On Semiclassical Operators

Semiclassical Analysis

Master's Thesis

Aalborg University
Department of Mathematical Sciences

Department of Mathematical Sciences
Aalborg University
http://www.math.aau.dk

AALBORG UNIVERSITY
STUDENT REPORT

Title:
On Semiclassical Operators

Theme:
Semiclassical Analysis

Project Period:
Spring Semester 2018

Project Group:
5.217a

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Page Numbers: 42

Date of Completion:
June 7, 2018

Abstract:
In this Master's thesis we study some results from semiclassical analysis. First we give the meaning to the anti-Wick quantization for a classical pseudodifferential operator and then extend this to semiclassical case. We utilize this to prove Gårding's inequality.
Thereafter, we give a formula for the Fourier transform of e^{icx^m}, when m is even, which for $m > 2$ involves generalized hypergeometric functions.
Lastly, we study oscillatory integrals, especially their behaviour when h goes to 0. This leads us to study stationary phase asymptotics, under different conditions, where we among other things have tried to use the aforementioned Fourier transform. In higher dimensions this required us to show Morse Lemma.

Danish Abstract

I dette kandidatspeciale har vi beskæftiget os med nogle resultater fra semiklassisk analyse. I den forbindelse har vi fundet frem til anti-Wick kvantificeringen af psudodifferential operatorer, og udvidet dette til semiklassiske psudodifferential operatorer. Dette har vi udnyttet til at bevise Gårdings ulighed.

Derefter har vi set på hvordan man udleder en formel for fouriertransformationen af e^{icx^m} hvor m er lige, og det viser sig at for $m > 2$ bliver det en linear kombination af generaliserede hypergeometriske funktioner.

Afrundingsvis har vi set på oscillerende integraler, især deres opførsel når h går mod 0. I den forbindelse er vi kommet ind på asymptotisk opførsel af stationær fase under forskellige betingelser, hvor vi bl.a. har forsøgt at bruge den førnævnte fouriertransformation. I forbindelse med at undersøge den asymptotiske opførsel af stationær fase i højere dimensioner har vi haft brug for at vise Morse Lemma.

Contents

Preface

The following Master's thesis is written as part of the master's programme in applied mathematical analysis at the Department of Mathematical Sciences at Aalborg University. It is expected of the reader to have extensive knowledge of mathematical analysis, especially in the areas of operator theory and Fourier analysis. The thesis is meant to be as self-contained as possible, though some of the minor result can be found in the appendix, mostly either with proof or a reference to one.

We would like to give a special thanks to our supervisor Professor Horia Cornean for guidance throughout this semester.

Aalborg University, June 7, 2018

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1. Anti-Wick

In the this chapter we study some result about the anti-Wick quantization. These results will be used to show Gårding's inequality. The following chapter is based on [Bouclet, 2015b].

1.1 Quantization

In this section we aim to deduce the anti-Wick quantization.

For $p, q \in \mathbf{R}^n$ the operator $U(q,p)\colon \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^{2n})$ is given by

$$[U(q,p)(\varphi)]\,(x) = e^{ix\cdot p}\varphi(x-q),$$

where $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Let $\eta \in \mathscr{S}(\mathbf{R}^n)$ such that $\|\eta\|_{L^2} = 1$ and define

$$\eta_{q,p} := [U(q,p)]\,(\eta). \tag{1.1}$$

Lemma 1.1.1 *The function $T(\varphi)$ is given by*

$$[T(\varphi)]\,(q,p) := \langle \varphi\,, \eta_{q,p}\rangle_{L^2} \in \mathscr{S}(\mathbf{R}^{2n}),$$

for any $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Furthermore, the linear map T is continuous.

Proof. We start by showing that $T(\varphi)$ is a $\mathscr{S}(\mathbf{R}^{2n})$. Let $\alpha, \beta \in \mathbf{N}^n$ then by integration by parts, we have

$$\begin{aligned} \partial_p^\beta \partial_q^\alpha \Big([T(\varphi)]\,(q,p) \Big) &= \partial_p^\beta \partial_q^\alpha \int_{\mathbf{R}^n} \varphi(x) e^{-ix\cdot p}\overline{\eta}(x-q)\,dx \\ &= (-i)^{|\beta|}(-1)^{|\alpha|}\int_{\mathbf{R}^n} x^\beta \varphi(x) e^{-ix\cdot p}(\partial^\alpha \overline{\eta})(x-q)\,dx. \end{aligned} \tag{1.2}$$

We need to check if this is bounded when we multiply by polynomials in both p and q. Let $\delta \in \mathbf{N}^n$ be a multiindex, then we can rewrite q^δ to be

$$q^\delta = (q - x + x)^\delta = \sum_{\delta' \leq \delta} (-1)^{|\delta'|} \binom{\delta}{\delta'} (x-q)^{\delta'} x^{\delta - \delta'}.$$

Hence letting $\mu \in \mathbf{N}^n$ and multiplying (1.2) by $p^\mu q^\delta$ we use integration by parts to look at the boundedness

$$\begin{aligned} \left| p^\mu q^\delta \partial_p^\beta \partial_q^\alpha \Big([T(\varphi)]\,(q,p) \Big) \right| &= \left| p^\mu q^\delta \int_{\mathbf{R}^n} x^\beta \varphi(x) e^{-ix\cdot p}(\partial^\alpha \overline{\eta})(x-q)\,dx \right| \\ = &\left| p^\mu \int_{\mathbf{R}^n} \sum_{\delta' \leq \delta} \binom{\delta}{\delta'} e^{-ix\cdot p} (-1)^{|\delta'|} x^{\beta + \delta - \delta'} \varphi(x)(x-q)^{\delta'} (\partial^\alpha \overline{\eta})(x-q)\,dx \right| \end{aligned}$$

$$= \left| \int_{\mathbf{R}^n} \sum_{\delta' \le \delta} \binom{\delta}{\delta'} \partial_x^\mu (e^{-ix\cdot p})(i)^{|\mu|}(-1)^{|\delta'|} x^{\beta+\delta-\delta'} \varphi(x)(x-q)^{\delta'}(\partial^\alpha \overline{\eta})(x-q)\, dx \right|$$
$$\le \sum_{\delta' \le \delta} \binom{\delta}{\delta'} \left| \int_{\mathbf{R}^n} e^{-ix\cdot p} \partial_x^\mu \left(x^{\beta+\delta-\delta'} \varphi(x)(x-q)^{\delta'}(\partial^\alpha \overline{\eta})(x-q) \right) dx \right|.$$

Now we use Leibniz' rule to obtain

$$\begin{aligned}
&\left| p^\mu q^\delta \partial_p^\beta \partial_q^\alpha \Big(\left[T(\varphi) \right](q,p) \Big) \right| \\
&\qquad \le \sum_{\delta' \le \delta} \binom{\delta}{\delta'} \sum_{\mu' \le \mu} \binom{\mu}{\mu'} \\
&\qquad\qquad \cdot \left| \int_{\mathbf{R}^n} e^{-ix\cdot p} \partial_x^{\mu'} \Big(x^{\beta+\delta-\delta'} \varphi(x) \Big) \partial_x^{\mu-\mu'} \Big((x-q)^{\delta'} (\partial^\alpha \overline{\eta})(x-q) \Big) dx \right| \\
&\qquad \le \sum_{\delta' \le \delta} \binom{\delta}{\delta'} \sum_{\mu' \le \mu} \binom{\mu}{\mu'} \\
&\qquad\qquad \cdot \int_{\mathbf{R}^n} \left| \partial_x^{\mu'} \Big(x^{\beta+\delta-\delta'} \varphi(x) \Big) \partial_x^{\mu-\mu'} \Big((x-q)^{\delta'} (\partial^\alpha \overline{\eta})(x-q) \Big) \right| dx \\
&\qquad =: \sum_{\delta' \le \delta} \binom{\delta}{\delta'} \sum_{\mu' \le \mu} \binom{\mu}{\mu'} I_1. \qquad (1.3)
\end{aligned}$$

By Lemma A.1.3 and Leibniz rule the estimate on the integral, I_1, becomes

$$\begin{aligned}
I_1 &\le 2^{|\mu|} \delta! \|\eta\|_{\max(|\delta|, |\alpha+\mu|)} \int_{\mathbf{R}^n} |\partial_x^{\mu'} (x^{\beta+\delta-\delta'} \varphi(x))|\, dx \\
&\le 2^{|\mu|} \delta! \|\eta\|_{\max(|\delta|, |\alpha+\mu|)} \sum_{\substack{\gamma \le \mu' \\ \gamma \le \beta+\delta-\delta'}} \binom{\mu'}{\gamma} \frac{(\beta+\delta)!}{(\beta+\delta-\delta'-\gamma)!} \int_{\mathbf{R}^n} |(x^{\beta+\delta-\delta'-\gamma} \partial_x^{\mu'-\gamma}) \varphi(x)|\, dx \\
&\le 2^{|\mu|} \delta! \|\eta\|_{\max(|\delta|, |\alpha+\mu|)} 2^{|\mu|} (\beta+\delta)! C_n \|\varphi\|_{\max(|\beta+\delta|+n+1, |\mu|)},
\end{aligned}$$

where we in the last inequality have used Lemma A.1.2 and Lemma A.1.4. Returning to the original estimate (1.3)

$$\begin{aligned}
\left| p^\mu q^\delta \partial_p^\beta \partial_q^\alpha \Big(\left[T(\varphi) \right](q,p) \Big) \right| &\le C_n 2^{|\delta|+3|\mu|} ((\beta+\delta)!)^2 \|\eta\|_{\max(|\delta|, |\alpha+\mu|)} \\
&\qquad \cdot \|\varphi\|_{\max(|\beta+\delta|+n+1, |\mu|)} \\
&\le C_{\alpha,\beta,\delta,\mu,n} \|\varphi\|_{\max(|\beta+\delta|+n+1, |\mu|)}.
\end{aligned}$$

Given $\varphi \in \mathscr{S}(\mathbf{R}^n)$ this shows that $T(\varphi) \in \mathscr{S}(\mathbf{R}^{2n})$, because all the seminorms of $T(\varphi)$ are bounded. If $\{\varphi_j\}$ is a sequence in $\mathscr{S}(\mathbf{R}^n)$ and we let $\varphi_j \to 0$, we have that $T(\varphi_j) \to 0$, and thereby T is continuous, this follows from Theorem A.1.5. ■

We would like to have an adjoint $T^* \colon \mathscr{S}(\mathbf{R}^{2n}) \to \mathscr{S}(\mathbf{R}^n)$, but to start with we are only able to consider a formal adjoint $T^* \colon \mathscr{S}'(\mathbf{R}^{2n}) \to \mathscr{S}'(\mathbf{R}^n)$ by

$$\langle T(\varphi), \Psi \rangle_{L^2(\mathbf{R}^{2n})} = \langle \varphi, T^*(\Psi) \rangle_{L^2(\mathbf{R}^n)},$$

where $\Psi \in \mathscr{S}'(\mathbf{R}^{2n})$, $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and the L^2-inner products are taken in the distributional sence. If instead of $\Psi \in \mathscr{S}'(\mathbf{R}^{2n})$ we consider $\psi \in \mathscr{S}(\mathbf{R}^{2n})$ the inner product becomes

$$\begin{aligned}
\langle T(\varphi), \psi\rangle_{L^2(\mathbf{R}^{2n})} &= \iint_{\mathbf{R}^{2n}} [T(\varphi)](p,q)\overline{\psi}(q,p)\, dq\, dp \\
&= \iint_{\mathbf{R}^{2n}} \left(\int_{\mathbf{R}^n} \varphi(x)\overline{\eta_{q,p}}(x)\, dx\right)\overline{\psi}(q,p)\, dq\, dp \\
&= \iiint_{\mathbf{R}^{3n}} \varphi(x)\overline{\eta_{q,p}}(x)\overline{\psi}(q,p)\, dx\, dq\, dp \\
&= \int_{\mathbf{R}^n} \varphi(x)\overline{\left(\iint_{\mathbf{R}^{2n}} \eta_{q,p}(x)\psi(q,p)\, dq\, dp\right)}\, dx,
\end{aligned}$$

by Fubini's Theorem, hence we are able to define $T^*\colon \mathscr{S}(\mathbf{R}^{2n}) \to \mathscr{S}(\mathbf{R}^n)$ as

$$[T^*(\psi)](x) := \iint_{\mathbf{R}^{2n}} \eta_{q,p}(x)\psi(q,p)\, dq\, dp. \tag{1.4}$$

The next lemma tells us that T^* is as well-behaved as T.

Lemma 1.1.2 *The adjoint T^* given by* (1.4) *is a continuous map from $\mathscr{S}(\mathbf{R}^{2n})$ to $\mathscr{S}(\mathbf{R}^n)$.*

PROOF. We look at the derivatives of $\eta_{p,q}$

$$\partial_\beta \eta_{q,p}(x) = \sum_{\beta'\le\beta} \binom{\beta}{\beta'} i^{|\beta'|} p^{\beta'} e^{ix\cdot p}\partial^{\beta-\beta'}\eta(x-q).$$

Now, by similar argument as those in the proof of Lemma 1.1.1, we get

$$\begin{aligned}
|[T^*(\psi)](x)| &= \Big|\iint_{\mathbf{R}^{2n}} \psi(q,p)\eta_{q,p}(x)dqdp\Big| \\
&= \Big|\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\iint_{\mathbf{R}^{2n}} q^{\alpha'}\psi(q,p)(x-q)^{\alpha-\alpha'}\partial^\beta\eta_{q,p}(x)dqdp\Big| \\
&= \Big|\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\sum_{\beta'\le\beta}\binom{\beta}{\beta'} i^{|\beta'|} \\
&\qquad \cdot \iint_{\mathbf{R}^{2n}} q^{\alpha'}p^{\beta'}\psi(q,p)e^{ix\cdot p}(x-q)^{\alpha-\alpha'}\partial^{\beta-\beta'}\eta(x-q)dqdp\Big| \\
&\le 2^{|\alpha+\beta|}C_{2n}\|\eta\|_{\max(|\alpha|,|\beta|)}\|\psi\|_{|\alpha+\beta|+2n+1},
\end{aligned}$$

which finishes the proof. ■

The following theorem presents a nice property for the operator T.

