The tightness of the inertia bound of graphs

When is the inertia bound not tight?

Master's Thesis

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Abstract:

This report on algebraic graph theory presents an upper bound for the independence number of a graph, and studies whether or not this bound is attainable for all graphs. This bound is known as the inertia bound.
Starting by introducing some preliminary graph and matrix theory, the report derives the inertia bound, as well as conditions, under which the bound is not tight, and examples of graphs that do attain the bound.
Next, it introduces the Paley graphs and presents some of their properties, with special focus on the Paley graph on 17 vertices and some of its induced subgraphs.
Using these properties, it is then proven that no matter what weights are assigned to the edges of a Paley 17 graph, it cannot attain the inertia bound. Thus, not all graphs have tight inertia bound.
The report concludes with a presentation of subjects apt for further study, such as further looking into what properties of graphs result in tight inertia bounds, or examining what happens with the inertia bound, when the weights of graphs are taken over different fields.

Contents

1. Introduction

In algebraic graph theory, there is a close connection between graphs and matrices. It is common to use matrices to define graphs, or use graphs to visualize matrices. Properties of the matrices can then be analysed and connected to properties of the associated graphs. An example of such a connection exists between the independence number $\alpha(G)$ of a graph G, and the eigenvalues of corresponding weight matrices W. These weight matrices are real and symmetric matrices such that the entries $w_{i,j} = 0$ if the edge $\{i, j\}$ doesn't exist in the graph G. From this definition, the weight matrix W has real eigenvalues, and so the amount of these can be denoted as $n^+(W)$ for the number of positive eigenvalues of W and $n^-(W)$ for the number of negative eigenvalues. Then, for any weight matrix W of a graph G of order n, the connection between $\alpha(G)$ and the eigenvalues of W is given as the so-called *inertia bound*:

$$\alpha(G) \leq \min\{n - n^+(W), n - n^-(W)\}. \tag{2.10}$$

This bound is attributed to Dragoš M. Cvetković, and appears in his 1971 Ph.D. thesis. Thus, it is also known as the *Cvetković bound*.

For inequalities and bounds such as the inertia bound, it is interesting to examine the conditions for which equality occurs, known as the bound being *tight* or the graph *attaining the bound*, and when the inequality is sharp. Specifically, one can pose the question: *Does each graph G of order n, have a weight matrix W, such that*

$$\alpha(G) = \min\{n - n^+(W), n - n^-(W)\}?$$

While equality in the inertia bound is possible for smaller graphs, with examples of this in Section 2.4, the purpose of this report is to show, that the answer to the above posed question is in fact *no*. This will be done using the graph $P(17)$, which is the Paley graph on 17 vertices, and by studying the properties of it and its weight matrices.

1.1 Overview

Chapter 2 will focus on the preliminary graph and matrix theory, which allows for the derivation of the inertia bound. Starting with definitions of graphs, their properties and associated adjacency matrices, it moves on to examine the properties of the eigenvalues of matrices of this type. Using this theory on eigenvalues and adjacency matrices, the inertia bound is presented as a logical consequence of their properties. The chapter concludes with a few examples of graphs, that attain the inertia bound, showing that equality can occur for some graphs.

Chapter 3 introduces the Paley graphs, and in particular the Paley graph on 17 vertices. It also present a number of properties for Paley 17, which will be used to show, that the inertia bound for the graph Paley 17 isn't tight. Two induced subgraphs of Paley 17, G_1 and G_2, are also introduced, and the conditions for which they have specific numbers of positive and negative eigenvalues are presented. These properties are needed for the final proof.

In Chapter 4 the sign of triangles present in Paley 17 are determined, together with the sign of all the edges of the graph. Thanks to specific diagonal matrices, there will be only two possible ways to distribute these signs. As will be shown in the final proof of this chapter, none of these ways, will lead to a tight weight matrix for Paley 17.

Finally, Chapter 5 looks to generalize the problem of the tight inertia bound to consider general fields, and not just the real numbers. This is done by introducing the isotropic bound - the generalized equivalent to the inertia bound. The chapter also looks at the conditions for equality in the bound in a general field, and under which conditions regular adjacency matrices for graphs can be used to attain the bound.

2. Graph and matrix theory

This chapter will introduce the basic graph theory, that allows for proving the existence of the inertia bound. It also provides the basis, on which can be discussed, whether the bound is tight for certain graphs.

2.1 Graphs and matrix representations

Graphs and matrices are closely connected in algebraic graph theory, with one being used to visualise the other in alternative ways. As there are different properties associated with each visualisation, graph or matrix, it is necessary to go through some of the most important here.

A graph G is defined from two sets. The set of vertices in the graph V, and the set of edges E.

Definition 2.1 (Graph): A graph is given by $G = (V, E)$, where $V = \{v_1, v_2, \dots\}$ is a non-empty, finite set of vertices on the graph and E is the set of edges in G. The elements of E specifically can be denoted as single elements $e_1, e_2, \dots$, or as two-element subsets of V, such that if $u, v \in V$ and an edge connects them, then $\{u, v\} \in E$. Two vertices u and v are adjacent, written $u \sim v$, if $\{u, v\} \in E(G)$. Otherwise, they are non-adjacent. △

The graphs that are examined in this report, will be exclusively simple, non-directed and, with the exception of Chapter 5, real graphs. This means that no two vertices can have more than one edge connecting them directly, no vertex can be connected directly to itself, in a so-called *loop*, and an edge $\{u, v\}$ can be considered the same as the edge $\{v, u\}$. These constrictions become more relevant when matrix representation of graphs is introduced later.

Automorphisms can exist on graphs. It is defined like so:

Definition 2.2 (Graph-automorphism): Let $G = (V, E)$ be a graph which has vertices $v_1, v_2, \dots \in V(G)$. An automorphism of G is a permutation $\sigma : V \to V$ of $V(G)$ such that $v_1 \sim v_2$ if and only if $\sigma(v_1) \sim \sigma(v_2)$. △

Given any graph $G = (V, E)$, the setup of its vertices and edges can be expressed in matrix form through the use of a so-called adjacency matrix.

Definition 2.3 (Adjacency matrix): Given a graph $G = (V, E)$ an *adjacency matrix* A can be constructed, by letting the entry $a_{i,j}$ be equal to the number of edges connecting the two vertices i and j. △

It is clear that the adjacency matrix will always be a square matrix. Also, as we are only concerned with simple graphs, A will consist of only 1's and 0's. It will also be symmetric, as any edge $\{i, j\}$ in the edge-set is equivalent to an edge $\{j, i\}$ as well. Finally, as a simple graph has no loops, the diagonal of A will be all 0's. The name "adjacency matrix" stems from two adjoined vertices being said to be adjacent.

If, instead, two vertices are non-adjacent, they are said to be independent. The independence between vertices in graphs will play a greater role later, specifically the independence number of a graph.

Definition 2.4 (Independence number): Given a graph G, the *independence number* $\alpha(G)$ of G is given as the cardinality of the lagest subset of vertices in G, for which all pairs of vertices of the subset are independent. △

Looking at Figure 2.1 and example of a graph G with $\alpha(G) = 3$ can be seen. G contains multiple independent sets of vertices, but the cardinality of the largest possible set is 3.

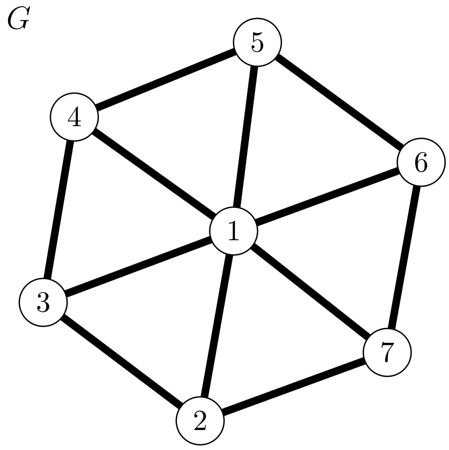


Figure 2.1: In the above graph G, vertices 2 and 5 make up an independent set. However, so does the set of vertices 2, 4 and 6. As such $\alpha(G) = 3$.

A special property of graphs depend on their independence number.

Definition 2.5 (α-crital graphs): A graph $G = (V, E)$ is *α-critical* if $\alpha(G) < \alpha(G - e)$ for all $e \in E(G)$. △

This means, that if you cannot remove any edge from G without increasing $\alpha(G)$, then G is α-critical.

Going back to matrices, if M is a matrix, then a specific type of submatrix is the *principal submatrix*, which is formed by omitting corresponding rows and columns from M. Formally, it will be defined as follows:

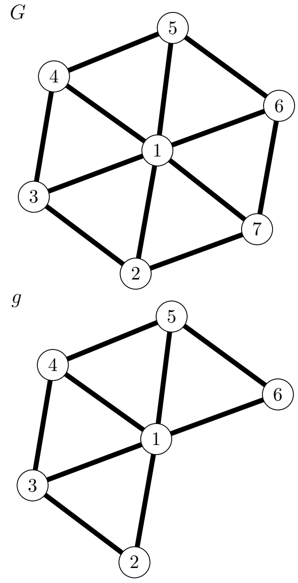
Definition 2.6 (Principal submatrix): An $m \times m$ matrix M is an $m \times m$ principal submatrix of an $n \times n$ matrix N, if M is obtained from N by omitting any $n - m$ rows and the corresponding $n - m$ columns from N. △

For matrices based on graphs, an induced subgraph, which takes a subset of the vertices of the graph and those of their edges, which adjoin only vertices in the subset, is analogue to forming a principal submatrix. This is because if vertex i and all connected edges are removed from the graph, then row i and column i is removed from the adjacency matrix of the graph. This forms a principal submatrix per Definition 2.6. An example of this process is shown in Figure 2.2.

The binary way of denoting vertex-neighbourhoods used in the adjacency matrix is not the only way of expressing graphs. Specifically, when considering real-life examples, it is often more evident to denote a connection between vertices by some value that symbolises a correlation between them, such as distance, difference or something similar. If this is the case, the graph is known as a *Weighted Graph*.

Definition 2.7 (Weighted graph): Let $G = (V, E)$ be a graph of order n, with vertex set V and edge set E. G is a *weighted graph* if the edges (i, j) has some weight $w_{ij} \in \mathbb{R}$ for all edges $(i, j) \in E(G)$. △

The goal of using this type of graph, is to ensure generality, as the edges of a regular, simple graph can be said to have a weight of 1. Thus, showing that a specific result holds for any



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

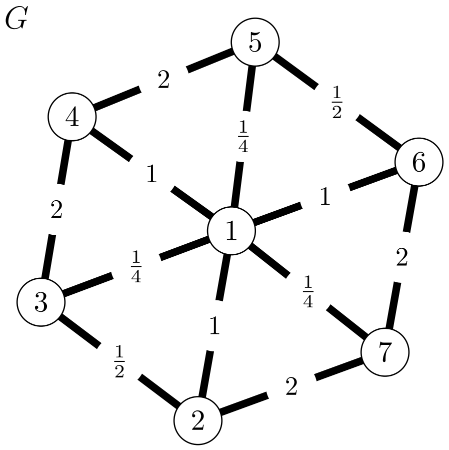
$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 2.2: Looking at the example graph G from Figure 2.1, by removing vertex 7 and all connected edges, an induced subgraph g is formed. The adjacency matrix B corresponding to g is also a principal submatrix of the adjacency matrix A corresponding to G, with row and column 7 removed.

weighted graph, means that it works in general for otherwise un-weighted graphs as well. A special adjacency matrix can be made for weighted graphs, using the weights of the edges as entries.

Definition 2.8 (Weighted adjacency matrix): The edge weights $w_{ij}, 1 \leq i, j \leq n$ of a weighted graph of order n form the (i, j)-elements of the *weighted adjacency matrix* W of that graph. For two non-adjacent vertices i and j, the matrix entry (i, j) is zero. △

In the same vein as for for the weighted graph, any adjacency matrix can be said to be weighted, with all "weights" having a value of 1 in the otherwise non-weighted adjacency matrix. An example of a weight matrix for a weighted graph is shown in Figure 2.3.



$$W = \begin{bmatrix} 0 & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 & \frac{1}{4} \\ 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 2 \\ \frac{1}{4} & \frac{1}{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 2 \\ \frac{1}{4} & 2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Figure 2.3: The graph G from Figure 2.1, when given the set of edge weights as above, has the matrix W as its corresponding weighted adjacency matrix.

