Master Thesis

Weighted Systems in Branching-Time

Behavioural Relations, Behavioural Distances, and their Logical Characterisations

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by

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Summary. This Thesis presents the mathematical foundations required for reasoning about weighted systems in branching-time. In particular, using Weighted Kripke Structures as our models, we produce a range of behavioural equivalence that are an analogue to the traditional notion of Branching Bisimulation.

We produce a behavioural equivalence relation for weighted behaviour in branching-time, namely Weighted Branching Bisimulation. We show that for a class of branching-compact WKS (ones WKS in which branching-behaviour accumulate weights only within closed sets of real numbers) are characterised by a suitable temporal logic, namely Weighted Branching Logic. This logic is based upon Computation Tree Logic without the next-operator, but with quantifiers that take weights into account as well.

We develop a notion of relaxing our bisimulation, thereby allowing us to compare almost similar weighted systems with one another. In parallel, we develop a similar notion of relaxing our logic. Again, we show that these relaxed bisimulation are characterised by the relaxation of our logic.

From this notion of relaxing our bisimulation we induce a distance between our weighted systems, by taking the greatest lower bounds of constants of relaxation. We show that this distance forms a behavioural pseudometric and that weighted systems that near each other also satisfy similar formulae.

We introduce what it means for weighted systems to be cheaper than each other in branching-time, thus producing two behavioural preorders. The first being Possibly Cheaper Than, in which the cheaper system is only required to have at least one way to be cheaper than the other system. The second being Always Cheaper Than, in which the cheaper system is required to always be cheaper than the other.

We also introduce a logic for reasoning about the bounds of a weighted system in branching-time. We show that fragments of this logic characterises our Possibly Cheaper Than relation and Always Cheaper Than relation on branching-compact WKS.

Similarly to the previous, we also develop a notion of relaxing our Possibly Cheaper than preorder, and induce a distance from this. We show that this distance forms a hemimetric and that weighted systems near each other again satisfy similar formulae.

Contents

1

Introduction

In an increasingly more complex, distributed, and communicating world of perplexing systems and interactions, our ability to simplify and abstractly represent this mountain of information is of utmost importance. Typically, we would like to create an abstract yet intuitive model of some aspect, some nook, of the real world, and then use this simplified model to make prediction about the real world.

Today, in computer science, there has been a trend towards the use of so called *model checking*, first pioneered by Clark and Emerson [EC80, CE08, CES86]. It is the process of enquiring an abstract model about whether it satisfies certain properties and specifications in an exhaustive and automatic manner. Often, for modelling reactive systems it is only required to inquire about qualitative aspects of systems, such as discrete actions, communication, handshakes, etc. These models however are incapable of capturing quantitative aspects of the real world, such as time, resource consumption, probabilistic and stochastic behaviour, etc. An example of application of a quantitative model could be the model checking of timed automata (TA) for a large class of real life applicable problems—ranging from error finding/correction to developing efficient schedules for reactive systems [RSV10, LPY01]—with the use of tools such as UPPAAL [LPY97].

In this Thesis we will focus on models in which we account for weights, more specifically we weight the transitions between possible states. Such models can be used to describe a large class of real world phenomena, such as resource consumption, time, length, energy, bandwith-usage, etc.

Example 1.1. Consider the model shown in Figure 1.1. Here we model a simplified version of a power plant, which can either produce some amount of energy (here measured in kilo-Watt-hours), thereby heating up and becoming more efficient; or shut off completely, resetting it all together. If the plant gets too hot, we enter a critical state and further power production would result in a meltdown, thereby denying future energy production.

One could be interested in model checking for certain properties, such as:

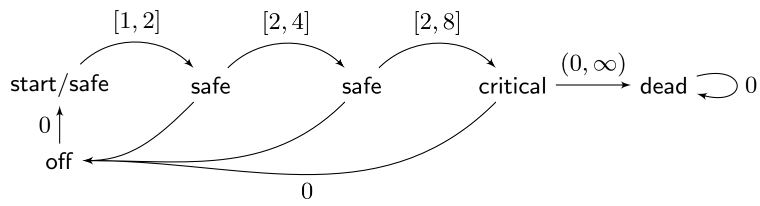


Fig. 1.1: Model of a power plant where the weights and intervals on the edges represent possible **kWh** produced by taking said edge.