Theorem 1.1.3 *The identity $T^*(T(\varphi)) = (2\pi)^n\varphi$, holds for every $\varphi \in \mathscr{S}(\mathbf{R}^n)$*

Proof. By using Fubini's Theorem and the definition of the Fourier transform we get

$$\begin{aligned}
\Big[T^*\Big([T(\varphi)]\,(q,p)\Big)\Big](y) &= \Big[T^*\Big(\int_{\mathbf{R}^n}\varphi(x)e^{-ix\cdot p}\overline{\eta}(x-q)dx\Big)\Big](y)\\
&= \iiint_{\mathbf{R}^{3n}}\varphi(x)e^{-ix\cdot p}\overline{\eta}(x-q)dx\,e^{iy\cdot p}\eta(y-q)dqdp\\
&= \iint_{\mathbf{R}^{2n}}e^{iy\cdot p}\int_{\mathbf{R}^n}e^{-ix\cdot p}\varphi(x)\overline{\eta}(x-q)dxdp\,\eta(y-p)dq\\
&= \int_{\mathbf{R}^n}\frac{(2\pi)^n}{(2\pi)^n}\int_{\mathbf{R}^n}e^{iy\cdot p}\left[\mathcal{F}_x(\varphi(x)\overline{\eta}(x-q))\right](p)dp\,\eta(y-q)dq\\
&= (2\pi)^n\int_{\mathbf{R}^n}\Big[\mathcal{F}_p^{-1}\left[\mathcal{F}_x(\varphi(x)\overline{\eta}(x-q))\right](p)\Big](y)\eta(y-q)dq\\
&= (2\pi)^n\int_{\mathbf{R}^n}\varphi(y)\overline{\eta}(y-q)\eta(y-q)dq\\
&= (2\pi)^n\varphi(y)\int_{\mathbf{R}^n}|\eta(y-q)|^2dq\\
&= (2\pi)^n\varphi(y),
\end{aligned}$$

hence the proof is finished. ■

The following corollary is a nice consequence of Theorem 1.1.3

Corollary 1.1.4 *For all $\varphi,\psi\in\mathscr{S}(\mathbf{R}^n)$,*

$$\begin{aligned}
\langle\varphi\,,\psi\rangle &= (2\pi)^{-n}\langle T(\varphi)\,,T(\psi)\rangle\\
&= (2\pi)^{-n}\iint_{\mathbf{R}^{2n}}\langle\varphi\,,\eta_{q,p}\rangle\overline{\langle\psi\,,\eta_{q,p}\rangle}dqdp,
\end{aligned}$$

especially

$$\|\varphi\|_{L^2}^2 = (2\pi)^{-n}\iint_{\mathbf{R}^{2n}}|\langle\varphi\,,\eta_{q,p}\rangle|^2dqdp.$$

For a function $a\in L^\infty(\mathbf{R}^{2n})$, we can define bilinear form B_a by

$$B_a(\varphi,\psi) := (2\pi)^{-n}\iint_{\mathbf{R}^{2n}}a(q,p)\langle\varphi\,,\eta_{q,p}\rangle\overline{\langle\psi\,,\eta_{q,p}\rangle}dqdp.$$

Corollary 1.1.4 and Cauchy-Schwarz inequality give that

$$|B_a(\varphi,\psi)|\le\|a\|_{L^\infty(\mathbf{R}^{2n})}\|\varphi\|_{L^2(\mathbf{R}^n)}\|\psi\|_{L^2(\mathbf{R}^n)}.$$

This and the fact that B_a is a sesquilinear form on $\mathscr{S}(\mathbf{R}^n)\times\mathscr{S}(\mathbf{R}^n)$ make us able to Riesz' Representation Theorem, since it can be shown by elementary calculations that is T an operator on L^2. Thus we can conclude, that there exists a unique bounded operator A on $L^2(\mathbf{R}^n)$ such that

$$B_a(\varphi,\psi) = \langle A\varphi\,,\psi\rangle,$$

for all $\varphi, \psi \in \mathscr{S}(\mathbf{R}^n)$. To be able to use this operator we want an explicit formula, thus we some formal calculations

$$\begin{aligned}\langle A\varphi\,,\psi\rangle &= B_a(\varphi,\psi)\\ &= (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a(q,p)\langle\varphi\,,\eta_{q,p}\rangle\overline{\langle\psi\,,\eta_{q,p}\rangle}dqdp\\ &= (2\pi)^{-n}\iiint_{\mathbf{R}^{3n}} a(q,p)\langle\varphi\,,\eta_{q,p}\rangle\overline{\psi}(x)\eta_{q,p}(x)dxdqdp\\ &= \int_{\mathbf{R}^n}(2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a(q,p)\langle\varphi\,,\eta_{q,p}\rangle\eta_{q,p}(x)dqdp\overline{\psi}(x)dx,\end{aligned}$$

by Fubini's Theorem, hence we are able to define the operator A by

$$[A(\varphi)](x) \coloneqq (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a(q,p)\langle\varphi\,,\eta_{q,p}\rangle\eta_{q,p}(x)dqdp.$$

Definition 1.1.5 The operator A is the *anti-Wick quantization* of a and is denoted by

$$A = Op^{\mathrm{aW}}(a).$$

By straightforward calculations, Definition 1.1.5 gives the following results:

(i) $\|Op^{\mathrm{aW}}(a)\|_B \leq \|a\|_{L^\infty(\mathbf{R}^n)}$,

(ii) $Op^{\mathrm{aW}}(1) = I$,

(iii) $Op^{\mathrm{aW}}(a)^* = Op^{\mathrm{aW}}(\overline{a})$ and

(iv) $a \geq 0 \Rightarrow Op^{\mathrm{aW}}(a) \geq 0$.

The notation $Op^{\mathrm{aW}}(a)$ is reminiscent of that of a pseudodifferential operator, $Op(a)$, from Definition A.1.7. In fact, we will later look at the relation between the anti-Wick quantization of some $\mathscr{S}(\mathbf{R}^{2n})$-function and the pseudodifferential operator of the same $\mathscr{S}(\mathbf{R}^{2n})$-function. To do this we have to introduce the following definition.

Definition 1.1.6 For fixed $\varphi, \psi \in \mathscr{S}(\mathbf{R}^n)$, the *Wigner function associated to* φ *and* ψ, $W_{\varphi,\psi}$ is given by

$$W_{\varphi,\psi}(x,\xi) = (2\pi)^{-n}e^{ix\cdot\xi}\hat{\varphi}(\xi)\overline{\psi(x)}. \tag{1.5}$$

Given a real and even function $\eta \in \mathscr{S}(\mathbf{R}^n)$, such that $\|\eta\|_{L^2} = 1$, we define the Wigner function, W, to be

$$W \coloneqq W_{\eta,\eta}, \tag{1.6}$$

which is also even.

1.2 Semiclassical scaling

We are now able to begin the investigation of semiclassical theory. We start by defining the semiclassical quantization of a pseudodifferential operator, with $a \in \mathscr{S}(\mathbf{R}^{2n})$ as a symbol.

Definition 1.2.1 Let $h \in]0,1]$ and $a \in \mathscr{S}(\mathbf{R}^{2n})$, then

$$[Op_h(a)(\varphi)](x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{\varphi}(\xi)\, d\xi,$$

is called a *semiclassical pseudodifferential operator.*

By denoting $a_h(x,\xi) := a(x,h\xi)$, we see that by Definition A.1.7 we have that $Op_h(a) = Op(a_h)$. We also want a semiclassically scaled Wigner function, hence we define a scaled η:

$$\eta_h(x) := h^{-\frac{n}{4}}\eta\left(\frac{x}{h^{\frac{1}{2}}}\right), \tag{1.7}$$

and thus we get a Wigner function from (1.5)

$$W_{\eta_h,\eta_h} = (2\pi)^{-n}e^{ix\cdot\xi}\widehat{\eta}_h(\xi)\overline{\eta_h}(x) = (2\pi)^{-n}e^{ix\cdot\xi}\widehat{\eta}(h^{1/2}\xi)\overline{\eta}(h^{-1/2}x). \tag{1.8}$$

If we define a new W_h as a scaling of W from (1.6)

$$W_h := h^{-n}W\left(h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}\xi\right),$$

we can formulate following lemma.

Lemma 1.2.2 *For every $a \in \mathscr{S}(\mathbf{R}^n)$ and every $h \in]0,1]$*

$$\left(a_h * \overline{W}_{\eta_h,\eta_h}\right)(x,\xi) = \left(a * \overline{W}_h\right)(x, h\xi).$$

Proof. The left hand side is given by

$$(a_h * \overline{W}_{\eta_h,\eta_h})(x,\xi) = (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a(x+q, h(\xi+p))e^{-iq\cdot p}\eta(-h^{-\frac{1}{2}}q)\hat{\eta}(-h^{\frac{1}{2}}p)dqdp.$$

Now we want to see if the right hand side is equal to the left

$$\begin{aligned}
(a * \overline{W}_h)(x,h\xi) &= \iint_{\mathbf{R}^{2n}} a(x+q, h\xi+p)\overline{W}_h(-q,-p)dqdp \\
&= h^{-n}\iint_{\mathbf{R}^{2n}} a(x+q, h\xi+p)\overline{W}(-h^{-\frac{1}{2}}q, -h^{-\frac{1}{2}}p)dqdp \\
&= (2\pi)^{-n}h^{-n}\iint_{\mathbf{R}^{2n}} a(x+q, h\xi+p)e^{\frac{-iq\cdot p}{h}}\eta(-h^{-\frac{1}{2}}q)\hat{\eta}(-h^{-\frac{1}{2}}p)dqdp \\
&= (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a(x+q, h(\xi+p))e^{-iq\cdot p}\eta(-h^{-\frac{1}{2}}q)\hat{\eta}(-h^{\frac{1}{2}}p)dqdp,
\end{aligned}$$

in the last equality we have use the change of variable $p \mapsto hp$, hence the proof is done. ■

It follows easily from (1.7) that η_h has L^2-norm equal to 1 and thus we can define $Op_h^{\mathrm{aW}}(a)$ by

$$\left[Op_h^{\mathrm{aW}}(a)(\varphi)\right](x) := (2\pi)^{-n} \iint_{\mathbf{R}^2} a(q,p)\langle(\eta_h)_{q,p}\,,\varphi\rangle(\eta_h)_{q,p}(x)dqdp,$$

for $a \in \mathscr{S}(\mathbf{R}^{2n})$ and $h \in]0,1]$, where $(\eta_h)_{q,p}$ is given by replacing η with η_h in (1.1). As teased in the previous section we are now able to state the connection between the anti-Wick quantization and a pseudodifferential operator, in a semiclassical sence.

Lemma 1.2.3 *For all $a \in \mathscr{S}(\mathbf{R}^{2n})$ and all $h \in]0,1]$*

$$Op_h^{\mathrm{aW}}(a_h) = Op_h(a * \overline{W}_h). \tag{1.9}$$

Proof. From Lemma 1.2.2 we know that the symbol of the operator on the right hand side can be expressed as

$$\begin{aligned}(a_h * \overline{W}_{\eta_h,\eta_h})(x,\xi) &= (2\pi)^{-n} \iint_{\mathbf{R}^2} a_h(x+q,\xi+p)\overline{e^{i(-q)\cdot(-p)}\eta_h(-q)\hat{\eta}_h(-p)}dqdp \\ &= (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} a_h(x+q,\xi+p)e^{-iq\cdot p}\eta_h(q)\hat{\eta}_h(p)dqdp,\end{aligned}$$

where we have used (1.8). Thus by Definition 1.2.1 we get

$$\begin{aligned}&\left[Op(a_h * \overline{W}_{\eta_h,\eta_h})\varphi\right](x) \\ &\quad= (2\pi)^{-2n} \iiint_{\mathbf{R}^{3n}} e^{ix\cdot\xi - iq\cdot p} a_h(x+q,\xi+p)\eta_h(q)\widehat{\eta}_h(p)\hat{\varphi}(\xi)dqdpd\xi. \end{aligned}\tag{1.10}$$

Elementary calculations give that

$$\begin{aligned}\langle\varphi\,,(\eta_h)_{q,p}\rangle &= (2\pi)^{-n}\langle\hat{\varphi}\,,\mathcal{F}(M_pT_q\eta_h)\rangle \\ &= (2\pi)^{-n}\langle\hat{\varphi}\,,T_pM_{-q}\widehat{\eta}_h\rangle \\ &= (2\pi)^{-n}\int_{\mathbf{R}^n}\hat{\varphi}(\xi)\overline{e^{-iq\cdot(\xi-p)}\widehat{\eta}_h(\xi-p)}d\xi \\ &= (2\pi)^{-n}\int_{\mathbf{R}^n} e^{iq\cdot(\xi-p)}\widehat{\eta}_h(\xi-p)\hat{\varphi}(\xi)d\xi.\end{aligned}$$

Looking at the left hand side of (1.9)

$$\begin{aligned}\left[Op_h^{\mathrm{aW}}(a_h)\varphi\right](x) &= (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} a_h(q,p)\langle\varphi\,,(\eta_h)_{q,p}\rangle(\eta_h)_{q,p}(x)dqdp \\ &= (2\pi)^{-2n}\iiint_{\mathbf{R}^{3n}} a_h(q,p)e^{iq\cdot(\xi-p)}\widehat{\eta}_h(\xi-p)\hat{\varphi}(\xi)e^{ip\cdot x}\eta_h(x-q)d\xi dqdp \\ &= (2\pi)^{-2n}\iiint_{\mathbf{R}^{3n}} a_h(q+x,p+\xi)e^{i(q+x)\cdot(-p)}\widehat{\eta}_h(-p)\hat{\varphi}(\xi)e^{i(p+\xi)\cdot x}\eta_h(-q)dqdpd\xi \\ &= (2\pi)^{-2n}\iiint_{\mathbf{R}^{3n}} e^{ix\cdot\xi-iq\cdot p}a_h(q+x,p+\xi)\eta_h(q)\widehat{\eta}_h(p)\hat{\varphi}(\xi)dqdpd\xi \end{aligned}\tag{1.11}$$

where in third line we change the variables from p to $p+\xi$ and from q to $q+x$, then from equations (1.10) and (1.11) the proof is finished. ■

1.3 Gårding's inequality

In order to prove Gårding's inequality for Schwartz functions, we first introduce two result. The first of these is a semiclassical variant of the Calderon-Vaillancourt Theorem.

Theorem 1.3.1 *For $\alpha, \beta \in \mathbf{N}^n$, there exists $C, \widetilde{N} \geq 0$ such that for every $a \in \mathscr{S}(\mathbf{R}^{2n})$, we get*

$$\|Op_h(a)\varphi\|_{L^2} \leq C \max_{|\alpha+\beta|\leq\widetilde{N}} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty} \|\varphi\|_{L^2},$$

every $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and every $h \in]0,1]$.

The proof is omitted but can be found in [Bouclet, 2015a] and the interested reader can look at the original result [Calderon and Vaillancourt, 1971].

In the second result we will need a special case of the seminorms on the space of Schwartz functions, found in Appendix A, which is given by

$$\|f\|_{d\leq m} \coloneqq \sup_{d\leq|\beta|\leq m} \sup_{x\in\mathbf{R}^n} |\langle x\rangle^{2(m-d)} \partial^\beta f(x)|,$$

for $f \in \mathscr{S}(\mathbf{R}^n)$.

Lemma 1.3.2 *There exists some $C > 0$ such that*

$$\|a * \overline{W}_h - a\|_{L^\infty} \leq Ch\|a\|_{2\leq 4},$$

for every $a \in \mathscr{S}(\mathbf{R}^{2n})$ and every $h \in]0,1]$.

