Magnetic Pseudodifferential Operators and Acoustic Black Holes

Differential Equations

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Abstract:
This thesis is an exposition of the work I have done the first two years as a 4+4 Phd-student at Aalborg University. It contains a summary of a published paper on magnetic pseudodifferential operators and some ongoing work on acoustic black holes. In the end I give some possible research directions for the remaining two years of my Phd. In the summary of the first article I focus on the spectral theory of the spectrums endpoint along with the case where the spectrum has a gap which does not close. In relation to the work on acoustic black holes, i quickly review some work done in the framework of Euler-Lagrange beam theory before moving on to try and generalize it to Timoshenko beam theory instead. Most of the work presented in this thesis belongs to the area of differential equations.

Contents

1. Summary in Danish

Dette speciale er skrevet som en oversigt over de første to år af min tid som 4+4 Phd-studerende, delt mellem Institut for Matematiske Fag og Institut for Materialer og Produktion på Aalborg universitet. Udover at være en oversigt over min hidtidige forskning, så afsluttes specialet med en kort oversigt over nogle forskningsområder/problemer der kunne have interesse for mig de sidste to år af mit studie. Jeg håber dette kan danne grobund for en givende discussion, som forhåbentlig kan give input til min kommende forskning.

I specialet gennemgår jeg en udgivet artikel, skrevet sammen med Horia Cornean, Henrik Garde og Benjamin Støttrup under emnet magnetiske pseudodifferential operatorer. I denne artikel definerer vi en klasse af magnetiske symboler der tillader en polynomial vækst i $x - x'$, i modsætning til de klassiske symboler der normalt tillader polynomial vækst i ξ. Dernæst definere vi, på baggrund af disse magnetiske symboler, magnetiske pseudodifferential operatorer og viser at de representeres ved en generaliseret matrix. Ved hjælp af denne generalisering viser vi, at spectrummet for disse operatorer er $\frac{1}{2}$-Hölder kontinuerte og at endepunkterne af spektrummet er Lipschitz kontinuerte. Jeg vil primært fokusere på at vise Lipschitz kontinuiten af endepunkterne i dette speciale, da det er den del af artiklen jeg har haft størst andel i.

Den anden del af specialet omhandler et iganværende forskningsprojekt, om akustiske sorte huller, som udføres i samarbejde med Sergey Sorokin. Dette handler, i al simpelthed om, at minimere reflektionen af en bølge der bevæger sig i en plade. Dette gør vi ved hjælp af form optimering. Vi starter ud med at beskrive hvordan dette kan gøres, hvis vi beskriver bølgen ud fra Euler-Bernoulli bølge teori og dernæst prøver vi at udvide det til den mere generelle Timoshenko teori. Det viser sig dog at der er nogle problemer i denne udvidelse, hvilket betyder at dette stadig er et igangværende projekt.

Udover disse to forskningsprojekter, som der bliver fokuseret på i dette speciale, har jeg også gang i et projekt i samarbejde med Horia Cornean, Ira Herbst, Jesper Møller og Benjamin Støttrup, under emnet sandsynlighedsregning. Mere specifict betragter vi singulære funktioner konstrueret ved hjælp af tidsrækker.

2. Introduction

The aim of this thesis is threefold. Firstly, it is an exposition of some of the work I have done during the first two years and some of the ideas I have for the remaining two years as a 4+4 Phd student jointly between the Department for Mathematical Sciences and Department of Materials and Production, both at Aalborg Univeristy. Secondly, it should serve as my master thesis and thirdly, it can hopefully be the basis for a rewarding discussion about future work.

The thesis covers primarily two articles, one published and one ongoing. The published paper is on the subject of magnetic pseudodifferential operators and is written in collaboration with Horia Cornean, Henrik Garde and fellow Phd student Benjamin Støttrup [Cornean et al., 2019b]. In this article, we define a class of magnetic symbols for which we allow polynomial growth in $x - x'$ instead of the usual growth in ξ which are allowed in classical symbols. To such symbols we then associate a magnetic pseudodifferential operator and show that these can be represented as a generalized matrix. Using this representation of the magnetic pseudodifferential operator, we are then able to show, that if the magnetic symbol is self-adjoint, then the spectrum is $\frac{1}{2}$-Hölder continuous. Furthermore, if the magnetic field is constant, then the minimum and maximum values of the spectrum is Lipschitz continuous and this extends to the case where the spectrum has a gap, which does not close, for which the gap edges also are Lipschitz continuous.

In this thesis we primarily focus on the proof of the Lipschitz continuity, since this is the part I have been most involved in. In the exposition here, i have tried not just to copy the proofs from the article, but instead to sometimes give sketches of the proofs and some places give more details than are given in the article.

The second article, which is ongoing work, is on the topic of acoustic black holes. The aim in the theory of acoustic black holes, is to minimize the reflection of a beam in a plate. This can be done in several ways, but we have considered shape optimization to reduce the reflection. In Chapter 3 there is a short review of how calculus of variation can be used to optimize the shape of a plate using the method of Lagrange multipliers, in the case where the beam is described by the Euler-Bernoulli beam theory.

Previously, to the best of my knowledge, acoustic black holes has only been considered using the Euler-Bernoulli beam theory and the aim of Chapter 5 is to extend this to beams described by Timoshenko beam theory. Using this theory has proved to be more complex and has temporarily put the work to a halt. A future alternative is to primarily put an emphasis on solving the problem numerically instead.

Besides the aforementioned work, which will be the focus of this thesis, I have also done some work in collaboration with Horia Cornean, Ira Herbst, Jesper Møller and fellow Phd-student Benjamin Støttrup in probability theory. In this work we consider singular functions i.e. functions $F\colon \mathbb{R} \to \mathbb{R}$ which are non-constant and satisfy that $F'(x) = 0$ for almost all $x \in \mathbb{R}$, and construct such by using time series. We let F denote the CDF of a random variable X on $[0,1]$, given by the base-q expansion

$$X \coloneqq (0, X_1 x_2 \ldots)_q \coloneqq \sum_{n=1}^{\infty} X_n q^{-n},$$

where $q \in \mathbb{N}$ and $\{X_n\}_{n\geq 1}$ is a time series with $X_n \in \{0, 1 \ldots, q-1\}$, for every n. A simple example of a singular function is if $q = 3$, the probability of getting 0 and 2 is equally likely and the probability of getting 1 is zero, then F is the famous Cantor function. The fundamental assumption in our work is stationarity of the time series and we show that this is equivalent with the corresponding CDF satisfying a functional equation. Using this we are then able to show that several well known time series are singular.

3. Preliminaries

This chapter is a short introduction to pseudodifferential operators and the theory of acoustic black holes which are used in Chapter 4 and 5, respectively. It contains almost no proofs and is just meant to serve as an introduction to notation, methods and ideas used in the later chapters.

3.1 Pseudodifferential Operators

The following introduction to the theory of pseudodifferential operators is based on [Hörmander, 1985]. The notation differs slightly from [Hörmander, 1985] and is instead based on the article [Cornean et al., 2019b] which we present in more details in Chapter 4

We begin by defining a class of functions, called symbols, to which we later associate pseudodifferential operators.

Definition 3.1.1 (Symbols) Let $m \in \mathbb{R}$. If $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^m, \tag{3.1}$$

where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$, for all $x, \xi \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{N}_0$, then a is called a *symbol* of order m.

We denote the space of all symbols of order m by $S^m(\mathbb{R}^d \times \mathbb{R}^d)$. This is in fact a specific case of a more general definition of symbols given in [Hörmander, 1983], where he considers symbols of order m and type ρ, δ. This is identical with our definition for $\rho = \delta = 0$.

Next we define classical pseudodifferential operators in weak sense. If $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$ we define the *pseudodifferential operator associated to a* by $\mathrm{Op}^c(a)\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ as

$$\langle \mathrm{Op}^c(a) f\,, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} e^{i\xi\cdot(x-x')} a(x,\xi) f(x') \overline{g(x)} \,\mathrm{d}x' \,\mathrm{d}x \,\mathrm{d}\xi.$$

To show that this is actually well-defined, note that an application of the Multinomial theorem gives $\langle x \rangle^{2d} = \sum_{|\alpha| \le d} C_\alpha x^{2\alpha}$ and recall that the seminorm on $\mathscr{S}(\mathbb{R}^d)$ is given by

$$\|f\|_N = \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} \langle x \rangle^{2N} |(D^\alpha f)(x)|.$$

Thus by using integration by parts we get

$$|\langle \mathrm{Op}^c(a) f\,, g \rangle| = \Big| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} e^{i\xi\cdot(x-x')} a(x,\xi) f(x') \overline{g(x)} \,\mathrm{d}x' \,\mathrm{d}x \,\mathrm{d}\xi \Big|$$

$$
\begin{aligned}
&= \Big| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} \frac{\langle \xi \rangle^{2d}}{\langle \xi \rangle^{2d}} e^{i\xi\cdot(x-x')} a(x,\xi) f(x') \overline{g(x)}\, \mathrm{d}x'\, \mathrm{d}x\, \mathrm{d}\xi \Big| \\
&= \Big| \frac{C_\alpha}{(2\pi)^d} \sum_{|\alpha|\le d} \int_{\mathbb{R}^{3d}} \frac{1}{\langle \xi \rangle^{2d}} e^{i\xi\cdot(x-x')} \partial_x^{2\alpha} \big[a(x,\xi) f(x') \overline{g(x)}\big]\, \mathrm{d}x'\, \mathrm{d}x\, \mathrm{d}\xi \Big| \\
&\le C \|f\|_{4d} \|g\|_{4d},
\end{aligned}
$$

where we in the last inequality have used Leibniz' formula and (3.1). Note, that the classical pseudodifferential operators are sometimes called Kohn-Nirenberg pseudodifferential operators after Joseph J. Kohn and Louis Nirenberg who introduced them in [Kohn and Nirenberg, 1965].