- Is it possible to produce 6 **kWh** while remaining in a safe state?
- Is it always the case that if I produce 15 **kWh** or more without going through an off state, it will result in a meltdown (i.e. ending up in a dead state)?
- Is it always the case that as long as I produce below 5 **kWh** I will remain in a safe state?

For this particular model, all of the above hold. △

However, in order to perform such model checking, a mathematically precise definition of what exactly a model is (typically an algebra or coalgebra) and what a specification/property is (typically a logical expression). Perhaps the most classical example is that of Calculus of Communicating Systems (CCS), presented Milner in [Mil80]; or Labelled Transition Systems (LTS), coupled together with Hennesy-Milner Logic (HML), presented by Hennessy and Milner in [HM85]. For quantitative systems there are also many examples of models and associated logics, e.g. discrete and continuous time Markov Chains and Markovian logics [Pan09], for probabilistic and stochastic systems respectively; timed automata and timed HML [AILS07], for timed systems; and Weighted Computation Tree Logic and Weighted Kripke Structures [CHM+15, JLSO16], for weighted systems.

Another important aspect—the one that will be the focus of this Thesis—is that of behavioural equivalence, i.e. when do we say that two systems are behaviourally equivalent. Today, the most often used notion of behavioural equivalence is that of bisimulation introduced by Hennesy, Milner [HM85] and Park [Par81]. Furthermore, they also showed, that for a class of LTS, that bisimulation is characterised by HML, meaning that two models are bisimilar if and only if they satisfy the same HML formulae. Again, there exists many analogues for bisimulation for quantitative systems, e.g. probabilistic and stochacstic bisimulation [LS91, DP03], and weighted bisimulation and general weighted bisimulation [HLM+17].

However, these classical notions of bisimulation can often be too restrictive. Firstly, sometimes we are only interested in observable behaviour and wish to abstract away from internal/hidden behaviour. Milners observational equivalence [Mil80] serves this purpose, as does Weiland and Glabbecks later

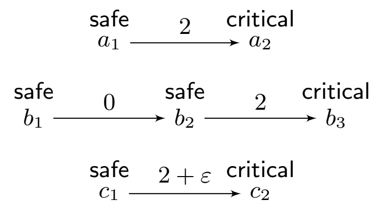


Fig. 1.2: Graph illustrating weighted branching behabiour and relaxation of said behaviour, where $\varepsilon > 0$.

and more refined notion of branching bisimulation [vGW89]. Branching bisimulation also has the property of being characterised by several temporal analogues to modal logics [DV95]—particularly of interest being Computation Tree Logic (CTL) without the Next-operater. In this Thesis we extend this notion of branching bisimulation to weighted systems, Weighted Branching Bisimulation (WBB), and show that it can be characterised by a weighted version of CTL, which we name Weighted Branching Logic (WBL).

Example 1.2. Consider the graph shown in Figure 1.2. Here we have two systems, a_1 and b_1, that both by moving through safe states and accumulating a weight of 2 reaches a critical state. However, the way in which they accomplish this is different. For a_1, only a single transition of weight 2 from the initial safe state to the final critical state is required. For b_2 however, we need to move to an intermediate safe state with weight 0, and then to our critical state with another move of weight 2.

In this case we would like to say that in branching-time, b_1 can simulate a_1 (or that b_1 is weighted branching similar to a_1), as any move a_1 can make, b_1 can match with a finite sequence of transitions of equal accumulated weight. △

Secondly, for the case of quantitative systems, small deviations in the measurable aspects of the model can render two otherwise equivalent systems non-equivalent, and as we often base our models upon real world data measured with a certain error, this renders our models almost useless. Luckily, a more robust definition of behavioural equivalence has been discovered, one in which we group systems that are almost (deviating within some given margin of error) behaviourally equivalent together. This was first done for probabilistic systems by Giacalone et al. [GJS90] and later expanded upon by Desharnais et al. [DGJP99], by relaxing the probabilistic bisimulation by some error. This in turn gives rise to a concept of distance between systems, the greatest lower bound of errors required for two systems two be relaxed bisimilar. In this Thesis we extend these results to that of weighted branching systems. We develop a similar notion of relaxed bisimulation for weighted branching bisimulation and show that it can be characterised by an appropriate relaxation of our WBL. From this we derive a notion of distance between weighted systems and show that systems that are close together are guaranteed to satisfy similar formulae.