Proof. First we look at the left hand side and apply Taylor's Formula A.2.3 to get

$$\begin{aligned}
\Big|\Big(a * \overline{W}_h - a\Big)(x,\xi)\Big| &= \Big|\iint_{\mathbf{R}^2} a(x+q,\xi+p)\overline{W}_h(-q,-p)dqdp - a(x,\xi)\Big| \\
&= \Big|\iint_{\mathbf{R}^{2n}} a(x,\xi)\overline{W_h(-q,-p)}dqdp - a(x,\xi) \\
&\quad + \iint_{\mathbf{R}^{2n}} \sum_{|\alpha+\beta|=1} q^\alpha p^\beta (\partial_1^\alpha \partial_2^\beta a)(x,\xi)\overline{W_h(-q,-p)}dqdp \\
&\quad + \iint_{\mathbf{R}^{2n}} \sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} q^\alpha p^\beta \int_0^1 (1-t)(\partial_1^\alpha \partial_2^\beta a)(x+tq,\xi+tp)dt\overline{W_h(-q,-p)}dqdp\Big| \\
&= \Big|\sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} \int_0^1 \iint_{\mathbf{R}^{2n}} (q^\alpha p^\beta e^{-\frac{i}{h}q\cdot p})(1-t)(\partial_1^\alpha \partial_2^\beta a)(x+tq,\xi+tp) \\
&\qquad \cdot (2\pi)^{-n} h^{-n} \eta(-h^{-\frac{1}{2}}q)\hat{\eta}(-h^{-\frac{1}{2}}p)dqdpdt\Big|,
\end{aligned}$$

where the last equality follows from the fact that W_h is even and integrates to 1. By integration by parts and Leibniz' rule we get

$$
\begin{aligned}
&\left|\left(a * \overline{W}_h - a\right)(x,\xi)\right| \\
&= \Big| \sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} \int_0^1 \iint_{\mathbf{R}^{2n}} -h^2 e^{-\frac{i}{h} q\cdot p} \partial_q^\beta \partial_p^\alpha \left[(1-t)(\partial_1^\alpha \partial_2^\beta a)(x+tq,\xi+tp)\right. \\
&\qquad \left. \cdot (2\pi)^{-n} h^{-n} \eta(h^{-\frac{1}{2}} q) \hat{\eta}(h^{-\frac{1}{2}} p)\right] dqdpdt \Big| \\
&= (2\pi)^{-n} \Big| h^{2-n} \sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} \int_0^1 \iint_{\mathbf{R}^{2n}} e^{-\frac{i}{h} q\cdot p} \sum_{\alpha'\le\alpha} \binom{\alpha}{\alpha'} \sum_{\beta'\le\beta} \binom{\beta}{\beta'} (1-t) \\
&\qquad \cdot \left(\partial_1^{\alpha+\beta'} \partial_2^{\beta+\alpha'} a\right)(x+tq,\xi+tp) t^{|\alpha'+\beta'|} (\partial^{\beta-\beta'}\eta)(h^{-\frac{1}{2}} q)(\partial^{\alpha-\alpha'}\hat{\eta}) \\
&\qquad \cdot (h^{-\frac{1}{2}} p) h^{\frac{-2+|\alpha'+\beta'|}{2}} dqdpddt \Big|.
\end{aligned}
$$

Then by the changes of variables $q \mapsto h^{\frac{1}{2}} q$ and $p \mapsto h^{\frac{1}{2}} p$, both with Jacobian determinant $h^{\frac{n}{2}}$, we obtain

$$
\begin{aligned}
&\left|\left(a*\overline{W}_h - a\right)(x,\xi)\right| \\
&= (2\pi)^{-n} \Big| \sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} \int_0^1 \iint_{\mathbf{R}^{2n}} e^{-iq\cdot p} \sum_{\alpha'\le\alpha} \binom{\alpha}{\alpha'} \sum_{\beta'\le\beta} \binom{\beta}{\beta'} (1-t) \\
&\qquad \cdot \left(\partial_1^{\alpha+\beta'} \partial_2^{\beta+\alpha'} a\right)(x+tqh^{1/2},\xi+tph^{1/2}) t^{|\alpha'+\beta'|} (\partial^{\beta-\beta'}\eta)(q) \\
&\qquad \cdot (\partial^{\alpha-\alpha'}\hat{\eta})(p) h^{\frac{2+|\alpha'+\beta'|}{2}} dqdpddt \Big| \\
&\le (2\pi)^{-n} \sum_{|\alpha+\beta|=2} \frac{2!}{\alpha!\beta!} \sum_{\alpha'\le\alpha} \binom{\alpha}{\alpha'} \sum_{\beta'\le\beta} \binom{\beta}{\beta'} h\|a\|_{2\le 4} \\
&\qquad \cdot \int_0^1 \iint_{\mathbf{R}^{2n}} |(\partial^{\alpha-\alpha'}\hat{\eta})(q)| dq |(\partial^{\beta-\beta'}\eta)(p)| dpdt \\
&\le (2\pi)^{-n} \max_{|\beta|\le 2} \|\partial^\beta \hat{\eta}\|_{L^1} \max_{|\beta|\le 2} \|\partial^\beta \eta\|_{L^1} (2n)^2 2^5 h\|a\|_{2\le 4} \\
&= Ch\|a\|_{2\le 4},
\end{aligned}
$$

hence the proof is done. ■

We now formulate the main result of this section.

Theorem 1.3.3 (Gårding's inequality) *There exists $C, N \ge 0$ such that for every non-negative $a \in \mathscr{S}(\mathbf{R}^{2n})$ we have*

$$\mathrm{Re}(\langle Op_h(a)\varphi\,,\varphi\rangle) \ge -Ch\|a\|_N \|\varphi\|_{L^2},$$

for every $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and every $h \in]0,1]$.

Proof. Form the linearity of the operators we get

$$\begin{aligned} Op_h(a) &= Op_h(a - a * \overline{W_h}) + Op_h(a * \overline{W_h}) \\ &= Op_h(a - a * \overline{W_h}) + Op_h^{\mathrm{aW}}(a_h), \end{aligned}$$

which follows by (1.9). Since $a \geq 0$ we have that $Op_h^{\mathrm{aW}}(a_h)$ is a positive operator, thus we check $Op_h(a)$ and observe that by Theorem 1.3.1 we have

$$\begin{aligned} \|Op_h(a - a * \overline{W_h})\varphi\|_{L^2} &\leq C \max_{|\alpha+\beta| \leq \tilde{N}} \|\partial_x^\alpha \partial_\xi^\beta (a - a * \overline{W_h})\|_{L^\infty} \|\varphi\|_{L^2} \\ &= C \max_{|\alpha+\beta| \leq \tilde{N}} \|(\partial_x^\alpha \partial_\xi^\beta a - (\partial_x^\alpha \partial_\xi^\beta a) * \overline{W_h})\|_{L^\infty} \|\varphi\|_{L^2} \\ &\leq Ch\|a\|_{2 \leq \tilde{N}+4} \|\varphi\|_{L^2}, \end{aligned} \tag{1.12}$$

where the last inequality follows from Lemma 1.3.2. Hence we can conclude that

$$\begin{aligned} \mathrm{Re}(\langle Op_h(a)\varphi \, , \varphi \rangle) &= \mathrm{Re}(\langle (Op_h(a - a * \overline{W_h}) + Op_h^{\mathrm{aW}}(a_h))\varphi \, , \varphi \rangle) \\ &= \mathrm{Re}(\langle Op_h(a - a * \overline{W_h})\varphi \, , \varphi \rangle) + \mathrm{Re}(\langle Op_h^{\mathrm{aW}}(a_h)\varphi \, , \varphi \rangle) \\ &\geq \mathrm{Re}(\langle Op_h(a - a * \overline{W_h})\varphi \, , \varphi \rangle) \\ &\geq -Ch\|a\|_{2 \leq \tilde{N}+4} \|\varphi\|_{L^2}^2, \end{aligned}$$

where the last inequality follows from (1.12) and Cauchy-Schwarz inequality, which finishes the proof. ■

2. Fourier Transform of e^{icx^m}

In this chapter we will show a formula for the Fourier transform of e^{icx^m}, where c is a reel constant and $m \in \mathbf{N}$ is even. These results become important when we consider oscillatory integrals.

2.1 Preliminary results

In this section we present some results used in the proof for the Fourier transform of e^{icx^m}, the first is an integral identity.

Lemma 2.1.1 *Let $m \in \mathbf{N}$, $-1 < \frac{a}{b} \in \mathbf{Q}$, $b \in \mathbf{N}$ and $c \in \mathbf{R} \setminus \{0\}$, then*

$$\int_0^\infty x^{\frac{a}{b}} e^{icx^m} dx = \frac{1}{(-ic)^{(a+b)/(mb)}} \int_0^\infty z^{\frac{a}{b}} e^{-z^m} dz, \tag{2.1}$$

where $(-ic)^{1/m}$ is the principal m'th root of $-ic$, same for the b'th root.

For $b = 1$ we have

$$\int_0^\infty x^a e^{icx^m} dx = \frac{1}{(-ic)^{(a+1)/(m)}} \int_0^\infty z^a e^{-z^m} dz. \tag{2.2}$$

The idea behind the proof of this lemma is based on techniques used in [Evans and Zworski, 2003, p. 26-27].

PROOF. Let $\varepsilon, M > 0$ then from the left hand side of (2.1) we have

$$\begin{aligned}
&\frac{(\varepsilon - ic)^{1/m + a/(bm)}}{(\varepsilon - ic)^{(a+b)/mb}} \lim_{M\to\infty} \int_0^M x^{a/b} e^{(ic-\varepsilon)x^m} dx \\
&= \frac{1}{(\varepsilon - ic)^{(a+b)/mb}} \lim_{M\to\infty} \int_0^M (\varepsilon - ic)^{\frac{1}{m}} \left((\varepsilon - ic)^{\frac{1}{m}} x\right)^{a/b} e^{(ic-\varepsilon)x^m} dx \\
&= \frac{1}{(\varepsilon - ic)^{(a+b)/mb}} \lim_{M\to\infty} \oint_{\gamma_1} z^{a/b} e^{-z^m} dz \qquad (2.3) \\
&= \frac{1}{(\varepsilon - ic)^{(a+b)/mb}} \lim_{M\to\infty} \left\{ \int_0^{M(c^2+\varepsilon^2)^{1/2m}} z^{a/b} e^{-z^m} dz + \oint_{\gamma_2} z^{a/b} e^{-z^m} dz \right\},
\end{aligned}$$

where $\gamma_1(x) = (\varepsilon - ic)^{1/m} x$, for $x \in [0, M]$. We have that $\gamma_1(M) = M(c^2 + \varepsilon^2)^{1/(2m)} e^{\pm\varphi i/m}$, where the plus sign is used if $c < 0$, and the minus if $c > 0$. In this manor we get that $0 < \varphi < \frac{\pi}{2}$, and thus define $\gamma_2(t) := M(c^2 + \varepsilon^2)^{1/2m} e^{\pm ti}$, for $t \in [0, \frac{\varphi}{m}]$. Making a norm estimate of the contour integral over γ_2 we get

$$\left| \oint_{\gamma_2} z^{\frac{a}{b}} e^{-x^m} dz \right| = \left| \int_0^{\frac{\varphi}{m}} iM(c^2 + \varepsilon^2)^{\frac{1}{2m}} e^{\pm ti} \left(M(c^2 + \varepsilon^2)^{\frac{1}{2m}} e^{\pm ti} \right)^{\frac{a}{b}} e^{-(M(c^2+\varepsilon^2)^{\frac{1}{2m}} e^{\pm ti})^m} dt \right|$$

$$\leq \frac{\pi}{2m}(c^2+\varepsilon^2)^{\frac{a+b}{2mb}} M^{a/b+1} \max_{t\in[0,\varphi/m]} \left| e^{-(M^m(c^2+\varepsilon^2)^{\frac{1}{2}}e^{mti}} \right|.$$

Since $\varphi < \frac{\pi}{2}$ and $0 < \mathrm{Re}(e^{mti}) \leq 1$ there exist positive constants, c_1, c_2, such that

$$\left| \oint_{\gamma_2} z^{a/b} e^{-x^m} dz \right| < c_1 M^{a/b+1} e^{-c_2 M^m}.$$

Letting M go to infinity in (2.3) we get

$$\int_0^\infty x^{a/b} e^{(ic-\varepsilon)x^m} = \frac{1}{(\varepsilon - ic)^{(a+b)/(bm)}} \int_0^\infty z^{a/b} e^{-zx^m} dz.$$

Taking the limit of ε going to 0, finishes the proof. ■

The formula for the Fourier transform of e^{icx^m} use the generalized hypergeometric function therefore we state a definition which is based upon [Andrews et al., 1999, p. 61-62].

Definition 2.1.2 Given $p, q \in \mathbf{N}_0$ then for sequences $\{a\}_q, \{b\}_q \subset \mathbf{R} \setminus (\{0\} \cup \mathbf{Z}^-)$ the *generalized hypergeometric function* is given by

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k, \ldots, (a_p)_k}{(b_1)_k, \ldots, (b_q)_k} \frac{x^k}{k!},$$

for $x \in \mathbf{C}$, where $(a)_k = a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$.

It is worth noting that convergence of the generalized hypergeometric functions is determined by the relation between p and q.

Theorem 2.1.3 *The function* ${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right)$ *converges absolutely for all* $x \in \mathbf{C}$ *if* $p \leq q$ *and for* $|x| < 1$ *if* $p = q+1$.
It diverges for all non-zero $x \in \mathbf{C}$ *if* $p > q+1$.

For a proof see [Andrews et al., 1999, p. 62].

2.2 Fourier Transform

With the definition of the generalized hypergeometric functions in mind we state the main result of this chapter.

Theorem 2.2.1 *Let* $2 \leq m \in \mathbf{N}$ *be even and* $c \neq 0$ *be a real constant, then given the set* $E = \{0, 2, \ldots, m-2\}$ *the Fourier transform of* e^{icx^m} *is given by*

$$\mathcal{F}\Big(e^{icx^m}\Big)(\xi) = \sum_{k\in E} a_k \xi^k \; {}_0F_{m-2}\left(; M_{m-k}; \frac{\xi^m}{c(i)^{m+3} m^m}\right), \tag{2.4}$$

for every $\xi \in \mathbf{R}$*, where* $M_{m-k} = \Big\{ \frac{k+s}{m} \in \mathbf{Q} \mid \forall s \in \{2,3,..,m\}$ *such that* $s \neq m-k \Big\}$ *and*

$$a_k = (-i)^k \frac{2}{m(k!)(-ic)^{(k+1)/m}} \Gamma\Big(\frac{k+1}{m}\Big),$$

with $(-ic)^{1/m}$ *as the principal m'th root of* $-ic$*.*

Proof. We define $f(\xi)$ by the right hand side (2.4) as

$$f(\xi) \coloneqq \mathcal{F}\Big(e^{icx^m}\Big)(\xi) = \int_{\mathbf{R}} e^{-ix\xi} e^{icx^m} \, dx,$$

for $\xi \in \mathbf{R}$. Now to find an appropriate differential equation we take the $(m-1)$'th derivative

$$f^{(m-1)}(\xi) = \int_{\mathbf{R}} \partial_\xi^{m-1}\Big(e^{-ix\xi}\Big) e^{icx^m} \, dx = (-i)^{m-1} \int_{\mathbf{R}} e^{-ix\xi} x^{m-1} e^{icx^m} \, dx.$$

To ensure that we can use integration by parts we rewrite the integrand

$$\begin{aligned} (-i)^{m-1} \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}} e^{-ix\xi} x^{m-1} e^{i(c+i\varepsilon)x^m} \, dx \\ &= (-i)^{m-1} \lim_{\varepsilon \to 0^+} \frac{1}{i(c+i\varepsilon)m} \int_{\mathbf{R}} e^{-ix\xi} \partial_x \Big(e^{i(c+i\varepsilon)x^m}\Big) \, dx \\ &= (-i)^m \lim_{\varepsilon \to 0^+} \frac{-\xi}{i(c+i\varepsilon)m} \int_{\mathbf{R}} e^{-ix\xi} e^{i(c+i\varepsilon)x^m} \, dx \\ &= -(-i)^{m+1} \frac{\xi}{cm} \int_{\mathbf{R}} e^{-ix\xi} e^{icx^m} \, dx, \end{aligned}$$

which gives that

$$f^{(m-1)}(\xi) = -\frac{(-i)^{m+1}}{cm} \xi f(\xi). \tag{2.5}$$

To solve this differential equation (2.5) we use the initial value conditions $f^{(j)}(0) = b_j$, for every $j \in \{0, 1, \ldots, m-1\}$, where

$$f^{(j)}(0) = (-i)^j \int_{\mathbf{R}} x^j e^{icx^m} \, dx. \tag{2.6}$$

Thus we observe that the integrand of (2.6) is even when j is even, and odd when j is odd, hence $b_j = 0$ for odd j. To find the initial values of b_j for j even, we rewrite (2.6) by applying Lemma 2.1.1

$$\begin{aligned} (-i)^j \int_{\mathbf{R}} x^j e^{icx^m} \, dx &= (-i)^j \frac{2}{(-ic)^{(j+1)/m}} \int_0^\infty z^j e^{-z^m} \, dz \\ &= (-i)^j \frac{1}{(-ic)^{(j+1)/m}} \frac{2}{m} \int_0^\infty t^{(j+1)/m-1} e^{-t} \, dt \end{aligned}$$

$$= (-i)^j \frac{2}{m(-ic)^{(j+1)/m}} \Gamma\Big(\frac{j+1}{m}\Big).$$

To solve the differential equation we write $f(\xi)$ as a Taylor series at 0

$$f(\xi) = \sum_{k=0}^{\infty} a_k \xi^k,$$

taking the $(m-1)$'th derivative of this, we get

$$\begin{aligned} f^{(m-1)}(\xi) &= \sum_{k=m-1}^{\infty} a_k \xi^{k-m+1} \prod_{l=n-m+2}^{k} l \\ &= \sum_{k=-1}^{\infty} a_{k+m} \xi^{k+1} \prod_{l=n+2}^{k+m} l. \end{aligned}$$