Next, we define the Weyl quantization of pseudodifferential operators in weak sense. If $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$ then the *Weyl quantization of the pseudodifferential operator associated to* a, denoted by $\mathrm{Op}^w(a)\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is given as

$$
\langle \mathrm{Op}^w(a) f\,, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} e^{i\xi\cdot(x-x')} a\Big(\frac{x+x'}{2}, \xi\Big) f(x') \overline{g(x)}\, \mathrm{d}x'\, \mathrm{d}x\, \mathrm{d}\xi,
$$

where $a((x+x')/2,\xi)$ is called the *Weyl symbol.* The Weyl quantization is often used in relation to mathematical physics since the symmetry of a and anti-symmetri of the exponential gives that $\mathrm{Op}^w(a)^* = \mathrm{Op}^w(\bar{x})$ and hence if the symbol a is real, then the operator is formally self-adjoint.

In Chapter 4 we study spectral properties of pseudodifferential operators associated to symbols which can be seen as a generalisation of the Weyl symbols, in the sense that they take the presence of a magnetic field into consideration.

3.2 Acoustic Black Holes

The idea of acoustic black holes in plates originates from M. A. Mironov in his paper [Mironov, 1988]. The idea is, that if the thickness of a plate various such as the height goes to 0, then a flexural wave can propagate without reflection. This comes with an obvious problem in applications, since it is not possible to create a plate for which the height goes to zero, there will always be a truncation on the plate (see Figure 3.1). Thus the idea in the theory acoustic black holes, is to minimize the reflection of a wave in a plate. To do so, different approaches has been suggested such as adding a thin layer of dampening material [Krylov, 2004] or changing the height profile [Feurtado et al., 2014]. In this section we consider some unpublished work, done by Benjamin Støttrup [Cornean et al., 2019a], about how to mathematically optimize the height profile using Lagrange multipliers.

We consider the Euler-Bernoulli beam theory, in which the behaviour of a flexural wave is governed by the differential equation

$$
EI \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} - \rho A \omega^2 w(x) = 0,
$$

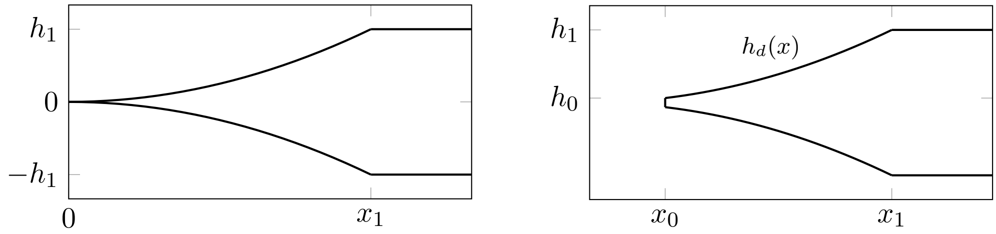


Figure 3.1: Non-truncated profile (left) and Truncated Profile (right).

where ρ is the density of the material, $A = bh(x)$ is the cross section area, $E = E_0(1-i\eta)$ is the elastic modulus with loss, ω is the angular frequency and $I = \frac{b(h(x))^3}{12}$ is the second moment area. If we assume that the solution is on the form $w(x) = We^{ik_dx}$, then we get the Euler-Bernoulli dispersion equation

$$EIk_d^4 - \rho A\omega^2 = 0.$$

Solving this equation for k_d, which we call the *dimensional local wave number* gives

$$k_d(x) = \frac{12^{1/4}\omega^{1/2}}{c^{1/2}(h(x))^{1/2}},$$

where $c^2 = \frac{E}{\rho}$. The efficiency of an acoustic black hole is measured by the *reflection coefficient*

$$R = \exp\Big(-2\int_{x_0}^{x_1} \mathrm{Im}(k_d(x))\,\mathrm{d}x\Big) \tag{3.2}$$

under the constraint

$$\left|\frac{1}{k_d^2}\frac{\mathrm{d}k_d}{\mathrm{d}x}\right| \ll 1. \tag{3.3}$$

We call the left hand side of (3.3) the *normalized wave number variation*. In [Feurtado and Conlon, 2015] it was, based on numerical experiments, suggested that (3.3) is satisfied if the normalized wave number is less than 0.3.

To ease notation in the rest of the section, we introduce non dimensional variables, given by the transformations

$$h(\xi) = \frac{h_d(x_0+\xi h_1)}{h_1}, \quad k(\xi) = k_d(x_0+\xi h_1)h_1, \quad h_\ell = \frac{h_0}{h_1}, \quad t = \frac{x_1-x_0}{h_1}, \ \Omega = \frac{\omega h_1}{c}.$$

This gives the profile in Figure 3.2.

Our aim is to find a height function $h\colon [0,t] \to [h_\ell, 1]$ which satisfies:

(i) h is differentiable with a continuous derivative.

(ii) $h(0) = h_\ell$ and $h(t) = 1$.

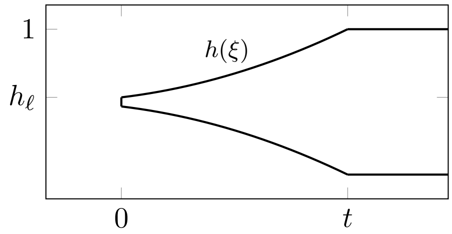


Figure 3.2: Non-dimensional truncated profile.

(iii) h minimizes

$$R = \exp\Big(-2\int_0^t \mathrm{Im}(k(\xi))\,\mathrm{d}\xi\Big) \tag{3.4}$$

under the constraint

$$\left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\right| \ll 1.$$

Note that h being a minimizer of (3.4) is equivalent to h being a maximizer of the integral

$$\int_0^t \mathrm{Im}(k(\xi))\,\mathrm{d}\xi. \tag{3.5}$$

In order to maximize this integral we use the method of Lagrange multipliers (see [Gelfand and Fomin, 2000; Shames and Dym, 1991] for more details). By this method, we can maximize (3.5) under the constraint

$$\int_0^t \left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\right|^{2n} \mathrm{d}\xi = C \ll 1.$$

Note, that the power $2n$ arises because if n is large then this constraint approximates a pointwise constraint. Thus we want to maximize the functional

$$J(h) := \int_0^t \mathrm{Im}(k(\xi)) + \lambda\left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\right|^{2n} \mathrm{d}\xi,$$

where λ is called the *Lagrange multiplier*. If we let $\delta = (-\lambda)^{-1/2n}$ then

$$J(h) = \int_0^t \mathrm{Im}(k(\xi)) - \Big(\delta^{-1}\left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\right|\Big)^{2n} \mathrm{d}\xi,$$

which can be interpreted as if $|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}| > \delta$ then J is penalized. Note that this idea is similar to the SIMP approach in topology optimization (see [Sigmund and Maute, 2013] for more details).

If we use the first order approximation of k with respect to η and approximate $|1-i\eta|^{1/4} \approx 1$, then we get that maximizing J is equal to maximizing the functional

$$I(h) = \int_0^t b\frac{1}{(h(\xi))^{1/2}} - \frac{(h'(\xi))^{2n}}{(h(\xi))^n}\,\mathrm{d}\xi,$$

where $b = (2\delta)^{2n}12^{(2n+1)/4}\Omega^{n+1/2}\frac{\eta}{8}$. We find the optimal height for a fixed Ω, which we denote by $\Omega^{(d)}$ and refer to as the *optimal profile*.

To maximize the functional I we define $F\colon \mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}\to\mathbb{R}$ by

$$F(\xi,u,v) := bu^{-1/2} - v^{2n}u^{-n}. \tag{3.6}$$

The Euler-Lagrange equation corresponding to the functional I is

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}v}F\Big)(\xi,h(\xi),h'(\xi)) = \Big(\frac{\mathrm{d}}{\mathrm{d}u}F\Big)(\xi,h(\xi),h'(\xi))$$

and since F does not explicitly depend on ξ it follows that

$$F(\xi,h(\xi),h'(\xi)) - h'(\xi)F(\xi,h(\xi),h'(\xi)) = a,$$

where $a\in\mathbb{R}$. By (3.6) we get that

$$h'(\xi) = (2n-1)^{-1/(2n)}(a(h(\xi))^n - b(h(\xi))^{n-1/2})^{1/(2n)} \tag{3.7}$$

and we now have that any maximizer of I must be a solution of (3.7). Furthemore, it follows by the Legendre condition that a solution to the Euler-Lagrange equation can't be a minimizer.

If $b = 0$ this is a separable equation which can be solved analytically, but it is not interesting in applications since it means that I simply is an integral over the normalized wave number variation.