Example 1.3. Consider again the graph shown in Figure 1.2. Here the two systems a_1 and c_1 divert in behaviour only by a small constant $\varepsilon > 0$. As such we would like for these two systems to be bisimilar if we relax our bisimulation by a constant of ε. △

Lastly, one may not always be interested in if two systems are behaviourally equivalent, in fact—and especially the setting of weighted systems—one would often be more interested in whether a particular model was cheaper than another or vice versa. We look into what exactly it means to be *cheaper than* in a weighted branching setting and devise two behavioural relations. We develop a logic for reasoning about the bounds of weights in branching-time based upon the Markovian-like logic of Hansen et al. [HLM$^+$17]. We then develop notions of relaxing these relations and induce a behavioural hemimetric upon our models, similar to the case of WBB.

1.1 Contributions

The following is an abbreviated list of the contributions of this Thesis.

- **Weighted Branching Bisimulation**: We re-introduce the concept of Weighted Branching Bisimulation, in a new and more mathematically beautiful flavour. Our notation is easier and more intuitive to read than that of [Jen18], and allows more impressive results. It is qualitative behaviour first, in the sense that we calculate the accumulated weight of runs based upon behavioural properties.
- **Weighted Branching Logic**: We re-introduce Weighted Branching Logic with a easier to read semantics, that is also qualitative behaviour first. We show that for a class of *branching-compact* systems that this logic characterises our bisimulation.
- **Behavioural Pseudometric** We relax our bisimulation using the Hausdorff distance between sets of accumulated weights of runs. We show that for *branching-compact* systems our relaxation can be characterised by a suitable relaxation of our logic. We also show that the distance induced by the infimum of this relaxation nicely forms a behavioural pseudometric and robustness results.
- **Cheaper Than Relations** We introduce to notions of what it means for weighted systems to be cheaper than in branching-time. The first is that of Potentially Cheaper Than, in which we say that a system is cheaper than another, if there exists some way for it to mimic the other in a cheaper way. The second is that of Always Cheaper Than, in which we say that a system is cheaper than another if no matter how it mimics the other, it is cheaper.

- **Bounded Branching Logic** We introduce a logic similar to the one in [HLM+17] for reasoning about the bounds of weighted systems, but in branching time. We show that our cheaper than relations are characterised by subsets of this logic on *branching-compact* systems.
- **Behavioural Hemi-Metric** We extend our notion of relaxing our bisimulation to our Potentially Cheaper Than relations. We show that again we can characterise this relaxation with a suitable relaxation of our logic. This time, we show that the distance induced by the infimum of this relaxation forms a hemi-metric.

1.2 Related Work

First of all, this Thesis is a continuation of the work done on the 9th semester in Computer Science at Aalborg University by the Author [Jen18]. While little of the actual theory remains from it, this Thesis is still heavily informed by the ideas presented there. We present similar, yet distinctively different, results as the ones presented in [Jen18], the main difference being that in this work, we take a qualitative behaviour first approach to our behavioural relations and logics. We however conjure analogues to ideas such as Weighted Branching Bisimulation and Branching-Finite that were first presented here.

The Authors work was initially inspired by (unpublished) ideas put forth by Foshammer et al. in [FLMX17] and can in a sense be seen as a continuation of their work. This work is in turn based upon previous work [FLM16] in which simulation distances for weighted branching systems are first presented, though in a parametric setting.

Core to this Thesis is the notion of branching-time and branching bisimulation, which were first introduced by Weiland and Glabbeek in [vGW89]. This in turn builds upon the notion of observational equivalence, also known as weak bisimulation, presented by Milner in [Mil89].

Efficient algorithms for model checking a logic very similar to the ones presented in this Thesis have been devised by Jensen et al. in [JLSO16]. This is done by using dependency graphs the encode behaviour and then finding minimal fixed points, a method initially presented by Liu et al in [LS98]. While the logics in this Thesis are not equivalent to the one presented by Jensen et al., we believe the similar methods can be used to model check the ones presented here. Parametric model checking for the logic in [JLSO16] has also been done by Hansen et al. in [CHM+15].

The concept of relaxing quantitative behavioural equivalences was first proposed by Giacalone et al. in [GJS90]. Later Desharnais et al. in [DGJP99] expanded upon this by work and developed the notion of a bisimulation metric, in which processes at distance 0 are behaviourally equivalent. A deeper analysis of of metrics for weighted systems is done by Fahrenberhg, Thrane, and Larsen in [TFL10, LFT11, FTL11].