The weigted adjacency matrix of a graph G is also known as the *weight matrix* of G.

Definition 2.9 (Tight weight matrix): A weight matrix W of a graph G is called *tight* if

$$\alpha(G) = \min\{|G| - n^+(W), |G| - n^-(W)\}.$$

Here $n^+(W)$ and $n^-(W)$ describe the number of positive and negative eigenvalues of W respectively, including their multiplicity. △

An important note for the weighted adjacency matrix is, that given G as a simple graph, containing no loops or double edges, W will be a both real and symmetric $n \times n$ matrix.

As W is a real, square matrix, it of course has eigenvalues. As these eigenvalues will play a role in finding the inertia bound, it is relevant to look into some results and techniques concerning them.

2.2 Eigenvalues

There are multiple ways of finding the eigenvalues of matrices. One such way, is to find the zeros for the characteristic polynomial of the matrix. Given a matrix A, the characteristic polynomial is given by

$$\phi(A, \lambda) = \det(\lambda I - A), \tag{2.1}$$

where λ is an eigenvalue to A, corresponding to an eigenvector $\mathbf{x}$, with I being the identity matrix. Another way of finding the eigenvalues of a matrix, is to look at eigenvectors, as, if

$$A\mathbf{x} = \lambda \mathbf{x}, \tag{2.2}$$

then λ is an eigenvalue of A.

For matrices based on simple, non-weighted graphs, this final method can be further refined. As the adjacency matrix A of a graph G has its rows and columns indexed by the vertices $V(G)$ of G, A can be viewed as a linear mapping on $\mathbb{R}^{V(G)}$, that is, the space of real functions on $V(G)$. If $f \in \mathbb{R}^{V(G)}$ then the vertices $u \in V(G)$ are assigned a value $f(u)$ $\forall u \in V(G)$. The image Af of f under A is given by

$$(Af)(u) = \sum A_{uv} f(v), \tag{2.3}$$

where u and v are vertices in $V(G)$. As A is a matrix with only entries of 0 and 1, (2.3) can be rewritten as

$$(Af)(u) = \sum_{u \sim v} f(v). \tag{2.4}$$

Here, $u \sim v$ means that the vertices u and v are adjacent, and so, by (2.4), the value of Af at vertex u is given by the sum of the values of f at the neighbours of u. Using (2.2), and if we suppose f is an eigenvector of A with eigenvalue λ, then

$$\lambda f(u) = \sum_{u \sim v} f(v), \tag{2.5}$$

meaning that the sum of the values of f at the neighbours of u is λ times the value of f on u. Conversely, if some f satisfies (2.5) for some λ, then f is an eigenvector of the graph, as is the case in Figure 2.4. If multiple eigenvalues are equal, this eigenvalue is said to have a higher multiplicity.

For a real, symmetric $n \times n$ matrix A, the set of eigenvalues of A will be denoted as $\text{ev}(A)$ Given a eigenvalue $\lambda \in \text{ev}(A)$, let E_λ, which is called a principal idempotent of A, be the matrix representing an orthogonal projection onto the eigenspace of λ. Then

$$E_\lambda^2 = E_\lambda,$$

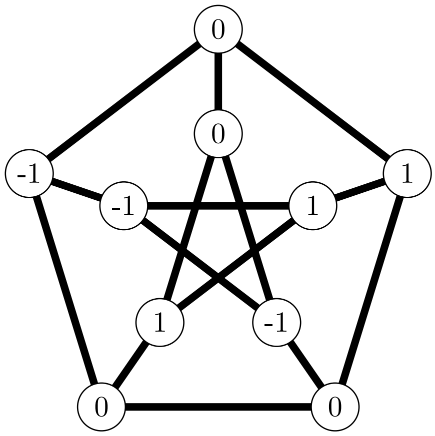


Figure 2.4: An eigenvector to the Petersen graph gives the following values to its vertices. As the sum of all neighbours to a vertex equals the value of the vertex itself, this eigenvector has corresponding eigenvalue 1. [Godsil and Royle, 2001]

and since two distinct eigenspaces of A are orthogonal as well, it follows that for two eigenvalues $\lambda, \tau \in \text{ev}(A),\ \lambda \neq \tau$,

$$E_\lambda E_\tau = 0.$$

As there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A,

$$I = \sum_{\lambda \in \text{ev}(A)} E_\lambda.$$

From this, one can see that

$$A = \sum_{\lambda \in \text{ev}(A)} \lambda E_\lambda. \tag{2.6}$$

Equation (2.6) is also known as the *Spectral Decomposition* of A. Generally, for any polynomial f, it follows from (2.6) that

$$f(A) = \sum_{\lambda \in \text{ev}(A)} f(\lambda) E_\lambda. \tag{2.7}$$

Now, since this f can be chosen, so that it vanishes for all but one of the eigenvalues of A, it follows from (2.7) that E_λ is a polynomial in A. The matrices E_λ are linearly independent as if $\sum_\lambda a_\lambda E_\lambda = 0$, then

$$0 = E_\tau \sum_\lambda a_\lambda E_\lambda = a_\tau E_\tau,$$

meaning that the principal idempotents form a basis for the vector space of all polynomials of A. Therefore this vector space has dimension equal to the number of distinct eigenvalues of A.

Equation (2.7) also holds for rational functions - functions that can be expressed as the ratio f/g of two other polynomials - as long as this rational function is defined at all eigenvalues of A. To see this, consider the function g, which from Equation (2.7) has an inverse function

$$g(A)^{-1} = \sum_{\lambda \in \text{ev}(A)} g(\lambda)^{-1} E_\lambda.$$

Then

$$\frac{f(A)}{g(A)} = f(A)g(A)^{-1} = \sum_{\lambda \in \mathrm{ev}(A)} f(\lambda)g(\lambda)^{-1}E_\lambda^2 = \sum_{\lambda \in \mathrm{ev}(A)} \frac{f(\lambda)}{g(\lambda)} E_\lambda.$$

This gives the special case of (2.7) for rational functions:

$$(xI - A)^{-1} = \sum_{\lambda \in \mathrm{ev}(A)} (x - \lambda)^{-1} E_\lambda, \tag{2.8}$$

which can be used to prove some results, improtant for the theory of interlacing, which will be presented in Section 2.3.

Theorem 2.10: Let A be a real, symmetric $n \times n$ matrix, and let B be the principal submatrix of A, obtained from deleting the ith row and column of A. Then

$$\frac{\phi(B, x)}{\phi(A, x)} = e_i^\top (xI - A)^{-1} e_i,$$

where e_i denotes the ith standard vector. Recall that $\phi(A, x)$, presented in (2.1), denotes the characteristic polynomial of A.

Proof: It is clear from the standard determinantal formula, that when it is used for the inverse matrix one gets

$$\left((xI - A)^{-1}\right)_{ii} = \frac{\det(xI - B)}{\det(xI - A)}.$$

Note that the left side here, is taken for the ith row and column. Thus,

$$\left((xI - A)^{-1}\right)_{ii} = e_i^\top (xI - A)^{-1} e_i,$$

which is enough to complete the proof. ∎

Corollary 2.11: For any graph G,

$$\phi'(G, x) = \sum_{u \in V(G)} \phi(G \setminus u, x).$$

Proof: From (2.8),

$$\mathrm{tr}(xI - A)^{-1} = \sum_\lambda (x - \lambda)^{-1} \, \mathrm{tr}(E_\lambda),$$

and by Theorem 2.10,

$$\mathrm{tr}(xI - A)^{-1} = \sum_{u \in V(G)} \frac{\phi(G \setminus u, x)}{\phi(G, x)}.$$

Now, by denoting the multiplicity of λ as a zero to the polynomial $\phi(G, x)$ by m_λ, one can reach the partial fraction expansion

$$\frac{\phi'(G, x)}{\phi(G, x)} = \sum_\lambda \frac{m_\lambda}{x - \lambda}. \tag{2.9}$$

As E_λ is a symmetric matrix with $E_\lambda^2 = E_\lambda$, its eigenvalues are all 0 or 1. The trace of E_λ is equal to its rank, but its rank is the dimension of the eigenspace associated with λ, and thus $\mathrm{tr}(E_\lambda) = m_\lambda$. This completes the proof. ∎

A rational function f/g is called *proper* if the degree of f is less than the degree of g. Proper rational functions has a partial fraction expansion. Given a proper rational function $p = f/g$, this can be seen by the following:

The function $f(x)$ can be expanded using its eigenvalues $\lambda_1, \dots, \lambda_r$ and their multiplicities $m_1, \dots, m_r$.

$$f(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \dots (x - \lambda_r)^{m_r}.$$

Then differentiating f and dividing $f'(x)$ with $f(x)$ yields

$$\frac{f'(x)}{f(x)} = \frac{m_1(x - \lambda_1)^{m_1 - 1}(x - \lambda_2)^{m_2} \dots + m_2(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2 - 1} \dots}{(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \dots (x - \lambda_r)^{m_r}}.$$

By denoting $f_i(x) = m_i(x - \lambda_i)^{m_i - 1}$ and reducing the fraction, we get the partial fraction expansion as presented in [Godsil and Royle, 2001, p. 188].

$$\sum_{i=1}^{r} \frac{f_i(x)}{(x - \lambda_i)^{m_i}}.$$

In the expansion m_i will be a positive integer, and $f_i(x)$ a non-zero polynomial of order less than m_i. The numbers λ_i are called the *poles* of p, and specifically, m_i is the order of the pole at λ_i. A λ_i for which $m_i = 1$ is called a *simple pole.* The expansion in (2.2) reduces to the form seen in (2.9), as the $f_i(x)$ can be further reduced in the fractions.

Theorem 2.12: Let A be a real, symmetric, $n \times n$ matrix and b a vector of length n. Define $\psi(x)$ as the rational function $b^\top(xI - A)^{-1}b$. Then all zeros and poles of ψ are simple, and ψ' is negative everywhere it is defined. If λ and τ are two consecutive poles of ψ, then the closed interval $[\lambda, \tau]$ contains exactly one zero of ψ.

Proof: By Equation (2.8)

$$b^\top(xI - A)^{-1}b = \sum_{\lambda \in \mathrm{ev}(A)} \frac{b^\top E_\lambda b}{x - \lambda},$$

implying that the poles of ψ are simple. Differentiating both sides yields

$$\psi'(x) = -\sum_{\lambda} \frac{b^\top E_\lambda b}{(x - \lambda)^2}.$$

Using (2.8), the right side above equals $-b^\top(xI - A)^{-2}b$, and thus

$$\psi'(x) = -b^\top(xI - A)^{-2}b$$

As $b^\top(xI - A)^{-2}b$ is the squared length of $(xI - A)^{-1}b$, this implies that $\psi' < 0$ for all x that are not poles of ψ. This, in turn, implies that each zero of ψ must be simple.

Now suppose that λ and τ are two consecutive poles of ψ. As they are simple poles, ψ must be a strictly decreasing function on the interval $[\lambda, \tau]$, and positive for values of x in this interval sufficiently close to λ, and negative for values of x sufficiently close to τ. Thus, accordingly, the interval must contain exactly one zero of ψ. ■

With this, we can introduce a special property of eigenvalues, which will allow us to derive the inertia bound.

2.3 Interlacing

Given a real, $n \times n$, symmetric matrix A, the different eigenvalues of A can be ordered according to their values. Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be one such, non-increasing ordering. If another matrix B is real and symmetric, but an $m \times m$ matrix, with $m \leq n$, then the eigenvalues of B *interlace* with the eigenvalues of A if

$$\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A), for\ i = 1, \ldots, m.$$

The next result will show, how the eigenvalues of a principal submatrix, introduced in Definition 2.6, of a symmetric matrix A interlace with the eigenvalues of A.

