes and considers optimization of the height profile of acoustic black holes.

Contents

1 Summary in Danish **1**

2 Introduction **3**

3 Magnetic Pseudodifferential Operators **5**
3.1 Preliminaries 6
3.1.1 Matrix-like Structure of Operators 7
3.1.2 Magnetic pseudodifferential operators 8
3.2 Main Results 9
3.3 Proofs 10
3.3.1 Proof of Theorem 3.2.2 11
3.3.2 Proof of Theorem 3.2.1(2) 12

4 Acoustic Black Holes **17**
4.1 Theoretical Background 18
4.1.1 Problem Statement 19
4.2 Approach 19
4.3 Results 20
4.3.1 The case $b = 0$ 21
4.3.2 The case $b > 0$ 21
4.3.3 Numerical Comparison With a Classical Profile 23

5 Future Work **25**

Bibliography **27**

1 | Summary in Danish

Dette speciale indeholder en redegørelse for de resultater, jeg har opnået gennem de første to år af mit 4+4 ph.d. forløb. Jeg vil i dette speciale præsentere to artikler; en som er publiceret, og en som er tæt på at være klar til indsendelse til en journal. Denne præsentation består af en redegørelse for baggrunden for det videnskabelige arbejde samt en gennemgang af de hovedresultater der er opnået. Ydermere vil de vigtigste dele af beviserne blive præsenteret samt motivationen bag fremgangsmåden i beviserne. Som afslutning vil dette speciale indeholde en gennemgang af potentielle retninger for min fremtidige forskning.

Den første artikel hedder “Magnetic pseudodifferential operators represented as generalized Hofstadter-like matrices”, og er skrevet i samarbejde med Horia Cornean, Henrik Garde og Kasper Studsgaard Sørensen. Artiklen er udgivet i Journal of Pseudo-Differential Operators and Applications og omhandler såkaldte magnetiske pseudodifferentialoperatorer, der generaliserer de klassiske Weyl pseudodifferential-operatorer. Disse magnetiske pseudodifferentialoperatorer blev oprindeligt indført som en funktionskalkyle for en partikel i $\mathbb{R}^d$ under inflydelse af et magnetisk felt. I vores artikel viser vi, at for en bestemt klasse af symboler kan magnetiske pseudodifferentialoperatorer betragtes på en matrixlignende form, som en såkaldt “generalized Hofstadter-like matrix”. Udfra denne matrixkonstruktion viser vi, at magnetiske pseudodifferentialoperatorer er begrænsede på $L^2(\mathbb{R}^2)$, samt at deres spektra er $\frac{1}{2}$-Hölder kontinuert med hensyn til styrken af det magnetiske felt. I artiklen vises også at når det magnetiske felt er konstant så vil spektraets endepunkter være Lipschitz kontinuert med hensyn til styrken af det magnetiske felt. I dette speciale vil jeg dog ikke komme nærmere ind på det sidste resultat, da mit bidrag til artiklen hovedsageligt omhandler de andre resultater.

Den anden artikel, som endnu ikke er publiceret, hedder “Optimal profile design for acoustic black holes” og er lavet i samarbejde med Horia Cornean og Sergey Sorokin. Artiklen omhandler såkaldte akustiske sorte huller, der laves ved at lade højden af en plade eller stang gå mod nul, på en glat måde, nær kanten. Gøres dette vil vibrationer i pladen eller stangen i høj grad ikke reflekteres fra denne kant. I literaturen bruges typisk en højdeprofil på formen $h(x) = \varepsilon x^m$ for $m \geq 2$. I vores artikel anvender vi variationsregning til at optimere højdeprofilen for et akustisk sort hul.

2 | Introduction

This thesis contains a summary of the research I have done as part of the first two years of my PhD study and a plan for the research I intend to conduct as part of the remaining two years of my 4+4 PhD. These two years of research have resulted in one published paper [2] and one paper [3] which is close to submission to a peer-reviewed journal. In addition to summarising these papers I will elaborate on details left out of the papers and emphasize the main ideas behind the proofs instead of just repeating the proofs verbatim. For this reason many smaller results are either excluded or included without proof in this thesis. For completeness I include both papers in the appendices of this thesis. These two papers concern very different topics; the published paper treats magnetic pseudodifferential operators and their spectral properties while the unpublished paper treats the acoustic phenomenon known as acoustic black holes. The first subject is purely mathematical with applications to mathematical physics, and the second subject is very much applied mathematics as it is based on a problem of mechanical engineering. Therefore the style in which the results are presented differs vastly between the two papers. For this thesis I have modified the way the results of [3] is presented to better emphasise the strategy behind obtaining the results.

Although these two subjects are unrelated, the methods used to treat them are all from the field of mathematical analysis. The reason for this disparity in the subjects treated in the two papers is that the funding for my 4+4 PhD-study is split between the Department of Mathematical Sciences and the Department of Materials and Production. Hence I conduct work which is in the interest of both departments. In the future it is my ambition to further study acoustic black holes, and hopefully tying together the subjects of the two papers presented in this thesis (see also Chapter 5). Before continuing with a more detailed introduction of the two papers I will briefly mention that I am also working on a third paper about the construction of singular functions in collaboration with Horia Cornean, Ira Herbst, Jesper Møller, and Kasper Studsgaard Sørensen. At the time of writing this thesis, the results of this third paper are not fully developed for which reason it is not included here (see Chapter 5 for a short exposition of this paper).

The published paper [2] is joint work with Horia Cornean, Henrik Garde and Kasper Studsgaard Sørensen and concerns so-called magnetic pseudodifferential operators. This type of operator was first introduced in [25] as a generalization of the classical Weyl quantization of pseudodifferential operators. The motivation for this introduction was to obtain a pseudodifferential calculus for a nonrelativistic quantum particle in $\mathbb{R}^n$ influenced by a magnetic field B such that the corresponding magnetic pseudodifferential operators becomes gauge covariant [25, 31]. Multiple well-known results for the classical Weyl quantization have been generalized to the magnetic case, e.g. Calderón-Vaillancourt type theorems [31] and Beals criterion [5, 17].

In the paper [2] we give a new proof of a Calderón-Vaillancourt type result, show that when the operators are self-adjoint, their spectra are $\frac{1}{2}$-Hölder continuous with respect to the magnetic field strength b in the Hausdorff distance, and prove that when the magnetic field is constant the spectral edges are Lipschitz continuous in b. These two types of spectral results have previously been established for so-called Harper-like operators (see [4, 27] and references therein).

The unpublished paper [3] is joint work with Horia Cornean and Sergey Sorokin. The paper applies well-known optimization results to obtain an optimal design for so-called acoustic black holes. The theory of acoustic black holes in bars and plates originate from paper [24]. In this paper M. Mironov establishes that if the height of a plate decreases sufficiently smoothly to 0 in a finite interval, then a flexural wave propagating towards the edge will not be reflected. The wave is trapped near the edge of the plate, which has motivated the term "acoustic black hole" for this phenomenon. In practice it is impossible to achieve zero reflection from the edge since real beams will be cut off at some non-zero height leading to reflection [24].

In [24] the specific height profile $h(x) = \varepsilon x^m$, for $m \geq 2$ and $\varepsilon > 0$ is proposed. Subsequently other authors have tried to develop this idea of Mironov to create efficient vibration dampening in beams and plates by combining the power law profile of the edge with other methods of vibration dampening [7, 20, 21, 29]. Most notable for our paper is [29] which uses multi-objective optimization methods to determine numerically the optimal (in terms of having a small reflection coefficient and not violating the underlying assumptions of the theory of acoustic black holes both at low frequencies) profile of the form $h(x) = \varepsilon x^m + h_0$ for an acoustic black hole.

In [3] we apply calculus of variations to solve a similar optimization problem for the height profile. A key difference between [3] and [29] is that we are not restricting our attention to profiles of the form $h(x) = \varepsilon x^m + h_0$. With our method we derive closed form expressions for the optimal height profile and obtain, as a special case, the classical profile $h(x) = \varepsilon x^2$ first considered in [24].

3 | Magnetic Pseudodifferential Operators

In quantum mechanics it is of interest to have a meaningful correspondence between classical and quantum observables [10, 14]. In the case of a spinless particle in $\mathbb{R}^d$, classical observables are functions on $\mathbb{R}^{2d}$, and quantum observables are self-adjoint operators on $L^2(\mathbb{R}^{2d})$ [25]. When no magnetic field is present, it is natural to require that the classical position and momenta observables $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ (i.e. the coordinates of the phase space $\mathbb{R}^{2d}$) correspond to their quantum counterparts $X_1, \ldots, X_d, D_1, \ldots, D_d$ (i.e. X_j is multiplication with the j'th variable and $D_j = -i\partial_j$). Establishing this correspondence is sometimes called the *quanitzation problem* [10]. One way of solving the quantization problem is to use the Weyl quantization which is a functional calculus for the operators $X_1, \ldots, X_d, D_1, \ldots, D_d$ that ascribes to a function f on $\mathbb{R}^d$ the operator (cf. [10, 14, 16, 25])

$$\operatorname{Op}^W(f)u(x) \coloneqq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}\xi\cdot(x-x')} f((x+x')/2, \xi)u(x')\,\mathrm{d}x'\,\mathrm{d}\xi, \tag{3.1}$$

for suitable functions f and u. The formal justification for this formula is that by the Fourier transform

$$f(x,\xi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}(q,p) e^{i(qx+p\xi)}\,\mathrm{d}q\,\mathrm{d}p,$$

and replacing $e^{i(qx+p\xi)}$ with the operator $e^{i(qX+pD)}$, where $qX = q_1X_1 + \cdots + q_dX_d$ and $pD = p_1D_1 + \cdots + p_dD_d$ we get

$$f(X,D) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}(q,p) e^{i(qX+pD)}\,\mathrm{d}q\,\mathrm{d}p.$$

One can show that $e^{i(qX+pD)}u(x) = e^{i(qp/2+qx)}u(x+p)$ from which it follows that $\operatorname{Op}^W(f) = f(X,D)$ [10].