The second part of this Thesis is heavily inspired by the work of Hansen et al. [HLM+17] with regards to reasoning about bounds in weighted systems. In fact, the logic we introduce for reasoning about bounds in branching-time can be seen as a temporal variation of their logic.

2

Preliminaries

In this chapter we introduce the basic preliminary notation and theory required for understanding the research presented in this Thesis. We also precisely define the mathematical model with which we will model weighted systems, namely Weighted Kripke Structures.

2.1 Set Theoretic Notation

In this section we will introduce some basic set theoretic notation and definitions.

Let A be a set. We denote the set of subsets of A, also called the powerset of A, as 2^A.

Binary Relations

As the majority of this thesis is a study of behavioural relations, the following are some basic definitions for binary relations.

Let A be a set, a binary relation on A, is a subset of the Cartesian product of A with itself, i.e. a subset of $A \times A$. In this thesis, we will typically use $\mathcal{R}$ to denote binary relations on some set A.

Let A be a set and $\mathcal{R} \subseteq A \times A$ be a binary relation on A. For denoting membership of $\mathcal{R}$, i.e. $(a_1, a_2) \in \mathcal{R}$ for some $a_1, a_2 \in A$, we will use the shorthand $a_1 \mathcal{R} a_2$.

We are only interested in certain binary relations that have a specific, yet intuitive, structure. These are known as preorders and equivalence relations. A preorder is a binary relation used to described how elements of a set are related to one another, an example being $\leq$ on the natural numbers. An equivalence relation describes which elements of a set are equivalent, an example here being $=$ on the natural numbers.

Let A be a set and $\mathcal{R} \subseteq A \times A$ a binary relation on A. We say that $\mathcal{R}$ is a *preorder* if it is

1. *reflexive*, i.e. for all $a \in A$ we have that $a\mathcal{R}a$, and
2. *transitive*, i.e. for all $a_1, a_2, a_3 \in M$, if $a_1\mathcal{R}a_2$ and $a_2\mathcal{R}a_3$ then $a_1\mathcal{R}a_3$.

We say that $\mathcal{R}$ is an *equivalence relation* if it is a preorder and if it is

3. *symmetric*, i.e. for all $a_1, a_2 \in A$ we have that if $a_1\mathcal{R}a_2$ then $a_2\mathcal{R}a_1$.

Let A be a set, $B \subseteq A$ a subset of A, and $\mathcal{R} \subseteq A \times A$ a binary relation on A. The *closure* of $\mathcal{R}$ on B, denoted $\mathcal{R}(B)$, is the set

$$\mathcal{R}(B) = \{a \in A \mid \exists b \in B : b\mathcal{R}a\}.$$

If $\mathcal{R}$ is a preorder, we say that $\mathcal{R}(B)$ is the *upper-set* of B.

2.2 Basic Analysis

In this section we will introduce some basic number theory and lemmas that will be required for some of the main results of this Thesis later on.

In this thesis we work with the following sets of numbers:

- $\mathbb{N}$: the set of natural numbers including 0, i.e. $\{0, 1, 2, 3, ...\}$
- $\mathbb{Q}$: the set of rational numbers.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}_{\geq 0}$: the set of real numbers greater than or equal to 0.

Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be subsets of the real numbers. The *Cartesian sum* of A and B, denoted $A + B$, is defined as follows:

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

Lemma 2.1. *If A is a* countable *set, then there exists an increasing sequence of* finite *subsets, $A_0 \subseteq A_1 \subseteq ...$, such that $\bigcup\limits_{k \in \mathbb{N}} A_k = A$.*

Proof. Since A is countable, we can enumerate the elements, i.e. $A = \{a_0, a_1, ...\}$. Consider now the sequence of subsets defined for all $k \in \mathbb{N}$ by $A_k = \{a_0, ..., a_i\}$. Clearly, for any $k \in \mathbb{N}$ the set A_k is finite and $A_k \subseteq A_{k+1}$. Furthermore, clearly we have that $\bigcup_{k \in \mathbb{N}} A_k = A$. $\square$