Since $f^{(m-1)}(0) = 0$ it follows that $a_{m-1} = 0$ which implies $a_{dm-1} = 0$ for all $d \in \mathbf{N}$. Furthermore, if we use the this Taylor series in the differential equation (2.5) we get

$$f^{(m-1)}(\xi) = \sum_{k=0}^{\infty} \frac{(-i)^{m+3}}{cm} a_k \xi^{k+1}.$$

Combining these two result for $f^{m-1}(\xi)$ we observe that

$$a_{k+m} = \frac{(-i)^{m+3}(k+1)}{cm \prod_{l=k+1}^{k+m} l} a_k,$$

for every $0 \leq k \leq m-2$. Thus the Taylor series can be rewritten as

$$f(\xi) = \sum_{k=0}^{m-2} a_k \Bigg(\sum_{d=0}^{\infty} \Big(\frac{(-i)^{m+3}}{cm} \Big)^d \frac{k!(\prod_{l=1}^{d} m(l-1)+k+1)}{(k+dm)!} \xi^{k+dm} \Bigg).$$

Comparing this to our initial value conditions for (2.5) we get that $a_j = b_j/(j!)$ and thus defining a set $E \coloneqq \{0, 2, \ldots, m-2\}$ our function is given by

$$f(\xi) = \sum_{k \in E} a_k \xi^k \Bigg(\sum_{d=0}^{\infty} \Big(\frac{1}{(i)^{m+3} cm} \Big)^d \frac{k! \prod_{l=1}^{d} (m(l-1)+k+1)}{(k+dm)!} \xi^{dm} \Bigg) =: \sum_{k \in E} a_k \xi^k g_k(\xi).$$

Looking at g_k we get that

$$\begin{aligned} g_k(\xi) &= \sum_{d=0}^{\infty} \Big(\frac{1}{(i)^{m+3} cm} \Big)^d \frac{k! \prod_{l=1}^{d} (m(l-1)+k+1)}{(k+dm)!} \xi^{dm} \\ &= \sum_{d=0}^{\infty} \Big(\frac{\xi^m}{(i)^{m+3} c} \Big)^d \frac{1}{m^d} \frac{k! \prod_{l=1}^{d} (m(l-1)+k+1)}{k! \prod_{l=1}^{d} (m(l-1)+k+1)} \prod_{s=2}^{m} \frac{1}{\prod_{l=1}^{d} (m(l-1)+k+s)} \end{aligned}$$

$$= \sum_{d=0}^{\infty}\left(\frac{\xi^m}{(i)^{m+3}c}\right)^d \frac{1}{m^d}\prod_{s=2}^{m}\frac{1}{m^d\prod_{l=1}^{d}\left(\frac{k+s}{m}+l-1\right)}$$
$$= \sum_{d=0}^{\infty}\left(\frac{\xi^m}{(i)^{m+3}cm^m}\right)^d \frac{1}{d!}\prod_{\substack{s=2\\ s\neq m-k}}^{m}\frac{1}{\left(\frac{k+s}{m}\right)_d}$$
$$= {}_0F_{m-2}\left(;M_{m-k};\frac{\xi^m}{c(i)^{m+3}m^m}\right),$$

where $M_{m-k} = \left\{\frac{k+s}{m} \in \mathbf{Q} \mid \forall s \in \{2,3,..,m\} \text{ such that } s \neq m-k\right\}$, hence the proof is done. ■

In the next chapter we study the behaviour of oscillatory integrals in multiple dimensions, thus we need the following result for the Fourier transform when $m = 2$.

Lemma 2.2.2 *Suppose $A \in \mathbf{R}^{n\times n}$ is symmetric and non-singular, then*

$$\left[\mathcal{F}\left(e^{\frac{i}{2}(Ax)\cdot x}\right)\right](\xi) = \frac{(2\pi)^{n/2}}{|\det A|^{1/2}} e^{\frac{i\pi}{4}\operatorname{sgn} A} e^{-\frac{i}{2}(A^{-1}(\xi))\cdot\xi}.$$

In one dimension the signature of A is the same as the sign function, hence we denote it by $\operatorname{sgn} A$.

Proof. Calculating the Fourier transform we get

$$\begin{aligned}\left[\mathcal{F}\left(e^{\frac{i}{2}(Ax)\cdot x}\right)\right](\xi) &= \int_{\mathbf{R}^n} e^{\frac{i}{2}(Ax)\cdot x - ix\cdot\xi}\,dx\\ &= \int_{\mathbf{R}^n} e^{\frac{i}{2}(Ax)\cdot x - ix\cdot\xi + \frac{i}{2}((A^{-1}\xi)\cdot\xi - (A^{-1}\xi)\cdot\xi)}\,dx\\ &= \int_{\mathbf{R}^n} e^{\frac{i}{2}(A(x-A^{-1}\xi))\cdot(x-A^{-1}\xi)) - \frac{i}{2}(A^{-1}\xi)\cdot\xi}\,dx\\ &= e^{-\frac{i}{2}(A^{-1}\xi)\cdot\xi}\int_{\mathbf{R}^n} e^{\frac{i}{2}(Ay)\cdot y}\,dy,\end{aligned}$$

where last equality follows from the change of variable $y := (x - A^{-1}\xi)$. Now using that A is reel and symmetric we see that A has eigenvalues $(\lambda_1,\ldots,\lambda_n)$, where $\lambda_1,\ldots,\lambda_r$ are the positive eigenvalues and $\lambda_{r+1},\ldots,\lambda_n$ are the negative eigenvalues and let $p_1,\ldots,p_n$ denote the corresponding eigenvector Then by spectral decomposition we get

$$\begin{aligned}\int_{\mathbf{R}^n} e^{\frac{i}{2}(Ay)\cdot y}\,dy &= \int_{\mathbf{R}^n} e^{\sum_{k=1}^{n}\frac{1}{2}(i\lambda_k)\|p_k\|^2}\,dp\\ &= \prod_{k=1}^{n} 2\int_0^\infty e^{\frac{1}{2}(i\lambda_k)\|p_k\|^2}\,dp_k\\ &= \prod_{k=1}^{n} 2\frac{1}{(-\frac{i}{2}\lambda_k)^{1/2}}\int_0^\infty e^{-z^2}dz\end{aligned}$$

$$= \prod_{k=1}^{n} \frac{\sqrt{2\pi}}{(-i\lambda_k)^{1/2}},$$

where the third equality follows from Lemma 2.1.1, and from this lemma we also know that we have to take the principal square root, which gives

$$(-i\lambda_k)^{1/2} = (-i(\operatorname{sgn}\lambda_k))^{1/2}|\lambda_k|^{1/2} = \frac{e^{\frac{i\pi}{4}\operatorname{sgn}\lambda_k}}{|\lambda_k|^{1/2}}.$$

Hence the Fourier transform becomes

$$\left[\mathcal{F}\left(e^{\frac{i}{2}(Ax)\cdot x}\right)\right](\xi) = \left(\prod_{k=1}^{n} \frac{\sqrt{2\pi}e^{\frac{i\pi}{4}\operatorname{sgn}\lambda_k}}{|\lambda_k|^{1/2}}\right)e^{-\frac{i}{2}(A^{-1}\xi)\cdot\xi} = \frac{(2\pi)^{n/2}e^{\frac{i\pi}{4}\operatorname{sgn}A}}{|\det A|^{1/2}}e^{-\frac{i}{2}(A^{-1}\xi)\cdot\xi},$$

which finishes the proof. ■

3. Oscillatory Integrals

The aim of this chapter is to get an understanding of how oscillatory integral behave when h goes to 0. The following chapter is based on [Evans and Zworski, 2003].

A motivating factor to study the behaviour of oscillatory integral is that the semiclassical pseudodifferential operators from Chapter 1 which are given by Definition 1.2.1 can be rewritten by the change of variable $\xi \mapsto \xi/h$

$$[Op_h(a)\varphi](x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{\varphi}(\xi)\, d\xi = (2\pi h)^{-n} \int e^{\frac{i}{h}x\cdot\xi} a(x,\xi)\widehat{\varphi}\left(\frac{\xi}{h}\right)\, d\xi,$$

where we see that the integral kernel looks to be an inverse semiclassical Fourier transform of the symbol with respect to the second variable.

Definition 3.0.1 The *semiclassical Fourier transform* is given for $h > 0$ by

$$[\mathcal{F}_h(\varphi)](\xi) := \int_{\mathbf{R}^n} e^{-(i/h)x\cdot\xi} \varphi(x)\, dx, \tag{3.1}$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$, and the inverse by

$$\left[\mathcal{F}_h^{-1}(\varphi)\right](x) := (2\pi h)^{-n} \int_{\mathbf{R}^n} e^{(i/h)x\cdot\xi} [\mathcal{F}_h(\varphi)](\xi)\, d\xi,$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$.

The semiclassical Fourier transform has properties similar to the ones of the classical Fourier transform.

Theorem 3.0.2 *Let $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and $\alpha \in \mathbf{N}^n$, then we get*

$$\begin{aligned}(-ih\partial_\xi)^\alpha [\mathcal{F}_h(\varphi)](\xi) &= [\mathcal{F}_h((-x)^\alpha \varphi)](\xi), \\ [\mathcal{F}_h((-ih\partial_x)^\alpha \varphi)](\xi) &= \xi^\alpha [\mathcal{F}_h(\varphi)](\xi), \\ (2\pi h)^{-n} \|\mathcal{F}_h(\varphi)\|_{L^2}^2 &= \|\varphi\|_{L^2}^2,\end{aligned}$$

for every $h > 0$.

The proof is omitted since it follows from Definition 3.0.1 by straightforward calculation.

3.1 Rapid decay

In order to understand behaviour of oscillatory integral, like the semiclassical Fourier transform from (3.1), when h goes to 0, we start our study of such integrals in the one dimensional case.

Definition 3.1.1 Let $h > 0$ be a real parameter, then the *oscillatory integral* is defined by

$$I_h(a,\varphi) := \int_{-\infty}^{\infty} e^{i\varphi(x)/h} a(x)\,dx, \tag{3.2}$$

for $a \in C_c^\infty(\mathbf{R})$ and $\varphi \in C^\infty(\mathbf{R})$.

When a and φ are given we will denote (3.2) by I_h, if there is no chance of confusion.
If there for every positive integer N, exists a constant $C_N > 0$ such that

$$|I_h| \leq C_N h^N, \tag{3.3}$$

for all $h \in]0,1]$, this property denotes as $I_h = O\left(h^\infty\right)$ *when* h *goes to* 0.

Lemma 3.1.2 *Assume* $a \in C_c^\infty(\mathbf{R})$, $\varphi \in C^\infty(\mathbf{R})$ *and let* $K := \operatorname{supp} a$. *If* $\varphi'(x) \neq 0$ *for every* $x \in K$, *then*

$$I_h = O\left(h^\infty\right),$$

as h *goes to* 0.

Proof. The aim of the proof is to obtain (3.3), by way of integration by parts. We define an operator L given by

$$[L(\cdot)](x) := \frac{h}{i}\frac{\partial_x(\cdot)(x)}{\varphi'(x)},$$

and this operator clearly acts as the identity on $e^{i\varphi(x)/h}$. We can define the formal adjoint of L, by using integration by parts

$$\begin{aligned}\langle L(f)\,, g\rangle &= \int_{\mathbf{R}} \frac{h}{i}\frac{\partial_x(f)(x)}{\varphi'(x)}\overline{g}(x)\,dx \\ &= \int_{\mathbf{R}} f(x) ih\partial_x\left(\frac{\overline{g}}{\varphi'}\right)(x)\,dx \\ &=: \langle f\,, L^*(g)\rangle,\end{aligned}$$

where $f \in C^\infty(\mathbf{R})$ and $g \in C_c^\infty(\mathbf{R})$, hence

$$[L^*(\cdot)](x) = (-ih)\partial_x\left(\frac{\cdot}{\left(\overline{\varphi'}\right)}\right)(x). \tag{3.4}$$

Thus by using the operator L multiple times in (3.2) and taking the absolute value, we get

$$|I_h| = \left|\int_K \left[L^N\left(e^{i\varphi(\cdot)/h}\right)\right](x)a(x)\,dx\right| = \left|\int_K e^{i\varphi(x)/h}\overline{[(L^*)^N(\overline{a})]}(x)\,dx\right|,$$

for every $N \in \mathbf{N}$. Hence by the triangle inequality and applying Lemma A.3.1 to L^*, we obtain

$$
\begin{aligned}
|I_h| &\leq \int_K \left| e^{i\varphi(x)/h} \overline{[(L^*)^N(\overline{a})]}(x) \right| dx \\
&\leq h^N \int_K \left| \sum_{k=0}^{N} a^{(k)}(x) \frac{A_{N,k}(x)}{(\varphi'(x))^{2N-j}} \right| dx \\
&\leq h^N (N+1)\mu(K) \max_{0\leq j\leq N} \left(\sup_{x\in K} \left| \frac{A_{N,j}(x)}{(\varphi'(x))^{2N-j}} \right| \right) \sum_{k=0}^{N} \sup_{x\in\mathbf{R}} |a^{(k)}(x)| \qquad (3.5) \\
&\leq h^N C_N,
\end{aligned}
$$

where we note that $A_{0,0} \equiv 1$, $A_{N,m} \equiv 0$ when $m < 0$ or when $m > N$, and $A_{N,m}(x) = A_{N-1,m-1}(x) + \varphi'(x)A'_{N-1,m}(x) + (m-2N-1)\varphi''(x)A_{N-1,m}(x)$. ■

Remark 3.1.3 As a consequence of (3.5) from the proof of Lemma 3.1.2, we see that the constant from (3.3) can be expressed as

$$C_N = \bar{C}_N \sum_{k=0}^{N} \sup_{x\in\mathbf{R}} |a^{(k)}(x)|.$$

The Definition 3.1.1 of one dimensional oscillatory integrals easily extends to higher dimensional spaces.

Definition 3.1.4 Let $h > 0$ be a real parameter, then the *oscillatory integral* is defined by

$$I_h(a,\varphi) := \int_{\mathbf{R}^n} e^{i\varphi(x)/h} a(x)\, dx,$$

for $a \in C_c^\infty(\mathbf{R}^n)$ and $\varphi \in C^\infty(\mathbf{R}^n)$.

Now we see that results of Lemma 3.1.2 and Remark 3.1.3, also extends nicely to higher dimensions.

Lemma 3.1.5 *Let $a \in C_c^\infty(\mathbf{R}^n)$ and $\varphi \in C^\infty(\mathbf{R}^n)$. If $\nabla\varphi(x) \neq 0$ for every $x \in$* supp(a)*, then*

$$I_h(a,\varphi) = O(h^\infty),$$

as $h \to 0$. Moreover, for each $N \in \mathbf{N}$

$$|I_h(a,\varphi)| \leq \bar{C}_N h^N \sum_{|\alpha|\leq N} \sup_{x\in\mathbf{R}^n} |(\partial^\alpha a)(x)|,$$

where $\bar{C}_N$ does only depend on supp(a)*, N and the derivatives of φ.*

This result follows by similar reasoning as in the proof of Lemma 3.1.2.