If $b > 0$ it is possible to show, that for every $n\in N$ there exists a number

$$b_n = \frac{2n-1}{t^{2n}}\Big(\int_{h_\ell}^1 (h_\ell^{-1/2}y^n - y^{n-1/2})^{-1/(2n)}\Big)^{2n},$$

such that

(i) If $b\le b_n$ then there exist a unique $a \ge bh_\ell^{-1/2}$ satisfying that (3.7) has a unique smooth solution satisfying the boundary conditions $h(0) = h_\ell$ and $h(t) = 1$.

(ii) If $b > b_n$ then there exists a unique $t_b\in[0,t]$ and a smooth function h_b which satisfies (3.7) along with the boundary conditions $h_b(t_b) = h_\ell$ and $h_b(t) = 1$, such that

$$h(\xi) = \begin{cases} h_\ell & \text{if } 0\le\xi\le t,\\ h_b(\xi) & \text{if } t_b<\xi\le t\end{cases}$$

is a C^1 function which is piecewise smooth and satisfies (3.7) (see Figure 3.3).

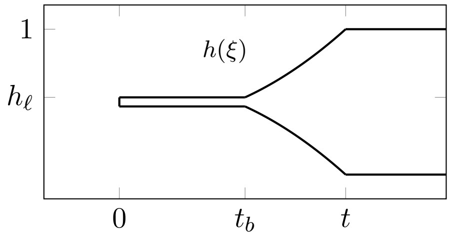


Figure 3.3: Non-dimensional optimal shape for $b > b_n$.

Note, that it is most often necessary to determine b_n and t_b using numerics (see [Cornean et al., 2019a] for more details).

If we let $n \to \infty$ and thus consider the normalized wave number variation as a pointwise estimate, it is possible to show that for

$$\Omega_0 = \Big(\frac{1-\sqrt{h_\ell}}{12^{1/4}\delta t}\Big)^2$$

we have:

(i) If $\Omega < \Omega_0$ then there exists no optimal profile.

(ii) if $\Omega \geq \Omega_0$ then there exists an optimal profile given by

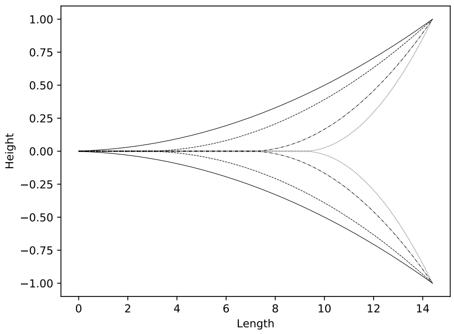
$$h(\xi) = \begin{cases} h_\ell & \text{if } 0 \leq \xi \leq t_\infty, \\ \Big(12^{1/4}\delta\Omega^{1/2}(\xi - t) + 1\Big)^2 & \text{if } t_\infty < \xi \leq t, \end{cases}$$

where

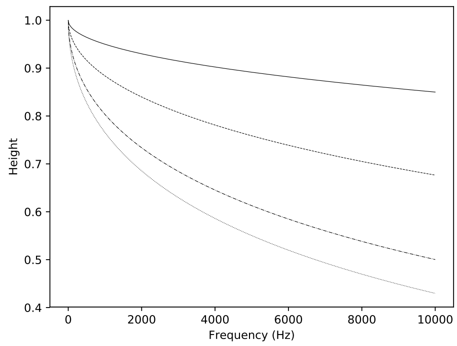
$$t_\infty = t - \frac{1-\sqrt{h_\ell}}{12^{1/4}\delta\Omega^{1/2}}.$$

We end this chapter by during some numerics. Let $x_0 = 2\text{cm}$, $x_1 = 20\text{cm}$, $h_0 = 0.002\text{cm}$, $h_1 = 1.25\text{cm}$, $c = 3000\text{m/s}$, $\eta = 0.01$ and $\delta = \frac{\sqrt{2}}{5}$. In Figure 3.4a we plot the optimal height with respect to design frequencies 1000Hz, 2500Hz, 5000Hz and the lowest frequency satsisfying that $\Omega \geq \Omega_0$. In Figure 3.4b we have plotted the associated reflection coefficient to each of the profiles in Figure 3.4a. We easily see that lower design frequencies gives a longer constant part of the profile and that the reflection coefficient is a decreasing function in the design frequency. In Figure 3.5 we have plotted the normalized wave number variation. Here we see that our constraint for the normalized wave number variation are invalid for low frequencies in the nonconstant part of the profile.

We have here only considered Euler-Bernoulli beam theory. In Chapter 5 we try, with a similar method, to find the optimal shape to minimize reflection of a wave in a plate, but instead of the Euler-Bernoulli theory we will consider Timoshenko beam theory. It is ongoing work and not even close to being finished.

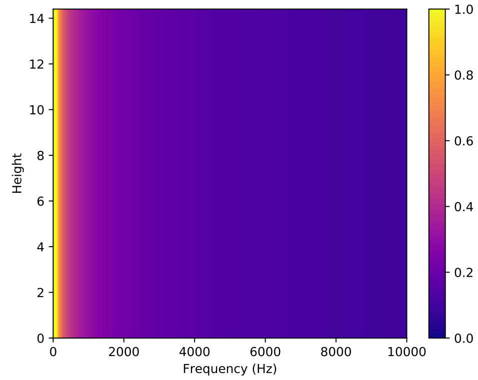


(a) Optimal profile design for the design frequencies 1000Hz (dashed), 2500Hz (dash-dotted), 5000Hz (dotted) and the minimal frequency (solid).

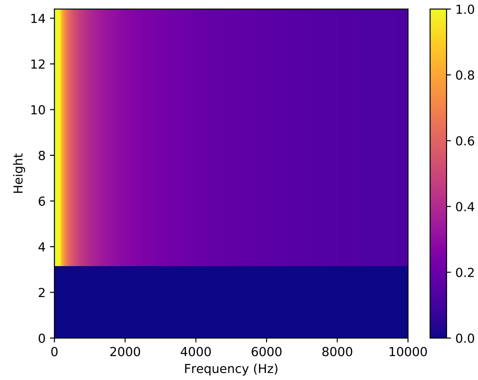


(b) Reflection coefficientsthe design frequencies 1000Hz (dashed), 2500Hz (dash-dotted), 5000Hz (dotted) and the minimal frequency (solid).

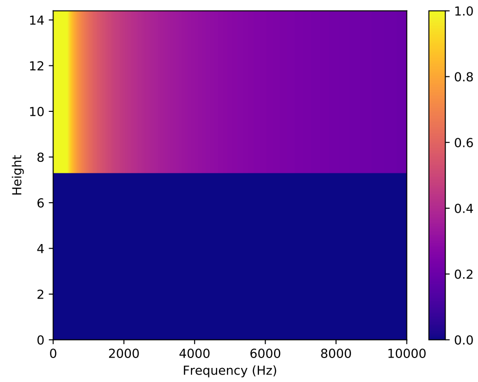
Figure 3.4: Optimal profiles and their corresponding reflection coefficients for varying b.



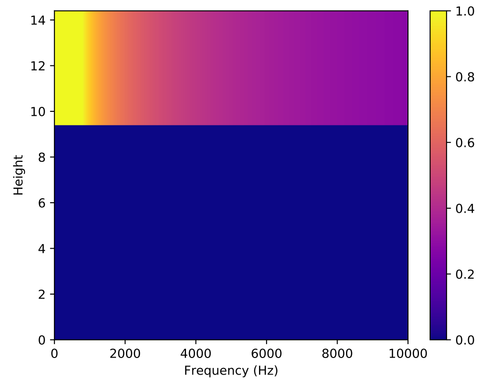
(a) NWV for the profile designed with the minimal frequency.



(b) NWV for the profile designed with a frequency of 1000Hz.



(c) NWV for the profile designed with a frequency of 2500Hz.



(d) NWV for the profile designed with a frequency of 5000Hz.

Figure 3.5: NWV for the profiles of Figure 3.4.

4. Magnetic Pseudodifferential Operators

The following is a more in depth walk through of some of the parts in [Cornean et al., 2019b]. Let $d \geq 2$ and

$$BC^\infty(\mathbb{R}^d) \coloneqq \Big\{ f \in C^\infty(\mathbb{R}^d;\mathbb{R}) \mid \sup_{x\in\mathbb{R}^d} |\partial^\alpha f(x)| < \infty, \quad \forall \alpha \in \mathbb{N}_0^d \Big\}.$$

In Chapter 3 we considered classical pseudodifferential operators. In this chapter we consider pseudodifferential operators in a magnetic field, given by a 2-form $B(x) = \sum_{i,j} B_{ij}(x)\mathrm{d}x_i \wedge \mathrm{d}x_j$ such that for all $i, j = 1, \dots, d$ we have:

(i) B_{ij} is antisymmetric, i.e. $B_{ij} = -B_{ji}$,

(ii) B_{ij} is smooth and all its partial derivatives are bounded, i.e. $B_{ij} \in BC^\infty(\mathbb{R})^d$,

(iii) B is a closed 2-form i.e. $\mathrm{d}B = \partial_k B_{ij} + \partial_j B_{ki} + \partial_i B_{jk} = 0$.