When considering a particle moving in $\mathbb{R}^d$ under the influence of a magnetic field B, the corresponding functional calculus should be developed for the operators $X_1, \ldots, X_d, \Pi_1, \ldots, \Pi_d$, where $\Pi_j = D_j - A_j$ are the magnetic momenta and A is a vector potential of B, i.e. $B = \mathrm{d}A$ [25, 31]. One approach is to replace $f(x,\xi)$

by $f(x, \xi - A(x))$ in the formula for $\mathrm{Op}^W(f)$ but this leads to operators which are not *gauge covariant*, i.e. different choices of vector potential leads to operators which are not unitary equivalent (see [31] and references therein). By adapting the above construction of the classical Weyl calculus to the magnetic case, [25] derived the magnetic Weyl calculus

$$\mathrm{Op}_b^{\mathrm{W}}(f)u(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}\xi\cdot(x-x')} \mathrm{e}^{\mathrm{i}b\varphi(x,x')} f((x+x')/2, \xi) u(x') \,\mathrm{d}x' \,\mathrm{d}\xi, \tag{3.2}$$

where $\varphi(x, x')$ denotes the flux of the B through the oriented triangle with vertices $x, x', 0$. This definition leads to the right gauge covariance. The theory of magnetic pseudodifferential operators have further been developed, hereby establishing for example magnetic pseudodifferential operators for the Hörmander symbol classes $S^m_{\rho,\delta}$ [31], Calderón-Vaillancourt-like theorems [31], Beals criterion [5, 17], and results on spectral theory [23, 26], see also references in our paper [2].

This chapter presents and discusses some results on magnetic pseudodifferential operators obtained in our paper [2]. The paper contains one main result composed of three parts and in this chapter I will cover the first two parts of the main result, since my contribution to the paper concerns these two. The first part established a Hofstadter-like matrix structure for magnetic pseudodifferential operators. As a consequence of this matrix structure, we obtain a Calderón-Vaillancourt type result for magnetic pseudodifferential operators. Similar results have been obtained in [31] but our method of proof is vastly different. The second part proves that when the magnetic pseudodifferential operator is self-adjoint then the spectrum is $\frac{1}{2}$-Hölder continuous. This type of result have previously been obtained for magnetic pseudo differential operators with elliptic symbol of class $S^m_{1,0}$, with $m > 0$ [6] and for the discrete Harper operators (see [4, 27] and references therein). It is worth noting that the discrete Harper operators and the Hofstadter-like matrix representation we obtain are in principle similar.

3.1 Preliminaries

Let $d \geq 2$ and for $x \in \mathbb{R}^d$ define $\langle x \rangle := (1 + |x|^2)^{1/2}$. Then by the trivial identity

$$(1 + |x_1|^2) \dots (1 + |x_d|^2) \leq \langle x \rangle^{2d}$$

and Tonelli's theorem it follows that

$$\int_{\mathbb{R}^d} \frac{1}{\langle x \rangle^{2d}} \,\mathrm{d}x < \infty, \quad \text{and} \quad \sum_{\gamma \in \mathbb{Z}^d} \frac{1}{\langle \gamma \rangle^{2d}} < \infty,$$

which are estimates that I use extensively throughout this thesis.

3.1.1 Matrix-like Structure of Operators

One of the main ideas behind the proofs in [2] is, in some sense, to discretize operators on $L^2(\mathbb{R}^d)$ to obtain matrix-like representations of such operators. In practice we define $\Omega =]-\frac{1}{2}, \frac{1}{2}[^d$ to be the open unit d-hypercube and note that for a.a. $x \in \mathbb{R}^d$ we have $x \in \Omega + \gamma$ for some $\gamma \in \mathbb{Z}^d$. Thus every $f \in L^2(\mathbb{R}^d)$ can be characterized by the sequence $(f_\gamma)_{\gamma\in\mathbb{Z}^d}$ where $f_\gamma \colon \Omega \to \mathbb{C}$ are given by

$$f_\gamma(\cdot) = \chi_\omega(\cdot) f(\cdot + \gamma) \tag{3.3}$$

and χ_Ω denotes the characteristic function on Ω. Intuitively one can say that we "cut out" the function f. Like-wise it is not hard to see how one can "stitch together" an $L^2(\mathbb{R}^d)$ function from a sequence $(f_\gamma)_{\gamma\in\mathbb{Z}^d}$ if $\sum_{\gamma\in\mathbb{Z}^d} \|f_\gamma\|^2_{L^2(\Omega)} < \infty$. This motivates the definition of the Hilbert space

$$\mathscr{H} := \bigoplus_{\gamma\in\mathbb{Z}^d} L^2(\Omega) = \Big\{ (f_\gamma)_{\gamma\in\mathbb{Z}^d} \subset L^2(\Omega) \mid \sum_{\gamma\in\mathbb{Z}^d} \|f_\gamma\|^2_{L^2(\Omega)} < \infty \Big\},$$

equipped with the inner product

$$\langle (f_\gamma), (g_\gamma) \rangle_{\mathscr{H}} := \sum_{\gamma\in\mathbb{Z}^d} \langle f_\gamma, g_\gamma \rangle_{L^2(\Omega)}.$$

We say that an operator A on $\mathscr{H}$ is a *generalized matrix* of the operators $(A_{\gamma,\gamma'})_{\gamma,\gamma'\in\mathbb{Z}^d} \subset B(L^2(\Omega))$ when

$$(Af)_\gamma = \sum_{\gamma'\in\mathbb{Z}^d} A_{\gamma,\gamma'} f_{\gamma'}$$

for all $f = (f_\gamma)_{\gamma\in\mathbb{Z}^d} \in \mathscr{H}$ and we denote this by $A = \{A_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$. One may also see that A acts on $\ell^2(\mathbb{Z}^d; L^2(\Omega))$. A natural question is then what conditions should be imposed on the "matrix elements" $A_{\gamma,\gamma'}$ to ensure that the generalized matrix $\{A_{\gamma,\gamma'}\}_{\gamma,\gamma'}$ is a bounded operator on $\mathscr{H}$. The conditions which suffices for this thesis are given by the following Schur-Holmgren type result.

Lemma 3.1.1 (Lemma 2.4 of [2]). *Suppose that there exists a constant C and operators $(T_{\gamma,\gamma'})_{\gamma,\gamma'\in\mathbb{Z}^d} \subset B(L^2(\Omega))$ such that*

$$\|T_{\gamma,\gamma'} f\|_{L^2(\Omega)} \le \frac{C\|f\|_{L^2(\Omega)}}{\langle \gamma - \gamma' \rangle^{2d}},$$

for every $\gamma, \gamma' \in \mathbb{Z}^d$ and $f \in C_0^\infty(\Omega)$. Then $T = \{T_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$ is a bounded operator on $\mathscr{H}$ with

$$\|T\| \le \sum_{\gamma\in\mathbb{Z}^d} \frac{C}{\langle \gamma \rangle^{2d}}.$$

3.1.2 Magnetic pseudodifferential operators

Let B be a magnetic field, i.e. a closed 2-form, with components in $BC^\infty(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d; \mathbb{R}) : \sup_{x\in\mathbb{R}^d} |\partial^\alpha f(x)| < \infty, \forall \alpha \in \mathbb{N}_0^d \right\}$ and let $\varphi(x, x')$ denote the magnetic flux through the oriented triangle with vertices 0, x, x'. In the subsequent sections we need the facts that for all $x, x', y, z \in \mathbb{R}^d$ and $\alpha, \alpha', \beta \in \mathbb{N}_0^d$ (see also [2]):

1. There exists a constant $C_{\alpha,\alpha'}$ such that
$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \varphi(x, x')| \le C_{\alpha,\alpha'} |x||x'|; \tag{3.4}$$
2. $\varphi(x, x') = -\varphi(x', x)$;
3. If $\Delta(x, y, z)$ denotes the area of the triangle with vertices $x, y, z \in \mathbb{R}^d$ then the map $\mathfrak{f}\colon \mathbb{R}^{3d} \to \mathbb{R}$ given by
$$\mathfrak{f}(x, y, z) \coloneqq \varphi(x, y) + \varphi(y, z) - \varphi(x, z)$$
is the magnetic flux through the triangle with vertices x, y, z and satisfies
$$|\partial_x^\alpha \partial_y^{\alpha'} \mathfrak{f}(x, y, z)| \le C_{\alpha,\alpha'} \Delta(x, y, z), \tag{3.5}$$
for some constant $C_{\alpha,\alpha'}$.

With this φ given I can define our symbols and the magnetic pseudodifferential operator I consider in this thesis.

Definition 3.1.2 ([2]). The *magnetic symbol class* $M_\varphi(\mathbb{R}^{3d})$ is the set of all functions of the form
$$a_b(x, x', \xi) = \mathrm{e}^{\mathrm{i}b\varphi(x,x')} a(x, x', \xi),$$
where $b \in \mathbb{R}$ and $a \in C^\infty(\mathbb{R}^{3d})$ is any function for which there exists $M \ge 0$ such that
$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \partial_\xi^\beta a(x, x', \xi)| \le C_{\alpha,\alpha',\beta} \langle x - x' \rangle^M, \tag{3.6}$$
for every $\alpha, \alpha', \beta \in \mathbb{N}_0^d$ and some constant $C_{\alpha,\alpha',\beta}$.

We remark that this definition of symbols differs from the classical symbols of class $S_{0,0}^0$ (cf. [16]) in two central ways. Firstly, the symbol contains a phase factor $\mathrm{e}^{b\varphi(x,x')}$ where b is the strength of the magnetic field. This factor is necessary for obtaining a gauge covariant magnetic pseudodifferential operator [25, 31]. By (3.4) it is clear that
$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \mathrm{e}^{\mathrm{i}b\varphi(x,x')}| \le p_{\alpha,\alpha'}(|x|, |x'|)$$
for some multivariate polynomial $p_{\alpha,\alpha'}$. Secondly, we allow that all derivatives of a have at most polynomial growth in $x - x'$. Since we already include the phase factor this growth in $x - x'$ does not introduce any additional complications when working with the corresponding magnetic pseudodifferential operator. In fact it is more convenient to work with this definition cf. [2, Remark 1.3].