Theorem 2.13: Let A be a real, symmetric, $n\times n$ matrix and let B be a principal submatrix of A with order $m \times m$. Then for $i = 1, \ldots, m$

$$\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A).$$

Proof: This proof uses induction on n. For $m = n$ it is clear that the inequality holds. Assume $m = n - 1$. By Theorem 2.10 the following holds for some i:

$$\frac{\phi(B, x)}{\phi(A, x)} = e_i^\top (xI - A)^{-1} e_i.$$

This is a rational function, as it is a ratio of two characteristic polynomials. Denote this rational function by ψ. By Theorem 2.12, $\psi(x)$ has only simple poles and zeros, and each consecutive pair of poles is seperated by only a single zero. The poles of ψ are zeros of A and the zeros of ψ are zeros of B. For a real, symmetric matrix M and a real number k, let $n(k, M)$ denote the number of indices i such that $\lambda_i(M) \geq k$. Look now at the behaviour of $n(k, A) - n(k, B)$ as k decreases. For k greater than the largest pole of ψ, the difference $n(k, A) - n(k, B)$ starts out at zero, but, since each pole is simple, the value of the difference will increase by one each time k passes through a pole of ψ as it decreases. Since each zero is simple as well, the difference will decrease by one as k passes through a zero. There is exactly one zero between each pair of consecutive poles, so the difference will alternate between 0 and 1. It follows that $\lambda_{i+1}(A) \leq \lambda_i(B) \leq \lambda_i(A)$ for all i.

Suppose now that $m < n - 1$. Then B is a principal submatrix of another principal submatrix C of A, with order $(n - 1) \times (n - 1)$. By induction

$$\lambda_{n-1-m+i}(C) \leq \lambda_i(B) \leq \lambda_i(C),$$

and as we have already shown that

$$\lambda_{i+1}(A) \leq \lambda_i(C) \leq \lambda_i(A),$$

it follows that the eigenvalues of B interlace the eigenvalues of A. ■

Using interlacing, a bound can be put on the size of the independence number introduced in Definition 2.4. For a symmetric matrix A, let $n^+(A)$ denote the number of positive eigenvalues of A, and $n^-(A)$ denote the number of negative eigenvalues of A.

Theorem 2.14: Let G be a graph with n vertices, and let A denote the symmetric $n \times n$ matrix, which has $A_{uv} = 0$ if the vertices u and v of G are not adjacent. Then

$$\alpha(G) \leq \min\{n - n^+(A), n - n^-(A)\} \tag{2.10}$$

Proof: Let S be an induced subgraph of G created from an independent set of vertices of size s. Let B be the adjacency matrix of S, which means it is a principal submatrix of A. B is then the zero matrix. From Theorem 2.13

$$\lambda_{n-s+i}(A) \leq \lambda_i(B) \leq \lambda_i(A),$$

but as $\lambda_i(B) = 0$ for all i we can infer that

$$0 \leq \lambda_s(A),$$

and thus that

$$n^-(A) \leq n - s.$$

The same argument can be applied using $-A$ in place of A to deduce that $n^+(A) \leq n - s$. As such

$$s \leq n - n^+(A)$$

and

$$s \leq n - n^-(A),$$

meaning that

$$s = \alpha(G) \leq \min n - n^+(A), n - n^-(A).$$ ■

Note that in the above, A is really the adjacency matrix of G. As the only requirement of A is symmetry, Theorem 2.14 also holds for any weight matrix W to a graph G.

Note also the similarity between Equation (2.10) and the definition of tight weight matrices in Definition 2.9. Equation (2.10) is one way to describe the inertia bound. Consequently, a graph has a tight weight matrix if and only if its inertia bound is tight - meaning that there is an equality in (2.10). Another, equivalent way of expressing this bound uses the multiplicity of zero as an eigenvector in the weight matrix W. Denote this value by $n^0(W)$, then

$$\alpha(G) \leq n^0(W) + \min\{n^+(W), n^-(W)\}. \tag{2.11}$$

In fact, this version of the inertia bound holds true as long as the matrix W has entries from an ordered field $\mathbb{F}$ [Elzinga, 2007]. This will be looked further into in Chapter 5.

The reason for the name *inertia bound*, stems from the definition of the inertia of a matrix.

Definition 2.15 (Matrix inertia): For an $n \times n$ matrix A, the *inertia* of A is given as the ordered set

$$(n^+(A), n^-(A), n^0(A)),$$

where $n^+(A)$ denotes the number of positive eigenvalues of A, $n^-(A)$ the number of negative eigenvalues of A and $n^0(A)$ the multiplicity of 0 as an eigenvector of A. △

As can be seen from (2.10) and (2.11), the inertia of a matrix plays a large role in determining the inertia bound.

Graphs for which Equation (2.10) or (2.11) are equalities, are also said to *attain* the inertia bound. As the inertia bound is attributed to Dragoš M. Cvetković, it is also known as the Cvetković bound [Sinkovic, 2018].

Recall the definition of a tight weight matrix given in Definition 2.9. By using Theorem 2.13 a special case can be found, where the weight matrix is never tight.

Theorem 2.16: Let W be a weight matrix for a graph G of order n. If W has a principal submatrix with $\alpha(G)+1$ positive eigenvalues and a principal submatrix with $\alpha(G)+1$ negative eigenvalues, then W is not tight.

Proof: A consequence of Theorem 2.13, is that W has at least $\alpha(G)+1$ positive eigenvalues and at least $\alpha(G)+1$ negative eigenvalues. So

$$n^-(W) \leq n - \alpha(G) - 1$$

and

$$n^+(W) \leq n - \alpha(G) - 1$$

and thus

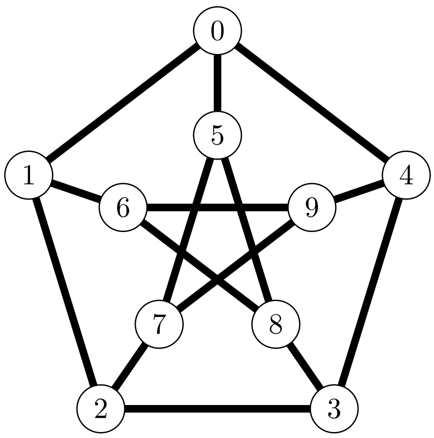
$$min\{n - n^-(W), n - n^+(W)\} \geq \alpha(G) + 1 > \alpha(G).$$

As this inequality is strict, W is not tight. ∎

This result will be used to prove the non-tightness of the Paley 17 graph. Many graphs, however, do possess tight weight matrices, which is also why this property of Paley 17 is important, as it shows that this is definitely not the case for all graphs. Some examples of graphs with tight inertia bound will be shown next.

2.4 Two graphs that attain the inertia bound

To show that there are cases where graphs can attain the inertia bound, this section will look closer at two smaller graphs, and see that equality occurs in Equation (2.11). First studied is the Petersen graph, which is seen in Figure 2.4. The Petersen graph, which will be denoted by P henceforth, has independence number $\alpha(P) = 4$. Using the adjacency matrix A_P of P, seen in Figure 2.5, the eigenvalues can be calculated.



$$A_P = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Figure 2.5: The Petersen graph P and its corresponding adjacency matrix A_P.

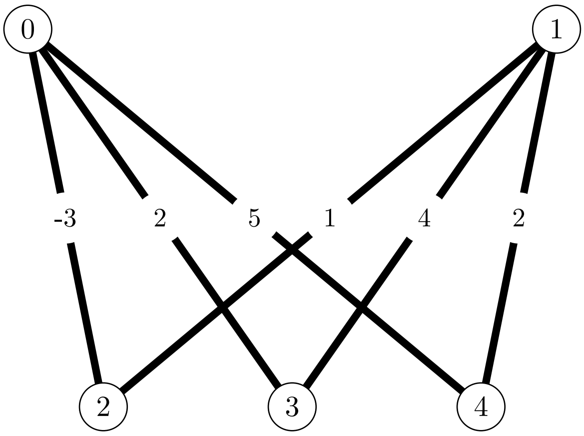
The matrix A_P has the eigenvalues 1 with multiplicity 5, 3 with multiplicity 1 and -2 with multiplicity 4. Thus, it has 6 positive eigenvalues and 4 negative, and the right half of Equation (2.11) becomes

$$n^0(A_P) + \min\{n^+(A_P), n^-(A_P)\} = 0 + \min\{6, 4\} = 4.$$

Thus, Equation (2.11) becomes an equality, as

$$\alpha(P) = 4 = n^0(A_P) + \min\{n^+(A_P), n^-(A_P)\}.$$

The second example, will be the complete bipartite graph $K_{2,3}$. Complete bipartite graphs $K_{m,n}$ are defined as having a partition of its vertices into two smaller sets of vertices of size m and n respectively, with each of these sets being independent sets, but with each vertex of one set connected to every vertex of the other. As such, it is clear that for complete bipartite graphs $K_{m,n},\ m \leq n$, $\alpha(K_{m,n}) = n$. Giving $K_{2,3}$ a set of edge-weights and a corresponding weight matrix $W_{K_{2,3}}$ as in Figure 2.6, the eigenvalues of the matrix can be calculated.



$$W_{K_{2,3}} = \begin{bmatrix} 0 & 0 & -3 & 2 & 5 \\ 0 & 0 & 1 & 4 & 2 \\ -3 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2.6: The complete bipartite graph $K_{2,3}$ with a set of edge-weights, and its corresponding weight matrix $W_{K_{2,3}}$.

These eigenvalues are found to be one instance of zero as an eigenvalue, two positive eigenvalues, and two negative. Thus

$$n^0(W_{K_{2,3}}) + \min\{n^+(W_{K_{2,3}}), n^-(W_{K_{2,3}})\} = 1 + \min\{2, 2\} = 3,$$

and so there is an equality in Equation (2.11), as

$$\alpha(K_{2,3}) = 3 = n^0(W_{K_{2,3}}) + \min\{n^+(W_{K_{2,3}}), n^-(W_{K_{2,3}})\}.$$

As it turns out, ways of constructing weight matrices exist for all graphs on 10 or fewer vertices, such that equality occurs in Equation (2.11) [Sinkovic, 2018]. This is in part, what leads to the question of whether this is possible for *all* graphs. As will be made apparent, that is not the case, as there exists a graph, for which Equation (2.11) has a sharp inequality. This graph is called the Paley 17 graph, the Paley graph on 17 vertices. It and its properties are the main focus of the next chapter.

3. Paley 17

It was previously shown, that the independence number of graphs are bounded by the inertia bound, with some graphs being able to attain the bound for given weight matrices. This chapter will introduce a special class of graphs, the Paley graphs, that will help show, that not all graphs can attain the inertia bound. Specifically, It is the properties of the Paley graph on 17 vertices, that makes the bound unattainable.

3.1 Paley graphs

First is the definition of the special class of graphs. The different Paley graphs are dependent on primes, and the numbering of their vertices.

Definition 3.1 (Paley graph): Let q be a prime power, such that $q \equiv 1 \mod 4$. Then the *Paley graph* $P(q)$ has as vertex set, the elements of the finite field $GF(q)$ with two vertices adjacent if and only if their difference is a non-zero square in $GF(q)$. △

In the definition, the condition of congruence on q means that -1 is a square in $GF(q)$. Thus any pairing $\{u, v\}$ of vertices are only adjacent if either $u - v$ or $v - u$ is a square in $GF(q)$. This means Paley graphs must be undirected.

One property that Paley graphs have, which will be used to great effect, is that they are transitive over vertices, edges and arcs in the graphs.

Definition 3.2 (Transitive graphs): The three types of graph transitivity are defined as follows:

> A graph G is *vertex-transitive* if its automorphism group acts transitively on $V(G)$. That is, for any two vertices of G there is an automorphism mapping one to the other.
>
> A graph G is *edge-transitive* if its automorphism group acts transitively on $E(G)$.
>
> A graph G is *arc-transitive* if its automorphism group acts transitively on the arcs of G. An *arc* in G is an ordered pair of adjacent vertices in G.