3.2 Stationary Phase

In the previous section we results where the first derivative of φ was non-zero, but in *stationary phase asymptotics* we study φ that has a non-degenerate critical point.

Theorem 3.2.1 *Let $a \in C_c^\infty(\mathbf{R})$, $\varphi \in C^\infty(\mathbf{R})$, $h > 0$ and $K \coloneqq \operatorname{supp}(a)$. Assume $x_0 \in K$ such that*

$$\varphi'(x_0) = 0, \quad \varphi''(x_0) \neq 0.$$

Furthermore, assume that φ' does not vanish on $K \setminus \{x_0\}$.

Then there exists differential operators $A_{2k}(x, D)$, of order less than or equal to $2k$, for $k \in \{0, 1, \dots\}$, such that for every $N \in \mathbf{N}$

$$\left| I_h - \Big(\sum_{j=0}^{N-1} [A_{2j} a] (x_0) h^{j+1/2} \Big) e^{i\varphi(x_0)/h} \right| \leq C_N h^{N+1/2} \sum_{m=0}^{2N+2} \sup_{x \in \mathbf{R}} |a^{(m)}(x)|.$$

Consequently, we see that

$$A_0 = (2\pi)^{1/2} |\varphi''(x_0)|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sgn} \varphi''(x_0)}$$

and thence

$$I_h = (2\pi h)^{1/2} |\varphi''(x_0)|^{-1/2} e^{i\frac{\pi}{4} \operatorname{sgn} \varphi''(x_0)} e^{i\varphi(x_0)/h} a(x_0) + O(h^{3/2})$$

when h goes to 0.

Proof. By Taylor's Formula A.2.3 we get

$$\varphi(x) = \varphi(x_0) + \varphi'(x_0)(x - x_0) + (x - x_0)^2 \int_0^1 (1 - t) \varphi''(x_0 + t(x - x_0)) \, dt,$$

and if we define

$$\Phi(x) \coloneqq 2 \int_0^1 (1 - t) \varphi''(x_0 + t(x - x_0)) \, dt,$$

then we can rewrite φ as

$$\varphi(x) = \varphi(x_0) + \frac{1}{2} \Phi(x)(x - x_0)^2. \tag{3.6}$$

An easy consequence of this is that $\varphi''(x_0) = \Phi(x_0)$.

We choose a characteristic function $\chi \in C^\infty(\mathbf{R})$ such that $\chi(x) \in [0, 1]$ for every $x \in \mathbf{R}$, $\chi \equiv 1$ in a small neighbourhood of x_0, and $\operatorname{sgn} \varphi''(x) = \operatorname{sgn} \varphi''(x_0) \neq 0$ on $\operatorname{supp}(\chi)$. Hence, we can rewrite the oscillatory integral

$$I_h = \int_{\mathbf{R}} e^{i\varphi(x)/h} a(x) \, dx$$

$$
\begin{aligned}
&= \int_{\mathbf{R}} e^{i\varphi(x)/h}\chi(x)a(x)\,dx + \int_{\mathbf{R}} e^{i\varphi(x)/h}(1-\chi(x))a(x)\,dx \\
&= e^{i\varphi(x_0)/h}\int_{\mathbf{R}} e^{i(\Phi(x)(x-x_0)^2)/2h}\chi(x)a(x)\,dx + O(h^\infty),
\end{aligned}
$$

when h goes to 0, by using (3.6) in the first integral and applying Lemma 3.1.2 to the second integral. For $x \in \operatorname{supp}(\chi)$ close to x_0 we make the change of variable

$$y(x) := |\Phi(x)|^{1/2}(x-x_0),$$

such that I_h becomes

$$
\begin{aligned}
I_h &= e^{i\varphi(x_0)/h}\int_{\mathbf{R}} e^{i(\Phi(x)(x-x_0)^2)/2h}\chi(x)a(x)\,dx + O(h^\infty) \\
&= e^{i\varphi(x_0)/h}\int_{\mathbf{R}} e^{i\frac{\operatorname{sgn}(\varphi''(x_0))}{2h}y^2}\chi(x(y))a(x(y))\Big|\frac{dx}{dy}\Big|\,dy + O(h^\infty),
\end{aligned}
$$

as h goes to 0. Now by defining $u(y) := \chi(x(y))a(x(y))|x'(y)|$ and noting that by the Inverse Function Theorem A.2.5 $u \in C^\infty(\mathbf{R})$, we are able to apply Plancherel's formula and get

$$
\begin{aligned}
I_h &= (2\pi)^{1/2}e^{i\varphi(x_0)/h}\int_{\mathbf{R}}\left[\mathcal{F}\left(e^{i\frac{\operatorname{sgn}(\varphi''(x_0))}{2h}y^2}\right)\right](\xi)\widehat{u}(\xi)\,d\xi + O(h^\infty) \\
&= \left(\frac{h}{2\pi}\right)^{1/2} e^{i\varphi(x_0)/h}e^{i\frac{\pi}{4}\operatorname{sgn}(\varphi''(x_0))}\int_{\mathbf{R}} e^{-i\frac{\operatorname{sgn}(\varphi''(x_0))h}{2}\xi^2}\widehat{u}(\xi)\,d\xi + O(h^\infty),
\end{aligned}
$$

when h goes to 0, which follows from Lemma 2.2.2. Defining a function J by

$$J(h,u) := \int_{\mathbf{R}} e^{-i\frac{\operatorname{sgn}(\varphi''(x_0))h}{2}\xi^2}\widehat{u}(\xi)\,d\xi,$$

our aim is to make a Taylor expansion of J and use it to make a norm estimate. Thus we look at the derivatives of J with respect to h and get the following identity

$$\partial_h J(h,u) = \int_{\mathbf{R}} e^{-i\frac{\operatorname{sgn}(\varphi''(x_0))h}{2}\xi^2}\left(\frac{-i\operatorname{sgn}(\varphi''(x_0))\xi^2}{2}\widehat{u}(\xi)\right)d\xi = J(h,Pu),$$

where $P(\cdot) := \frac{-i\operatorname{sgn}(\varphi''(x_0))}{2}\frac{d^2}{dy^2}(\cdot)$. Hence the Taylor expansion becomes

$$J(h,u) = \sum_{k=0}^{N-1}\frac{h^k}{k!}J(0,P^k u) + \frac{h}{N!}R_N(h,u),$$

and the remainder is given by $R_N(h,u) := N\int_0^1(1-t)^{N-1}J(th,P^N u)\,dt$. Now returning to I_h and subtracting the non-remainder terms from both sides we get

$$
\begin{aligned}
I_h - &\left(\sum_{k=0}^{N-1}\left(\frac{e^{\frac{i\pi}{4}\operatorname{sgn}(\varphi''(x_0))}}{(2\pi)^{1/2}k!}J(0,P^k u)\right)h^{k+1/2}\right)e^{i\varphi(x_0)/h} \\
&= h^{N+1/2}\frac{e^{i\varphi(x_0)/h}e^{\frac{i\pi}{4}\operatorname{sgn}(\varphi''(x_0))}}{(2\pi)^{1/2}N!}R_N(h,u) + O(h^\infty).
\end{aligned}
$$

Since we want a norm estimate of this we start by estimation of the remainder term

$$\begin{aligned}
|R_N(h,u)| &= \left| N \int_0^1 (1-t)^{N-1} J(th, P^N u) dt \right| \\
&\le N \int_0^1 (1-t)^{N-1} \int_{\mathbf{R}} \left| \left[\mathcal{F}(P^N u) \right] (\xi) \right| d\xi dt \\
&= \widetilde{C}_N \| \mathcal{F}(P^N u) \|_{L^1} \\
&\le \widetilde{C}_N \sum_{k=0}^{2} \sup_{x \in \mathbf{R}} \left| \partial^k \left[(P^N u)(x) \right] \right|,
\end{aligned}$$

where the last inequality follows from Lemma A.2.4. Hence, combining this estimate with Remark 3.1.3 we obtain

$$\begin{aligned}
&\left| I_h - \left(\sum_{k=0}^{N-1} \left(\frac{e^{\frac{i\pi}{4} \operatorname{sgn}(\varphi''(x_0))}}{(2\pi)^{1/2} k!} J(0, P^k u) \right) h^{k+1/2} \right) e^{i\varphi(x_0)/h} \right| \\
&\le h^{N+1/2} \frac{1}{(2\pi)^{1/2} N!} \widetilde{C}_N \left(\sum_{k=0}^{2} \sup_{x \in \mathbf{R}} \left| \partial^k \left[(P^N u)(x) \right] \right| \right) \\
&\quad + h^{2N+2} \bar{C}_{2N+2} \sum_{k=0}^{2N+2} \sup_{x \in \mathbf{R}} |\partial^k (a)(x)| \\
&\le (1 + h^{N+3/2}) h^{N+1/2} \widehat{C}_N \sum_{k=0}^{2N+2} \sup_{x \in \mathbf{R}} |\partial^k (a)(x)| \\
&\le h^{N+1/2} C_N \sum_{k=0}^{2N+2} \sup_{x \in \mathbf{R}} |\partial^k (a)(x)|,
\end{aligned}$$

since $h^{N+3/2} \le h \le 1$, for $h \in]0,1]$ and $N \in \mathbf{N}$, and thus the proof is finished. ■

Thus far we have looked at stationary phase asymptotics where $m = 2$ was the smallest natural number such that $\varphi^{(m)}(x_0) \neq 0$, but if we want to look further we need to study what happens for $m > 2$.

Theorem 3.2.2 *Let $a \in C_c^\infty(\mathbf{R})$, $\varphi \in C^\infty(\mathbf{R})$, $m > 2$, $h > 0$ and $K := \operatorname{supp}(a)$. Assume $x_0 \in K$ such that*

$$\varphi^{(i)}(x_0) = 0, \quad \varphi^{(m)}(x_0) \neq 0,$$

for every $i \in \{0, 1, \dots, m-1\}$ Furthermore, assume that $\varphi^{(i)}$ does not vanish on $K \setminus \{x_0\}$, for every $i \in \{0, 1, \dots, m-1\}$. Then there exists differential operators $A_k(x, D)$, of order less than or equal to k, for $k \in \{0, 1, \dots\}$, such that for every $N \in \mathbf{N}$, it holds

$$\left| I_h - \sum_{k=0}^{N-1} \left[A_k a \right] (x_0) h^{(k+1)/m} \right| \le C_N h^{(N+1)/m} \sum_{j=0}^{N} \sup_{x \in \mathbf{R}} |a^{(j)}(x)|.$$

In particular, A_0 is

$$
\begin{aligned}
A_0 = {} & \frac{1}{m}\int_0^1 \frac{e^{i\frac{\varphi^{(m)}(x_0)}{m!}\xi} + e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}\xi}}{\xi^{1-1/m}}\,d\xi \\
& + \frac{i(m-1)!}{\varphi^m(0)}\Bigg(e^{i\frac{\varphi^{(m)}(x_0)}{m!}} + (-1)^m e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}} \\
& + \left(\frac{1}{m}-1\right)\int_1^\infty \frac{e^{i\frac{\varphi^{(m)}(x_0)}{m!}\xi} + (-1)^m e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}\xi}}{\xi^{2-1/m}}\,d\xi\Bigg)
\end{aligned} \tag{3.7}
$$

and A_1 is given by

$$
\begin{aligned}
[A_1(a)](0) = {} & \left(a'(0) - a(0)\frac{2\varphi^{m+1}(0)}{(m^2+m)\varphi^m(0)}\right)\Bigg\{\frac{1}{m}\int_0^1 \frac{e^{i\frac{\varphi^{(m)}(x_0)}{m!}\xi} - e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}\xi}}{\xi^{1-2/m}}\,d\xi \\
& + \frac{i(m-1)!}{\varphi^m(0)}\Bigg(e^{i\frac{\varphi^{(m)}(x_0)}{m!}} + (-1)^{m+1} e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}} \\
& + \left(\frac{2}{m}-1\right)\int_1^\infty \frac{e^{i\frac{\varphi^{(m)}(x_0)}{m!}\xi} + (-1)^{m+1} e^{i(-1)^m\frac{\varphi^{(m)}(x_0)}{m!}\xi}}{\xi^{2-2/m}}\,d\xi\Bigg)\Bigg\}.
\end{aligned} \tag{3.8}
$$

PROOF. We assume that $x_0 = 0$ then by repeated use of the Fundamental Theorem of Calculus and Integration by Parts we get that

$$\varphi(x) = x\int_0^1 \varphi'(sx)\,ds = x^m\int_0^1 \frac{(1-s)^{m-1}}{(m-1)!}\varphi^{(m)}(sx)\,ds.$$

From this we define

$$y(x) := x\left(m\int_0^1 (1-s)^{m-1}\frac{\varphi^{(m)}(sx)}{\varphi^{(m)}(0)}\,ds\right)^{1/m}.$$

Note that $y'(0) = 1 \neq 0$, thus from the Inverse Function Theorem A.2.5 we get that there exist some $\tilde{x} \in C^\infty([-\delta,\delta])$, for small enough $\delta > 0$ such that $y(\tilde{x}(x)) = x$ and by the chain rule $\tilde{x}'(0) = 1$.

By introducing a characteristic function χ which is supported in the interval $[\tilde{x}(-\delta), \tilde{x}(\delta)]$, with similar properties as the characteristic function from the proof of Theorem 3.2.1, then utilizing Lemma 3.1.2, as in Theorem 3.2.1, the oscillatory integral becomes

$$I_h = \int_{\mathbf{R}} e^{iy^m\varphi^{(m)}(0)/m!h} g(y)\,dy + O(h^\infty), \tag{3.9}$$

when h goes to 0, where $g(y) := \chi(\tilde{x}(y))a(\tilde{x}(y))\tilde{x}'(y) \in C_c^\infty(\mathbf{R})$ with $\operatorname{supp}(g) \subset [-\delta,\delta]$. We see that the integral can be rewritten as

$$
\begin{aligned}
\int_{\mathbf{R}} e^{iy^m\varphi^{(m)}(0)/m!h} g(y)\,dy = {} & \int_0^\infty e^{iy^m\varphi^{(m)}(0)/m!h} g(y)\,dy \\
& + \int_0^\infty e^{i(-1)^m y^m\varphi^{(m)}(0)/m!h} g(-y)\,dy.
\end{aligned}
$$

Letting $K_m := \varphi^{(m)}(0)/m!$, $\widetilde{K_m} := (-1)^m\varphi^{(m)}(0)/m!$ and $\widetilde{g}(y) := g(-y)$ we define a functions

$$F(h) := \int_0^\infty e^{iy^m K_m/h} g(y)\, dy, \tag{3.10}$$

and

$$\widetilde{F}(h) := \int_0^\infty e^{iy^m \widetilde{K}_m/h} \widetilde{g}(y)\, dy. \tag{3.11}$$

First we look at $F(h)$ by making a change of variable $\xi = \frac{y^m}{h}$ in (3.10) and get

$$\begin{aligned} F(h) &= \frac{h^{1/m}}{m}\int_0^\infty e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi \\ &= \frac{h^{1/m}}{m}\int_0^1 e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi + \frac{h^{1/m}}{m}\int_1^\infty e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi \\ &=: F_1(h) + F_2(h). \end{aligned}$$

We observe that $F_1(h)$ makes sense for all m when h goes to 0, thus to ensure that the limit $F(h)$ exists when h goes to 0 we write the integrand as

$$\lim_{\varepsilon\to 0^+} e^{[iK_m-\varepsilon]\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}$$

and observe that for any fixed ε this is $L^1([0,\infty])$. Hence we get

$$\frac{m}{h^{1/m}}F(h) = \int_0^1 e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi + \lim_{\varepsilon\to 0^+}\int_1^\infty e^{[iK_m-\varepsilon]\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi.$$