Definition 3.1.3 ([2])**.** For each $a_b \in M_\varphi(\mathbb{R}^{3d})$ define the *magnetic pseudodifferential operator* $\mathrm{Op}(a_b)\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by

$$\langle \mathrm{Op}(a_b)f, g\rangle := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}\xi\cdot(x-x')}\mathrm{e}^{\mathrm{i}b\varphi(x,x')}a(x,x',\xi)f(x')\overline{g(x)}\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}\xi, \qquad (3.7)$$

for $f, g \in \mathscr{S}(\mathbb{R}^d)$.

Note that we define pseudo-differential operators in the weak sense. This is to ensure that $\mathrm{Op}(a_b)$ is well defined even though we have not required any decay of a_b with respect to ξ. To see this, one uses integration by parts to introduce the factor $\langle\xi\rangle^{-2d}$ in the integral. The growth of a_b in x and x' is then counteracted by the two Schwartz functions f and g. Note also that if $a(x,x',\xi) = a((x+x')/2,\xi)$ we obtain the magnetic Weyl pseudodifferential operator (3.2) and if $b = 0$ we obtain the classical Weyl quantisation (3.1). It is worth noting that a consequence of the main result of [2] is that for all $a_b \in M_\varphi(\mathbb{R}^{3d})$ there exists a symbol $\tilde{a} \in S^0_{0,0}(\mathbb{R}^{2d})$ such that $\mathrm{Op}(a_b) = \mathrm{Op}^W(\tilde{a})$ [2, Remark 1.3]. This follows from [2, Theorem 1.1] and an application from the magnetic Beals criterion, [5, 17] utilizing the growth in $x - x'$ allowed in (3.6).

3.2 Main Results

The operator defined in Definition 3.1.3 is the object of interest in [2]. As mentioned above, this thesis focuses on the two first main result of [2] which are stated in the following theorem.

Theorem 3.2.1 (Part of Theorem 1.1 of [2])**.** *If $a_b \in M_\varphi(\mathbb{R}^{3d})$ with $b \in [0, b_{\max}]$ for some $b_{\max} > 0$, then:*

(1) The operator $\mathrm{Op}(a_b)$ in (3.7) extends to a bounded operator on $L^2(\mathbb{R}^d)$.

Additionally, if $a(x,x',\xi) = \overline{a(x',x,\xi)}$ then $\mathrm{Op}(a_b)$ is self-adjoint and in this case:

(2) The spectrum of $\mathrm{Op}(a_b)$ is $\frac{1}{2}$-Hölder continuous in b on the interval $[0, b_{\max}]$, i.e. there exists a constant C such that

$$d_{\mathrm{H}}(\sigma(\mathrm{Op}(a_b)), \sigma(\mathrm{Op}(a_{b'}))) \le C|b - b'|^{1/2}, \qquad (3.8)$$

for all $b, b' \in [0, b_{\max}]$. Here d_H denotes the Hausdorff distance: For two compact sets $X, Y \subset \mathbb{R}$

$$d_{\mathrm{H}}(X,Y) := \max\{\sup_{x\in X}\mathrm{dist}(x,Y), \sup_{y\in Y}\mathrm{dist}(y,X)\}.$$

In [31, Theorem 3.1] it was established that magnetic Weyl pseudodifferential operators are bounded operators on $L^2(\mathbb{R}^d)$ for symbols of class $S^0_{\rho,\rho}(\mathbb{R}^d)$ for $\rho \in [0,1)$. However we use a different definition of magnetic pseudodifferential operators and the remark following Definition 3.1.3 concerning correspondence between our definition and the one used in e.g. [17, 25, 31] rely on the first item in Theorem 3.2.1. Furthermore, our proof of the Calderón-Vaillancourt type result of the first item in Theorem 3.2.1 is new.

In the proof of Theorem 3.2.1, we show that $\mathrm{Op}(a_b)$ is unitary equivalent to a generalized matrix (cf. Subsection 3.1.1). Specifically, for any $b \in \mathbb{R}$, let $U_b \colon L^2(\mathbb{R}^d) \to \mathscr{H}$ be given by

$$(U_b f)_\gamma(\cdot) := \mathrm{e}^{-\mathrm{i}b\varphi(\cdot+\gamma,\gamma)}\chi_\Omega(\cdot)f(\cdot+\gamma). \tag{3.9}$$

Then

$$[U_b^*(f_\gamma)_{\gamma\in\mathbb{Z}^d}](\cdot) = \sum_{\gamma\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}b\varphi(\cdot,\gamma)}\chi_\Omega(\cdot-\gamma)f_\gamma(\cdot-\gamma) \tag{3.10}$$

and we have the following result which is a excerpt of [2, Theorem 1.1].

Theorem 3.2.2 (Part of Theorem 1.1 of [2])**.** *For each $\gamma,\gamma' \in \mathbb{Z}^d$ there exists $A_{\gamma,\gamma',b} \in B(L^2(\Omega))$ such that*

$$U_b\mathrm{Op}(a_b)U_b^* = \{\mathrm{e}^{\mathrm{i}b\varphi(\gamma,\gamma')}A_{\gamma,\gamma',b}\}_{\gamma,\gamma'\in\mathbb{Z}^d}. \tag{3.11}$$

Moreover, for every $N \in \mathbb{N}$ there exists a constant C_N such that

$$\|A_{\gamma,\gamma',b}\| \le C_N\langle\gamma-\gamma'\rangle^{-N}, \tag{3.12}$$

and

$$\|A_{\gamma,\gamma',b} - A_{\gamma,\gamma',b'}\| \le C_N\langle\gamma-\gamma'\rangle^{-N}|b-b'|, \quad \text{for } b,b' \in [0,b_{\max}], \tag{3.13}$$

for all $\gamma,\gamma' \in \mathbb{Z}^d$.

Note that the first item of Theorem 3.2.1 is a direct consequence of combining Theorem 3.2.2 with Lemma 3.1.1. The title of the paper [2] is inspired by (3.11) since in addition to being a generalized matrix each “matrix element” contains a magnetic phase factor $\mathrm{e}^{\mathrm{i}b\varphi(\gamma,\gamma')}$ similar to the Harper-like operators considered in [4, 27] which are generalizations of the classical Harper operator, known for its Hofstadter butterfly spectrum [1, 15].

3.3 Proofs

In this section I give an overview of the proof of Theorem 3.2.1 as it is given in [2]. As mentioned above, the first item of Theorem 3.2.1 follows from Theorem 3.2.2. The proof of the second item of Theorem 3.2.1 relies heavily on the representation of $U_b\mathrm{Op}(a_b)U_b^*$ as a “generalized Hofstadter-like matrix” as well as the inequality (3.13). Therefore I start by summarizing the proof of Theorem 3.2.2.

3.3.1 Proof of Theorem 3.2.2

Working with a magnetic symbol $a_b \in M_\varphi(\mathbb{R}^{3d})$ is difficult since we have no knowledge of the behaviour of a_b with respect to ξ. We mitigate this problem by considering the regularized symbol $a_{b,\varepsilon}(x,x',\xi) = a(x,x',\xi)\mathrm{e}^{-\varepsilon\langle\xi\rangle}$, for $\varepsilon > 0$. For simplicity we assume also $a \in S^0_{0,0}$ (cf. Remark 2.7 of [2])

The fast decay of $a_{b,\varepsilon}$ when $|\xi| \to \infty$ makes it possible to apply Schur-Holmgrens lemma and Fubini's theorem to show that $\mathrm{Op}(a_{b,\varepsilon})$ is a bounded integral operator on $L^2(\mathbb{R}^d)$ [2, Lemma 2.1]. By using (3.9) and (3.10) we then show through explicit calculations that the operator $A_{b,\varepsilon} := U_b \mathrm{Op}(a_{b,\varepsilon}) U_b^*$ is a generalized matrix of the form

$$A_{b,\varepsilon} = \{\mathrm{e}^{\mathrm{i}b\varphi(\gamma,\gamma')} A_{\gamma,\gamma',b,\varepsilon}\}_{\gamma,\gamma'\in\mathbb{Z}^d},$$

where each $A_{\gamma,\gamma',b,\varepsilon}$ is an integral operator in $L^2(\Omega)$ with integral kernel

$$K_{\gamma,\gamma'}(\underline{x},\underline{x}') := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot(\underline{x}+\gamma-\underline{x}'-\gamma')}\mathrm{e}^{\mathrm{i}b\mathfrak{f}_{\gamma,\gamma'}(\underline{x},\underline{x}')}\mathrm{e}^{-\varepsilon\langle\xi\rangle} a(\underline{x}+\gamma,\underline{x}'+\gamma',\xi)\,\mathrm{d}\xi, \tag{3.14}$$

where we use an underline to indicate variables in Ω and where $\mathfrak{f}_{\gamma,\gamma'}(x,x') := \mathfrak{f}(x+\gamma,\gamma',\gamma) + \mathfrak{f}(x+\gamma,x'+\gamma',\gamma')$.