[Godsil and Royle, 2001] △

Of these three transitive properties arc-transitivity is the strongest one, as, necessarily, arc-transitive graphs are both vertex- and edge transitive. The opposite doesn't always hold true, however [Godsil and Royle, 2001, pp. 35-36].

That Paley graphs possess these transitive properties, comes as a consequence of the following theorem.

Theorem 3.3: The function $\sigma_{ab} : V \to V$, $\sigma_{ab}(v) = av + b$, where $v \in V$, a is a non-zero square in $GF(q)$ and $b \in GF(q)$, is an automorphism of $P(q)$.

Proof: It is easily noted, that σ_{ab} is bijective. Let now $v_1, v_2 \in V(P(q))$, with their images $\sigma_{ab}(v_1), \sigma_{ab}(v_2)$ and let a be a non-zero square in $GF(q)$. Since $v_1 \sim v_2$ if and only if $v_1 - v_2$ is a non-zero square in $GF(q)$, and since a is a non-zero square as well, then $v_1 - v_2$ is a non-zero square if and only if $a(v_1 - v_2)$ is a non-zero square. Further, $a(v_1 - v_2)$ is a non-zero square if and only if $av_1 - b - (av_2 - b)$ is a non-zero square, which it is if and only if $\sigma_{ab}(v_1) \sim \sigma_{ab}(v_2)$. This proves the automorphism. ■

The automorphism of Theroem 3.3 provides the necessary conditions for vertex-, edge- and arc-transitivity of Paley graphs. The arc-transitivity, and thus the vertex- and edge-transitivity as well, can be shown as follows. Let an arc in a Paley graph $P(q)$ be given as $(u_1, v_1), \{u_1, v_1\} \in E(P(q))$. Assume that the arc (u_2, v_2) also exists in $P(q)$, then an automorphism σ_{ab}, as in the above theorem, for which $\sigma_{ab}(u_1) = u_2$ and $\sigma_{ab}(v_1) = v_2$ can be constructed. As

$$\begin{aligned} au_1 + b &= u_2, \\ av_1 + b &= v_2 \end{aligned} \quad \Rightarrow a = \frac{u_2 - v_2}{u_1 - v_1},$$

letting $b = u_2 - au_1$ constructs an automorphism σ_{ab} for which if (u_1, v_1) is an arc in $P(q)$, then $(\sigma(u_1), \sigma(v_1)) = (u_2, v_2)$ is an arc. As such, $P(q)$ is arc-transitive.

Besides their transitive properties, Paley graphs are also noted for being self-complementary and strongly regular [Bollobás, 2001, p. 316]. One especially interesting Paley graph is $P(17)$, for, as will be shown, its inertia bound is not tight.

Example 3.4 (Constructing Payley 17): For $q = 17$ the field $GF(17)$ is constructed by the integer arithmetic modulo 17, meaning that its vertices span the range $[0, 16]$. The different numbers with square roots modulo 17 are

$$\begin{aligned} &\pm 1 \quad \text{with square roots } \pm 1 \text{ for } +1 \text{ and } \pm 4 \text{ for } -1; \\ &\pm 2 \quad \text{with square roots } \pm 6 \text{ for } +2 \text{ and } \pm 7 \text{ for } -2; \\ &\pm 4 \quad \text{with square roots } \pm 2 \text{ for } +4 \text{ and } \pm 8 \text{ for } -4; \\ &\pm 8 \quad \text{with square roots } \pm 5 \text{ for } +8 \text{ and } \pm 3 \text{ for } -8. \end{aligned}$$

Thus in P(17), each vertex x is adjacent to the vertices $x \pm 1 \pmod{17}$, $x \pm 2 \pmod{17}$, $x \pm 4 \pmod{17}$ and $x \pm 8 \pmod{17}$. $P(17)$ is depicted in Figure 3.1.

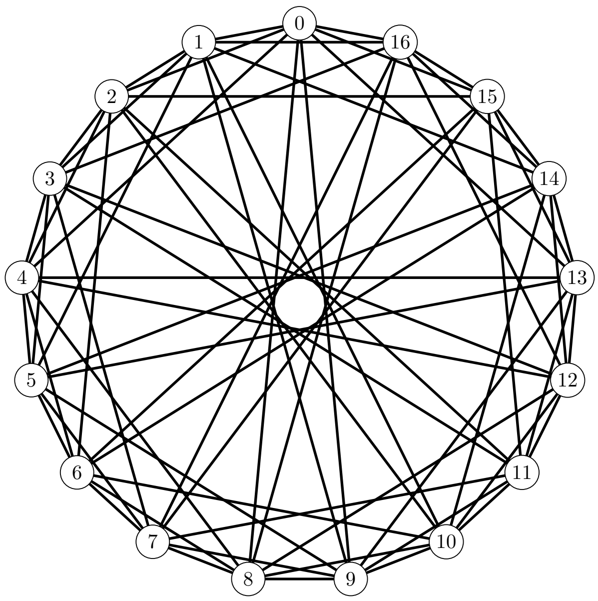


Figure 3.1: The Paley 17 graph $P(17)$.

To properly show, that the inertia bound of $P(17)$ is not tight, other properties of Paley 17 and its subgraphs will be presented.

Property 1: Paley 17 has independence number $\alpha(P(17)) = 3$.

One can see this for one self using, for example, Figure 3.1. Here the vertices 1,7 and 12 are one set of independent vertices, which are otherwise adjacent to all other vertices of the graph. Other sources, [Shearer] and [Exoo], have computed the independence numbers of Paley graphs spanning up to 10 000 vertices and also found the independence number of $P(17)$ to be 3.

Property 2: The graph $P(17)$ is α-critical.

As $P(17)$ is arc-transitive, the symmetry of the graph means, that if removing any edge increases $\alpha(P(17))$, then it is α-critical. Deleting the edge $\{0,1\}$ means that vertex 0 is independent from all the vertices 1, 7 and 12 mentioned above. Thus a new independent set $[0,1,7,12]$ is created, and so $\alpha\left(P(17)-\{0,1\}\right)=4$. Thus $P(17)$ is α-critical.

As mentioned in Example 3.4 the non-zero squares modulo 17 are $\pm1,\pm2,\pm4$ and ±8. A k-edge of $P(17)$ is an edge $\{v_1,v_2\}$ for which $v_1-v_2=\pm k \pmod{17}, k\in\{1,2,4,8\}$. For any of these values of k, the set of k-edges form cycles of length 17 as follows:

k	$k-$ edge cycles
1	$0-1-2-\cdots-16-0$
2	$0-2-4-\cdots-14-16-1-3-\cdots-13-15-0$
4	$0-4-8-12-16-3-7-11-15-2-6-10-14-1-5-9-13-0$
8	$0-8-16-7-15-6-14-5-13-4-12-3-11-2-10-1-9-0$

This gives rise to the next property of $P(17)$.

Property 3: $P(17)$ has a 2-factorization consisting of four cycles of length 17.

We will use $a-b-c$ triangles to show isomorphism between subgraphs of $P(17)$.

Definition 3.5: Let G be a graph of order n. An $a-b-c$ triangle in G consists of an a-edge, a b-edge and a c-edge, that form a triangle $\triangle(u,v,w),\ u,v,w\in V(G)$ with three mutually adjacent vertices of G. △

For example, in $P(17)$ the sets of vertices $\triangle(0,1,2)$, $\triangle(0,2,4)$, $\triangle(0,4,8)$ and $\triangle(0,8,9)$ form respectively a $1-1-2$ triangle, a $2-2-4$ triangle, a $4-4-8$ triangle and an $8-8-1$ triangle.

Theorem 3.6 (Property 4): There exists an automorphism of $P(17)$ which maps the triangle $\triangle(0,1,2)$ to any other triangle. Meaning that the group of automorphisms on $P(17)$ act transitively on its triangles.

Proof: The $P(17)$ graph has 68 triangles, which can be found from its characteristic polynomial. These are 17 of each of the example triangles mentioned above. 17 $1-1-2$ triangles, 17 $2-2-4$ triangles, 17 $4-4-8$ triangles and 17 $8-8-1$ triangles.

Recall Theorem 3.3. The automorphisms given by $\sigma_{1b}, b\in\{0,1,\ldots,16\}$ will map $\triangle(0,1,2)$ to any $1-1-2$ triangle depending on b. Similarly, the automorphisms σ_{ab} for $a\in\{2,4,8\}, b\in\{0,1,\ldots,16\}$ maps $\triangle(0,1,2)$ to any of the $2-2-4$, $4-4-8$ or $8-8-1$ triangles in $P(17)$. ■

The *sign* of the triangles will also serve a function in showing, whether the weight matrix of $P(17)$ is tight. It will be defined as follows:

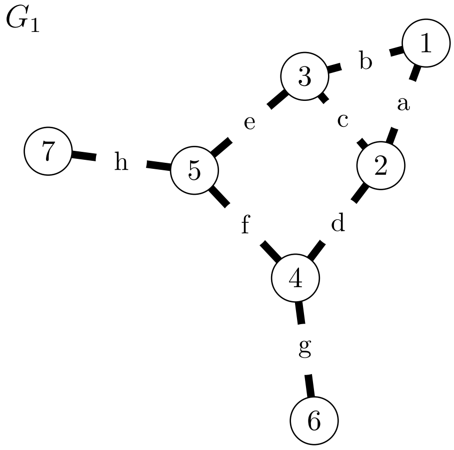
Definition 3.7 (Sign of a triangle): Given a graph G with weight matrix W, the *sign* of a triangle in G is the sign of the product of the entries of W which correspond to the edges of that triangle. △

These properties of $P(17)$ and its triangles will be used in Chapter 4 to show that it does not attain the inertia bound. Also needed for this, will be some specific subgraphs of $P(17)$.

3.2 The induced subgraphs of $P(17)$

In the following, two specific induced subgraphs of $P(17)$, G_1 and G_2, and their principal submatrices will be examined. When the entries in the submatrices are of non-zero weight, their determinants will be non-zero as well. This, together with the inertia bound (2.10) will be used to show how the eigenvalues of the principal submatrices of G_1 and G_2 are distributed. Either 3 positive to 4 negative, or 4 positive to 3 negative eigenvalues.

Starting with G_1, let it be given with weights as in Figure 3.2, with the corresponding weight matrix W_1.



$$W_1 = \begin{bmatrix} 0 & a & b & 0 & 0 & 0 & 0 \\ a & 0 & c & d & 0 & 0 & 0 \\ b & c & 0 & 0 & e & 0 & 0 \\ 0 & d & 0 & 0 & f & g & 0 \\ 0 & 0 & e & f & 0 & 0 & h \\ 0 & 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & 0 & 0 \end{bmatrix}$$

Figure 3.2: The principal subgraph G_1 of $P(17)$ and its weight matrix W_1.

Property 5: G_1 is an induced subgraph of $P(17)$.

This is most easily shown, by reverting the vertex-numbering of G_1 back, so it is consistent with the numbering in $P(17)$. This is done in Figure 3.3, where, also, the vertices and edges are shown as they appear in $P(17)$.