By this we are able to define a function φ which describes the magnetic flux through a triangle with one of the vertices being 0 (see [Cornean et al., 2019b] for more details). If $x, x', y, z \in \mathbb{R}^d$ and $\alpha, \alpha', \beta \in \mathbb{N}_0^d$ then we have the following properties of φ:

(i) For some constant $C_{\alpha,\alpha'}$, we have

$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \varphi(x, x')| \leq C_{\alpha,\alpha'} |x||x'| \tag{4.1}$$

(ii) $\varphi(x, x') = -\varphi(x', x)$.

(iii) Let $\Delta(x, y, z)$ denote the area of triangle with vertices x, y, z. Then $\mathfrak{f}\colon \mathbb{R}^{3d} \to \mathbb{R}$ given by

$$\mathfrak{f}(x, y, z) \coloneqq \varphi(x, y) + \varphi(y, z) - \varphi(x, z)$$

describes the magnetic flux through the triangle with vertices x, y, z. Furthermore, we have that

$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \mathfrak{f}(x, y, z)| \leq C_{\alpha,\alpha'} \Delta(x, y, z),$$

for some constant $C_{\alpha,\alpha'}$.

Using φ we can now define a kind of generalization of the Weyl symbols. Let $M_\varphi(\mathbb{R}^{3d})$ the set of functions

$$a_b(x,x',\xi) = e^{ib\varphi(x,x')}a(x,x',\xi),$$

where $b \in \mathbb{R}$ and $a \in C^\infty(\mathbb{R}^{3d})$ satisfies that for some $M \geq 0$ we have

$$|\partial_x^\alpha \partial_{x'}^{\alpha'} \partial_\xi^\beta a(x,x',\xi)| \leq C_{\alpha,\alpha',\beta}\langle x - x'\rangle^M, \tag{4.2}$$

for all multindices $\alpha, \alpha', \beta \in \mathbb{N}_0^d$ and some constant $C_{\alpha,\alpha',\beta}$. We call this class of symbols for *magnetic symbols*.

Note, that in opposition to classical symbols, which allow a polynomial growth in ξ, we instead allow polynomial growth in $x - x'$. If a depends on x, x' as $(x - x')/2$, $M = 0$ and $\alpha' = 0$, then the magnetic symbol is the same as a Weyl symbol.

We are now ready to introduce *magnetic pseudodifferential operators* defined in a weak sense. To do so, let $a \in M_\varphi(\mathbb{R}^{3d})$ and define the oprator $\mathrm{Op}(a_b)\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by

$$\langle \mathrm{Op}(a_b)f\,, g\rangle := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{3d}} e^{i\xi\cdot(x-x')}e^{ib\varphi(x,x')}a(x,x',\xi)f(x')\overline{g(x)}\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}\xi, \tag{4.3}$$

where $f, g \in \mathscr{S}(\mathbb{R}^d)$. That this is well-defined, can be shown by a calculation similar to the one we did in Chapter 3, to show that the classical pseudodifferential operator was well-defined.

Before we state our main theorem, we introduce the setting. Let $\Omega =]-1/2, 1/2[^d$ and define the space

$$\mathscr{H} := \bigoplus_{\gamma\in\mathbb{Z}^d} L^2(\Omega) = \Big\{(f_\gamma)_{\gamma\in\mathbb{Z}^d} \subset L^2(\Omega) \mid \sum_{\gamma\in\mathbb{Z}^d} \|f_\gamma\|^2_{L^2(\Omega)} < \infty\Big\}.$$

If we equip $\mathscr{H}$ with the inner product

$$\langle (f_\gamma)\,, (g_\gamma)\rangle_{\mathscr{H}} := \sum_{\gamma\in\mathbb{Z}^d} \langle f_\gamma\,, g_\gamma\rangle_{L^2(\Omega)},$$

then we get a Hilbert space. The idea with this space, is to show that up to a unitary transformation, which we will soon define, our magnetic pseudodifferential operators can be identified with a bounded generalized matrix on $\mathscr{H}$. Thus instead of studying the operators on the whole space, we split the space into small boxes and study them on each of them individually.

Let $b \in \mathbb{R}$ and define $U_b\colon L^2(\Omega) \to \mathscr{H}$ as

$$(U_b f)_\gamma(\cdot) := e^{-ib\varphi(\cdot+\gamma,\gamma)}\chi_\Omega(\cdot)f(\cdot + \gamma),$$

where χ_Ω is the characteristic function on Ω. Then U_b is unitary with inverse

$$[U_b^*(f_\gamma)_{\gamma\in\mathbb{Z}^d}](\cdot) = \sum_{\gamma\in\mathbb{Z}^d} e^{ib\varphi(\cdot,\gamma)}\chi_\Omega(\cdot - \gamma)f_\gamma(\cdot - \gamma).$$

Let $\mathcal{A}$ be an operator on $\mathscr{H}$. We call $\mathcal{A}$ a *generalized matrix* of the operators $(\mathcal{A}_{\gamma,\gamma'})_{\gamma,\gamma'\in\mathbb{Z}^d} \subset B(L^2(\Omega))$ if

$$\mathcal{A} = \{\mathcal{A}_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\mathbb{Z}^d} \quad \text{and} \quad (\mathcal{A}f)_\gamma = \sum_{\gamma'\in\mathbb{Z}^d} \mathcal{A}_{\gamma,\gamma'} f_{\gamma'},$$

for every $f \in \mathscr{H}$. The last notation we need, before we can introduce our main theorem is the Hausdorff distance which we denote by

$$d_{\mathrm{H}}(X,Y) := \max\{\sup_{x\in X} \operatorname{dist}(x,Y), \sup_{y\in Y} \operatorname{dist}(y,X)\},$$

where $X, Y \subset \mathbb{R}$ are compact sets.

We are now ready to present our main theorem.

Theorem 4.0.1 *If $a_b \in M_\varphi(\mathbb{R}^{3d})$ with $b \in [0, b_{\max}]$ for some $b_{\max} > 0$, then:*

(i) The operator $\mathrm{Op}(a_b)$ extends to a bounded operator on $L^2(\mathbb{R}^d)$ and for each $\gamma, \gamma' \in \mathbb{Z}^d$ there exists $\mathcal{A}_{\gamma\gamma',b} \in B(L^2(\Omega))$ such that

$$U_b \mathrm{Op}(a_b) U_b^* = \{\mathrm{e}^{\mathrm{i}b\varphi(\gamma,\gamma')} \mathcal{A}_{\gamma\gamma',b}\}_{\gamma,\gamma'\in\mathbb{Z}^d}. \tag{4.4}$$

Moreover, for every $N \in \mathbb{N}$ there exists a constant C_N such that

$$\|\mathcal{A}_{\gamma\gamma',b}\| \le C_N \langle \gamma - \gamma' \rangle^{-N}, \tag{4.5}$$

and

$$\|\mathcal{A}_{\gamma\gamma',b} - \mathcal{A}_{\gamma\gamma',b'}\| \le C_N \langle \gamma - \gamma' \rangle^{-N} |b - b'|, \quad \text{for } b, b' \in [0, b_{\max}], \tag{4.6}$$

for all $\gamma, \gamma' \in \mathbb{Z}^d$.

Additionally, if $a(x, x', \xi) = \overline{a(x', x, \xi)}$ then $\mathrm{Op}(a_b)$ is self-adjoint and in this case:

(ii) The spectrum of $\mathrm{Op}(a_b)$ is $\frac{1}{2}$-Hölder continuous in b on the interval $[0, b_{\max}]$, i.e. there exists a constant C such that

$$d_{\mathrm{H}}(\sigma(\mathrm{Op}(a_b)), \sigma(\mathrm{Op}(a_{b'}))) \le C|b - b'|^{1/2}, \tag{4.7}$$

for all $b, b' \in [0, b_{\max}]$.

(iii) Assume that φ comes from a constant magnetic field, i.e. $\varphi(x, x') = \frac{1}{2} x^\top B x'$ where B is an antisymmetric matrix. If E_b denotes the maximum (minimum) of $\sigma(\mathrm{Op}(a_b))$, then it is Lipschitz continuous in b on $[0, b_{\max}]$. Furthermore, if e_b denotes an edge of a spectral gap which remains open when b varies in some interval $[b_1, b_2] \subset [0, b_{\max}]$, then e_b is Lipschitz continuous on $[b_1, b_2]$.

Remark 4.0.2 It can be shown, that our class of magnetic pseudodifferential operators actually agrees with the class of magnetic Weyl operators. The one inclusion is clear, while the other follows by an application of the Beals criterion for magnetic pseudodifferential operators as given in [Cornean et al., 2018] (see Remark 1.3 in [Cornean et al., 2019b] for more details).

In this report, we are going to shortly go through the proof of *(i)* and place an emphasize on the proof of *(iii)*.

Proof of Theorem 1.1*(i)*. In our definition of magnetic pseudodifferential operators (4.3) we have no control over the behaviour of ξ, thus we begin by adding some decay in ξ to our symbol.