Proceeding from (3.14) we want to take ε to zero but the kernel $K_{\gamma,\gamma'}$ is not well-defined when $\varepsilon = 0$. We solve this problem by utilizing the fact that we are only interested in $K_{\gamma,\gamma'}$ on Ω^2. This allows us to replace $\mathrm{e}^{\mathrm{i}b\mathfrak{f}_{\gamma,\gamma'}(\underline{x},\underline{x}')} a(\underline{x}+\gamma,\underline{x}'+\gamma',\xi)$ in (3.14) with a Fourier series. To simplify the notation of the subsequent part of the proof we use a tilde accent to indicate functions and operators depending on m, m', γ, γ'. With this notation the aforementioned Fourier series is of the form

$$\sum_{m,m'\in\mathbb{Z}^d} \frac{\mathrm{e}^{\mathrm{i}m\cdot x + m'\cdot x'}}{\langle m\rangle^{2d}\langle m'\rangle^{2d}} \tilde{a}_b(\xi), \tag{3.15}$$

for functions $\tilde{a}_b$ satisfying

$$|\partial_\xi^\beta \tilde{a}_b| \le C_\beta \langle\gamma-\gamma'\rangle^{4d}. \tag{3.16}$$

The functions $\tilde{a}_b$ are simply Fourier coefficients multiplied with the factor $\langle m\rangle^{2d}\langle m'\rangle^{2d}$. The inequality (3.16) is obtained from integration by parts. The significance of replacing $\mathrm{e}^{\mathrm{i}b\mathfrak{f}_{\gamma,\gamma'}(\underline{x},\underline{x}')} a(\underline{x}+\gamma,\underline{x}'+\gamma',\xi)$ with (3.15) in $K_{\gamma,\gamma'}$ is that it allows us to rewrite the operator $A_{\gamma,\gamma',b,\varepsilon}$ in a way which is also well-defined for $\varepsilon = 0$. For $\varepsilon > 0$ we define the operators $\tilde{A}_{b,\varepsilon} \colon C_0^\infty(\Omega) \to \mathscr{S}(\mathbb{R}^d)$ by

$$(\tilde{A}_{b,\varepsilon}h)(x) := \mathrm{e}^{\mathrm{i}m\cdot x}\mathscr{F}^{-1}\Big[\mathrm{e}^{\mathrm{i}(*)\cdot(\gamma-\gamma')}\tilde{a}_b(*)\mathrm{e}^{-\varepsilon\langle *\rangle}\mathscr{F}\Big(\mathrm{e}^{\mathrm{i}m'\cdot(\cdot)}h(\cdot)\Big)(*)\Big](x), \tag{3.17}$$

for all $\varepsilon \geq 0$. Then using Fubini's theorem to rearrange the order of integration we get

$$(A_{\gamma,\gamma',b,\varepsilon}h)(\underline{x}) = \sum_{m,m'\in\mathbb{Z}^d} \frac{1}{(\langle m\rangle\langle m'\rangle)^{2d}}(\tilde{A}_{b,\varepsilon}h)(\underline{x}), \tag{3.18}$$

for all $h \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. Note how writing this formula with a Fourier transform and an inverse Fourier transform is made possible by choosing h to have compact support on Ω and the Fourier series expansion in (3.15). Since the Fourier transform is a bijection on $\mathscr{S}(\mathbb{R}^d)$ we have that $\tilde{A}_{b,\varepsilon}$ is well-defined even when $\varepsilon = 0$ and thus we define $A_{\gamma,\gamma',b}$ on $C_0^\infty(\Omega)$ by

$$(A_{\gamma,\gamma',b}h)(\underline{x}) := (A_{\gamma,\gamma'b,0}h)(\underline{x}) = \sum_{m,m'\in\mathbb{Z}^d} \frac{1}{(\langle m\rangle\langle m'\rangle)^{2d}}(\tilde{A}_{b,0}h)(\underline{x}). \tag{3.19}$$

From (3.17), (3.18) and (3.19) it is almost exclusively a matter of applying Parseval's identity, (3.16) and Lebesgues dominated convergence theorem to obtain Theorem 3.2.2.

3.3.2 Proof of Theorem 3.2.1(2)

To prove the second item of Theorem 3.2.1 we work with the generalized matrix structure in (3.11) instead of $\mathrm{Op}(a_b)$ as given in Definition 3.1.3. To simplify the notation, let $H_b := U_b\mathrm{Op}(a_b)U_b^* = \{\mathrm{e}^{\mathrm{i}b\varphi(\gamma,\gamma')}A_{\gamma,\gamma',b}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$ for any $b \in [0, b_{\max}]$. Recall that for this part of Theorem 3.2.1 we assume that $\mathrm{Op}(a_b)$, and hence H_b is self-adjoint for all $b \in [0, b_{\max}]$. Let $b_0 \in [0, b_{\max}]$ be arbitrary and let δb be such that $b_0 + \delta b \in [0, b_{\max}]$. Instead of working directly with $d_{\mathrm{H}}(\sigma(H_{b_0}), \sigma(H_{b_0+\delta b}))$ it is more convenient to introduce the following bounded self-adjoint operators:

$$\begin{aligned}
H_{b_0}^{\delta b} &:= \{\mathrm{e}^{\mathrm{i}(b_0+\delta b)\varphi(\gamma,\gamma')}A_{\gamma,\gamma',b_0}\}_{\gamma,\gamma'\in\mathbb{Z}^d},\\
H_{\delta b,b_0}^{\delta b} &:= \{\mathrm{e}^{\mathrm{i}(b_0+\delta b)\varphi(\gamma,\gamma')}A_{\gamma,\gamma',b_0}\}_{|\gamma-\gamma'|<|\delta b|^{-1/2}},\\
H_{\delta b,b_0} &:= \{\mathrm{e}^{\mathrm{i}b_0\varphi(\gamma,\gamma')}A_{\gamma,\gamma',b_0}\}_{|\gamma-\gamma'|<|\delta b|^{-1/2}}.
\end{aligned}$$

Note how the matrix elements of the two first operators have a different b value for the phase factor and the operator and note also that the two last operators are band diagonal generalized matrices.

By the triangle inequality it suffices to find C such that

$$d_{\mathrm{H}}(\sigma(H_{b_0+\delta b}), \sigma(H_{b_0}^{\delta b})) \leq C|\delta b|, \tag{3.20}$$

$$d_{\mathrm{H}}(\sigma(H_{b_0}^{\delta b}), \sigma(H_{\delta b,b_0}^{\delta b})) \leq C|\delta b|, \tag{3.21}$$

$$d_{\mathrm{H}}(\sigma(H_{\delta b,b_0}^{\delta b}), \sigma(H_{\delta b,b_0})) \leq C|\delta b|^{1/2}, \tag{3.22}$$

$$d_{\mathrm{H}}(\sigma(H_{\delta b,b_0}), \sigma(H_{b_0})) \leq C|\delta b|. \tag{3.23}$$

The inequality (3.20) follows from Lemma 3.1.1 and (3.13). The inequalities (3.21) and (3.23) also follow from Lemma 3.1.1 and additionally the fact that both inequalities concern a generalized matrix A and a band diagonal generalized matrix B such that $A_{\gamma,\gamma'} = B_{\gamma,\gamma'}$ whenever $|\gamma - \gamma'| < |\delta b|^{-1/2}$.

The remaining part of the proof is to prove (3.22) for which we use ideas based on [6]. Our proof shows that item 1. of the following general lemma holds with $T_1 = H_{\delta b,b_0}$ and $T_2 = H^{\delta b}_{\delta b,b_0}$.

Lemma 3.3.1 (Lemma 3.3 of [2]). *Let T_1, T_2 be bounded operators on some Hilbert space. The following assertions are equivalent:*

1. *There exists a constant C such that if $z \in \rho(T_j)$ with $\operatorname{dist}(z, \sigma(T_j)) > C$ then $z \in \rho(T_k)$, for $j, k = 1, 2$ and $j \neq k$.*
2. *There exists a constant C such that $d_{\mathrm{H}}(\sigma(T_1), \sigma(T_2)) \leq C$.*

Note that since we consider self-adjoint operators we only need to show that 1. of Lemma 3.3.1 holds for real numbers. We do so in the following way.

Suppose that $x \in \mathbb{R}$ with

$$\operatorname{dist}(x, \sigma(H_{\delta b,b_0})) > c|\delta b|^{1/2} \tag{3.24}$$

for some constant $c > 0$, and choose $\delta_0 > 0$ such that $z \in \rho(H_{\delta b,b_0})$ whenever $|z - x| < \delta_0$. For any $\delta \in \mathbb{R}$ with $0 < |\delta| < \delta_0$ we define $z_\delta = x + \mathrm{i}\delta$ and note that $z_\delta \in \rho(H^{\delta b}_{\delta b,b_0})$ by self-adjointness. A simple factorization then gives

$$H^{\delta b}_{\delta b,b_0} - x = (\mathrm{id} + \mathrm{i}\delta(H^{\delta b}_{\delta b,b_0} - z_\delta)^{-1})(H^{\delta b}_{\delta b,b_0} - z_\delta) \tag{3.25}$$

and if we can find a family of uniformly bounded operators S_{z_δ} for $0 < |\delta| < \delta_0$, such that

$$(H^{\delta b}_{\delta b,b_0} - z_\delta)S_{z_\delta} = \mathrm{id} + \mathcal{O}\Big(\frac{|\delta b|^{\frac{1}{2}}}{\operatorname{dist}(z_\delta, \sigma(H_{\delta b,b_0}))}\Big), \tag{3.26}$$

then choosing c and δ sufficiently small gives that $x \in \rho(H^{\delta b}_{\delta b,b_0})$. To complete the proof of 1. of Lemma 3.3.1 we repeat the arguments but with $H_{\delta b,b_0}$ and $H^{\delta b}_{\delta b,b_0}$ interchanged, but for simplicity we do not consider this case (see also [2, Remark 3.7]).

We explicitly construct S_z as generalized matrices. Our starting point, which is the same as in [6], is to consider a function $g \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq g \leq 1$ and $\operatorname{supp} g \subset B_r(0)$, $r > 0$ such that $\sum_{\gamma \in \mathbb{Z}^d} g^2(x - \gamma) = 1$ for every $x \in \mathbb{R}^d$. Furthermore, for any $n \in \mathbb{Z}^d$ define $g_{n,\delta b}(x) := g(|\delta b|^{1/2}x - n)$, and for each $n, \gamma \in \mathbb{Z}^d$ define the scalars

$$g^{\pm}_{\gamma,n,\delta b} := \mathrm{e}^{\pm \mathrm{i}\delta b \varphi(\gamma, n|\delta b|^{-1/2})} g_{n,\delta b}(\gamma).$$

We then define the operator $W_{\delta b}$ on $B(\mathscr{H})$ by

$$W_{\delta b}(R) := \Big\{ \sum_{n \in \mathbb{Z}^d} g^{+}_{\gamma,n,\delta b} R_{\gamma,\gamma'} g^{-}_{\gamma',n,\delta b} \Big\}_{\gamma,\gamma' \in \mathbb{Z}^d},$$

for $R \in B(\mathscr{H})$. By the definition of $g_{n,\delta b}$ it follows that $g_{n,\delta b} g_{n',\delta b} = 0$ if $n' \notin B_{2r}(n)$. Working from the definition of $W_{\delta b}$ and applying this identity we show that $W_{\delta b}$ is a bounded operator on $B(\mathscr{H})$.