To find a limit function that is $L^1([1,\infty])$ rewrite the exponential function of the second integral and then apply Integration by Parts

$$\begin{aligned} \frac{m}{h^{1/m}}F_2(h) &= \lim_{\varepsilon\to 0^+}\int_1^\infty \frac{\frac{d}{d\xi}\left(e^{[iK_m-\varepsilon]\xi}\right)}{iK_m-\varepsilon}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi \\ &= \lim_{\varepsilon\to 0^+}\left[\frac{-1}{iK_m-\varepsilon}e^{iK_m-\varepsilon}g(h^{1/m}) - \int_1^\infty \frac{e^{[iK_m-\varepsilon]\xi}}{iK_m-\varepsilon}\frac{d}{d\xi}\left(\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi\right)\right] \\ &= \frac{i}{K_m}\Bigg[e^{iK_m}g(h^{1/m}) \\ &\qquad + \int_1^\infty e^{iK_m\xi}\left(\frac{h^{1/m}}{m}\frac{g'(\xi^{1/m}h^{1/m})}{\xi^{2-2/m}} + \left(\frac{1}{m}-1\right)\frac{g(\xi^{1/m}h^{1/m})}{\xi^{2-1/m}}\right)d\xi\Bigg]. \end{aligned}$$

Now it is possible to take the limit because both $2-1/m$ and $2-2/m$ are greater than 1 since $m>2$, and thus the limit function is $L^1([1,\infty])$ and we have

$$\begin{aligned} \frac{m}{h^{1/m}}F(h) &= \int_0^1 e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\, d\xi + \frac{i}{K_m}e^{ik_m}g(h^{1/m}) \\ &\quad + \frac{i}{K_m}\int_1^\infty e^{iK_m\xi}\left(\frac{h^{1/m}}{m}\frac{g'(\xi^{1/m}h^{1/m})}{\xi^{2-2/m}} + \left(\frac{1}{m}-1\right)\frac{g(\xi^{1/m}h^{1/m})}{\xi^{2-1/m}}\right)d\xi. \end{aligned} \tag{3.12}$$

We want to find differential operators B_k for each $k \in \mathbf{N}_0$ thus we start the limit of h to 0 in (3.12) and obtain B_0:

$$\lim_{h\to 0+}\left(\frac{m}{h^{1/m}}F(h)\right) = \int_0^1 e^{iK_m\xi}\frac{g(0)}{\xi^{1-1/m}}\,d\xi + \frac{i}{K_m}e^{iK_m}g(0)$$
$$+\frac{i}{K_m}\int_1^\infty e^{iK_m\xi}\left(\frac{1}{m}-1\right)\frac{g(0)}{\xi^{2-1/m}}\,d\xi$$
$$=: m[B_0(g)](0).$$

To find the general B_k we define

$$J_\gamma(\xi,h) := \sum_{j=0}^{\gamma-1}\frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{\gamma-(1+j)(m)}}\left(\frac{h^{1/m}}{m}\right)^j C_{\gamma,j},$$

where $C_{\gamma,j}$ is defined as

$$C_{\gamma,j} := \begin{cases}\displaystyle\sum_{E_{\gamma,j}}\prod_{p=1}^{\gamma-j-1}\left(\frac{x_p-p+1}{m}-x_p\right) & \text{for } j<\gamma-1\\ 1 & \text{for } j=\gamma-1,\end{cases}$$

and the set $E_{\gamma,j} = \{(x_1,\dots,x_{\gamma-j-1}) \in \mathbf{Z}^{\gamma-j-1} \mid 0 < x_1 < \dots < x_{\gamma-j-1} < \gamma\}$. We do this to write $F_2(h)$ as

$$\frac{m}{h^{1/m}}F_2(h) = \int_1^\infty e^{iK_m\xi}J_1(\xi,h)\,d\xi,$$

in order to ease calculations. Observing that

$$e^{iK_m\xi} = \left(\frac{-i}{K_m}\right)^q \partial_\xi^q(e^{iK_m\xi}),$$

we apply integration by parts q times to obtain

$$\int_1^\infty\left(\frac{-i}{K_m}\right)^q\partial_\xi^q(e^{iK_m\xi})J_1(\xi,h)\,d\xi$$
$$= e^{iK_m}\sum_{\gamma=1}^q\left[\left(\frac{i}{K_m}\right)^\gamma\partial_\xi^{\gamma-1}(J_1(1,h))\right] + (-1)^q\left(\frac{-i}{K_m}\right)^q\int_1^\infty e^{iK_m\xi}\partial_\xi^q\left(J_1(\xi,h)\right)\,d\xi$$
$$= e^{iK_m}\sum_{\gamma=1}^q\left[\left(\frac{i}{K_m}\right)^\gamma(J_\gamma(1,h))\right] + \left(\frac{i}{K_m}\right)^q\int_1^\infty e^{iK_m\xi}J_{q+1}(\xi,h)\,d\xi,$$

where the last equality follows from Lemma A.3.2. By multiplying both sides of $F(h)$ in (3.12) with $h^{1/m}m^{-1}$ we see that

$$F(h) = \frac{h^{1/m}}{m}\left\{\int_0^1 e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\,d\xi + e^{iK_m}\sum_{\gamma=1}^q\left[\left(\frac{i}{K_m}\right)^\gamma(J_\gamma(1,h))\right]\right.$$
$$\left.+\left(\frac{i}{K_m}\right)^q\int_1^\infty e^{iK_m\xi}J_{q+1}(\xi,h)\,d\xi\right\}$$

$$
\begin{aligned}
&= \frac{h^{1/m}}{m}\Bigg\{\int_0^1 e^{iK_m\xi}\frac{g(\xi^{1/m}h^{1/m})}{\xi^{1-1/m}}\,d\xi \\
&\quad + e^{iK_m}\sum_{\gamma=1}^{q}\left[\left(\frac{i}{K_m}\right)^{\gamma}\sum_{j=0}^{\gamma-1}C_{\gamma,j}\left(\frac{h^{1/m}}{m}\right)^{j} g^{(j)}(h^{1/m})\right] \\
&\quad + \left(\frac{i}{K_m}\right)^{q}\int_1^{\infty} e^{iK_m\xi}\sum_{j=0}^{q}C_{q+1,j}\left(\frac{h^{1/m}}{m}\right)^{j}\frac{g^{(j)}((\xi h)^{1/m})}{\xi^{q+1-(1+j)/m}}\,d\xi\Bigg\}.
\end{aligned}
$$

Choosing $(N+1)/m < q \leq N$ then by Taylor's Formula A.2.3 for every derivative of g of at most degree $N-1$ we get

$$
\begin{aligned}
F(h) = \frac{h^{1/m}}{m}\Bigg\{&\int_0^1 e^{iK_m\xi}\frac{\sum_{w=0}^{N-1}\frac{\xi^{w/m}h^{w/m}}{w!}g^{(w)}(0)}{\xi^{1-1/m}}\,d\xi \\
&+ e^{iK_m}\sum_{\gamma=1}^{q}\sum_{j=0}^{\gamma-1}\frac{1}{m^j}\left(\frac{i}{K_m}\right)^{\gamma}C_{\gamma,j}\left(\sum_{w=j}^{N-1}\frac{1}{(w-j)!}g^{(w)}(0)h^{w/m}\right) \\
&+ \left(\frac{i}{K_m}\right)^{q}\sum_{j=0}^{q}\frac{1}{m^j}C_{q+1,j}\sum_{w=j}^{N-1}\left(\frac{1}{(w-j)!}\int_1^{\infty}\frac{e^{iK_m\xi}}{\xi^{q+1-(w+1)/m}}\,d\xi g^{(w)}(0)h^{w/m}\right) \\
&+ \int_0^1 e^{iK_m\xi}\frac{\frac{N}{N!}\xi^{N/m}h^{N/m}}{\xi^{1-1/m}}\int_0^1(1-s)^{N-1}g^{(N)}(s\xi^{1/m}h^{1/m})\,ds\;\;d\xi \\
&+ e^{iK_m}\sum_{\gamma=1}^{q}\sum_{j=0}^{\gamma-1}\frac{1}{m^j}\left(\frac{i}{K_m}\right)^{\gamma}C_{\gamma,j}\frac{N-j}{(N-j)!} \\
&\quad\cdot\left(\int_0^1(1-s)^{N-j-1}g^{(N)}(sh^{1/m})\,ds\right)h^{N/m} \\
&+ \left(\frac{i}{K_m}\right)^{q}\sum_{j=0}^{q}\frac{1}{m^j}C_{q+1,j}\frac{N-j}{(N-j)!} \\
&\quad\cdot\int_1^{\infty}\int_0^1(1-s)^{N-j-1}g^{(N)}(s(\xi h)^{1/m})\frac{e^{iK_m\xi}}{\xi^{q+1-(N+1)/m}}\,ds\,d\xi\;h^{N/m}\Bigg\}.
\end{aligned}
$$

From this we define B_k to be given by

$$
\begin{aligned}
[B_k(g)](0) :=&\frac{1}{m}\int_0^1\frac{e^{iK_m\xi}}{\xi^{1-(k+1)/m}}\,d\xi\frac{g^{(k)}(0)}{k!} \\
&+ e^{iK_m}\sum_{\gamma=1}^{q}\left(\frac{i}{K_m}\right)^{\gamma}\left(\sum_{j=0}^{\min(k,\gamma-1)}\frac{C_{\gamma,j}}{(k-j)!m^{j+1}}\right)g^{(k)}(0) \\
&+ \left(\frac{i}{K_m}\right)^{q}\left(\sum_{j=0}^{\min(k,q)}\frac{C_{q+1,j}}{(k-j)!m^{j+1}}\right)\int_1^{\infty}\frac{e^{iK_m\xi}}{\xi^{q+1-(1+k)/m}}\,d\xi g^{(k)}(0).
\end{aligned}
$$

Note that for $q=1$ the operator $[B_0(g)](0)$ given by these formulas is the same as the one derived earlier.

To see that these operators has the desired properties we make a norm estimate of $F(h)$ where we subtract a sum of these operators

$$\left|F(h) - \sum_{k=0}^{N-1}[B_k(g)](0)h^{(k+1)/m}\right| \leq h^{(N+1)/m}C_1 \sup_{x\in\mathbf{R}}|\sum_{k=0}^{N} a^{(k)}(x)|,$$

which follows from the fact that g is compactly supported. By similar arguments for $\widetilde{F}(h)$ from (3.11) we can find differential operators $\widetilde{B}_k$ given by

$$\begin{aligned}\left[\widetilde{B}_k(\widetilde{g})\right](0) :=&(-1)^k\Bigg\{\frac{1}{m}\int_0^1 \frac{e^{i\widetilde{K}_m\xi}}{\xi^{1-(k+1)/m}}\,d\xi\frac{g^{(k)}(0)}{k!}\\ &+ e^{i\widetilde{K}_m}\sum_{\gamma=1}^{q}\left(\frac{i}{\widetilde{K}_m}\right)^{\gamma}\left(\sum_{j=0}^{\min(k,\gamma-1)}\frac{C_{\gamma,j}}{(k-j)!m^{j+1}}\right)g^{(k)}(0)\\ &+\left(\frac{i}{\widetilde{K}_m}\right)^{q}\left(\sum_{j=0}^{\min(k,q)}\frac{C_{q+1,j}}{(k-j)!m^{j+1}}\right)\int_1^\infty \frac{e^{i\widetilde{K}_m\xi}}{\xi^{q+1-(1+k)/m}}\,d\xi g^{(k)}(0)\Bigg\},\end{aligned}$$

hence we have

$$\left|\widetilde{F}(h) - \sum_{k=0}^{N-1}[\widetilde{B}_k(\widetilde{g})](0)h^{(k+1)/m}\right| \leq h^{(N+1)/m}\widetilde{C}_1\sum_{k=0}^{N}\sup_{x\in\mathbf{R}}|a^{(k)}(x)|.$$

Since B_k and $\widetilde{B}_k$ are all linear operators we define $A_k := B_k + \widetilde{B}_k$, looking at the estimate of the difference between the oscillatory integral I_h and an N-term sum of A_k by use of the two previous estimates and Remark 3.1.3

$$\begin{aligned}&\left|I_h - \sum_{k=0}^{N-1}[A_k(g)](0)h^{(k+1)/m}\right|\\ &\quad= \left|F(h) + \widetilde{F}(h) + O(h^\infty) - \sum_{k=0}^{N-1}\Big([B_k(g)](0) + [\widetilde{B}_k(\widetilde{g})](0)\Big)\,h^{(k+1)/m}\right|\\ &\quad\leq \left|F(h) - \sum_{k=0}^{N-1}[B_k(g)](0)h^{(k+1)/m}\right| + \left|\widetilde{F}(h) - \sum_{k=0}^{N-1}[\widetilde{B}_k(\widetilde{g})](0)h^{(k+1)/m}\right|\\ &\qquad + |O(h^\infty)|\\ &\quad\leq \Big((C_1+\widetilde{C}_1)h^{(N+1)/m} + \bar{C}_N h^N\Big)\sum_{k=0}^{N}\sup_{x\in\mathbf{R}}|a^{(k)}(x)|\\ &\quad\leq (C_1+\widetilde{C}_1+\bar{C}_N)h^{(N+1)/m}\sum_{k=0}^{N}\sup_{x\in\mathbf{R}}|a^{(k)}(x)|,\end{aligned}$$

since for $h\in\,]0,1]$ then $h^N \leq h^{(N+1)/m}$, and hence the proof is done. ■

The idea behind this proof is different from the proof of Theorem 3.2.1. If instead we had used this approach from (3.9) onward for m even, we could utilize Theorem 2.2.1 to get

$$\begin{aligned} I_h &= \int_{\mathbf{R}} e^{iy^m \varphi^{(m)}(0)/m!h} g(y)\, dy + O(h^\infty) \\ &= \frac{1}{2\pi} \sum_{k=0}^{m-2} a_k h^{(k+1)/m} \int_{\mathbf{R}} \xi^k \, {}_0F_{m-2}\left(; M_{m-k}; \frac{h\xi^m}{K_m (i)^{m+3} m^m}\right) \mathcal{F}\left(g(y)\right)(\xi)\, d\xi \\ &\quad + O(h^\infty), \end{aligned}$$

where $M_{m-k} = \left\{\frac{k+s}{m} \in \mathbf{Q} \mid \forall s \in \{2, 3, .., m\} \text{ such that } s \neq m-k\right\}$ and

$$a_k = (-i)^k \frac{2}{m(k!)(-iK_m)^{(k+1)/m}} \Gamma\left(\frac{k+1}{m}\right).$$

By defining

$$J_k(h, g) := \int_{\mathbf{R}} \xi^k \, {}_0F_{m-2}\left(; M_{m-k}; \frac{h\xi^m}{K_m (i)^{m+3} m^m}\right) \mathcal{F}\left(g(y)\right)(\xi)\, d\xi$$

we want to use Taylor's Formula A.2.3 for h at 0 then for each k we get

$$J_k(h, g) = \sum_{j=0}^{N-1} \left(\frac{h^j}{j!} \partial_h^j (J_k)(0, g)\right) + \frac{N}{N!} h^N R_k(h, g),$$

where $R_k(h, g) = \int_0^1 (1-t)^{N-1} \partial_h^N (J_k)(th, g)\, dt$ and $\partial_h^j (J_k)$ is given by

$$\begin{aligned} \partial_h^j (J_k)(h, g) = \int_{\mathbf{R}} &\left(\frac{1}{i^{m+3} m^m K_m}\right)^j \left(\prod_{\substack{s=2 \\ s \neq m-k}}^{m} \left(\frac{k+s}{m}\right)_j \right)^{-1} \\ &\cdot \, {}_0F_{m-2}\left(; M_{m-k} + j; \frac{h\xi^m}{K_m (i)^{m+3} m^m}\right) \\ &\cdot \xi^{mj+k} \mathcal{F}\left(g(y)\right)(\xi)\, d\xi. \end{aligned}$$

We observe that $\partial_h^j (J_k)(0, g)$ simplifies to

$$\begin{aligned} \partial_h^j (J_k)(0, g) &= \left(\frac{1}{i^{m+3} m^m K_m}\right)^j \left(\prod_{\substack{s=2 \\ s \neq m-k}}^{m} \left(\frac{k+s}{m}\right)_j \right)^{-1} \int_{\mathbf{R}} \xi^{mj+k} \mathcal{F}\left(g(y)\right)(\xi)\, d\xi \\ &= 2\pi \left(\frac{1}{i^{m+3} m^m K_m}\right)^j \left(\prod_{\substack{s=2 \\ s \neq m-k}}^{m} \left(\frac{k+s}{m}\right)_j \right)^{-1} D^{mj+k}\left(g\right)(0). \end{aligned}$$

This gives that A_0 and A_1 are given by $a_0 \partial_h^j (J_0)(0, g)/j!$ and $a_1 \partial_h^j (J_1)(0, g)/j!$ respectively, for $j = 0$ since $m > 2$. First of we see that $A_1 = 0$ because $a_1 = 0$, which agrees with the formula for A_1 from (3.8), when m is even. To see that A_0 agrees with (3.7) we use Lemma 2.1.1 to get

$$\begin{aligned} a_0(J_0)(0,g) &= \frac{2\Gamma\left(\frac{1}{m}\right)}{m(-iK_m)^{1/m}} g(0) \\ &= \frac{2}{m(-iK_m)^{1/m}} \int_0^\infty t^{1/m-1} e^{-t}\, dt\; a(0) \\ &= \frac{2}{m} \int_0^\infty \frac{e^{iK_m \xi}}{\xi^{1-1/m}}\, d\xi a(0), \end{aligned}$$

where the last integral is the exact integral from (3.10) that via integration by parts gave us B_0. By noting that $\tilde{B_0} = B_0$ for all even $m > 2$, and that $A_0 = B_0 + \tilde{B_0}$, we know that this A_0 agrees with (3.7).