Lemma 4.0.3 *Let $a_b \in M_\varphi(\mathbb{R}^{3d})$. For $\varepsilon > 0$ define $a_{b,\varepsilon} \colon \mathbb{R}^{3d} \to \mathbb{C}$ by*

$$a_{b,\varepsilon}(x, x', \xi) := a_b(x, x', \xi)\mathrm{e}^{-\varepsilon\langle\xi\rangle}$$

and $K_{b,\varepsilon} \colon \mathbb{R}^{2d} \to \mathbb{C}$ by

$$K_{b,\varepsilon}(x, x') := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot(x-x')} a_{b,\varepsilon}(x, x', \xi)\,\mathrm{d}\xi.$$

Then the integral operator with kernel $K_{b,\varepsilon}$ is a bounded operator on $L^2(\mathbb{R}^d)$ and for $f \in \mathscr{S}(\mathbb{R}^d)$ we have

$$(\mathrm{Op}(a_{b,\varepsilon})f)(x) = \int_{\mathbb{R}^d} K_{b,\varepsilon}(x, x') f(x')\,\mathrm{d}x'. \tag{4.8}$$

Our aim is to show that $U_b\mathrm{Op}(a_b)U_b^*$ can be written as a generalized matrix. To do so we first show that $\mathcal{A}_{b,\varepsilon} := U_b\mathrm{Op}(a_{b,\varepsilon})U_b^*$ can be written as a generalized matrix. If we underline variables to indicate they lies in Ω, then

$$(\mathcal{A}_{b,\varepsilon}(f_{\gamma'}))_\gamma(\underline{x}) = \sum_{\gamma'\in\mathbb{Z}^d}\int_\Omega K_{b,\varepsilon}(\underline{x}+\gamma, \underline{x}'+\gamma') e^{ib(\varphi(\underline{x}'+\gamma',\gamma')-\varphi(\underline{x}+\gamma,\gamma))} f_{\gamma'}(\underline{x}')\,\mathrm{d}\underline{x}',$$

for $(f_{\gamma'}) \in \mathscr{H}$. Defining

$$K_{\gamma,\gamma'}(\underline{x}, \underline{x}') := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot(\underline{x}+\gamma-\underline{x}'-\gamma')} e^{ib\mathfrak{f}_{\gamma,\gamma'}(\underline{x},\underline{x}')} e^{-\varepsilon\langle\xi\rangle} a(\underline{x}+\gamma, \underline{x}'+\gamma', \xi)\,\mathrm{d}\xi,$$

where $\mathfrak{f}_{\gamma,\gamma'}(x, x') := \mathfrak{f}(x+\gamma, \gamma', \gamma) + \mathfrak{f}(x+\gamma, x'+\gamma', \gamma')$ then leads to

$$(\mathcal{A}_{b,\varepsilon}(f_{\gamma'}))_\gamma(\underline{x}) = \sum_{\gamma'\in\mathbb{Z}^d} e^{ib\varphi(\gamma,\gamma')}\int_\Omega K_{\gamma,\gamma'}(\underline{x}, \underline{x}') f_{\gamma'}(\underline{x}')\,\mathrm{d}\underline{x}',$$

which shows that $\mathcal{A}_{b,\varepsilon} = \{e^{ib\varphi(\gamma,\gamma')}\mathcal{A}_{\gamma\gamma',b,\varepsilon}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$. Using Fourier series, it is then possible to show that the integral operators $\mathcal{A}_{\gamma,\gamma',b,\varepsilon}$ actually is well-defined when letting $\varepsilon \to 0$ and that $\mathcal{A}_{\gamma\gamma',b,\varepsilon} \to \mathcal{A}_{\gamma\gamma',b}$ strongly on $C_0^\infty(\Omega)$. (see [Cornean et al., 2019b] for more details)).

We introduce the following technical lemma, which gives a sufficient condition for a generalized matrix to be bounded.

Lemma 4.0.4 *Suppose that there exists a constant C and operators $(T_{\gamma,\gamma'})_{\gamma,\gamma'\in\mathbb{Z}^d} \subset B(L^2(\Omega))$ such that*

$$\|T_{\gamma,\gamma'}f\|_{L^2(\Omega)} \leq \frac{C\|f\|_{L^2(\Omega)}}{\langle\gamma-\gamma'\rangle^{2d}},$$

for every $\gamma,\gamma' \in \mathbb{Z}^d$ and $f \in C_0^\infty(\Omega)$. Then $T = \{T_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$ is a bounded operator on $\mathscr{H}$ with

$$\|T\| \leq \sum_{\gamma\in\mathbb{Z}^d} \frac{C}{\langle\gamma\rangle^{2d}}.$$

By (4.5) it now follows that $H_b := \{e^{ib\varphi(\gamma,\gamma)}\mathcal{A}_{\gamma\gamma',b}\}_{\gamma,\gamma'\in\mathbb{Z}^d}$ is a bounded operator on $L^2(\mathbb{R}^d)$ and it can be shown that $\mathcal{A}_{b,\varepsilon} \to H_b$ strongly as $\varepsilon \to 0$.

It only remains to show that $\mathrm{Op}(a_b)$ has a continuous extension on $L^2(\mathbb{R}^d)$. Since $\mathcal{A}_{b,\varepsilon} \to H_b$ strongly it follows that $\mathrm{Op}(a_{b,\varepsilon}) \to U_b^* H_b U_b$ strongly. Furthermore, using Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0}\langle \mathrm{Op}(a_{b,\varepsilon})f\,,g\rangle = \langle \mathrm{Op}(a_b)f\,,g\rangle,$$

for every $f,g \in \mathscr{S}(\mathbb{R}^d)$, which shows that $U_b^* H_b U_b$ is a continuous extension to $L^2(\mathbb{R}^d)$. ■

Proof of Theorem 1.1*(iii)* Recall, that we denote the maximum of $\sigma(\mathrm{Op}(a_b))$ by E_b and that φ comes from a constant magnetic field. This makes φ bilinear and we have the relation

$$\varphi(x,y) + \varphi(y,z) = \varphi(x,z) + \varphi(x-y, y-z), \tag{4.9}$$

for every $x,y,z \in \mathbb{R}^d$. Furthermore, for $s \in \mathbb{R}$ we define

$$H_b^s := \{e^{i(b+s)\varphi(\gamma,\gamma')}\mathcal{A}_{\gamma\gamma',b}\}_{\gamma,\gamma'\in\mathbb{R}}.$$

If $s = 0$ we simplify and just write H_b.

Let $b_0, b_0 + \delta b \in [0, b_{\max}]$, for arbitrary b_0 and sufficiently small δb. To show Lipschitz continuity of E_b, we would like to show

$$|E_{b_0+\delta b} - E_{b_0}| \leq C|\delta b|.$$

By the triangle inequality we get

$$|E_{b_0+\delta b} - E_{b_0}| \leq |E_{b_0+\delta b} - \sup\sigma(H_{b_0}^{\delta b})| + |\sup\sigma(H_{b_0}^{\delta b}) - E_{b_0}|. \tag{4.10}$$

We consider the two absolute values on the right-hand side independently.

Note, that if S, T are bounded and self-adjoint operators on a Hilbert space, then by Theorem 4.10 in chapter V of [Kato, 1995] it follows that

$$d_{\mathrm{H}}(\sigma(S), \sigma(T)) \leq \|S - T\|. \tag{4.11}$$

By (4.6) we get that the assumptions in Lemma 4.0.4 is satisfied and by (4.11) it then follows that

$$|E_{b_0+\delta b} - \sup \sigma(H^{\delta b}_{b_0})| \leq d_{\mathrm{H}}(\sigma(H_{b_0+\delta b}), \sigma(H^{\delta b}_{b_0})) \leq \|H_{b_0+\delta b} - H^{\delta b}_{b_0}\| \leq C|\delta b| \quad (4.12)$$

Regarding the second absolute value in (4.10), we show that

$$\sup(\sigma H^{\delta b}_{b_0}) \leq \sigma(H_{b_0}) + C|\delta b|, \quad (4.13)$$

$$\sup(\sigma H_{b_0}) \leq \sigma(H^{\delta b}_{b_0}) + C|\delta b|. \quad (4.14)$$

To do so, we make a short detour, to introduce some properties of the fundamental solution to the heat equation, which will play an important part in the proof. We recall that the fundamental solution to the heat equation is

$$G(y, y', t) = \frac{1}{(4\pi t)^{d/2}} e^{-|y-y'|^2/4t},$$

which we immediately see is symmetric in the spatial coordinates. Furthermore, by applying semi-group theory we get the relation

$$G(y, y', 2t) = \int_{\mathbb{R}^d} G(y, y', t) G(y', y'', t) \,\mathrm{d}y',$$

which for $y = y''$ gives

$$\int_{\mathbb{R}^d} |G(y, y', t)|^2 \,\mathrm{d}y' = G(y, y', 2t) = \frac{1}{(8\pi t)^{d/2}}.$$

Next we define the linear functional $\Lambda_{\gamma,\gamma',t}$ by

$$\Lambda_{\gamma,\gamma',t} f := \int_{\mathbb{R}^d} f(y') G(\gamma, y', t) G(y', \gamma', t) \,\mathrm{d}y'.$$

Applying this functional on $e^{i\delta b\varphi(\gamma,\cdot)} e^{i\delta b\varphi(\cdot,\gamma')}$, using (4.9), the above properties of the fundamental solution to the heat equation and rearranging we get the relation (see [Cornean et al., 2019b] for the specific steps)