Lemma 2.2. *Given an index set I, let $\{(A_\alpha, B_\alpha) \mid A_\alpha, B_\alpha \subseteq \mathbb{R}, \alpha \in I\}$ be a paired family of sets of real numbers such that* $\inf A_\alpha \geq \inf B_\alpha$ *for any $\alpha \in I$, then*

$$\inf \bigcup_{\alpha \in I} A_\alpha \geq \inf \bigcup_{\alpha \in I} B_\alpha$$

Proof. Assume towards a contradiction that $\inf \bigcup_{\alpha \in I} A_\alpha < \inf \bigcup_{\alpha \in I} B_\alpha$, then there exists some $u \in \mathbb{R}$ such that $\inf \bigcup_{\alpha \in I} A_\alpha < u < \inf \bigcup_{\alpha \in I} B_\alpha$.

Since $u < \inf \bigcup_{\alpha \in I} B_\alpha$ we get that $\forall \alpha \in I : u < \inf B_\alpha$. Furthermore, since $\inf \bigcup_{\alpha \in I} A_\alpha < u$ we get that $\exists \alpha \in I : \inf A_\alpha < u$. This in turn implies that there exists some $\alpha \in I$ such that $\inf A_\alpha < u < \inf B_\alpha$, thereby contradiciting our assumption that $A_\alpha \geq \inf B_\alpha$ for all $\alpha \in I$. $\square$

Lemma 2.3. *Let* $A_0 \supseteq A_1 \supseteq ...$ *and* $B_0 \supseteq B_1 \supseteq ...$ *be countable decreasing sequences of compact sets of real numbers, such that* $\inf A_n \geq \inf B_n$ *for any* $n \in \mathbb{N}$*, then*

$$\inf \bigcap_{n \in \mathbb{N}} A_n \geq \inf \bigcap_{n \in \mathbb{N}} B_n$$

Proof. If $\inf \bigcap_{n \in \mathbb{N}} A_n = \infty$ then the inequality trivially holds.

If $\inf \bigcap_{n \in \mathbb{N}} B_n = \infty$ then $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$, and as B_n is a decreasing sequence of compact sets we have by the contrapositive of Cantor's Intersection Theorem that there must exist some $N \in \mathbb{N}$ for which we have that $B_N = \emptyset$. Therefore, we get that $A_N = \emptyset$ as $\inf A_N \geq \inf B_N$, and hence $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ which in turn implies $\inf \bigcap_{n \in \mathbb{N}} A_n = \infty$.

If $\inf \bigcap_{n \in \mathbb{N}} A_n \neq \infty$ and $\inf \bigcap_{n \in \mathbb{N}} B_n \neq \infty$ then, as A_n and B_n are decreasing sequences of compact sets, we get that $\inf \bigcap_{n \in \mathbb{N}} A_n = \sup_{n \in \mathbb{N}} \inf A_n$ and $\inf \bigcap_{n \in \mathbb{N}} B_n = \sup_{n \in \mathbb{N}} \inf B_n$. Since for all $n \in \mathbb{N}$, $\inf A_n \geq \inf B_n$, we get that $\inf \bigcap_{n \in \mathbb{N}} A_n \geq \inf \bigcap_{n \in \mathbb{N}} B_n$. $\square$

Distances.

Another major topic of this thesis is that of behavioural distances, and as such we now introduce some basic definitions regarding distance functions.

Let A be a set. A *distance* on A is a function, $d : A \times A \to \mathbb{R}$, assigning to each pair of elements of A a greater than or equal to zero value. For some $a_1, a_2 \in A$, $d(a_1, a_2)$ can be intuitively thought of as the distance from a_1 to a_2.

As with binary relations, we are only interested in distance function that exhibit certain properties. In this thesis we make use of hemimetrics and pseudometrics. An example of a pseudometric is the absolute difference between real numbers, i.e. $d(x, y) = |x - y|$ for any $x, y \in \mathbb{R}$.

Let A be a set and $d : A \times A \to \mathbb{R}$ be a function from $A \times A$ to the real numbers. We say that d is a *hemimetric* if, for arbitrary $m, n, o \in M$, it satisfies the following axioms:

1. $d(m, m) = 0$ (identity), and
2. $d(m, o) \leq d(m, n) + d(n, o)$ (triangular inequality).

We say that d is a *pseudometric* if it is a hemimetric and if, for arbitrary $m, n \in M$, it satisfies the following axiom:

3. $d(m, n) = d(n, m)$ (symmetry).