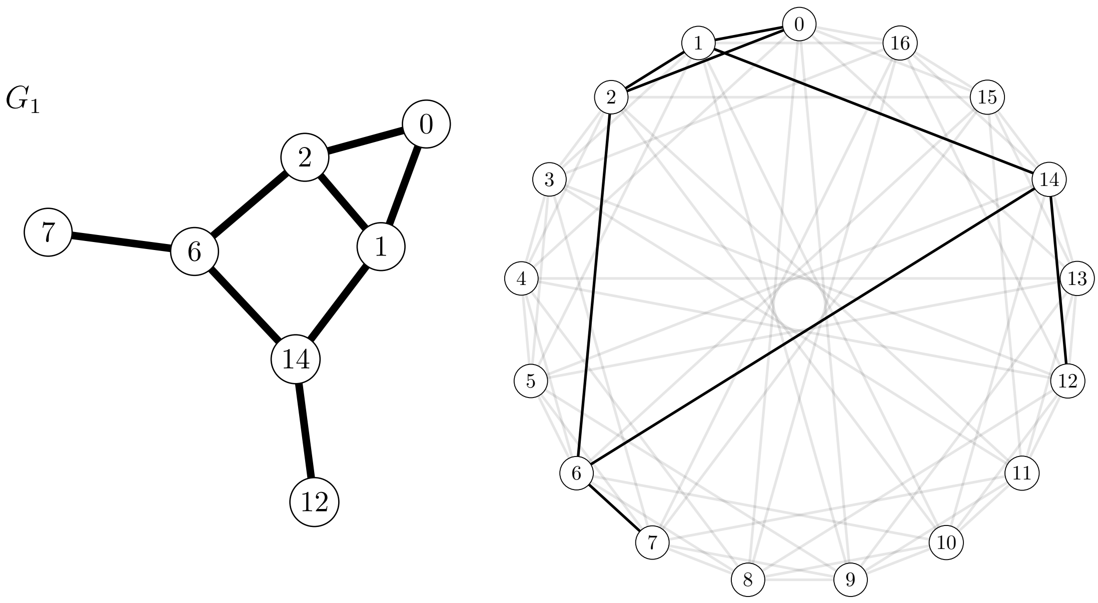


Figure 3.3: G_1 with the same vertex-numbering as in $P(17)$. It is indeed an induced subgraph of $P(17)$.

Property 6: For the weight graph W_1

$$\det(W_1) = 2abcg^2h^2.$$

While this is easily calculated, it is a long process, and so the proof of this statement will be skipped. Instead, if one feels it necessary, it can be verified through the use of calculation tools.

Theorem 3.8 (Property 7): Let the product $abcgh \neq 0$.

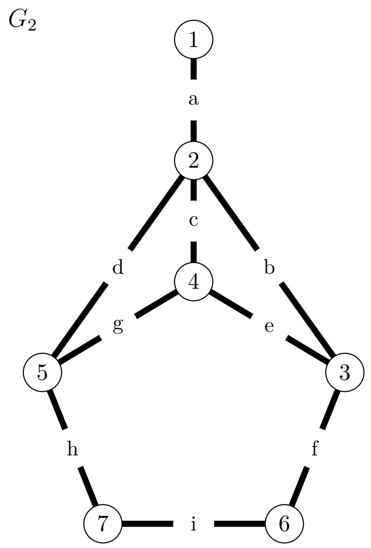
- If $abc > 0$, then $n^+(W_1) = 3$ and $n^-(W_1) = 4$.
- If $abc < 0$, then $n^+(W_1) = 4$ and $n^-(W_1) = 3$.

Proof: Equation (2.10) will be used on G_1 and W_1. Note that G_1 has 7 vertices, so $n = 7$. Since $\alpha(G_1) = 3$ - which is easily verified - Equation (2.10) becomes

$$3 \leq \min\{7 - n^+(W_1), 7 - n^-(W_1)\}.$$

This means that $n^+(W_1) \leq 4$ and $n^-(W_1) \leq 4$. As $abcgh \neq 0$ it follows that $\det(W_1) \neq 0$, meaning that all eigenvalues are non-zero. Thus either $n^+(W_1) = 3$ and $n^-(W_1) = 4$ or $n^+(W_1) = 4$ and $n^-(W_1) = 3$. Now, if $abc > 0$ then $\det(W_1) > 0$, making the product of the eigenvalues of W_1 positive and so $n^-(W_1) = 4$. For $abc < 0$ it is the opposite case, with $\det(W_1) < 0$, and so $n^-(W_1) = 3$. ■

Next is the induced subgraph G_2. Let it be given as in Figure 3.4, and with weight matrix W_2.



$$W_2 = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & b & c & d & 0 & 0 \\ 0 & b & 0 & e & 0 & f & 0 \\ 0 & c & e & 0 & g & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & h \\ 0 & 0 & f & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & h & i & 0 \end{bmatrix}$$

Figure 3.4: The principal subgraph G_2 of $P(17)$ and its weight matrix W_2.

Property 8: The graph G_2 is an induced subgraph of $P(17)$.

In the same style as for G_1, the proof of this is presented in Figure 3.5.

It will also be shown for G_2, how the eigenvalues are distributed. Once again it pertains to the determinant of W_2.

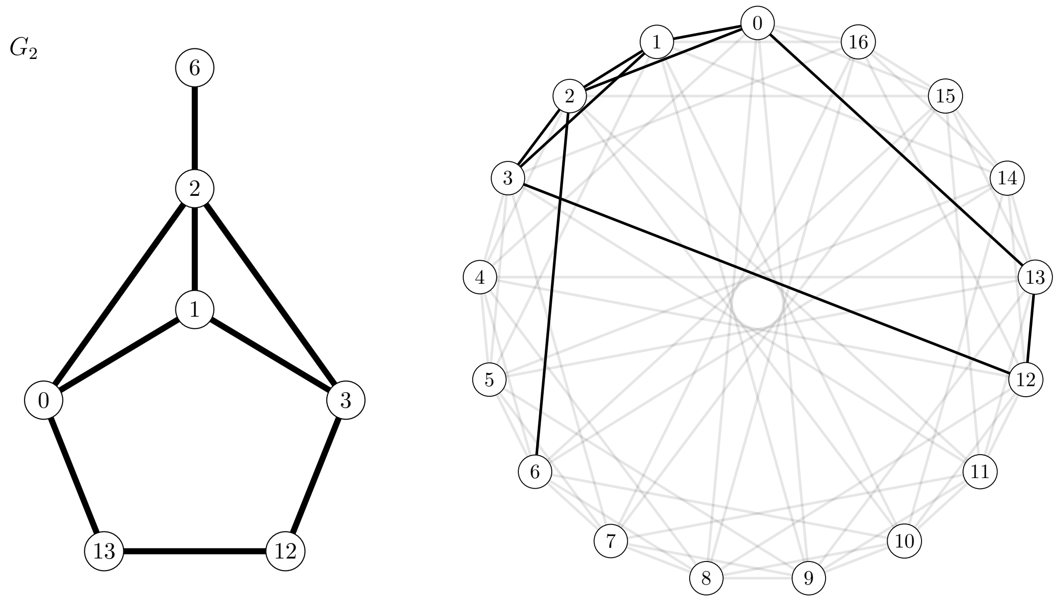


Figure 3.5: G_2 with the same vertex-numbering as in $P(17)$. It is indeed an induced subgraph of $P(17)$.

Property 9: For the weight graph W_2,

$$\det(W_2) = -2a^2efghi.$$

As with the determinant of W_1, this is easily proven, or otherwise shown through the use of computational tools.

Theorem 3.9 (Propterty 10): Let the product $aefghi \neq 0$.

- If $efghi < 0$, then $n^+(W_2) = 3$ and $n^-(W_2) = 4$.
- If $efghi > 0$, then $n^+(W_2) = 4$ and $n^-(W_2) = 3$.

Proof: The proof of this is analogue to that of Theorem 3.8. It is easily verified that G_2 has $\alpha(G_2) = 3$, and it has 7 vertices so $n = 7$. Thus, by (2.10),

$$3 \leq \min\{7 - n^+(W_2), 7 - n^-(W_2)\}.$$

This means that $n^+(W_2) \leq 4$ and $n^-(W_2) \leq 4$. As $aefghi \neq 0$, $\det(W_2) \neq 0$, meaning that either $n^+(W_2) = 3$ and $n^-(W_2) = 4$ or $n^+(W_2) = 4$ and $n^-(W_2) = 3$. If $efghi < 0$ then $\det(W_2) > 0$ and so $n^-(W_2) = 4$. For $efghi > 0$ it is the opposite case, with $\det(W_1) < 0$, and so $n^-(W_1) = 3$. ■

The goal of introducing the induced subgraphs G_1 and G_2 is to use Theorem 3.6 to show that the triangles of $P(17)$, introduced in Section 3.1, are part of isomorphic copies of the subgraphs. This will be used to deduce the connection between the signs of the triangles of $P(17)$, and whether the weight matrix of $P(17)$ is tight or not.

4. The inertia bound of Payley 17

Some final results, pertaining to the weight matrices W of graphs, are needed before it can be shown, that the weight matrix of $P(17)$ is never tight. It will be shown what conditions are needed to obtain a tight weight matrix, and then proven that these condition cannot exist for $P(17)$ and its weight matrices.

4.1 Tight weight matrices

One problem stemming from the use of the weighted adjacency matrix, is that, in theory, the weight of a given edge could be zero. Usually, this is not the case, and the zero weight is more often reserved for the case where the vertices are not adjacent, but this problem can be avoided for α-critical graphs with tight weight matrices, by using the following property.

Theorem 4.1: Let G be a α-critical graph with a tight weight matrix W. Then $w_{ij} \neq 0$ for all edges $\{i, j\} \in E(G)$.

Proof: Assume that W is a tight weight matrix of a graph G of order n, so that

$$\alpha(G) = \min\{n - n^+(W), n - n^-(W)\},$$

and that $w_{ij} = 0$ for some edge $\{i, j\}$ in G. Then W is also the weight matrix for $G - \{i, j\}$, and so $\alpha(G - \{i, j\}) \leq \min\{n - n^+(W), n - n^-(W)\}$ by the inertia bound. Then $\alpha(G - \{i, j\}) \leq \alpha(G)$ which is a contradiction to the definition of α-critical graphs. ∎

As it has previously been stated, in Property 2, that $P(17)$ is α-critical, this gives rise to the following property:

Property 11: Any tight weight matrix W of Paley 17 has $w_{ij} \neq 0$ for all of the edges $\{i, j\} \in E(P(17))$.

This property allows the usage of the two theorems concerning the eigenvalues of the induced subgraphs G_1 and G_2 of $P(17)$, Theorems 3.8 and 3.9. The proof of the following theorem, regarding the triangles that make up $P(17)$, is also a consequence of that property.

Theorem 4.2: Let W be a tight weight matrix of $P(17)$. Every triangle of $P(17)$ has the same sign.

Proof: Let W be a tight weight matrix of $P(17)$. Suppose now, for the sake of contradiction, that some triangle $\triangle_1$ exists in $P(17)$, which has a different sign than the triangle $\triangle(0, 1, 2)$. Without loss of generality, assume the sign of $\triangle(0, 1, 2)$ to be negative. By Theorem 3.3 an automorphism σ exists on $P(17)$, which maps $\triangle(0, 1, 2)$ to $\triangle_1$. This means that σ maps G_1 to some isomorphic subgraph which contains $\triangle_1$. Call this isomorphic subgraph H_1. By Property 11 the entries of W that correspond to edges in $P(17)$ are non-zero.

Then Theorem 3.8, and the different signs of the triangles, imply that the principal submatrix of W corresponding to G_1 will have 4 positive and 3 negative eigenvalues. Meanwhile the principal submatrix of W corresponding to H_1 will have 3 positive and 4 negative eigenvalues.

Recall that $\alpha(G) = 3$, then, according to Theorem 2.16, W is not tight, which is a contradiction. Thus, all triangles must have the same sign. ∎

This fact for the triangles of $P(17)$ will be used to determine the possible distributions for the signs of the edges of $P(17)$. To reduce the number of cases that need to be examined, the following theorem is introduced.

Theorem 4.3: Let A be a symmetric $n \times n$ matrix with non-zero sub- and superdiagonal, meaning, respectively, the entries of the matrix that are just below and just above the main diagonal. There exists a matrix D, with diagonal entries $d_{ii} \in \{-1, 1\}$, such that the sub- and superdiagonals of DAD are positive.

Proof: The proof of this theorem is by induction on n. If $n = 2$ and the entries $a_{12} = a_{21} > 0$ then simply letting $D = I_2$, the unit matrix, gives the necessary result. So assume instead that $a_{12} = a_{21} < 0$. Let D be a diagonal matrix with just one entry 1 and one entry -1. Then the off-diagonal entries of matrix DAD will be positive.