Lemma 3.3.2 (Lemma 3.4 of [2])**.** *The operator* $W_{\delta b}$ *is bounded with* $\|W_{\delta b}\| \leq v_r^{1/2}$, *where* $v_r := |B_{2r}(0) \cap \mathbb{Z}^d|$.

By using a well-known identity for the resolvent norm together with Lemma 3.3.2 it is clear that $W_{\delta b}((H_{\delta b,b_0} - z_\delta)^{-1})$ is uniformly bounded in δ.

It remains to show that $W_{\delta b}((H_{\delta b,b_0} - z_\delta)^{-1})$ satisfies (3.26). We do this not only for z_δ but for any $z \in \rho(H_{\delta b,b_0})$.

Lemma 3.3.3 (Lemma 3.6 of [2])**.** *There exists a constant* C *such that for all* $z \in \rho(H_{\delta b,b_0})$ *the operator*

$$T_z = (H^{\delta b}_{\delta b,b_0} - z)W_{\delta b}((H_{\delta b,b_0} - z)^{-1}) - \mathrm{id}$$

is bounded on $\mathscr{H}$ *with*

$$\|T_z\| \leq \frac{C|\delta b|^{1/2}}{\operatorname{dist}(z, \sigma(H_{\delta b,b_0}))}.$$

The proof of Lemma 3.3.3 relies on two important facts:

(a) The matrix elements $(H^{\delta b}_{\delta b,b_0} - z)_{\gamma,\gamma' \in \mathbb{Z}^d}$ and $(H_{\delta b,b_0} - z)_{\gamma,\gamma' \in \mathbb{Z}^d}$ are equal up to the factor $\mathrm{e}^{\mathrm{i}\delta b \varphi(\gamma,\gamma')}$.

(b) Both $H^{\delta b}_{\delta b,b_0} - z$ and $H_{\delta b,b_0} - z$ are band diagonal generalized matrices with all non-zero entries having indices satisfying $|\gamma - \gamma'| < |\delta b|^{-1/2}$.

For every $\gamma, \gamma'' \in \mathbb{Z}^d$ let $\Delta g = g_{n,\delta b}(\gamma'') - g_{n,\delta b}(\gamma)$. By using some standard "tricks" together with (a), the definition of $g^+_{\gamma,n,\delta b}$ and $\mathfrak{f}$ we obtain that

$$\begin{aligned}
&[(H^{\delta b}_{\delta b,b_0} - z)W_{\delta b}((H_{\delta b,b_0} - z)^{-1})]_{\gamma,\gamma'} = \\
&\quad \sum_{\gamma'' \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\delta b \varphi(\gamma,\gamma'')} (H_{\delta b,b_0} - z)_{\gamma,\gamma''} \sum_{n \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\delta b \varphi(\gamma'', n|\delta b|^{-1/2})} \Delta g (H_{\delta b,b_0} - z)^{-1}_{\gamma'',\gamma'} g^-_{\gamma',n,\delta b} \\
&\quad + \sum_{\gamma'' \in \mathbb{Z}^d} (H_{\delta b,b_0} - z)_{\gamma,\gamma''} \sum_{n \in \mathbb{Z}^d} (\mathrm{e}^{\mathrm{i}\delta b \mathfrak{f}(\gamma,\gamma'', n|\delta b|^{-1/2})} - 1) g^+_{\gamma,n,\delta b} (H_{\delta b,b_0} - z)^{-1}_{\gamma'',\gamma'} g^-_{\gamma',n,\delta b} \\
&\quad + \sum_{\gamma'' \in \mathbb{Z}^d} (H_{\delta b,b_0} - z)_{\gamma,\gamma''} \sum_{n \in \mathbb{Z}^d} g^+_{\gamma,n,\delta b} (H_{\delta b,b_0} - z)^{-1}_{\gamma'',\gamma'} g^-_{\gamma',n,\delta b} \\
&=: [R_1]_{\gamma,\gamma'} + [R_2]_{\gamma,\gamma'} + [R_3]_{\gamma,\gamma'}.
\end{aligned}$$

Note first that R_3 is just the identity operator. Note also that by (b) none of the sums above contain any contribution for indices with $|\gamma - \gamma''| > |\delta b|^{-1/2}$. This property is very important when we prove that

$$\max\{\|R_1\|, \|R_2\|\} \leq \frac{C|\delta b|^{1/2}}{\operatorname{dist}(z, \sigma(H_{\delta b,b_0}))}, \tag{3.27}$$

for some constant C. Recall that $\operatorname{supp} g \subset B_r(0)$ and let $\tilde{g} \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq \tilde{g} \leq 1$, $\operatorname{supp} \tilde{g} \subset B_{r+2}(0)$ and $\tilde{g} = 1$ on $B_{r+1}(0)$. Furthermore, we define $\tilde{g}_{n,\delta b}(x) = \tilde{g}(|\delta b|^{1/2}x - n)$. Then $\tilde{g}_{n,\delta b}(x)g_{n,\delta b}(y) = g_{n,\delta b}(y)$ whenever $|x-y| \leq |\delta b|^{-1/2}$ and by (b) we can replace Δg and $g^+_{\gamma,n\delta b}$ in the definition of R_1 and R_2 with $\Delta g \tilde{g}_{n,\delta b}(\gamma'')$ and $g^+_{\gamma,n,\delta b}\tilde{g}_{n,\delta b}(\gamma'')$, respectively.

When considering the norm of R_1 we use the inequality $|g_{n,\delta b}(x) - g_{n,\delta b}(y)| \leq |\delta b|^{1/2} C_g |x-y|$ and a Schur-Holmgren type argument for $\ell^2(\mathbb{Z}^d)$ to get the desired bound (3.27) (see [2, Lemma 3.5 and Lemma 3.6]). For R_2 we use instead the inequality $|\gamma'' - n|\delta b|^{-1/2}|\tilde{g}_{n,\delta b}(\gamma'') \leq (r+2)|\delta b|^{-1/2}\tilde{g}_{n,\delta b}(\gamma'')$ and the same Schur-Holmgren type argument to get (3.27) (see [2, Lemma 3.5 and Lemma 3.6]).

4 | Acoustic Black Holes

In [24] it was demonstrated that a flexural wave does not reflect from the edge of a beam or plate if the edge is shaped as a wedge with height given by $h(x) = \varepsilon x^m$, for $m \geq 2$ and ε small (see Figure 4.1). This phenomenon has been coined the "acoustic black hole effect" and the part of beam with height given by h is generally referred to as an acoustic black hole. The main difficulty in using the acoustic black hole effect for practical purposes is that edge truncation always is present in real beams and plates (see Figure 4.1). In [24] it was shown that a truncation of the edge can lead to significantly increased reflection. The first step towards making the acoustic black hole effect practically viable was done by Krylov [19, 20, 21] who proposed adding a thin layer of dampening material to an acoustic black hole. This combination of two well-known methods for vibration dampening have been investigated experimentally and shown to be very efficient compared to just the truncated acoustic black hole [12, 13, 22].

Later it will be clear that the profile shape h of an acoustic black hole affects how well flexural waves are dampened. Hence one way to improve acoustic black holes is to consider different profiles. In the literature the profile $h(x) = \varepsilon x^m$ was originally considered in [24] and has been considered in many subsequent papers, see e.g. [12, 19, 20, 21, 32]. Furthermore, [21] considered profiles of the form $h(x) = \varepsilon \sin^m(x)$ and [8, 29] considered profiles given by $h(x) = \varepsilon x^m + h_0$. More profiles are considered in [18]. The paper [29] uses numerical optimization methods to determine parameters which make an acoustic black hole with profile of the form $h(x) = \varepsilon x^m + h_0$ optimal in some sense.

In this chapter I will discuss the results obtained in our own paper [3] concerning optimal profile design for acoustic black holes. In contrast to [29] we do this analytically using methods from calculus of variations. As a consequence we are not restricting our attention to e.g. profiles of the form $h(x) = \varepsilon x^m + h_0$. We obtain simple closed form expressions for profiles which we refer to as optimal. Numerical comparison between optimal profiles and classical profiles is also conducted.

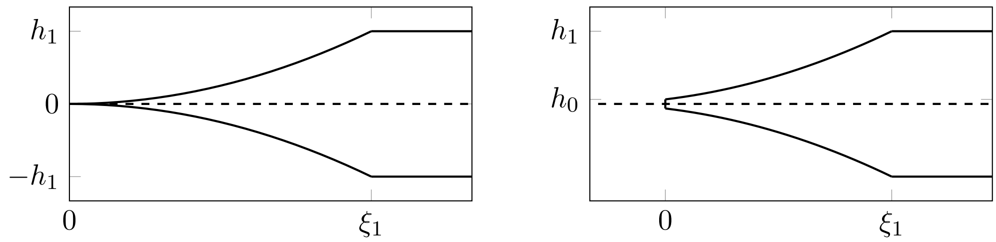


Figure 4.1: Non-truncated profile (left) and truncated profile (right) [3, Figure 1].

4.1 Theoretical Background

The motion of a flexural wave in an Euler-Bernoulli beam is governed by the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big[EI\frac{\mathrm{d}^2}{\mathrm{d}x^2}w\Big] - \rho A\omega^2 w = 0, \tag{4.1}$$

where E is the Young's modulus, I is the moment of inertia, ρ is the density, ω is the angular frequency, A is the area of the cross section and w denotes the displacement. In [3] energy dissipation is taken into account, which means that E is of the form $E(1 - i\eta)$ where $\eta > 0$ is the loss factor.