But the reason why we can not utilize this technique is that we are unable to find bounds for any of the remainder terms.

3.2.1 Higher dimensions

In order to study the higher dimensional case we have introduce Morse Lemma, which will be useful later.

Theorem 3.2.3 (Morse Lemma) *Suppose that $\varphi \in C^\infty(\mathbf{R}^n)$ has a critical point at x_0, where $\det H_\varphi(x_0) \neq 0$. Then there exist neighborhoods U and V of 0 and x_0 respectively, and a diffeomorphism $\kappa\colon V \to U$ such that.*

$$\left(\varphi \circ \kappa^{-1}\right)(x) = \varphi(x_0) + \frac{1}{2}\Big(\sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{n} x_i^2\Big),$$

where r is the number of positive eigenvalues of $\det H_\varphi(x_0)$.

Proof. After a linear change of variables B and a translation τ_{-x_0} we can define

$$h(x) := \varphi(\tau_{-x_0}(B(x))) - \varphi(x_0) \tag{3.13}$$

and then obtain

$$h(x) = \frac{1}{2}\Big(\sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{n} x_i^2\Big) + O(|x|^3).$$

By Taylor's Formula it follows that

$$\begin{aligned} h(x) &= \int_0^1 \partial_t(h(tx))\, dt \\ &= \int_0^1 (1-t)\partial_t^2(h(tx))\, dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (1-t) x^T H_h(tx) x \, dt \\
&= \frac{1}{2} \langle x \, , Q(x) x \rangle,
\end{aligned}
$$

where $Q(x) = 2\int_0^1 (1-t) H_h(tx)\, dt$ and

$$
Q(0) = \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}.
$$

Let $\varepsilon > 0$ and $W \subset \mathbf{R}^n$ be an open neighborhood of 0 then we will search for a smooth function $A \colon \mathbf{R}^n \to \mathbb{S}^{n\times n}$ such that $A(0) = I$ and satisfies

$$
\langle x \, , Q(x) x \rangle = \langle A(x) x \, , Q(0) A(x) x \rangle,
$$

for every $x \in W$ and then define a smooth function

$$
\tilde{\kappa}(x) = A(x) x. \tag{3.14}
$$

To find this A it suffices to show that

$$
Q(x) = A^T(x) Q(0) A(x), \tag{3.15}
$$

hence we suppose $F : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$, given by $F(A) = A^T Q(0) A$, is C^1. We want to apply the the Inverse Function Theorem A.2.5 to find an inverse G such that

$$
F \circ G = I.
$$

Thus to apply the theorem we need to find a mapping $C \in \mathfrak{B}(\mathbb{S}^{n\times n}, \mathbb{S}^{n\times n})$ such that

$$
\partial F(I) C = I,
$$

near Q(0). Note that each element of $F(A)$ can be written as

$$
[F(A)]_{i,j} = \sum_{s=1}^n \sum_{t=1}^n a_{t,i} q_{t,s} a_{s,j} = \sum_{k=1}^n \eta_k a_{k,i} a_{k,j},
$$

where $\eta_k = \begin{cases} 1 & \text{if } k \le r \\ -1 & \text{if } k \ge r+1 \end{cases}$. Thus looking at the derivatives element-wise we get

$$
\frac{\partial \, [F(A)]_{i,j}}{\partial a_{k,l}} = [\partial F(A)]_{(i,j),(k,l)} = \begin{cases} 0 & \text{if } l \ne j \text{ and } l \ne i \\ \eta_k a_{k,j} & \text{if } l = i \ne j \\ \eta_k a_{k,i} & \text{if } l = j \ne i \\ 2\eta_k a_{k,i} & \text{if } l = i = j. \end{cases}
$$

Taking $B \in \mathbb{S}^{n\times n}$, we get

$$\begin{aligned}
[\partial F(A)B]_{i,j} &= \sum_{k=1}^{n}\sum_{l=1}^{n} [\partial F(A)]_{(i,j),(k,l)}\, b_{k,l} \\
&= \sum_{k=1}^{n} [\partial F(A)]_{(i,j),(k,i)}\, b_{k,i} + \sum_{k=1}^{n} [\partial F(A)]_{(i,j),(k,j)}\, b_{k,j} \\
&= \sum_{k=1}^{n} \eta_k (a_{k,j} b_{k,i} + a_{k,i} b_{k,j}).
\end{aligned}$$

Thus if we let $A = I$ we obtain

$$[\partial F(I)B]_{i,j} = \eta_j b_{j,i} + \eta_i b_{i,j}$$

Furthermore, we have that

$$\begin{aligned}
\left[B^T Q(0)\right]_{i,j} &= \sum_{k=1}^{n} b_{l,i} q_{l,j} = \eta_j b_{j,i} \\
[Q(0)B]_{i,j} &= \sum_{k=1}^{n} q_{i,l} b_{l,j} = \eta_i b_{i,j}
\end{aligned}$$

we can conclude that $\partial F(I)B = B^T Q(0) + Q(0)B$. Defining

$$C(S) \coloneqq \frac{1}{2} Q^{-1}(0) S,$$

for $S \in \mathbb{S}^{n\times n}$, we get

$$\partial F(I)C(S) = \frac{1}{2}\left((Q^{-1}(0)S)^T Q(0) + Q(0)(Q^{-1}(0)S)\right) = S.$$

Thence we have satisfied the conditions for the Inverse Function Theorem A.2.5, such that there exists a $G \in C^\infty(\mathbb{S}^{n\times n}, \mathbb{S}^{n\times n})$ such that

$$F(G) = id,$$

in an open neighborhood of $Q(0)$, denoted as Ω_Q. We can find an $\varepsilon > 0$ such that for every $x \in B_\varepsilon(0)$, $Q(x)$ belongs to Ω_Q. Thus we can define

$$A(x) \coloneqq G(Q(x)),$$

for every $x \in B_\varepsilon(0)$, which gives that

$$F(A(x)) = F(G(Q(x))) = Q(x),$$

but from the definition of F we that $F(A(x)) = A^T(x)Q(0)A(x)$ and thus we have (3.15), note that because $Q \in C^\infty(B_\varepsilon(0), \Omega_Q)$ and $G \in C^\infty(\Omega_Q, G(\Omega_Q))$ we get that $A \in C^\infty(B_\varepsilon(0), G(\Omega_Q))$, and $A(0) = I$.

Lastly, to show the existence of the diffeomorphism κ we start by showing that $\tilde{\kappa}$, which defined by (3.14), has a smooth inverse. We have to know that it is invertible at 0, thus we differentiate the component functions of $\tilde{\kappa}$

$$[D\tilde{\kappa}(x)]_{t,s} = \Big(\sum_{i=1}^{n} \partial_s(a_{t,i}(x))x_i\Big) + a_{t,s}(x),$$

hence evaluation at 0 gives that

$$D\tilde{\kappa}(0) = A(0) = I,$$

thus it is invertible at 0. From the Inverse Function Theorem A.2.5 we now know that there exist open sets $\tilde{U}$ and $\tilde{V}$ in $\mathbf{R}^n$ such that $\tilde{V} \subset W$ and

$$\tilde{\kappa}^{-1} \colon \tilde{U} \to \tilde{V}$$

is in C^∞. Recalling (3.13) we have

$$\varphi(\tau_{-x_0}(B(\tilde{\kappa}^{-1}(x)))) - \varphi(x_0) = h(\tilde{\kappa}^{-1}(x)) = \frac{1}{2}\langle x\,, Q(0)x\rangle$$

which implies that

$$\varphi(\tau_{-x_0}(B(\tilde{\kappa}^{-1}(x)))) = \frac{1}{2}\langle x\,, Q(0)x\rangle + \varphi(x_0).$$

Choosing $U \coloneqq \tilde{U}$ and $V \coloneqq \tau_{-x_0}(B(\tilde{\kappa}^{-1}(\tilde{V})))$ we can now define a smooth function $\kappa^{-1} \colon U \to V$ given by

$$\kappa^{-1}(x) \coloneqq \tau_{-x_0}(B(\tilde{\kappa}^{-1}(x))).$$

Since it is defined by the invertible functions B, τ_{x_0} and $\tilde{\kappa}^{-1}$ we are able to define its inverse κ as

$$\kappa(x) \coloneqq \tilde{\kappa}(B^{-1}(\tau_{x_0}(x))),$$

hence the proof is done. ∎

Having proven Morse Lemma we can go on to the higher dimensional case of stationary phase asymptotics, but we return to the case $m = 2$.

Theorem 3.2.4 *Suppose that $a \in C_c^\infty(\mathbf{R}^n)$, $\varphi \in C^\infty(\mathbf{R}^n)$, $h > 0$ and let $K \coloneqq \operatorname{supp}(a)$. Assume that there exists $x_0 \in K$ such that $\nabla\varphi(x_0) = 0$, $H_\varphi(x_0) \neq 0$, and $\nabla\varphi(x) \neq 0$ for $x \in K \setminus \{x_0\}$.*

Then for each $k \in \mathbf{N}_0$ there exists differential operators $A_{2k}(x, D)$ of at most order $2k$, such that for every N:

$$\left| I_h - \Big(\sum_{j=0}^{N-1} [A_{2j}a]\,(x_0)h^{j+n/2}\Big)e^{i\varphi(x_0)/h} \right| \leq C_N h^{N+n/2} \sum_{|\alpha|\leq 2N+n+1} \sup_{x\in\mathbf{R}^n} |\partial^\alpha a(x)|.$$

Moreover,

$$A_0 = (2\pi)^{n/2} |\det H_\varphi(x_0)|^{-1/2} e^{i\pi \operatorname{sgn}(H_\varphi(x_0))/4}.$$

Proof. Without loss of generality we may assume that $\varphi(x_0) = 0$. Let $\chi \in C_c^\infty(\mathbf{R}^n)$ such that $\chi(x) = 1$ when $|x| \le 1$, $\chi(x) = 0$ when $|x| > 2$, and $0 < \chi(x) \le 1$ for $|x| \in]1, 2]$. Writing the oscillatory integral, $I_h \coloneqq I_h(a, \varphi)$, we have by Lemma 3.1.5

$$I_h = \int_{\mathbf{R}^n} e^{i\varphi(x)/h} a(x) dx = \int_{\mathbf{R}^n} e^{i\varphi(x)/h} \chi(x) a(x) dx + O(h^\infty),$$

as h goes to 0. By applying the change of variable $x \mapsto \kappa^{-1}(x)$, then by Theorem 3.2.3 (Morse Lemma) we get

$$I_h = \int_{\mathbf{R}^n} e^{i\langle Qx\,,x\rangle/2h} u(x) dx + O(h^\infty),$$

as h goes to 0, where $Q \coloneqq diag(\eta_1, \eta_2, \dots, \eta_n)$, for $\eta_k = \begin{cases} 1 & \text{if } k \le r \\ -1 & \text{if } k \ge r+1 \end{cases}$, where r is the number of positive eigenvalues of $H_\varphi(x_0)$,
and $u(x) \coloneqq \chi(\kappa^{-1}(x)) a(\kappa^{-1}(x)) |\det{(D\kappa^{-1})(x)}|$, hence $u \in C_c^\infty(\mathbf{R}^n)$. Thus by noting that $\operatorname{sgn} Q = \operatorname{sgn} H_\varphi(x_0)$ and that $|\det Q| = 1$ we can apply Lemma 2.2.2 and Plancherel's Formula to obtain

$$I_h = \left(\frac{h}{2\pi}\right)^{n/2} e^{i\pi \operatorname{sgn} Q/4} \int_{\mathbf{R}^n} e^{ih\langle Q^{-1}\xi\,,\xi\rangle/2} \widehat{u}(\xi)\, d\xi + O(h^\infty), \tag{3.16}$$

as h goes to 0. Defining

$$J(h, u) \coloneqq \int_{\mathbf{R}^n} e^{ih\langle Q^{-1}\xi\,,\xi\rangle/2} \widehat{u}(\xi)\, d\xi,$$

then by integration by parts we have

$$\begin{aligned} \partial_h J(h,u) &= \int_{\mathbf{R}^n} \partial_h \left(e^{ih\langle Q^{-1}\xi\,,\xi\rangle/2}\right) \widehat{u}(\xi)\, d\xi \\ &= \int_{\mathbf{R}^n} e^{ih\langle Q^{-1}\xi\,,\xi\rangle/2} \left(-\frac{i}{2}\langle Q^{-1}\xi\,,\xi\rangle \widehat{u}(\xi)\right) d\xi \\ &= J(h, P(u)), \end{aligned}$$

where $P(\cdot) = -\frac{i}{2}\langle Q^{-1}D_x\,, D_x(\cdot)\rangle$. Thus by using Taylor's Formula A.2.3 for h near 0:

$$J(h,u) = \sum_{k=0}^{N-1} \frac{h^k}{k!} J(0, P^k u) + \frac{h}{N!} R_N(h,u),$$

and the remainder is given by $R_N(h,u) \coloneqq N \int_0^1 (1-t)^{N-1} J(th, P^N u)\, dt$. We want a norm estimate of this we start by norm estimating the remainder term

$$|R_N(h,u)| = \left| N \int_0^1 (1-t)^{N-1} J(th, P^N u) dt \right|$$

$$\leq N \int_0^1 (1-t)^{N-1} \int_{\mathbf{R}^n} \left| \left[\mathcal{F}(P^N u) \right] (\xi) \right| d\xi dt$$
$$= \widetilde{C}_N \|\mathcal{F}(P^N u)\|_{L^1}$$
$$\leq \widetilde{C}_N \sum_{|\alpha| \leq n+1} \sup_{x \in \mathbf{R}^n} \left| \partial^\alpha \left[(P^N u)(x) \right] \right|,$$

where the last inequality follows from Lemma A.2.4. By subtracting the non-remainder terms from both sides of (3.16) and making a norm estimate we get

$$\left| I_h - \left(\sum_{k=0}^{N-1} \left(\frac{e^{\frac{i\pi}{4} \operatorname{sgn}(H_\varphi(x_0))}}{(2\pi)^{n/2} k!} J(0, P^k u) \right) h^{k+n/2} \right) \right|$$
$$\leq h^{N+n/2} \frac{1}{(2\pi)^{n/2} N!} \widetilde{C}_N \left(\sum_{|\alpha| \leq n+1} \sup_{x \in \mathbf{R}^n} \left| \partial^\alpha \left[(P^N u)(x) \right] \right| \right)$$
$$+ h^{2N+n+1} \bar{C}_{2N+n+1} \sum_{|\alpha| \leq 2N+n+1} \sup_{x \in \mathbf{R}^n} |\partial^\alpha (a)(x)|$$
$$\leq (1 + h^{N+(n+2)/2}) h^{N+n/2} \widehat{C}_N \sum_{|\alpha| \leq 2N+n+1} \sup_{x \in \mathbf{R}^n} |\partial^\alpha (a)(x)|$$
$$\leq h^{N+n/2} C_N \sum_{|\alpha| \leq 2N+n+1} \sup_{x \in \mathbf{R}^n} |\partial^\alpha (a)(x)|,$$

hence the proof is done. ■

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A. Apendix

In this Appendix we will state results that doesn't have to directly link up with the themes of the report, but are needed. Most of these will either be with proofs or references to where a proof can be found.