Assume the hypothesis to be true for $n = k$. Let A be a symmetric matrix of order $k + 1$, and with non-zero entries in the sub- and superdigaonals. Let A_1 be the principal submatrix of A which omits row and column $k + 1$. As A_1 is a $k \times k$ symmetric matrix, there exists a diagonal matrix D_1, so that $D_1 A_1 D_1$ has positive sub- and superdiagonals. By using these submatrices, we can construct the following equation

$$DAD = \begin{bmatrix} D_1 & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} A_1 & \mathbf{a} \\ \mathbf{a}^\top & a_{k+1,k+1} \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} D_1 A_1 D_1 & d D_1 \mathbf{a} \\ d \mathbf{a}^\top D_1 & d^2 a_{k+1,k+1} \end{bmatrix}.$$

Note here, that an appropriate choice of d will determine the sign of the matrix entries $(DAD)_{k,k+1} = (DAD)_{k+1,k}$. This concludes the proof. ∎

No matter the choice of entries in D, D^2 will always be the unit matrix. Thus the matrices A and DAD are similar, and with the same eigenvalues.

Example 4.4 (Constructing *D* for a Paley 13 graph): An example of the construction of a matrix D as in Theorem 4.3 will be given here. The tight weight matrix for the Paley 13 graph, presented in [Sinkovic, 2018, p. 40], will be used, as it is a symmetric matrix with non-zero sub- and superdiagonal, and with these containing both negative and positive values. This matrix will be denoted by A.

$$A = \begin{bmatrix} 0 & -2 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & -4 & 3 & 0 & 2 \\ -2 & 0 & 2 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 4 & 3 & 0 \\ 0 & 2 & 0 & -4 & 0 & -2 & 3 & 0 & 0 & 0 & 0 & -4 & 2 \\ -4 & 0 & -4 & 0 & 3 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 2 \\ 2 & 3 & 0 & 3 & 0 & 3 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 3 & 0 & 4 & 0 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 4 & 0 & 3 & 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & 0 & 3 & 0 & 3 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 3 & 0 & -4 & 0 & -4 & 2 \\ -4 & 0 & 0 & 0 & 0 & 3 & -2 & 0 & -4 & 0 & 2 & 0 & 2 \\ 3 & 4 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 2 & 0 & -2 & 0 \\ 0 & 3 & -4 & 0 & 0 & 0 & 0 & 2 & -4 & 0 & -2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 \end{bmatrix}.$$

Thus, a diagonal matrix D, such that DAD has positive sub- and superdiagonal, must be constructed so that DAD changes the sign of entries $a_{1,2} = a_{2,1}$, $a_{3,4} = a_{4,3}$, $a_{9,10} = a_{10,9}$ and

$a_{11,12} = a_{12,11}$. The following matrix D fulfils this purpose.

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

As can be computed

$$DAD = \begin{bmatrix} 0 & 2 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 4 & -3 & 0 & 2 \\ 2 & 0 & 2 & 0 & -3 & -3 & 0 & 0 & 0 & 0 & 4 & -3 & 0 \\ 0 & 2 & 0 & 4 & 0 & 2 & -3 & 0 & 0 & 0 & 0 & 4 & -2 \\ -4 & 0 & 4 & 0 & 3 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 2 \\ 2 & -3 & 0 & 3 & 0 & 3 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 3 & 0 & 4 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 4 & 0 & 3 & 0 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & 0 & 3 & 0 & 3 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 3 & 0 & 4 & 0 & -4 & 2 \\ 4 & 0 & 0 & 0 & 0 & -3 & 2 & 0 & 4 & 0 & 2 & 0 & -2 \\ -3 & 4 & 0 & 0 & 0 & 0 & -3 & -3 & 0 & 2 & 0 & 2 & 0 \\ 0 & -3 & 4 & 0 & 0 & 0 & 0 & 2 & -4 & 0 & 2 & 0 & 2 \\ 2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 2 & 0 \end{bmatrix},$$

and as can be seen, the sub- and superdiagonals of DAD have only positive entries. Further, the eigenvalues of each of the matrices A and DAD can be confirmed to be the same, and as such, their inertia are the same as well.

As a consequence of Theorem 4.3, any weight matrix of $P(17)$ can be assumed to have positive entries on its sub- and superdiagonal. These entries correspond to all 1-edges in $P(17)$ with the exception of the $\{0, 16\}$-edge. That leaves only two cases to consider when using Theorem 4.2 to determine the signs of the edges of $P(17)$. The first case, where all 1-edges of $P(17)$ are positive, and the second case, where all the 1-edges are positive except the edge $\{0, 16\}$, which is negative.

Theorem 4.5: Let W be a tight weight matrix of $P(17)$ and assume that all 1-edges are positive. Then all edges of $P(17)$ are positive and the sign of every triangle of $P(17)$ is positive.

Proof: Let W be a tight weight matrix for $P(17)$. By Theorem 4.2, all triangles in $P(17)$ have the same sign. Assume that sign to be negative. As every 2-edge in $P(17)$ belong to a $1-1-2$ triangle, and as all 1-edges have positive sign, the 2-edges must have negative sign. As every 4-edge belong to a $2-2-4$ triangle, every 4-edge must have negative sign as well. So too for the 8-edges, as they all belong to a $4-4-8$ triangle, the 8-edges must be negative. But then the $8-8-1$ triangles of $P(17)$ will have positive sign, meaning that it is not possible for all 1-edges in $P(17)$ to have positive sign while all triangles have negative sign.

Thus, all triangles must have positive sign. By a similar argument to above, this forces all 2- 4- and 8-edges in $P(17)$ to be positive as well. 2-edges beacuse of the $1-1-2$ triangle, 4-edges because of the $2-2-4$ triangle and 8-edges because of the $4-4-8$ triangle. ■

Theorem 4.6: Let W be a tight weight matrix of $P(17)$ and assume that all 1-edges are positive with the exception of the $\{0,16\}$-edge, which is negative. Then the remaining edges are negative except the 2-edges $\{0,15\}$ and $\{1,16\}$, the 4-edges $\{0,13\}$, $\{1,14\}$, $\{2,15\}$ and $\{3,16\}$, and the 8-edges $\{0,9\}$, $\{1,10\}$, $\{2,11\}$, $\{3,12\}$, $\{4,13\}$, $\{5,14\}$ $\{6,15\}$ and $\{7,16\}$. Also, the sign of every triangle is negative.

Proof: Let W be a tight weight matrix for $P(17)$. By Theorem 4.2, all triangles in $P(17)$ have the same sign. Assume that sign to be positive. As every 2-edge in $P(17)$ belong to a $1-1-2$ triangle, and as all 1-edges except edge $\{0,16\}$ have positive sign, all 2-edges, except $\{1,16\}$ and $\{0,15\}$, must have positive sign. As every 4-edge belong to a $2-2-4$ triangle, and all 2-edges except $\{1,16\}$ and $\{0,15\}$ are positive, every 4-edge, except $\{0,13\}$, $\{1,14\}$, $\{2,15\}$ and $\{3,16\}$, must have positive sign as well. Since every 8-edge belong to a $4-4-8$ triangle, and all 4-edges except $\{0,13\}$, $\{1,14\}$, $\{2,15\}$ and $\{3,16\}$ are positive, all 8-edges, except $\{0,9\}$, $\{1,10\}$, $\{2,11\}$, $\{3,12\}$, $\{4,13\}$, $\{5,14\}$ $\{6,15\}$ and $\{7,16\}$ must be positive. Look now at the $8-8-1$ triangle $\triangle(0,8,9)$. As the edge $\{0,9\}$ is the only one with negative sign, this triangle has negative sign as well. Thus, a contradiction to every triangle having positive sign. This means that every triangle must have negative sign. Note that by changing this property also switches the sign of all edges in the argument above. Therefore, all 2-edges are negative with the exception of $\{1,16\}$ and $\{0,15\}$, all 4-edges are negative with the exception of $\{0,13\}$, $\{1,14\}$, $\{2,15\}$ and $\{3,16\}$ and all 8-edges are negative with the exception of $\{0,9\}$, $\{1,10\}$, $\{2,11\}$, $\{3,12\}$, $\{4,13\}$, $\{5,14\}$ $\{6,15\}$ and $\{7,16\}$. ■

With these results it can be proven that the inertia bound of $P(17)$ is not tight.

4.2 The inertia bound is not tight

The strategy for proving that the inertia bound of Paley 17 is not tight, is to prove that, given the results of Section 3.2 and 4.1, $P(17)$ cannot have a tight weight matrix.

Theorem 4.7: The Paley graph on 17 vertices has no tight weight matrix.

Proof: Suppose for contradiction, that W is a tight weight matrix of Paley 17. By Theorem 4.3, there exists a diagonal matrix D, such that the entries of the sub- and superdiagonal of DWD are positive. As D is a diagonal matrix, non-zero entries of W stay non-zero in DWD, and entries of zero in W remain zero in DWD. Furthermore, as D^2 is the unit matrix, DWD is similar to W. This means that DWD is a tight weight matrix for $P(17)$. Let $DWD = \tilde{W}$.

The entries of the sub- and superdiagonal of $\tilde{W}$ correspond to the 1-edges in $P(17)$, except the edge $\{0,16\}$. By Property 11, all entries of $\tilde{W}$ corresponding to edges in $P(17)$ are non-zero. This will allow the use of Theorems 3.8 and 3.9. There are now two cases to consider, the first case where $\{0,16\}$ is positive, and the second where $\{0,16\}$ is negative.

In the first case, all edges of $P(17)$ will be positive, as per Theorem 4.5. Thus, all entries of $\tilde{W}$ will be positive as well. According to Property 8 and Theorem 3.9, G_2 is an induced subgraph of $P(17)$, and $\tilde{W}$ has a principal submatrix corresponding to G_2, with four *positive* eigenvalues. At the same time, according to Property 5 and Theorem 3.8, G_1 is an induced subgraph of $P(17)$, and $\tilde{W}$ has a principal submatrix corresponding to G_1, which has four *negative* eigenvalues.

Thus, in the case where the edge $\{0, 16\}$ of $P(17)$ is positive, $\tilde{W}$ has principal submatrices which have four positive and four negative eigenvalues. By Theorem 2.16, as $\alpha\left(P(17)\right) = 3$, $\tilde{W}$ is not tight. A contradiction.

In the second case, all the 1-edges of $P(17)$ are positive, except for $\{0, 16\}$. By Theorem 4.6, all other edges of $P(17)$ are negative, with the exception of the 2-edges $\{0, 15\}$ and $\{1, 16\}$, the 4-edges $\{0, 13\}$, $\{1, 14\}$, $\{2, 15\}$ and $\{3, 16\}$, and the 8-edges $\{0, 9\}$, $\{1, 10\}$, $\{2, 11\}$, $\{3, 12\}$, $\{4, 13\}$, $\{5, 14\}$ $\{6, 15\}$ and $\{7, 16\}$. Now, the induced subgraph G_2, which has vertices 0,1,2,3,6,12 and 13 (see Figure 3.5), has a 5-cycle consisting of the edges $\{0, 1\}$, $\{1, 3\}$, $\{3, 12\}$, $\{12, 13\}$ and $\{13, 0\}$. Of these, the edges $\{0, 1\}$, $\{3, 12\}$, $\{12, 13\}$ and $\{13, 0\}$ are positive, while $\{1, 3\}$ is negative. Thus, by Theorem 3.9, $\tilde{W}$ has a principal submatrix corresponding to G_2, which has four negative eigenvalues. The induced subgraph G_1, with vertices 0,1,2,6,7,12 and 14 (see Figure 3.3), has a triangle, which consists of the edges $\{0, 1\}$, $\{0, 2\}$ and $\{1, 2\}$. By Theorem 4.6, this triangle will have negative sign, and so, by Theorem 3.8, $\tilde{W}$ has a principal submatrix corresponding to G_1. which has four positive eigenvalues.