In the field of acoustic black holes, one often considers geometrical acoustics approximation of the solution to (4.1) [20, 21, 24]. In fact this is nothing but the first order WKB approximation to the solution of (4.1) [18]. In [3] we also restrict our attention to this first order approximation. As we are working with a WKB approximation, the validity of the solution depends on certain applicability conditions. In the theory of geometrical acoustics approximation (see [21] and references therein) one often uses the condition

$$|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}x}| \ll 1, \tag{4.2}$$

where k denotes the local wave number. In [8] the term normalized wave number variation (NWV) was used for the left hand side of (4.2). Through numerical considerations, [9] established 0.3 as a satisfactory value of the NWV for practical applications.

The effectiveness of an acoustic black hole is measured by its reflection coefficient, i.e. the ratio of the wave reflected from the truncated edge of the acoustic black hole and the wave travelling towards the edge [18]. Using the first order WKB approximation this number is given by [18, 24]

$$R = \exp\Big(-2\int_{x_0}^{x_1} \mathrm{Im}(k(x))\,\mathrm{d}x\Big) \tag{4.3}$$

when the acoustic black hole is situated in the interval $[x_0, x_1]$.

4.1.1 Problem Statement

As in [3], I will continue this exposition using mainly dimensionless variables. To avoid confusion, all dimensional variables which appear in the following are indicated with a tilde. It is straightforward to convert between dimensional and non-dimensional variables using Remark 1.1 of [3]. To ease the reading experience I will simplify the notation compared to [3].

In non-dimensional variables we use the following first order approximation to the local wave number

$$k(\xi) = \frac{12^{1/4}\Omega^{1/2}}{\sqrt{2}h^{1/2}(\xi)}\Big(1 + i\frac{\eta}{4}\Big), \tag{4.4}$$

where Ω is the non-dimensional frequency and η is the loss factor. This formula can be derived directly from (4.1). The problem considered in [3] can then be stated as follows:

Given $\Omega \geq 0$ and $\eta > 0$, find a C^1 function $h\colon [0,\xi_1] \to [h_0, 1]$, where $0 < h_0 < 1$ and $\xi_1 > 0$, satisfying the boundary conditions

$$h(0) = h_0, \quad \text{and} \quad h(\xi_1) = 1, \tag{4.5}$$

such that

$$R = \exp\Big(-2\int_0^{\xi_1} \frac{12^{1/4}\Omega^{1/2}}{\sqrt{2}h^{1/2}(\xi)}\frac{\eta}{4}\,\mathrm{d}\xi\Big), \tag{4.6}$$

is minimized while

$$\frac{\sqrt{2}}{2\cdot 12^{1/4}\Omega^{1/2}}\frac{h'(\xi)}{h^{1/2}(\xi)} \ll 1. \tag{4.7}$$

Note that (4.6) and (4.7) are simply (4.3) and (4.2), respectively, but stated in non-dimensional variables with k given by (4.4).

4.2 Approach

The statement of the problem in Section 4.1.1 and especially (4.6) suggests that it can be solved by using calculus of variations. The main complication is the condition imposed by (4.7). By imposing instead the condition that the integral of the left hand side of (4.7) over $[0, \xi_1]$ has to be small, say

$$\int_0^{\xi_1} \frac{\sqrt{2}}{2\cdot 12^{1/4}\Omega^{1/2}}\frac{h'(\xi)}{h^{1/2}(\xi)}\,\mathrm{d}\xi = C \tag{4.8}$$

for some small $C > 0$, we obtain an isoperimetric problem which can be solved by the well-known of Lagrange multipliers [11, 28]. Thus replacing the condition (4.7) with (4.8) makes the problem solvable by use of well-known methods. However, this approach is not satisfactory as (4.8) does not imply anything about the pointwise behaviour of the NWV. A more refined choice would be to replace (4.7) by

$$\int_0^{\xi_1} \Big| \frac{\sqrt{2}}{2 \cdot 12^{1/4}\Omega^{1/2}} \frac{h'(\xi)}{h^{1/2}(\xi)} \Big|^{2n} \,\mathrm{d}\xi \tag{4.9}$$

for some large $n \in \mathbb{N}$, since taking the $2n$th root of this integral approximates the essential supremum norm of the integrand. Replacing (4.7) with (4.9) still gives a isoperimetric problem which can be solved by maximizing the functional (see [11, 28])

$$h \mapsto \int_0^{\xi} \frac{12^{1/4}\Omega^{1/2}}{\sqrt{2}h^{1/2}(\xi)} \frac{\eta}{4} - \Big| \frac{\sqrt{2}}{2\delta 12^{1/4}\Omega^{1/2}} \frac{h'(\xi)}{h^{1/2}(\xi)} \Big|^{2n} \,\mathrm{d}\xi.$$

Note that this functional is slightly different but equivalent to the one considered in the method of Lagrange multipliers. We prefer this functional since δ and n have clear intuitive interpretations (cf. [3] for details). Intuitively the δ should be chosen small and n should be chosen large to approximate (4.7).

By a simple scaling of the above functional we obtain instead the functional

$$h \mapsto \int_0^{\xi_1} b\frac{1}{h^{1/2}(\xi)} - \frac{(h'(\xi))^{2n}}{h^n(\xi)} \,\mathrm{d}\xi, \tag{4.10}$$

where

$$b = (2\delta)^{2n} 12^{(2n+1)/4} \Omega^{n+1/2} \frac{\eta}{8}. \tag{4.11}$$

By maximizing this functional over the set $\{h \in C^1([0, \xi_1], [h_0, 1]) \mid h(0) = h_0, h(\xi_1) = 1\}$ for suitable values of δ and n, we obtain an approximate solution to the problem of Section 4.1.1. In [3] such functions are called optimal profiles.

To find maximizers of (4.10) we consider the corresponding Euler-Lagrange equation. By defining $F_{b,n} \colon \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}$ as

$$F_{b,n}(\xi, u, v) = bu^{-1/2} - v^{2n}u^{-n}$$

and using that $F_{b,n}$ does not explicitly depend on ξ, it follows that the Euler-Lagrange equation for (4.10) is a first order differential equation given by [11, pp. 18-19]

$$h'(\xi) = (2n-1)^{-1/(2n)} (ah^n(\xi) - bh^{n-1/2}(\xi))^{1/(2n)}, \quad \xi \in (0, \xi_1). \tag{4.12}$$

Note that the Legendre condition [11, p. 103] implies that no solution of (4.12) can be a minimizer of (4.10).

4.3 Results

In [3] the two cases $b = 0$ and $b > 0$ are considered separately. From (4.10) it appears that the former case is unimportant for practical applications as the term of the reflection coefficient vanishes. However it turns out that this case is related to the profile $h(x) = \varepsilon x^2$ originally considered in [24].

4.3.1 The case $b = 0$

When $b = 0$ it only requires a straightforward application of separation of variables to show that the function

$$h(\xi) = \left(\left(1 - \sqrt{h_0}\right)\frac{\xi}{\xi_1} + \sqrt{h_0}\right)^2 \tag{4.13}$$

is the unique solution to (4.12) satisfying (4.5) for all $n \in \mathbb{N}$. Note that as $b = 0$ this solution is also independent of δ. The following remark shows how the case $b = 0$ generalizes the classical profile $h(x) = \varepsilon x^2$ which has been used extensively in the study of acoustic black holes e.g. [8, 13, 20, 21, 24, 29].

Remark 4.3.1 (Paraphrasing Remark 3.1 in [3]). When considering non-dimensional variables, the height profile $\tilde{h}(\tilde{x}) = \varepsilon\tilde{x}^m$ discussed in the beginning of this chapter takes the form $h(\xi) = \tilde{h}_1^{-1}\varepsilon(\tilde{x}_0 + \xi\tilde{h}_1)^m$. This function is a solution to (4.12) with the boundary conditions (4.5) if and only if $m = 2$ and

$$\varepsilon := \frac{\tilde{h}_0}{\tilde{x}_0^2} = \frac{\tilde{h}_1}{\tilde{x}_1^2}. \tag{4.14}$$

This shows that (4.13) is a generalization of the well-known profile function $\tilde{h}(\tilde{x}) = \varepsilon\tilde{x}^2$.

Note that none of the other profiles listed in the beginning of this chapter solve the Euler-Lagrange equation.

4.3.2 The case $b > 0$

The Euler-Lagrange equation (4.12) is an autonomous differential equation and thus it is not difficult to obtain a local solution [30]. For $a \geq bh_0^{-1/2}$ we let $F_{a,b,n}$ be defined on $[h_0, \infty)$ by

$$F_{a,b,n}(s) = \int_1^s \frac{1}{(2n-1)^{-1/(2n)}(ay^n - by^{n-1/2})^{1/(2n)}}\,\mathrm{d}y, \tag{4.15}$$

and note that this function is well-defined [3, Lemma A.1].

The inverse $(F_{a,b,n})^{-1}$ exists and $h_{a,b,n}(\xi) := (F_{a,b,n})^{-1}(\xi - \xi_1)$ is the unique solution of (4.12) satisfying $h_{a,b,n}(\xi_1) = 1$. Recall that we desire a solution h satisfying also the boundary condition $h(0) = h_0$. Thus the problem is reduced to determining $a \geq bh_0^{-1/2}$ (if it exists) such that $h_{a,b,n}(0) = h_0$. Clearly this is equivalent with a being a solution to the equation

$$F_{a,b,n}(h_0) = -\xi_1. \tag{4.16}$$

Thus we have to ensure that $-\xi_1$ is in the domain of the function $a \mapsto F_{a,b,n}(h_0)$. The following lemma is helpful in this regard.

Lemma 4.3.2 (Lemma A.2 of [3]). *For fixed $b > 0$ and $n \in \mathbb{N}$ the map*

$$[bh_0^{-1/2}, \infty) \ni a \mapsto F_{a,b,n}(h_0)$$

is a strictly increasing continuous function which goes to 0 as a goes to ∞.

Lemma 4.3.2 shows that for every $n \in \mathbb{N}$ (4.16) has a unique solution if and only if $F_{bh_0^{-1/2},b,n}(h_0) \leq -\xi_1$. Next we will establish for which b this inequality holds.