A.1 Schwartz functions

Definition A.1.1 A function $f \in C^\infty$ is called Schwartz if

$$|x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}$$

for all $\alpha, \beta \in \mathbf{N}_0^n$, we say that $f \in \mathscr{S}(\mathbf{R}^n)$

For the Schwartz space, $\mathscr{S}(\mathbf{R}^n)$ we use two equivalent seminorms

$$\|f\|_m = \sup_{|\beta| \leq m} \sup_{x \in \mathbf{R}^n} |\langle x \rangle^{2m} \partial^\beta f(x)| \quad \text{and} \quad \|f\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}^n} |x^\alpha \partial^\beta f(x)|.$$

Note that for $n > m$ we have $\|f\|_n \geq \|f\|_m$. For the reader that is interested in a more detailed look at the seminorm topology on the Schwartz space we recommend [Grubb, 2009] or [Rudin, 1991].

Lemma A.1.2 *Let* $\alpha \in \mathbf{N}^n$ *then*

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} = 2^{|\alpha|}.$$

The proof is trivial and hence left out.

Lemma A.1.3 *Let* $f \in \mathscr{S}(\mathbf{R}^n)$ *then*

$$\partial^\mu (x^\alpha \partial^\beta f(x)) \leq 2^{|\mu|} \alpha! \|f\|_{\max(|\alpha|, |\mu+\beta|)},$$

for all $\alpha, \beta, \mu \in \mathbf{N}_0$.

PROOF.

$$\begin{aligned}
\partial^\mu (x^\alpha \partial^\beta f(x)) &= \sum_{\substack{\mu' \leq \mu \\ \mu' \leq \alpha}} \binom{\mu}{\mu'} \frac{\alpha}{(\alpha - \mu')!} x^{\alpha - \mu'} \partial^{\beta + \mu - \mu'} f(x) \\
&\leq \sum_{\mu' \leq \mu} \binom{\mu}{\mu'} |\alpha! \langle x \rangle^{2|\alpha|} \partial^{\beta + \mu - \mu'} f(x)| \\
&\leq 2^{|\mu|} \alpha! \|f\|_{\max(|\alpha|, |\beta + \mu|)}.
\end{aligned}$$

■

Lemma A.1.4 *Let $f \in \mathscr{S}(\mathbf{R}^n)$, then*

$$\int |x^\alpha \partial^\beta f(x)| dx \leq \|f\|_{\max(|\alpha|+n+1,|\beta|)} C_n,$$

for all $\alpha, \beta, \mu \in \mathbf{N}_0$.

Proof.

$$\begin{aligned}\int |x^\alpha \partial^\beta f(x)| dx &= \int |\langle x\rangle^{-2(n+1)} \langle x\rangle^{2(n+1)} x^\alpha \partial^\beta f(x)| dx \\ &\leq \|f\|_{\max(|\alpha|+n+1,|\beta|)} \int |\langle x\rangle^{-2(n+1)}| dx \\ &= \|f\|_{\max(|\alpha|+n+1,|\beta|)} C_n.\end{aligned}$$

■

Theorem A.1.5 *Assume that X, Y are topological vector spaces, that X is metrizable and that the map $T\colon X \to Y$ is linear. Furthermore, let $\{x_n\}$ be a sequence in X. Then the following properties are equivalent:*

(i) T is continuous.

(ii) T is bounded.

(iii) If $x_n \xrightarrow{n\to\infty} 0$, then Tx_n is bounded for every $n \in \mathbf{N}$.

(iv) If $x_n \xrightarrow{n\to\infty} 0$, then $Tx_n \xrightarrow{n\to\infty} 0$.

For a proof see [Rudin, 1991, p. 24-25].

The following definition is from [Hörmander, 1983, p. 236].

Definition A.1.6 Let $r \in \mathbf{R}$, $\delta, \rho \in [0,1]$ and let $S^r_{\rho,\delta}(\mathbf{R} \times \mathbf{R})$ be the space of all $a \in C^\infty(\mathbf{R}^{2n})$ which satisfy that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1 + \|\xi\|)^{r-\rho|\alpha|+\delta|\beta|},$$

for all $x, \xi \in \mathbf{R}^n$. Then a is called a *symbol* of order r.

Definition A.1.7 Let $r \in \mathbf{R}$ and $a \in S^r_{\rho,\delta}(\mathbf{R} \times \mathbf{R})$. The operator $Op(a)\colon \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n)$ given by

$$(Op(a)\varphi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} a(x,\xi) \widehat{\varphi}(\xi)\, d\xi,$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$ is called a *pseudodifferential operator* of order r.

Remark A.1.8 Any $f \in \mathscr{S}(\mathbf{R}^{2n})$ is a symbol of order 0, since

$$|\partial_\xi^\beta \partial_x^\alpha f(x,\xi)| \leq \langle x\rangle^{-2|\mu|} \langle \xi\rangle^{-2|\nu|} C_{\alpha,\beta} \leq C_{\alpha,\beta},$$

for every $\alpha, \beta, \mu, \nu \in \mathbf{N}^n$.

A.2 Classical Analysis

Definition A.2.1 Let M_p and T_q denote the modulation, and translation operators, such that

$$M_p\varphi(x) = e^{ix\cdot p}\varphi(x) \quad \text{and} \quad T_q\varphi(x) = \varphi(x-q).$$

Theorem A.2.2 *Let $\varphi \in L^1(\mathbf{R}^n)$ and $p \in \mathbf{R}^n$, then*

$$\mathcal{F}(M_p\varphi) = T_p\mathcal{F}\varphi \quad \text{and} \quad \mathcal{F}(T_p\varphi) = M_{-p}\mathcal{F}\varphi.$$

Theorem A.2.3 (Multivariate Taylor's formula) *Let $f \in C^j(\mathbf{R}^n)$, for $j \in \mathbf{N}$. Then it follows that*

$$f(x+y) = \sum_{|\alpha|<j} \partial^\alpha f(x)\frac{y^\alpha}{\alpha!} + j\int_0^1 (1-t)^{j-1} \sum_{|\alpha|=j} \partial^\alpha f(x+ty)\frac{y^\alpha}{\alpha!}dt,$$

for every $\alpha \in \mathbf{N}^n$.

For a proof see [Hörmander, 1983, p. 12-13]

Lemma A.2.4 *Assume $f \in L^2(\mathbf{R}^n)$. Then there exists a constant C, such that*

$$\|\widehat{f}\|_{L^1} \leq C \sup_{|\alpha|\leq n+1} \|\partial^\alpha f\|_{L^1}.$$

For a proof see [Evans and Zworski, 2003, p. 23-24]

Theorem A.2.5 (Inverse Function Theorem) *Let $\Omega \subset \mathbf{R}^n$ be open and $a \in \Omega$. Assume $f\colon \Omega \to \mathbf{R}^n$ is C^k, $\nabla f(a)$ is invertible and $b = f(a)$. Then there exist open sets in $\mathbf{R}^n$, U and V, such that $a \in U$, $b \in V$, f is injective on U and $f(U) = V$.*

If $g\colon V \to U$ is the inverse of f given by

$$g(f(x)) = x,$$

for $x \in U$, then g is C^k.

For a proof see [Rudin, 1976, p. 221-223].

A.3 Various results

Recall that L^* from (3.4) is given by

$$L^*(f) = (-ih)\partial\left(\frac{f(x)}{\overline{\varphi'}(x)}\right).$$

Lemma A.3.1 *If $a \in C_c^\infty(\mathbf{R})$ and $\varphi \in C^\infty(\mathbf{R})$, then*

$$\left(L^{*N}\right)(f) = (ih)^N \sum_{m=0}^{N} f^{(m)}(x) \frac{A_{N,m}(x)}{\overline{\varphi'}(x)^{2N-m}},$$

where

$$A_{n+1,m}(x) = A_{n,m-1}(x) + \overline{\varphi'}(x)A'_{n,m}(x) + (m-2n-1)\overline{\varphi''}(x)A_{n,m}(x)$$

given that $A_{0,0}(x) = 1$ and $A_{k,l}(x) = 0$ if $l > k$ or $l < 0$.

Proof. We will prove this by induction, for $n = 0$ we have $f(x) = \left[(L^*)^0(f)\right](x) = f^{(0)}(x)\frac{A_{0,0}(x)}{\overline{\varphi'}(x)^0}$ since $A_{0,0}(x) = 1$. Let's assume that the lemma holds for $N = n$, we get that

$$\begin{aligned}
\left[(L^*)^{n+1}(f)\right](x) &= \left[L^*(L^{*n}(f))\right](x) \\
&= L^*\left((-ih)^n \sum_{m=0}^{n} f^{(m)}(x)\frac{A_{n,m}(x)}{\overline{\varphi'}(x)^{2n-m}}\right) \\
&= -ih\partial\left((-ih)^n \sum_{m=0}^{n} f^{(m)}(x)\frac{A_{n,m}(x)}{\overline{\varphi'}(x)^{2n-m+1}}\right) \\
&= (-ih)^{n+1}\left(\sum_{m=1}^{n+1} f^{(m)}(x)\frac{A_{n,m-1}(x)}{\overline{\varphi'}(x)^{2n-m+2}}\right. \\
&\qquad + \sum_{m=0}^{n} f^{(m)}(x)\frac{\overline{\varphi'}(x)A'_{n,m}(x)}{\overline{\varphi'}(x)^{2n-m+2}} \\
&\qquad \left. + \sum_{m=0}^{n} f^{(m)}(x)\frac{(m-2n-1)\overline{\varphi''}(x)A_{n,m}(x)}{\overline{\varphi'}(x)^{2n-m+2}}\right).
\end{aligned}$$

Since $A_{n,-1}(x) = A_{n,n+1}(x) = 0$, we get

$$\begin{aligned}
&\left[(L^*)^{n+1}(f)\right](x) \\
&= (-ih)^{n+1}\left(\sum_{m=0}^{n+1} f^{(m)}(x)\frac{A_{n,m-1}(x) + \overline{\varphi'}(x)A'_{n,m}(x) + (m-2n-1)\overline{\varphi''}(x)A_{n,m}(x)}{\overline{\varphi'}(x)^{2(n+1)-m}}\right) \\
&= (-ih)^{n+1}\sum_{m=0}^{n+1} f^{(m)}(x)\frac{A_{n+1,m}(x)}{\overline{\varphi'}(x)^{2(n+1)-m}},
\end{aligned}$$

hence the proof is done. ■

Lemma A.3.2 *Let $l, m \in \mathbf{N}$, $m \geq 2$, $g \in C_c^\infty(\mathbf{R})$, $h > 0$ and*

$$J_l(\xi, h) = \sum_{j=0}^{l-1} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l-\frac{1+j}{m}}}\left(\frac{h^{1/m}}{m}\right)^j C_{l,j},$$

$$\text{where } C_{l,j} = \begin{cases} \sum_{E_{l,j}} \prod_{p=1}^{l-j-1} \left(\frac{x_p - p + 1}{m} - x_p \right) & \textit{for } j < l-1 \\ 1 & \textit{for } j = l-1, \end{cases}$$

where $E_{l,j} = \{(x_1, \ldots, x_{l-j-1}) \in \mathbf{Z}^{l-j-1} \mid 0 < x_1 < \ldots < x_{l-j-1} < l\}$, *then* $\partial_\xi J_l(\xi, h) = J_{l+1}(\xi, h)$.

PROOF. By straightforward calculations we have that

$$\begin{aligned}
\partial_\xi J_l(\xi,h) &= \partial_\xi \sum_{j=0}^{l-1} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l-\frac{1+j}{m}}} \left(\frac{h^{1/m}}{m}\right)^j C_{l,j} \\
&= \sum_{j=0}^{l-1} \frac{g^{(j+1)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1+j+1}{m}}} \left(\frac{h^{1/m}}{m}\right)^{j+1} C_{l,j} \\
&\quad + \sum_{j=0}^{l-1} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1+j}{m}}} \left(\frac{h^{1/m}}{m}\right)^j \left(\frac{1+j}{m} - l\right) C_{l,j} \\
&= \sum_{j=1}^{l-1} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1+j}{m}}} \left(\frac{h^{1/m}}{m}\right)^j \left[\left(\frac{1+j}{m} - l\right) C_{l,j} + C_{l,j-1}\right] \\
&\quad + \sum_{j=0}^{0} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1}{m}}} \left(\frac{h^{1/m}}{m}\right)^j \left(\frac{1+j}{m} - l\right) C_{l,0} \\
&\quad + \sum_{j=l}^{l} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{j+1}{m}}} \left(\frac{h^{1/m}}{m}\right)^j C_{l,j-1} \\
&= \sum_{j=1}^{l-1} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1+j}{m}}} \left(\frac{h^{1/m}}{m}\right)^j C_{l+1,j} \\
&\quad + \sum_{j=0}^{0} \frac{g^{(0)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{1}{m}}} \left(\frac{h^{1/m}}{m}\right)^j C_{l+1,j} \\
&\quad + \sum_{j=l}^{l} \frac{g^{(j)}(\xi^{1/m}h^{1/m})}{\xi^{l+1-\frac{j+1}{m}}} \left(\frac{h^{1/m}}{m}\right)^j C_{l+1,j} \\
&= J_{l+1}(\xi,h),
\end{aligned}$$

because we have that $C_{l,l-1} = C_{l+1,l} = 1$, $(\frac{1}{m} - l)C_{l,0} = (\frac{1}{m} - l)\prod_{p=1}^{l-1}(\frac{1}{m} - p) = \prod_{p=1}^{l}(\frac{1}{m} - p) = C_{l+1,0}$. To see the last equality we look at the set $E^*_{l,j} = \{(x_1, \ldots, x_{l-j}) \in \mathbf{Z}^{l-j} \mid 0 < x_1 < \ldots < x_{l-j-1} < l, x_{l-j} = l\}$, then we have

$$\begin{aligned}
&\left(\frac{1+j}{m} - l\right) C_{l,j} + C_{l,j-1} \\
&\qquad = \sum_{E_{l,j}} \left(\frac{1+j}{m} - l\right) \prod_{p=1}^{l-j-1} \left(\frac{x_p - p + 1}{m} - x_p\right) + \sum_{E_{l,j-1}} \prod_{p=1}^{l-j} \left(\frac{x_p - p + 1}{m} - x_p\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{E^*_{l,j} \cup E_{l,j-1}} \prod_{p=1}^{l-j} \left(\frac{x_p - p + 1}{m} - x_p \right) \\
&= C_{l+1,j}.
\end{aligned}
$$

■