This means that in the case where the edge $\{0, 16\}$ of $P(17)$ is negative, $\tilde{W}$ still has principal submatrices with four positive and four negative eigenvalues. Again, by Theorem 2.16, $\tilde{W}$ is not tight.

Thus, there does not exist a tight weight matrix W for $P(17)$ ∎

This means, that although graphs can attain the inertia bound given in Equations (2.10) and (2.11), one cannot assume that all graphs can achieve equality always. Thus, when examining the independence number of specific graphs, it is either necessary to compute the independence number, or otherwise determine which properties of the graphs lead to tight or non-tight weight matrices.

With the proof, that the inertia bound of $P(17)$ is not tight, it is also relevant to look closer at the induced subgraphs of $P(17)$. Specifically, [Sinkovic, 2018] shows, that for the subgraph obtained by deleting any one vertex of $P(17)$ together with its associated edges, the inertia bound of the resulting subgraph is also not tight, thus providing another example of a graph with this property.

One can also look further into the properties that lead to attaining the bound, and whether specific subgraphs of $P(17)$ has these properties. Further, the inertia bound can be generalized, to be over any field $\mathbb{F}$, instead of just over the real numbers, as has been the case in this report. This problem will be presented in the following, final chapter, together with some conditions of equality in the inertia bound.

5. Equality of the isotropic bound

In this chapter the inertia bound derived in Chapter 2 will be generalized over a field $\mathbb{F}$. Further properties for equality in the bound will also be presented, both in the general case and for graphs with weights from $\mathbb{R}$. This chapter is based on [Elzinga, 2007], and as it intents to give a short presentation of some of the results of this work, those with relevance to the inertia bound, it will not go into the same amount of detail. For further study, the sourced paper is recommended.

5.1 Conditions for attaining the bound

Define $\mathbb{F}^n$ as the vector space of all $n \times 1$ column vectors with entries from a field $\mathbb{F}$, and let $e_i,\ i \in [n]$ denote the i'th standard basic vector. A matrix, the entries of which are elements of $\mathbb{F}$, is called an $\mathbb{F}$-matrix.

If S is a subset of a set V of vertices to a graph G, let then $U_S = \text{span}\{e_i | i \in S\}$ the subspace of $\mathbb{F}^n$ spanned by the vectors $e_i,\ i \in S$. If S is an independent set, then the principal submatrix of an adjacency matrix A corresponding to S is a zero matrix, and so $x^\top Ay = 0$ for all $x, y \in U_S$. Denote by $\iota_{\mathbb{F}}(A)$ the maximum dimension of a subspace U, such that $x^\top Ay = 0$ for all $x, y \in U$. Then

$$\alpha(G) = |S| = \dim(U_S) \leq \iota_{\mathbb{F}}(A)$$

$$\implies \alpha(G) \leq \iota_{\mathbb{F}}(A). \tag{5.1}$$

Similarly to the definition of a weight matrix with entries in $\mathbb{R}$, a weight matrix can be defined with entries in the $\mathbb{F}$-space.

Definition 5.1 ($\mathbb{F}$-weight matrix): An $\mathbb{F}$-weight matrix of a graph G is a symmetric $\mathbb{F}$-matrix $\hat{A}$ such that $\hat{A}_{ij} = 0$ whenever $\{i, j\} \notin E(G)$. △

Now, a subspace U of $\mathbb{F}^n$ is said to be $\hat{A}$-*isotropic*, if $x^\top \hat{A} y = 0$ for all $x, y \in U$. By the above argument, the maximum dimension of an $\hat{A}$-isotropic subspace $\iota_{\mathbb{F}}(\hat{A})$ is an upper bound to the independence number of a graph G. That is,

$$\alpha(G) \leq \iota_{\mathbb{F}}(\hat{A}). \tag{5.2}$$

This parameter, $\iota_{\mathbb{F}}(\hat{A})$, is called the *Witt index* of the matrix $\hat{A}$ [Elzinga, 2007, p. 6]. Let

$$\iota^*_{\mathbb{F}}(G) = \min_{\hat{A}} \iota_{\mathbb{F}}(\hat{A})$$

denote the *minimum Witt index* of G over all $\mathbb{F}$-weight matrices $\hat{A}$. As each $\mathbb{F}$-weight matrix $\hat{A}$ serve as an upper bound for $\alpha(G)$, we have

$$\alpha(G) \leq \iota^*_{\mathbb{F}}(G). \tag{5.3}$$

This bound is called the *isotropic bound* on $\alpha(G)$ over the space $\mathbb{F}$.

Thus, this is an alternative way to bound the size of the independence number of graphs. Similar to the work done by [Sinkovic, 2018], the properties of the Witt index can be found and examined to determine for which cases the isotropic bound holds equalities or inequalities. This is done in [Elzinga, 2007].

In his work, Elzinga expands on some of the results presented in [Artin, 1957], among them Witt's Theorem, in order to show the conditions necessary for a tight isotropic bound over a given field. The presentation of these conditions rely on a theorem, for which some notation will be introduced.

If a vector $x = (x_i)_{i\in[n]}$, and a subset S of $[n]$ are given, then x_S denotes the restriction of x to S, given by $(x_i)_{i\in S}$.

Two square matrices A and B of the same order are *permutation similar*, which is written as $A \sim B$, if $B = P^\top AP$ for some permutation matrix P. If A and B are symmetric matrices, then they are also said to be *congruent*. Further, for a field $\mathbb{F}$ with $2 \neq 0$, one can always obtain a diagonal matrix from a symmetric matrix A, by performing the correct simultaneous elemental row and column operations on A, as shown by [Newman, 1972, pp. 62-63]. As such, for fields $\mathbb{F}$ with $2 \neq 0$, there exists a diagonal matrix D, such that $A = P^\top DP$ for some invertible matrix P, and so A is congruent to a diagonal matrix D.

Theorem 5.2: Let A be a symmetric matrix of order n with entries from a field $\mathbb{F}$ with $2 \neq 0$ and let S be the set of indices of a principal zero submatrix of A. Then $\iota_{\mathbb{F}}(A) \geq |S|$ with equality if and only if the associated partitioned matrix

$$\begin{array}{c} \\ S \\ \\ \end{array}\begin{array}{c} S \\ \left[\begin{array}{cc} O & B \\ B^\top & C \end{array}\right] \\ \end{array} \sim A \tag{5.4}$$

satisfies the following condition:

$$\text{If } Bx = 0, \text{ and } x^\top Cx = 0 \text{ then } x = 0.$$

Proof: The proof of the inequality $\iota_{\mathbb{F}}(A) \geq |S|$ is similar to the arguments that lead to Equation (5.1), but using S as the index set of a maximum principal zero submatrix of A.

Assume now that S is the index set of a principal zero submatrix of A, and $|S| = \iota_{\mathbb{F}}(A)$, and let x be a vector such that $Bx = 0$ and $x^\top Cx = 0$. Denote the complement of S in $[n]$ by $\bar{S}$ and let W be the subspace of vectors $w \in \mathbb{F}^n$ with w_S arbitrary and $w_{\bar{S}} = tx$ for some $t \in \mathbb{F}$. Then $w^\top Aw = 2tw_S^\top Bx + t^2x^\top Cx = 0$ for all $w \in W$. Thus W is A-isotropic, and $x = 0$, because otherwise $\iota_{\mathbb{F}}(A) \geq \dim W > |S|$, which is a contradiction.

Now, assume that the conditions on vectors x holds for a set S of indices of a principal zero submatrix of A, but $\iota_{\mathbb{F}}(A) > |S|$. As the subspace $U_S = \operatorname{span}\{e_i | i \in S\}$ is A-isotropic and $\dim U_S = |S| < \iota_{\mathbb{F}}(A)$, a theorem of Witt [Artin, 1957, p. 122] implies that there is an A-isotropic subspace properly containing U_S. Thus there is a vector $w \neq 0$ with $w_S = 0$ such that $u + w$ is A-isotropic for all $u \in U_S$. Let $x = w_{\bar{S}}$. Then, for the partition in (5.4), $u + w = u_S \oplus x$ so $(u + w)^\top A(u + w) = 2u_S^\top Bx + x^\top Cx$ for all $u \in U_S$. Taking $u = 0$ implies that $x^\top Cx = 0$. Then $2u_S^\top Bx = 0$ for all $u \in U_S$. Because $2 \neq 0$ in $\mathbb{F}$, and because the entries in u_S are arbitrary, this implies that $Bx = 0$. But $x \neq 0$, meaning there is a contradiction. ∎

Recalling Equation (5.3) implies the following corollary to Theorem 5.2, describing the conditions for attaining the isotropic bound.

Corollary 5.3: Let G be a graph with n vertices and let $\mathbb{F}$ be a field with $2 \neq 0$. Then $\alpha(G) = \iota^*_{\mathbb{F}}(G)$ if and only if for some maximum independent set S of vertices in G and some $\mathbb{F}$-weight matrix $\hat{A}$ of G (partitioned as in (5.4)), the only C-isotropic vector in the nullspace of $\hat{B}$ is the zero vector.

Meaning that for equality to appear in (5.3) for a graph G, the $\mathbb{F}$-weight matrix of G must have a partition as in (5.4), for which both $Bx = 0$ and $x^\top Cx = 0$ only if $x = 0$. An example of graphs that fulfils the conditions of Corollary 5.3 will be presented below.

Example 5.4 (Bipartite graphs fulfil Corollary 5.3): Let $G = (V_1, V_2, E)$ denote a bipartite graph with the partition of its vertices into the sets V_1 and V_2, with $|V_1| \leq |V_2|$, and edge set E. The maximum independent set of G is then the partitioned vertex set V_2. A set of vertex disjoint edges, edges that have no end point vertices in common, can be constructed going from V_1 into V_2. Denote this vertex disjoint set of edges by M. Such a set M is referred to as a *matching* in G [Elzinga, 2007, p. 13]. Let the entries of M be of non-zero weight, then a weight matrix $\hat{A}$ can be constructed, which has zeroes everywhere but the entries of M. Then $\hat{A}$ can be partitioned as in (5.4) by

$$\hat{A} \sim \begin{matrix} & V_2 & \\ V_2 & \begin{bmatrix} O & \hat{B} \\ \hat{B}^\top & \hat{C} \end{bmatrix} \end{matrix} = \begin{bmatrix} O_1 & M \\ M^\top & O_2 \end{bmatrix},$$

by letting the entries in $\hat{B}$ correspond to only the vertex disjoint edges in M. As O_1 will correspond to the largest principal zero submatrix of $\hat{A}$, the conditions of Corollary 5.3 still apply to $\hat{A}$.

To show equality in Equation (5.3) it will be enough to show that the only vector in the nullspace of $\hat{B}$ is the zero vector. As the disjoint edges of M make up $\hat{B}$, the columns of $\hat{B}$ are independent, and so $\hat{B}x = 0$ only for $x = 0$. Thus equality exists in (5.3) for all bipartite graphs - an example of which was seen earlier, in Section 2.4.

Next will be some properties of the isotropic bound when looking at an ordered field such as $\mathbb{R}$.

5.2 The isotropic bound over ordered fields and real numbers

As [Elzinga, 2007] mentions in his thesis, the inertia bound, as presented in Chapter 2, is really a special case of the isotropic bound, where the space $\mathbb{F}$ is replaced by the space of real numbers $\mathbb{R}$. Specifically, for a weight matrix W of a graph G, with entries of an ordered field $\mathbb{F}$,

$$\iota_{\mathbb{F}}(W) = n^0(W) + \min\{n^+(W), n^-(W)\}, \tag{5.5}$$

as the inertia of a matrix is well-defined, given that its entries are from an ordered field. This means, that it is possible to write Equation (2.11) using the minimum Witt index:

$$\alpha(G) \leq \iota^*_{\mathbb{F}}(G). \tag{5.6}$$

As $\mathbb{R}$ specifically is an ordered field, this does not have any effect on the results of previous chapters, but it means that the equations (2.11) and (5.6) are equivalent ways of expressing the inertia bound. However, working in unspecified ordered fields makes it possible to show the equality in Equation (5.5).