Lemma 4.3.3 (Lemma A.3 of [3]). *For fixed $n \in \mathbb{N}$ we have that*

$$(0, \infty) \ni b \mapsto F_{bh_0^{-1/2},b,n}(h_0)$$

is a strictly increasing continuous function which goes to $-\infty$ as b goes to 0 and to 0 as b goes to ∞.

Lemma 4.3.3 shows that for every $n \in \mathbb{N}$ there exists a unique solution b_n to the equation $F_{bh_0^{-1/2},b,n}(h_0) = -\xi_1$ and this gives two distinct cases (taken directly from [3]):

1. If $b \leq b_n$ there exists a unique a with $a \geq bh_0^{-1/2}$ such that (4.12) has a unique $C^\infty([0, \xi_1], [h_0, 1])$ solution satisfying (4.5).
2. If $b > b_n$ there exists $\xi(b) \in [0, \xi_1]$ and a function $h_b \in C^\infty([\xi(b), \xi_1], [h_0, 1])$ satisfying (4.12) with the boundary conditions $h_b(\xi(b)) = h_0$ and $h_b(\xi_1) = 1$ such that

$$h(\xi) = \begin{cases} h_0, & \text{if } 0 \leq \xi \leq \xi(b), \\ h_b(\xi), & \text{if } \xi(b) < \xi \leq \xi_1 \end{cases}$$

is a $C^1([0, \xi_1], [h_0, 1])$ and piecewise $C^\infty([0, \xi_1], [h_0, 1])$ solution to (4.12).

As n should be large, we consider next what happens to the two cases above when n goes to ∞.

Lemma 4.3.4 (Lemma A.4 of [3]). *For any $a \geq bh_0^{-1/2}$ the function $F_{a,b,n}$ converges uniformly to*

$$F_\infty(s) = \frac{1}{2\delta 12^{1/4}\Omega^{1/2}} \int_1^s \frac{1}{\sqrt{y}}\, \mathrm{d}y = \frac{\sqrt{s} - 1}{12^{1/4}\delta\Omega^{1/2}},$$

on $[0, \xi_1]$ as n goes to ∞.

Clearly, F_∞ is invertible and, similarly to before, we call $h(\xi) = F_\infty^{-1}(\xi - \xi_1)$ an optimal profile if it solves the boundary conditions (4.5). By defining

$$\Omega_0 = \Big(\frac{1 - \sqrt{h_0}}{12^{1/4}\delta\xi_1}\Big)^2,$$

we obtain the following version of the cases (1) and (2) above (taken directly from [3]):

(a) If $\Omega < \Omega_0$ then no optimal profile exists.

(b) If $\Omega \geq \Omega_0$ then the optimal profile is given by

$$h(\xi) = \begin{cases} h_0, & \text{if } 0 \leq \xi \leq \xi_\infty, \\ \left(12^{1/4}\delta\Omega^{1/2}(\xi - \xi_1) + 1\right)^2, & \text{if } \xi_\infty < \xi \leq \xi_1 \end{cases} \tag{4.17}$$

where

$$\xi_\infty = \xi_1 - \frac{1 - \sqrt{h_0}}{12^{1/4}\delta\Omega^{1/2}}.$$

Recall that the Ω appearing in (4.17) is fixed (cf. Section 4.1.1) and in the following we denote it by $\Omega^{(d)}$ to stress this fact. We refer to $\Omega^{(d)}$ as the design frequency. This is to avoid ambiguity when considering R and the NWV of an acoustic black hole with profile h, as both these quantities are dependent on the frequency Ω.

When inserting the function of (4.17) in (4.7) we have that

$$\left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\right| = \begin{cases} 0, & \text{if } 0 \leq \xi \leq \xi_\infty, \\ \sqrt{2}\delta\left(\frac{\Omega^{(d)}}{\Omega}\right)^{1/2}, & \text{if } \xi_\infty < \xi \leq \xi_1, \end{cases} \tag{4.18}$$

which clearly shows how δ affects the NWV. When inserting (4.17) in (4.6) we obtain

$$R = \left(\exp\frac{\eta\Omega^{1/2}}{2\sqrt{2}}\right)^{\frac{\ln(h_0)}{2\delta(\Omega^{(d)})^{1/2}} - \frac{12^{1/4}\xi_\infty}{h_0^{1/2}}} \tag{4.19}$$

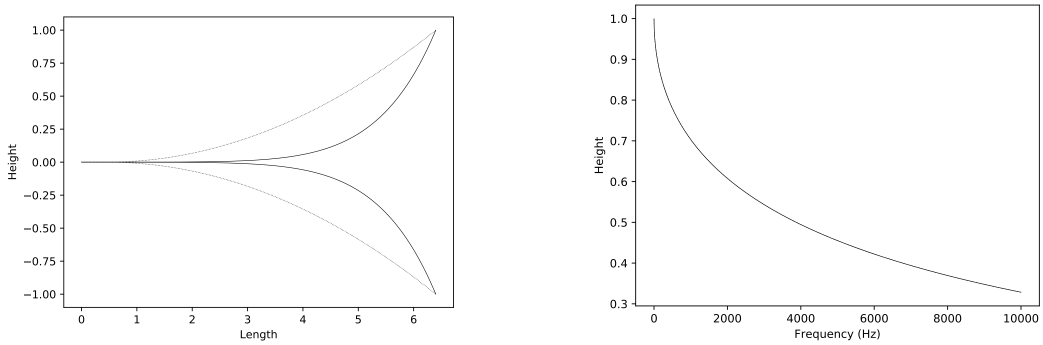
which shows that R is a decreasing function of $\Omega^{(d)}$. Note also that ξ_∞ is an increasing function of $\Omega^{(d)}$. This means that at higher design frequencies the acoustic black hole will have a longer part with constant height.

4.3.3 Numerical Comparison With a Classical Profile

It is well-known that increasing the power m of a classical profile $\tilde{h}(\tilde{x}) = \varepsilon\tilde{x}^m$ decreases the reflection coefficient but increases the NWV for a wide range of frequencies [8]. Increasing m has a similar effect on the profile shape as increasing Ω in (4.17).

To illustrate the advantages of the profile (4.17) compared to a classical profile of high power m, we consider a numerical example with the following dimensional quantities: $\tilde{x}_0 = 2$cm, $\tilde{x}_1 = 10$cm, $\tilde{h}_0 = 3.2 \cdot 10^{-6}$ cm, $\tilde{h}_1 = 1.25$cm, $\tilde{c} = 3000$m/s. Furthermore, we assume a loss factor of $\eta = 0.01$. Note that using dimensional variables makes comparisons with previous work easier.

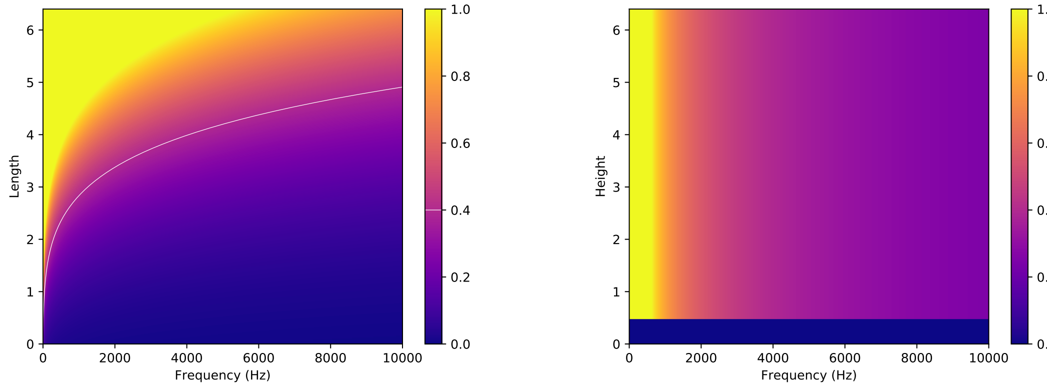
For these choices of parameters we construct a classical profile with power $m = 8$. For $\delta = \frac{\sqrt{2}}{5}$ we find an optimal profile given by (4.17) with similar reflection coefficient and compare shape and NWV. The profiles and their reflection coefficients are plotted in Figure 4.2.



(a) The classical (solid) and optimal (dotted) profile of Subsection 4.3.3.

(b) Reflection coefficients for the profiles in Figure 4.2a. The same line specefications as in Figure 4.2a have been used.

Figure 4.2: Classical and optimal profile and their reflection coefficients [3, Figure 4.3].



(a) NWV for the classical profile in Figure 4.2a. A white contour line has been added to indicate where the NWV equals 0.4.

(b) NWV for the optimal profile in Figure 4.2a.

Figure 4.3: NWV for the profiles of Figure 4.2a [3, Figure 4.3].

Note that the profiles have approximately identical reflection coefficients but the classical profile has a much larger derivative near the end of the acoustic black hole. This difference in derivative is clearly seen in the NWV in Figure 4.3.

Note that the NWV of the classical profile violates (4.2) for a large range of frequencies in the interval $[0, 10^4]$Hz at the end of the acoustic black hole. However, the optimal profile only violates (4.2) for low frequencies. This is one of the key features of the optimal profile, namely that the reflection coefficient can be reduced without violating (4.2) for a large range or frequencies.

5 | Future Work

In this chapter I discuss possible future work. My future research will reflect the way my 4+4 PhD is funded. As mentioned in Chapter 2 the funding is split between the Department of Mathematical Sciences and the Department of Materials and Productions. Furthermore, part of the Department of Mathematical Sciences share is supplied by GSGB. For this reason I, regrettably, do not anticipate to be able to conduct more research as technical as [2]. In the near future I expect to work primarily on the subject of acoustic black holes and on the paper concerning singular functions briefly mentioned in Chapter 2.

Soon I expect to finish the work on the paper [3] concerning optimal profile for acoustic black holes. Currently no new results are expected to be added to the is paper. After submitting the paper [3] to a peer-reviewed journal I intend to carry on the research in the field of acoustic black holes. As mentioned in Chapter 4 acoustic black holes can be improved by adding a thin layer of dampening material to the profile [19, 20, 21]. This changes the local wave number and the analysis in Chapter 4 could, in principle, be carried out for this new local wave number. Another way to build upon the method of [3] is to optimize the profile shape when considering higher order WKB approximations.