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\gamma')} &= (8\pi t)^{d/2} \Lambda_{\gamma,\gamma',t}(\mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\cdot)} \mathrm{e}^{\mathrm{i}\delta b\varphi(\cdot,\gamma')}) - \mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\gamma')} \Big[\Big(\mathrm{e}^{-|\gamma-\gamma'|^2/8t} - 1 \Big) \\ &\quad + (8\pi t)^{d/2} \Lambda_{\gamma,\gamma',t} \Big(\mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1 \Big) \Big] \\ &= \mathrm{I} - \mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\gamma')} [\mathrm{II} + \mathrm{III}], \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} \mathrm{I} &:= (8\pi t)^{d/2} \Lambda_{\gamma,\gamma',t}(\mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\cdot)} \mathrm{e}^{\mathrm{i}\delta b\varphi(\cdot,\gamma')}), \\ \mathrm{II} &:= \mathrm{e}^{-|\gamma-\gamma'|^2/8t} - 1, \end{aligned}$$

$$\mathrm{III} := (8\pi t)^{d/2}\Lambda_{\gamma,\gamma',t}\Big(\mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1\Big).$$

With this relation, we are now ready to prove (4.13). Recall, that a bounded and self-adjoint operator T on a separable Hilbert space satisfies

$$\sup_{\|x\|=1} \langle Tx\,, x\rangle = \sup\sigma(T). \tag{4.16}$$

Thus, if $f \in \mathscr{H}$ with $\|f\|_{\mathscr{H}} = 1$ then by using (4.15) we get

$$\begin{aligned}\langle H_{b_0}^{\delta b} f\,, f\rangle_{\mathscr{H}} &= \sum_{\gamma,\gamma'\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\gamma')}\mathrm{e}^{\mathrm{i}b_0\varphi(\gamma,\gamma')}\langle \mathcal{A}_{\gamma\gamma',b_0} f_{\gamma'}\,, f_\gamma\rangle_{L^2(\Omega)}\\ &= \sum_{\gamma,\gamma'\in\mathbb{Z}^d} (\mathrm{I} - \mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma,\gamma')}[\mathrm{II}+\mathrm{III}])\mathrm{e}^{\mathrm{i}b_0\varphi(\gamma,\gamma')}\langle \mathcal{A}_{\gamma\gamma',b_0} f_{\gamma'}\,, f_\gamma\rangle_{L^2(\Omega)}. \end{aligned} \tag{4.17}$$

Next we consider the series involving I, II and III separately. First, by defining

$$(\Phi_{\delta b,y',t})_\gamma := e^{i\delta b\varphi(y',\gamma)} G(y',\gamma,t) f_\gamma \in \mathscr{H},$$

and recalling that G is symmetric in the spatial coordinates and φ is anti-symmetric, it follows that

$$\sum_{\gamma,\gamma'\in\mathbb{Z}^d} \mathrm{I}\mathrm{e}^{\mathrm{i}b_0\varphi(\gamma,\gamma')}\langle \mathcal{A}_{\gamma\gamma',b_0} f_{\gamma'}\,, f_\gamma\rangle_{L^2(\Omega)} = (8\pi t)^{d/2}\int_{\mathbb{R}^d} \langle H_{b_0}\Phi_{\delta b,y',t}\,, \Phi_{\delta b,y',t}\rangle_{\mathscr{H}}\,\mathrm{d}y'.$$

If we normalize $\Phi_{\delta b,y',t}$ and apply (4.16) it follows that

$$\begin{aligned}\sum_{\gamma,\gamma'\in\mathbb{Z}^d} \mathrm{I}\mathrm{e}^{\mathrm{i}b_0\varphi(\gamma,\gamma')}\langle \mathcal{A}_{\gamma\gamma',b_0} f_{\gamma'}\,, f_\gamma\rangle_{L^2(\Omega)} &= (8\pi t)^{d/2}\int_{\mathbb{R}^d} \langle H_{b_0}\Phi_{\delta b,y',t}\,, \Phi_{\delta b,y',t}\rangle_{\mathscr{H}}\,\mathrm{d}y'\\ &\le \sup\sigma(H_{b_0})(8\pi t)^{d/2}\int_{\mathbb{R}^d}\sum_{\gamma\in\mathbb{Z}^d} |G(y',\gamma,t)|^2\|f_\gamma\|^2_{L^2(\Omega)}\,\mathrm{d}y'\\ &= \sup\sigma(H_{b_0}),\end{aligned}$$

where we in the last equality have used that $\int_{\mathbb{R}^d}|G(y,y',t)|^2\,\mathrm{d}y' = (8\pi t)^{-d/2}$, which was one of the properties we showed for the fundamental solution to the heat equation.

Secondly, regarding II we note that the inequality $|e^{-x}-1| \le x$, which is true for every $x \ge 0$, gives

$$\mathrm{II} \le \left|\mathrm{e}^{-|\gamma-\gamma'|^2/8t} - 1\right| \le \frac{|\gamma-\gamma'|^2}{8t}.$$

Lastly, we consider III. To do so, note that the coordinate change $x = y' - (\gamma+\gamma')/2$ gives

$$\Lambda_{\gamma,\gamma',t}(\varphi(\gamma-\cdot,\cdot-\gamma')) = \int_{\mathbb{R}^d} \varphi(\gamma-y',y'-\gamma')G(\gamma,y',t)G(y',\gamma',t)\,\mathrm{d}y'$$

$$\begin{aligned}&= \int_{\mathbb{R}^d} \varphi\Big(\frac{\gamma-\gamma'}{2} - x, x + \frac{\gamma-\gamma'}{2}\Big) G\Big(\gamma, x + \frac{\gamma+\gamma'}{2}, t\Big) G\Big(x + \frac{\gamma+\gamma'}{2}, \gamma', t\Big)\, \mathrm{d}x \\ &= \frac{1}{(4\pi t)^d} \int_{\mathbb{R}^d} \varphi\Big(\frac{\gamma-\gamma'}{2} - x, x + \frac{\gamma-\gamma'}{2}\Big) e^{-(|\frac{\gamma-\gamma'}{2} - x|^2 + |x + \frac{\gamma-\gamma'}{2}|^2)}\, \mathrm{d}x.\end{aligned}$$

Remembering, that φ is anti-symmetric and that the exponential factor is symmetric, it follows, by a change of variable $y = -x$, that

$$\Lambda_{\gamma,\gamma',t}(\varphi(\gamma - \cdot, \cdot - \gamma')) = 0.$$

This together with the inequality

$$|e^{i\delta bx} - 1 - i\delta bx| \leq |\delta bx|^2,$$

which holds for all $x \in \mathbb{R}$, gives

$$\begin{aligned}\Lambda_{\gamma,\gamma',t}(e^{i\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1) &\leq |\Lambda_{\gamma,\gamma',t}(e^{i\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1) - i\delta b\Lambda_{\gamma,\gamma',t}(\varphi(\gamma - \cdot, \cdot - \gamma'))| \\ &\leq \int_{\mathbb{R}^d} |\delta b\varphi(\gamma - y', y' - \gamma')|^2 G(\gamma, y', t) G(y', \gamma', t)\, \mathrm{d}y' \\ &= (\delta b)^2 \Lambda_{\gamma,\gamma',t}(|\varphi(\gamma - \cdot, \cdot - \gamma')|^2).\end{aligned}$$

Furthermore, by the bilinearity of φ it follows that

$$\varphi(\gamma - y', y' - \gamma') = \varphi(\gamma - y' + \gamma' - \gamma', y' - \gamma') = \varphi(\gamma - \gamma', y' - \gamma')$$

and by using Cauchy-Schwarz inequality we get

$$|\varphi(\gamma - \gamma', y' - \gamma')|^2 \leq C(|\gamma - \gamma'||B(y' - \gamma')|)^2 \leq C|\gamma - \gamma'|^2 |y' - \gamma'|^2.$$

Combining these two inequalities gives

$$\Lambda_{\gamma,\gamma',t}(e^{i\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1) \leq C(\delta b)^2 |\gamma - \gamma'|^2 \Lambda_{\gamma,\gamma',t}(|y' - \gamma'|^2).$$

By the trivial inequality $|y' - \gamma'|^2 \leq |\gamma - y'|^2 + |y' - \gamma'|^2$, changing the integral to polar coordinates and recalling that the gamma function is given by $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x$ we can bound this further

$$\begin{aligned}C(\delta b)^2 |\gamma - \gamma'|^2 \Lambda_{\gamma,\gamma',t}(|y' - \gamma'|^2) &\leq C(\delta b)^2 |\gamma - \gamma'| \frac{1}{(4\pi t)^d} \int_{\mathbb{R}^d} e^{-|y'-\gamma'|^2/4t} |y' - \gamma'|^2\, \mathrm{d}y' \\ &= C(\delta b)^2 |\gamma - \gamma'| \frac{t}{t^{d/2}}.\end{aligned}$$

Thus we have shown that

$$\mathrm{III} = (8\pi t)^{d/2} \Lambda_{\gamma,\gamma',t}(\mathrm{e}^{\mathrm{i}\delta b\varphi(\gamma-\cdot,\cdot-\gamma')} - 1) \leq C t^{d/2} (\delta b)^2 |\gamma - \gamma'|^2 \frac{t}{t^{d/2}} = C(\delta b)^2 |\gamma - \gamma'|^2 t.$$

If we define the operator $\widetilde{S}\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by

$$(\widetilde{S}x)_\gamma \coloneqq \sum_{\gamma' \in \mathbb{Z}^d} |\gamma - \gamma'|^2 \|\mathcal{A}_{\gamma\gamma',b_0}\| x_{\gamma'},$$

and apply a Schur-Holmgren type result, we get that $\widetilde{S}$ is bounded. Thus to sum up, we have shown that

$$\begin{aligned}\langle H_{b_0}^{\delta b} f\,, f\rangle_{\mathscr{H}} &\leq \sup\sigma(H_{b_0}) + \sum_{\gamma,\gamma'\in\mathbb{Z}^d}\Big[\frac{|\gamma-\gamma'|^2}{8t} + C(\delta b)^2|\gamma-\gamma'|^2 t\Big]\|\mathcal{A}_{\gamma,\gamma',b_0}\|\|f_{\gamma'}\|\|f_\gamma\| \\ &\leq \sup\sigma(H_{b_0}) + C\Big[\frac{1}{t} + (\delta b)^2 t\Big]\end{aligned}$$

and by choosing $t = 1/|\delta b|$ we have shown (4.13).