As observed earlier, for a field $\mathbb{F}$ with $2 \neq 0$, there exists a diagonal matrix D, such that $A = P^\top DP$ for a symmetric $\mathbb{F}$-matrix A. By [Newman, 1972, p. 69], if $\mathbb{F}$ is an ordered field, then the number of positive, negative and zero diagonal entries of D are the same for all other diagonal matrices congruent to D. For now, denote by $n^+(A)$, $n^-(A)$ and $n^0(A)$ the number of positive, negative and zero elements of the diagonal entries of D. Then the inertia of A over the ordered field F is the triple $(n^+(A), n^-(A), n^0(A))$, similarly to the inertia over $\mathbb{R}$. Denote the number of nonnegative and nonpositive diagonal entries of a matrix D by respectively $n_0^+(A) = n^0(A) + n^+(A)$ and $n_0^-(A) = n^0(A) + n^-(A)$.

If a matrix A is of order n, a subspace U of the field $\mathbb{F}^n$ is said to be respectively *positive semidefinite* or *negative semidefinite* with respect to A if, respectively, $u^\top Au \geq 0$ or $u^\top Au \leq 0$ for all $u \in U$.

Theorem 5.5: Let A be an $n \times n$ symmetric matrix with entries from an ordered field $\mathbb{F}$. If U is respectively positive semidefinite or negative semidefinite with respect to A, then, respectively, $\dim U \leq n_0^+$ or $\dim U \leq n_0^-(A)$, and in each case there exists a subspaces U such that equality is attained.

Proof: Let P be an invertible matrix and D a diagonal matrix such that $A = P^\top DP$. Then $x^\top Ax = (Px)^\top D(Px)$ and it follows, that a subspace U is positive semidefinite with respect to A if and only if the subspace $W = PU = \{Pu | u \in U\}$ is positive semidefinite with respect to D. Thus, it is sufficient to prove the result for a subspace W that is positive semidefinite with respect to a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$.

Let $V^- = V^-(D)$ be the subspace of $\mathbb{F}^n$ spanned by the $n^-(D)$ standard basis vectors e_i for which $d_i < 0$. Then $\dim V^- = n^-(D)$ and $x^\top Dx < 0$ for all nonzero vectors $x \in V^-$. Therefore, if W is positive semidefinite with respect to D, then $W \cap V^- = \{0\}$. Thus $\dim W + \dim V = \dim(W + V^-) \leq n$ so $\dim W \leq n - n_0^-(D) = n_0^+(A)$. Equality is then attained by the D-isotropic subspace $V_0^+(D)$ spanned by the $n_0^+(D)$ standard basis vectors e_i for which $d_i \geq 0$.

Replacing A by $-A$ in the above gives the corresponding proof for U negative semidefinite. ■

A corollary to the above theorem gives the case of equality in Equation (5.5).

Corollary 5.6: If $(n^+(A), n^-(A), n^0(A))$ is the inertia of a symmetric $n \times n$ matrix A with entries from an ordered field $\mathbb{F}$, then

$$\iota_{\mathbb{F}}(A) \leq n^0(A) + \min\{n^+(A), n^-(A)\}.$$

Equality holds if each positive element of $\mathbb{F}$ is a square.

Proof: If U is an A-isotropic subspace, then U is both positive semidefinite and negative semidefinite with respect to A. Then, by Theorem 5.5,

$$\iota_{\mathbb{F}}(A) \leq \min\{n_0^+(A), n_0^-(A)\} = n^0(A) + \min\{n^+(A), n^-(A)\}.$$

Now let P be an invertible matrix, and $D = \operatorname{diag}(d_1, \ldots, d_n)$, such that $P^\top AP = D$. If each positive element of $\mathbb{F}$ is a square, then there exists positive field elements a_i such that $d_i = a_i^2$ when $d_i > 0$ and $d_i = -a_i^2$ when $d_i < 0$. Let $\hat{D} = \operatorname{diag}(a_i^{-1}, \ldots, a_n^{-1})$ where $a_i = 1$ if $d_i = 0$. Then $\hat{D}^\top D\hat{D}$ is a diagonal matrix with diagonal entries 1, -1 or 0. Thus, by replacing P with $P\hat{D}$, we may assume that D is a diagonal matrix with $n^+(A)$ entries equal to 1, $n^-(A)$ entries equal to -1 and all other entries 0.

Let $p = \min\{n^+(A), n^-(A)\}$ and suppose that the diagonal entries are ordered so that $d_i = (-1)^i$, for $i = 1, \ldots 2p$. Let W be the subspace consisting of all vectors x with $x_{i-1} = x_i$ for even i, $2 \leq i < 2p$ arbitrary when $d_i = 0$ and $x_i = 0$ otherwise. Then W is a D-isotropic subspace and $U = PW$ is an A-isotropic subspace. Also $\dim U = \dim W = n^0(A) + p = n^0(A) + \min\{n^+(A), n^-(A)\}$. ■

Thus, for ordered fields $\mathbb{F}$, of which $\mathbb{R}$ is one, by Corollary 5.6,

$$\alpha(G) \leq \iota_{\mathbb{F}}(W) \leq n^0(W) + \min\{n^+(W), n^-(W)\},$$

for all weight matrices W of a graph G. Further, the congruence between the diagonal matrices D, with the number of positive, negative and zero values of the diagonal entries being the same, is what leads to the possibility of looking specifically at the eigenvalues, in the case of the real numbers. This is because a symmetric matrix A with elements of $\mathbb{R}$, such as a weight matrix for graph, is always diagonalisable. This means, that there exists a diagonal matrix D with its entries being the eigenvalues of A, and an invertible matrix P with its columns the eigenvectors of A, such that $A = P^{-1}DP$.

A final result for the inertia bound is relevant to show here, as it pertains to when graphs attain the inertia bound from just their regular adjacency matrix.

5.3 The inertia bound attained with adjacency matrices

Some graphs will attain the inertia bound even from their ordinary adjacency matrices. This property is dependent on which types of subgraphs can be obtained from the main graph, and how a decomposition of the graph can be made. A series of new properties must be presented before the result can be proven, however.

A *vertex cover* V' of a graph G, is a subset of $V(G)$, such that if an edge $\{u, v\} \in E(G)$ then $u \in V' \vee v \in V'$. This means, that V' is a set, such that every edge in $E(G)$ has an endpoint in V'. For an example of a vertex cover, see Figure 5.1.

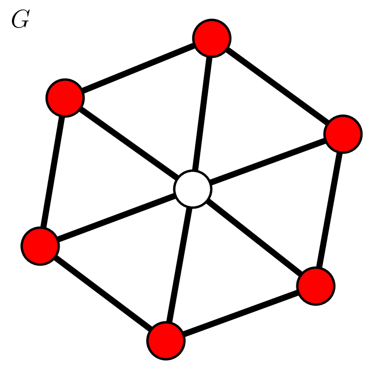


Figure 5.1: A vertex cover of the graph G is marked in red.

Further, a *minimum vertex cover* is a vertex cover of the smallest possible size. The *vertex cover number* of a graph G, denoted by $\tau(G)$, is then the cardinality of the minimum vertex cover. The vertex cover in Figure 5.1 is not the minimum vertex cover, as a cover of smaller size can be made. That is shown in Figure 5.2, from which it can be seen that $\tau(G) = 4$.

It is clear, that any vertices not included in a vertex covering of a graph must be independent. From this insight can be seen, that for a graph G of order n

$$n = \tau(G) + \alpha(G), \tag{5.7}$$

as the smallest vertex covering necessarily leads to the largest possible independent set of vertices.

A *biclique* in a graph G is a complete bipartite subgraph of G, that is, a subgraph $K_{m,n}$ with $|G| \geq m+n, m \leq n$. A *star* in G is a biclique determined by a single vertex and some or all of its incident edges, meaning a complete bipartite subgraph of the type $K_{1,n}$. A *biclique*

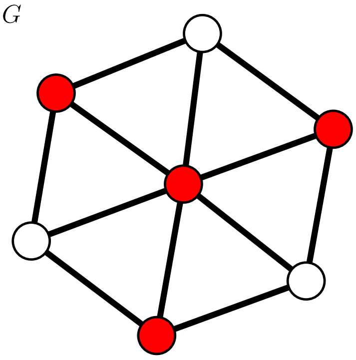


Figure 5.2: A minimum vertex cover of the graph G is marked in red. Note that then $\tau(G) = 4$.

decomposition of G is a partition of the edge set of G by bicliques in G. Meaning the edge sets of the bicliques span the edge set of G.

By [Kratzke et. al., 1988], if a graph G has adjacency matrix A, then a biclique decomposition of G has at least $\max\{n^+(A), n^-(A)\}$ bicliques, and if the decomposition has precisely $\max\{n^+(A), n^-(A)\}$, then G is called *eigensharp*. Note, intuitively, that the vertex cover number $\tau(G)$, is equal to the minimum number of stars needed to partition the edge set of G.

Theorem 5.7: Let G be a graph of order n with ordinary adjacency matrix A. Then

$$\alpha(G) = n^0(A) + \min\{n^+(A), n^-(A)\}$$

if and only if G has an eigensharp decomposition of stars.

Proof: From (5.7), $\tau(G) = n - \alpha(G)$. Then $\alpha(G) = n^0(A) + \min\{n^+(A), n^-(A)\}$ if and only if $\tau(G) = \max\{n^+(A), n^-(A)\}$, which it is if and only if G has an eigensharp decomposition of stars. ■

There are, of course, many other properties pertaining to the inertia bound, both over ordered fields and general fields, and the above results are simply ones with relevancy to preceding chapters. The sources below provide a more in-depth view of these many properties, as well as tackle more questions regarding the inertia bound.

6. Conclusion

Using preliminary theory of graphs and matrices, specifically that of weight matrices and eigenvalues, it has been shown, that the inertia bound cannot always be tight. The inertia bound, derived in Chapter 2 as

$$\alpha(G) \leq \min\{n - n^+(W), n - n^-(W)\}, \tag{2.10}$$

for a graph G of order n with weight matrix W, is indeed not tight for the specific graph Paley 17. This is because, given a theoretical tight weight matrix of Paley 17, an induced subgraph can always be found, which fulfils the criteria of Theorem 2.16, causing a contradiction to occur.

Chapter 3 introduced these induced subgraphs for Paley 17, the subgraphs G_1 and G_2, together with results determining the amount of positive and negative eigenvalues for their weight matrices, depending on the weights of their edges. Important to note is, that no matter these weights, G_1 and G_2 will have either four positive or four negative eigenvalues.

These results are used in Chapter 4 to determine that no matter what tight weight matrix W is given for Paley 17, these two principal submatrices G_1 or G_2 will fulfil the criteria of Theorem 2.16, with one having $\alpha(P(17)) + 1$ positive eigenvalues, and the other $\alpha(P(17)) + 1$ negative eigenvalues. Thus, for all weight matrices of Paley 17, the inertia bound is not tight, and the answer to the question posed in the introduction, of if there always is a way to attain the inertia bound, is *no*.

This question, however, is not the only point of interest given the inertia bound. Although equality cannot be found for any given graph G, one can still look further into the properties that lead to attaining the bound for given graphs, as is done in Chapter 5. Further, one can study if, and which, of the subgraphs of Paley 17, might attain the inertia bound, or one can study how large of a gap in the bound is present for different graphs.

While this report focuses on weight matrices with real values, a point of study would also be what other properties are present for weight matrices over other fields. For example, the results of interlacing, presented in Section 2.3, also hold true for Hermitian matrices, matrices with entries from $\mathbb{C}$ that are equal to their own conjugate transpose. So it might be possible, that a tight *Hermitian* weight matrix for Paley 17 exist. Further, as referenced in Chapter 5, [Elzinga, 2007] goes deeper into the theory behind the inertia bound, and generalizes it for any field, not just $\mathbb{R}$, with more properties and conditions for attaining the bound.

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