The literature on acoustic black holes uses almost exclusively first order WKB approximations to the Euler-Bernoulli beam equation (4.1) and the validity of the approximation is determined by the condition (4.2) [18, 21, 24]. Therefore I see the need for analysing, from a rigorous mathematical perspective, conditions of the validity of the WKB approximation in relation to acoustic black holes. Considering the acoustic black hole effect in Timoshenko beams also seems like a natural way to further develop the theory.

The acoustic black holes studied in this thesis and [3] are all placed at the edge of a beam or plate. In the literature so-called two dimensional acoustic black holes have been studied [13]. This is simply a cavity in the plate obtained rotating a power-law profile as in Figure 4.1 around the y-axis. To my knowledge the no other geometries for the two dimensional acoustc black hole have been considered. This paves the way for generalizing the approach of [3] to higher dimensions.

As a last possibility for future work with regards to acoustic black holes I hope to bridge the gap between the two papers [2] and [3] by studying the acoustic black hole effect in magnetic fields.

Concurrently with the preparation of this thesis I have been working on a paper on singular functions constructed from stationary time series and point processes. This is joint work with Horia Cornean, Ira Herbst, Jesper Møller, and Kasper Studsgaard Sørensen. Given a time series $(X_n)_{n\geq 1}$ of stochastic variables $X_n \in \{0, 1, \ldots, q-1\}$, $q \geq 2$ we consider the CDF F of a stochastic variable X defined as

$$X = \sum_{j=1}^{\infty} X_j q^{-j}.$$

At the present time we have shown that if $(X_n)_{n\geq 1}$ is assumed stationary then F satisfies a certain functional equation. Using this functional equation enables us to give a detailed description of the Lebesgue decomposition of F which generalizes previously established results. Furthermore, we consider concrete examples of time series and point processes for which we prove that the corresponding F is singular continuous. Although we already have some nice results this paper is not yet ready for publication. For this reason this paper has not been included in thesis.

Bibliography

[1] Jean Bellissard. “Lipshitz continuity of gap boundaries for Hofstadter-like spectra”. In: *Communications in Mathematical Physics* 160 (Mar. 1994), pp. 599–613. DOI: 10.1007/BF02173432.

[2] Decebal Horia Cornean et al. “Magnetic pseudodifferential operators represented as generalized Hofstadter-like matrices”. English. In: *Journal of Pseudo-Differential Operators and Applications* 10.2 (2019), pp. 307–336. ISSN: 1662-9981. DOI: 10.1007/s11868-018-0271-y.

[3] Horia Cornean, Sergey Sorokin, and Benjamin Støttrup. “Optimal Profile Design for Acoustic Black Holes”. Newest draft is included as appendix. 2019.

[4] Horia D. Cornean. “On the Lipschitz Continuity of Spectral Bands of Harper-Like and Magnetic Schrödinger Operators”. In: *Annales Henri Poincaré* 11.5 (Oct. 2010), pp. 973–990. ISSN: 1424-0661. DOI: 10.1007/s00023-010-0048-1.

[5] Horia D. Cornean, Bernard Helffer, and Radu Purice. “A Beals criterion for magnetic pseudo-differential operators proved with magnetic Gabor frames”. In: *Communications in Partial Differential Equations* 43.8 (2018), pp. 1196–1204. DOI: 10.1080/03605302.2018.1499777.

[6] Horia D. Cornean and Radu Purice. “On the Regularity of the Hausdorff Distance Between Spectra of Perturbed Magnetic Hamiltonians”. In: *Spectral Analysis of Quantum Hamiltonians*. Ed. by Rafael Benguria, Eduardo Friedman, and Marius Mantoiu. Basel: Springer Basel, 2012, pp. 55–66. ISBN: 978-3-0348-0414-1.

[7] Jacques Cuenca. “Wave Models For The Flexural Vibrations of Thin Plates - Model of the vibrations of polygonal plates by the image source method - Vibration damping using the acoustic black hole effect”. PhD thesis. Université du Maine, 2009.

[8] P. A. Feurtado, S. C. Conlon, and F. Semperlotti. “A normalized wave number variation parameter for acoustic black hole design”. In: *The Journal of Acoustical Society of America* 136.2 (2014), pp. 149–152.

[9] Philip A. Feurtado and Stephen C. Conlon. “Investigation of boundary-taper reflection for acoustic black hole design”. In: *J. Noise Cont. Eng.* 63.5 (2015), pp. 460–466. DOI: 10.3397/1/376341.

[10] Gerald B. Folland. *Harmonic Analysis in Phase Space. (AM-122)*. Princeton University Press, 1989. ISBN: 9780691085289.

[11] I.M. Gelfand and S.V. Fomin. *Calculus of Variations*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486135014.

[12] Vasil B. Georgiev et al. "Recent progress in vibration reduction using Acoustic Black Hole effect". In: *10ème Congrès Français d'Acoustique*. Ed. by Société Française d'Acoustique SFA. Lyon, France, Apr. 2010, pp. –.

[13] V.B. Georgiev et al. "Damping of structural vibrations in beams and elliptical plates using the acoustic black hole effect". In: *Journal of Sound and Vibration* 330.11 (2011), pp. 2497 –2508. ISSN: 0022-460X. DOI: `https://doi.org/10.1016/j.jsv.2010.12.001`.

[14] A. Grossmann, Guy Loupias, and Elias M. Stein. "An algebra of pseudo-differential operators and quantum mechanics in phase space". en. In: *Annales de l'Institut Fourier* 18.2 (1968), pp. 343–368. DOI: `10.5802/aif.305`.

[15] Douglas R. Hofstadter. "Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields". In: *Phys. Rev. B* 14 (6 Sept. 1976), pp. 2239–2249. DOI: `10.1103/PhysRevB.14.2239`.

[16] Lars Hörmander. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. eng. Classics in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007. ISBN: 978-3-540-49937-4.

[17] Viorel Iftimie, Marius Măntoiu, and Radu Purice. "Commutator Criteria for Magnetic Pseudodifferential Operators". In: *Communications in Partial Differential Equations* 35.6 (2010), pp. 1058–1094. DOI: `10.1080/03605301003717118`.

[18] Angelis Karlos, Stephen J. Elliott, and Jordan Cheer. "Higher-order WKB analysis of reflection from tapered elastic wedges". In: *Journal of Sound and Vibration* 449 (2019), pp. 368 –388. ISSN: 0022-460X. DOI: `https://doi.org/10.1016/j.jsv.2019.02.041`.

[19] Victor Krylov. "Laminated plates of variable thickness as effective absorbers for flexural vibrations". In: *Proc. 17th Int. Congress on Acoustics*. Vol. 1. Sept. 2001, pp. 270–271.

[20] Victor Krylov. "New type of vibration dampers utilising the effect of acoustic 'black holes'". In: *Acta Acustica united with Acustica* 90 (Aug. 2004), pp. 830–837.

[21] V.V. Krylov and F.J.B.S. Tilman. "Acoustic 'black holes' for flexural waves as effective vibration dampers". In: *Journal of Sound and Vibration* 274.3 (2004), pp. 605 –619. ISSN: 0022-460X. DOI: `https://doi.org/10.1016/j.jsv.2003.05.010`.

[22] V.V. Krylov and R.E.T.B. Winward. “Experimental investigation of the acoustic black hole effect for flexural waves in tapered plates”. In: *Journal of Sound and Vibration* 300.1 (2007), pp. 43 –49. ISSN: 0022-460X. DOI: https://doi.org/10.1016/j.jsv.2006.07.035.

[23] M. Lein, Marius Măntoiu, and Serge Richard. “Magnetic pseudodifferential operators with coefficients in C⋆-algebras”. In: *Publ. RIMS, Kyoto Univ.* 46 (2010), pp. 755 –788. URL: https://hal.archives-ouvertes.fr/hal-00355982.

[24] M. Mironov. “Propagation of a flexural wave in a plate whose thickness decreases smoothly to zero in a finite interval”. In: *Sov. Phys. Acoust.* 34 (1988), pp. 318–319.

[25] Marius Măntoiu and Radu Purice. “The magnetic Weyl calculus”. In: *Journal of Mathematical Physics* 45.4 (2004), pp. 1394–1417. DOI: 10.1063/1.1668334.

[26] Marius Măntoiu, Radu Purice, and Serge Richard. “Spectral and propagation results for magnetic Schrödinger operators; A C*-algebraic framework”. In: *Journal of Functional Analysis* 250.1 (2007), pp. 42 –67. ISSN: 0022-1236. DOI: https://doi.org/10.1016/j.jfa.2007.05.020.

[27] Gheorghe Nenciu. “On the smoothness of gap boundaries for generalized Harper operators”. In: *Advances in Operator Algebras and Mathematical Physics, Theta Series in Advanced Mathematics* (Oct. 2005), pp. 173–182.

[28] Irving H Shames. *Energy and Finite Element Methods in Structural Mechanics.* eng. New York, N.Y: Taylor & Francis, 2003. ISBN: 9780891169420.

[29] Micah R. Shepherd, Philip A. Feurtado, and Stephen C. Conlon. “Multi-objective optimization of acoustic black hole vibration absorbers”. In: *The Journal of the Acoustical Society of America* 140.3 (2016), EL227–EL230. DOI: 10.1121/1.4961735.

[30] G. Teschl. *Ordinary Differential Equations and Dynamical Systems.* Graduate studies in mathematics. American Mathematical Soc., 2012.

[31] M. Măntoiu V. Iftimie and R. Purice. “Magnetic pseudodifferential operators”. In: *Publications of RIMS* 43.3 (2007), pp. 585–623.

[32] Liuxian Zhao, Stephen C Conlon, and Fabio Semperlotti. “Broadband energy harvesting using acoustic black hole structural tailoring”. In: *Smart Materials and Structures* 23.6 (May 2014), p. 065021. DOI: 10.1088/0964-1726/23/6/065021.