We say that d is a *metric* if it is a pseudometric and if, for abitrary $m, n \in M$ it satisfies the following axiom:

4. $d(m,n) = 0 \Leftrightarrow m = n$ (Identity of Indiscernibles).

Sometimes we are not interested in the distance between particular points, but rather sets thereof. For this, the Hausdorff distance is well suited. In our case, we will only need it on the space of real numbers, for which it is defined as follows:

Definition 2.1 (Hausdorff Distance)**.** Given two sets of real, $A, B \subseteq \mathbb{R}$, the Hausdorff distance, $H : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to \mathbb{R}_{\geq 0}$, between the two sets is defined as follows:

$$H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|b-a|\right\}$$

△

Lemma 2.4. *The Hausdorff distance on* closed sets *real numbers is a* metric*.*

Proof. The proof is left up to the reader. □

2.3 Weighted Kripke Structures

In this section we introduce the coalgebraic structure, which we will use to model weighted systems. As this Thesis is a study of branching behaviour in weighted systems, we have chosen to use Weighted Kripke Structures, as Kripke structures [Kri07] have been shown to be well suited for reasoning about temporal behaviour [BCG88]. Weighted Kripke Structure (WKS) are just the straight forward extension of adding weights to the edges of a regular Kripke structure.

We also introduce a number of concepts and helper functions for reasoning about branching behaviour in WKS, such as the concept of runs, and accumulated weight along runs.

First is the definition of what a WKS is. Taking inspiration from [HLM+17], we have also decided to represent transitions in the form of a function assigning each pair of states a set of weights, instead of defining singular transitions.

Definition 2.2 (Weighted Kripke Structure)**.** Given a set of atomic propositions AP, a WKS is a tuple, $\mathcal{M} = \langle M, \theta, \ell\rangle$, where

- M is a set of states,
- $\theta : M \to [M \to 2^{\mathbb{R}}]$ is the transition function, and
- $\ell : M \to 2^{AP}$ is the labelling function.

△

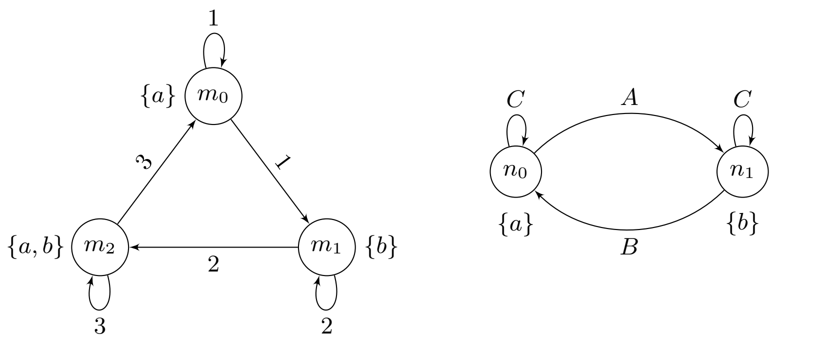


Fig. 2.1: Graphical representation of a WKS, where $a, b \in AP$ and $A, B, C \subseteq \mathbb{R}$.

Intuitively, the transition function can be read as follows: for an arbitrary WKS, $\mathcal{M} = \langle M, \theta, \ell \rangle$; and states in $\mathcal{M}$, $m, n \in M$; the set $\theta(m)(n)$ represents the possible weights of transitioning from m to n. If we have that $\theta(m)(n) = \emptyset$, then we say that m cannot transition to n.

For arbitrary $m \in M$, we extend $\theta(m)$ to sets of states, i.e. for $N \subseteq M$, we define

$$\theta(m)(N) = \bigcup_{n \in N} \theta(m)(n)\,.$$

Thus, $\theta(m)(N)$ represents the possible weights of going from m to an arbitrary state $n \in N$.

Example 2.1. Consider the WKS illustrated in Figure 2.1. Here the transition function θ is given by the values on the edges. E.g. $\theta(m_0)(m_0) = \theta(m_0)(m_1) = \{1\}$, as m_0 can transition to either m_0 or m_1 with a cost 1, however $\theta(m_0)(m_2) = \emptyset$ as m_0 cannot transition to m_2.