Lastly, we consider the case where the spectrum of H_b has a gap which does not close, when b varies in some interval $[b_1, b_2] \subset [b_0, b_{\max}]$. Thus we assume that $\sigma(H_b) = \sigma_1 \cup \sigma_2$ were $\sup\sigma_1 < \inf\sigma_2$ holds when b varies. We consider the case where $e_b = \inf\sigma_2$ and show that e_b is Lipschitz continuous in b. Up to adding a constant we have that $\sigma(H_b) \subset]-\infty, 0[$ when $b \in [b_1, b_2]$. We fix $b_0 \in]b_1, b_2[$ and let δb satisfy that $b_0 + \delta b \in [b_1, b_2]$. For δb sufficiently small, it is possible to make a contour $\mathscr{C}$ around σ_2 which satisfies that the distance between $\mathscr{C}$ and $\sigma(H_{b_0+\delta b})$ is positive, uniformly in δb. Note that the operator

$$T_b \coloneqq \frac{i}{2\pi}\int_{\mathscr{C}} z(H_b - z)^{-1}\,\mathrm{d}z$$

satisfies that $\sigma(T_b) = \sigma_2 \cup \{0\}$ and therefore $\inf\sigma(T_b) = e_b$. Our aim is now to show that

$$|e_{b_0+\delta b} - e_{b_0}| \leq C|\delta b|.$$

To do so, we first note that

$$\begin{aligned}\mathrm{d}_H\Big(\sigma(T_{b_0+\delta b}), \sigma\Big(\frac{i}{2\pi}\int_{\mathscr{C}} z(H_{b_0}^{\delta b} - z)^{-1}\,\mathrm{d}z\Big)\Big) &\leq \Big\|T_{b_0+\delta b} - \frac{i}{2\pi}\int_{\mathscr{C}} z(H_{b_0}^{\delta b} - z)^{-1}\,\mathrm{d}z\Big\| \\ &\leq C\int_{\mathscr{C}} |z|\|(H_{b_0+\delta b} - z)^{-1} - (H_{b_0}^{\delta b} - z)^{-1}\|\,\mathrm{d}z \\ &\leq C\int_{\mathscr{C}} |z|\|(H_{b_0+\delta b} - z)^{-1}(H_{b_0}^{\delta b} - H_{b_0+\delta b})(H_{b_0}^{\delta b} - z)^{-1}\|\,\mathrm{d}z \\ &\leq C|\delta b|,\end{aligned}$$

where we have used (4.12), the second resolvent identity

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1}$$

and the bound

$$\|(A - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \sigma(A))}.$$

The following technical lemma, shows that the resolvent can be written as a generalized matrix.

Lemma 4.0.5 *Let $z \in \mathscr{C}$ and let $b = b_0 + \delta b$ as above. Seen as an operator in $\mathscr{H} = \ell^2(\mathbb{Z}^d; L^2(\Omega))$, the resolvent $(H_b - z)^{-1}$ is also written*

$$(H_b - z)^{-1} = \{[(H_b - z)^{-1}]_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \mathbb{Z}^d},$$

with matrix elements $[(H_b - z)^{-1}]_{\gamma,\gamma'} \in B(L^2(\Omega))$. For every $N \in \mathbb{N}$ there exists a constant C_N independent of b and z such that

$$\|[(H_b - z)^{-1}]_{\gamma,\gamma'}\| \leq C_N \langle \gamma - \gamma' \rangle^{-N}.$$

Next we define

$$\begin{aligned}
[\tilde{T}_{b_0}^{\delta b}]_{\gamma,\gamma'} &:= e^{i\delta b \varphi(\gamma,\gamma')}[T_{b_0}]_{\gamma,\gamma'}, \\
[S_{\delta b}(z)]_{\gamma,\gamma'} &:= e^{i\delta b \varphi(\gamma,\gamma')}[(H_{b_0} - z)^{-1}]_{\gamma,\gamma'}.
\end{aligned}$$

By the decay of arbitrary order away from the diagonal, we have that

$$\begin{aligned}
[(H_{b_0}^{\delta b} - z)S_{\delta b}(z)]_{\gamma,\gamma''} &= \sum_{\gamma' \in \mathbb{Z}^d} e^{i\delta b \varphi(\gamma,\gamma'')} e^{i\delta b \varphi(\gamma-\gamma',\gamma'-\gamma'')}[H_{b_0} - z]_{\gamma,\gamma'}[(H_{b_0} - z)^{-1}]_{\gamma',\gamma''} \\
&= [\mathrm{id} + \mathcal{O}(\delta b)]_{\gamma,\gamma''}.
\end{aligned}$$

For sufficiently small $|\delta b|$ it follows from the Neumann series, that $\mathrm{id} + \mathcal{O}(\delta b)$ is invertible and hence

$$(H_{b_0}^{\delta b} - z)^{-1} = S_{\delta b}(z)(\mathrm{id} - \mathcal{O}(\delta b))^{-1} = S_{\delta b}(z) + \mathcal{O}(\delta b),$$

uniformly in $z \in \mathscr{C}$. By this it follows that

$$\frac{i}{2\pi} \int_{\mathscr{C}} z(H_{b_0}^{\delta b} - z)^{-1} \, \mathrm{d}z = \frac{i}{2\pi} \int_{\mathscr{C}} z S_{\delta b}(z) \, \mathrm{d}z + \mathcal{O}(\delta b) = \tilde{T}_{b_0}^{\delta b} + \mathcal{O}(\delta b),$$

which shows the inequality

$$\|\tilde{T}_{b_0}^{\delta b} - \frac{i}{2\pi} \int_{\mathscr{C}} z(H_{b_0}^{\delta b} - z)^{-1} \, \mathrm{d}z\| \leq C|\delta b|.$$

To summarize we now have

$$\begin{aligned}
|e_{b_0+\delta b} - e_{b_0}| &\leq |e_{b_0+\delta b} - \inf \sigma(\tilde{T}_{b_0}^{\delta b})| + |\inf \sigma(\tilde{T}_{b_0}^{\delta b}) - e_{b_0}| \\
&\leq d_{\mathrm{H}}(\sigma(T_{b_0+\delta b}), \sigma(\tilde{T}_{b_0}^{\delta b})) + d_{\mathrm{H}}(\sigma(\tilde{T}_{b_0}^{\delta b}), \sigma(T_{b_0})) \\
&\leq \|T_{b_0+\delta b} - \tilde{T}_{b_0}^{\delta b}\| + d_{\mathrm{H}}(\sigma(\tilde{T}_{b_0}^{\delta b}), \sigma(T_{b_0})) \\
&\leq C|\delta b| + d_{\mathrm{H}}(\sigma(\tilde{T}_{b_0}^{\delta b}), \sigma(T_{b_0})).
\end{aligned}$$

We note that $\tilde{T}_{b_0}^{0} = T_{b_0}$ and that the family $\tilde{T}_{b_0}^{\delta b}$ is similar to the one in (4.4), thus by Lemma 4.0.5 we can apply the first part of Theorem 4.0.1*(iii)*, to conclude that e_b is Lipschitz continuous in b. ■

5. Acoustic Black Holes

In this chapter we consider, as in Chaper 3, a truncated plate using non-dimensional coordinates (see Figure 3.1), but instead of the Euler-Bernoulli beam theory, we consider Timoshenko beam theory. The set of differential equations describing the motion of a Timoshenko beam is given as

$$\begin{aligned}-\rho A\frac{\partial^2 w}{\partial t^2}(x,t)+\kappa GA\Big(\frac{\partial^2 w}{\partial x^2}(x,t)-\frac{\partial\psi}{\partial x}(x,t)\Big)+q(x,t)=0,\\ -\rho I\frac{\partial^2\psi}{\partial t^2}(x,t)+EI\frac{\partial^2\psi}{\partial x^2}(x,t)+\kappa GA\Big(\frac{\partial w}{\partial x}(x,t)-\psi(x,t)\Big)=0,\end{aligned} \tag{5.1}$$

where $G=\frac{E}{2(1+\nu)}$ is the shear modulus with ν the poisson ratio, κ is the Timoshenko shear coefficient and q is the distributed load (the rest of the variables was introduced in Chapter 3).

If we assume that the solutions to (5.1) is of the form

$$w(x,t)=\widetilde{W}he^{ik_dx-i\omega t}\qquad\text{and}\qquad\psi(x,t)=\Psi e^{ik_dx-i\omega t},$$

and that there is no load on the plate, i.e. $q(x,t)\equiv 0$, we get the system of linear equations

$$\begin{aligned}\Big(-k^2+\frac{2(1+\nu)}{\kappa}\Omega^2\Big)\widetilde{W}-ik\Psi=0,\\ \frac{6ik\kappa h_1^2}{(1+\nu)(h(\xi))^2}\widetilde{W}+\Big(-k^2+\Omega^2-\frac{6\kappa h_1^2}{(1+\nu)(h(\xi))^2}\Big)\Psi=0.\end{aligned}$$

Taking the determinant of this system of equations and equating to 0 gives the Timoshenko dispersion equation

$$k^4-\Big(1+\frac{2(1+\nu)}{\kappa}\Big)k^2\Omega^2-\frac{12h_1^2}{(h(\xi))^2}\Omega^2+\frac{2(1+\nu)}{\kappa}\Omega^4=0. \tag{5.2}$$

From this equation it is possible to find the wave number. In order to do so, let $u=k^2$, then we get a second order equation in u, which has the solutions

$$u=\frac{(\kappa+2(1+\nu))\Omega^2\pm\sqrt{(\kappa-2(1+\nu))\Omega^4+\frac{48h_1^2\kappa^2}{(h(\xi))^2}\Omega^2}}{2\kappa}.$$

Thus the wave numbers are given by

$$k=\pm\frac{(\kappa+2(1+\nu))\Omega^2\pm\sqrt{(\kappa-2(1+\nu))\Omega^4+\frac{48h_1^2\kappa^2}{(h(\xi))^2}\Omega^2}}{\sqrt{2\kappa}}.$$

Recall, that the wave number is the magnitude of the wave vector, i.e. k must be positive and that Ω is a complex number, thus k is also a complex number. If we make a first order approximation of k with respect to η, then we get

$$k = \Big(\frac{\omega^2 h_1^2 \rho}{2E_0\kappa}\Big)^{1/2}\sqrt{\kappa^+(1+i\eta) \pm \sqrt{\kappa^- + \tilde{h}_\xi + i\Big(2\kappa^-\eta E_0 + \tilde{h}_\xi E_0 \eta\Big)}},$$

where $\kappa^\pm = \kappa \pm 2(1+\nu)$ and $\tilde{h}_\xi = \frac{48\kappa^2}{(h(\xi))^2\omega^2\rho}$. The squareroot is on the form $\sqrt{a+bi\pm\sqrt{c+di}}$, which by a first order Taylor approximation can be written as

$$\sqrt{a+bi\pm\sqrt{c+di}} = \sqrt{a\pm\sqrt{c}} + i\frac{b\pm\frac{d}{2\sqrt{c}}}{\sqrt{a\pm\sqrt{c}}}.$$

Thus we can approximate the imaginary part of k by

$$\mathrm{Im}(k) = \Big(\frac{\omega^2 h_1^2 \rho}{2E_0\kappa}\Big)^{1/2}\frac{\kappa^+\eta \pm \frac{2\kappa^-\eta E_0 + \tilde{h}_\xi E_0 \eta}{2(\kappa^+ + \tilde{h}_\xi)^{1/2}}}{(\kappa^- \pm (\kappa^- + \tilde{h}_\xi)^{1/2})^{1/2}}$$

We recall, that our aim is to find a height function $h\colon [0,t] \to [h_\ell, 1]$ which satisfies that:

(i) h is differentiable with a continuous derivative,

(ii) $h(0) = h_\ell$, $h(t) = 1$,

(iii) h is the minimizer of (3.4) under the constraint (3.3).

As in Chapter 3, we use the method of Lagrange multipliers and consider the functional

$$J(h) = \int_0^t \mathrm{Im}(k(\xi)) - \Big(\delta^{-1}\Big|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}\xi}\Big|\Big)^{2n}\,\mathrm{d}\xi,$$

where, $\delta = (-\lambda)^{-1/2n}$. We can explicitly calculate the normalized wave number variation but, as with $\mathrm{Im}(k)$, it will be much more involved than in the Euler-Bernoulli theory case. This is still ongoing work, and a possibility is to take a more numerical approach here, than in the Euler-Bernoulli beam theory case.

6. Future Plans

During the first two years of study, several other research ideas have come up, apart from the already considered questions. I will here shortly describe some of the problems which could be interesting for me the next two years.

Of course the study of acoustic black holes using Timoshenko theory is one of the first thing I am going to continue with. It could both be to try and continue with the analytic side and get some estimates or if that leads nowhere, then a more numerical investigation could also be interesting. Another thing that has been suggested by Sergey Sorokin is to consider acoustic black holes (either in Euler-Bernoulli beam theory or Timoshenko Beam theory) in more advanced geometries than a simple plate, or in higher dimensions. To do so, will naturally lead to a further study of the basic ideas of mechanics and acoustics.

Another interesting problem which stems from discussions with Horia Cornean and Shu Nakamura lies in the field of spectral theory. Let $d \geq 2$ and S^{d-1} denote the d-dimensional unit sphere. Furthermore, let $H := -\Delta + V$, where $V\colon \mathbb{R} \times S^{d-1} \to \mathbb{R}$ satisfies that for every $\omega \in S^{d-1}$ there exists a T_ω-periodic function $v_\omega\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{r\to\infty} |V(r,\omega) - v_\omega(r)| = 0,$$

i.e. the potential is periodic in the limit. Then we believe that:

(i) If $h_\omega := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_\omega \in H^2(\mathbb{R})$, then

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{\omega \in S^{d-1}} \{\sigma(h_\omega) \cup [0,\infty)\}.$$

(ii) The spectrum has a band structure with absolutely continuous and dense pure point spectrum (and probably even no embedded eigenvalues?).

(iii) The absolute continuous spectrum is stable under small perturbations.

(iv) It is possible to find and asymptotic estimate of the number of eigenvalues of $f(H)$.

(v) There is a limiting absorption principle in 0 i.e. if $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\psi_\varepsilon := (H + i\varepsilon)^{-1}\varphi$ then $\lim_{\varepsilon\to 0} \psi_\varepsilon$ is a solution to a Helmholtz equation.

To verify these statements a more involved study of spectral theory and perturbation theory is necessary.

The last possible problem which I will describe here is about determinantal point processes and has been suggested by Jesper Møller. If $C\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is a function, then we denote by $[C](x_1,\dots,x_n)$ an $n\times n$ matrix with entries $C(x_i,x_j)$. If X is a locally finite spatial point process on $\mathbb{R}^d$, then the *n'th order intensity function* $\rho^{(n)}\colon \mathbb{R}^{dn}\to[0,\infty)$, for $n=1,2,\dots$, if it exists, is given by

$$E\sum_{x_1,\dots,x_n}^{\neq} h(x_1,\dots,x_n)=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\rho^{(n)}(x_1,\dots,x_n)h(x_1,\dots,x_1)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n,$$

where $h\colon\mathbb{R}^{dn}\to[0,\infty)$ is an arbitrary Borel function and $\neq$ symbolize that $x_1,\dots,x_n$ are pairwise distinct. If

$$\rho^{(n)}(x_1,\dots,x_n)=\det[C](x_1,\dots,x_n),$$

for $(x_1,\dots,x_n)\in\mathbb{R}^{dn}$ and $n=1,2,\dots$, then we call X a *determinantal point process* with kernel C.

Let $S\subset\mathbb{R}^d$ denote a generic compact set. Then under suitable assumptions, C restricted to $S\times S$ has a spectral representation given by

$$C(x,y)=\sum_{k=1}^{\infty}\lambda_k\varphi_k(x)\overline{\varphi_k(y)},$$

where $(x,y)\in S\times S$ and the series is both absolutely and uniformly convergence. Furthermore, $\{\varphi_k\}$ forms an orthonormal basis of $L^2(S)$ and the set of eigenvalues $\{\lambda_k\}$ are unique, the only possible accumulation point is 0 and if $\lambda_k\neq 0$ then it is real and have finite multiplicity.

The idea of this problem is then to study properties of determinental point processes by considering the spectral representation.

It is currently the plan that I have a stay abroad in the spring 2020. The destination has not yet been decided, but several possibilities have been discussed. Sergey Sorokin has some connections in the United Kingdom, namely John Chapman at Keele University and Nigel Peake at Cambridge University both doing research in applied mathematics, more specific in the area of acoustics. Another option is to visit Adrien Pelat at the Le Mans University. Adrien Pelat gave a talk in 2018 at Aalborg University on acoustic black holes, from which the ideas presented in this thesis stems from. The last option, suggested by Horia Cornean, is to visit Eric Cances at the Ecole des Ponts ParisTech. Eric Cances is doing research on numerical methods in quantum chemistry and material sciences.

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