Similarly, $\theta(n_0)(n_1) = A$, as n_0 can transition to n_1 with any weight in the set of weights A. If we use the extended notation of θ, we have that $\theta(n_0)(\{n_0, n_1\}) = A \cup C$, as n_0 can transition to either n_0 with any weight in A or to n_1 with any weight in C. △

We now introduce the concept of non-blocking WKS. Non-blocking WKS are WKS in which all states at least has one transition. This is done for purely notational reasons, and any blocking WKS can easily be made into a non-blocking variant, by adding zero weighted self-loops to any blocking states.

Definition 2.3 (Non-Blocking)**.** We say that a WKS $\mathcal{M} = \langle M, \theta, \ell \rangle$ is non-blocking if for all $m \in M$, we have that $\theta(m)(M) \neq \emptyset$. △

Remark 2.1. From this point onwards, we implicitly assume that all WKS used in this Thesis are *non-blocking*.

The immediate convenience of only inquiring about non-blocking can already be seen when we define runs. Runs are infinite computations, here defined as an infinite sequence of states, where it is possible to transition from each state in the sequence to the next. Had the WKS not been non-blocking, some we would have had to include computations of finite length as well.

Definition 2.4 (Runs)**.** Let $\langle M, \theta, \ell \rangle$ be a WKS, and $m_0 \in M$. A run starting in m_0 is an infinite sequence,

$$\sigma = m_0 m_1 ... m_{i-1} m_i m_{i+1} ... \tag{2.1}$$

where for all $i \in \mathbb{N}$, $m_i \in M$ and $\theta(m_i)(m_{i+1}) \neq \emptyset$. We use $\sigma[i]$ to denote the i-th state of σ. Furthermore, let $\texttt{Runs}$ be the set of all runs in $\langle M, \theta, \ell \rangle$ and let $\texttt{Runs}(m)$ denote the set of runs starting in the state $m \in M$. △

Runs are required when we wish to reason about branching behaviour, as they allow us to build a computation tree from each state in a mathematical concise way.

We also need a way to reason about accumulated weight over a run. This is done by taking the Cartesian sum of each possible weighted transition from the current state to the next state in the run up till the desired position.

Definition 2.5 (Accumulated Weight)**.** Let $\langle M, \theta, \ell \rangle$ be a WKS and $\sigma \in \texttt{Runs}$. The set of possible accumulated weights of σ at position k is defined recursively as,

$$\mathcal{W}(\sigma)(k) = \begin{cases} \{0\} & \text{if } k = 0 \\ \theta(\sigma[k-1])(\sigma[k]) + \mathcal{W}(\sigma)(k-1) & \text{otherwise} \end{cases} .$$

△

Example 2.2. Consider the WKS illustrated in Figure 2.1. Let σ be the run starting in m_0 that just cycles through m_0, m_1 and m_2 ad infinitum, i.e.

$$\sigma = m_0 m_1 m_2 m_0 m_1 m_2 m_0 m_1 m_2 ...$$

The accumulated weight of σ at position 4 would be

$$\mathcal{W}(\sigma)(4) = \{0 + 1 + 2 + 3 + 1 + 2\} = \{9\} .$$

△

As the focus of this thesis is that of weighted *branching* behaviour, we need an additional helper function. Like the accumulated weight function, which for a given run returns the possible ways a run can accumulate a weight up to some position, the branching weight of a state is defined as all possible accumulated weights of starting in some initial state, moving through only a designated set of states, and ending up in some terminal state.

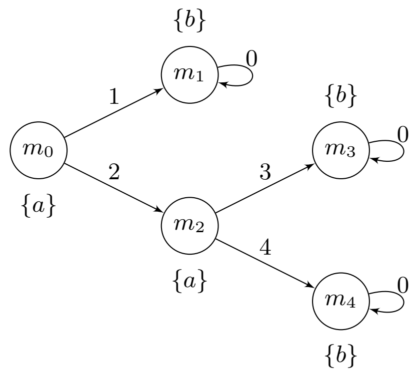


Fig. 2.2: WKS for illustrating weighted branching behaviour.

Definition 2.6 (Branching Weight)**.** For an arbitrary WKS $\mathcal{M} = \langle M, \theta, \ell \rangle$, $m \in M$, and $S, T \subseteq M$ let

$$\Theta(m)(S,T) = \left\{ w \in \mathcal{W}(\sigma)(k) \mid \sigma \in \texttt{Runs}(m), k \in \mathbb{N}, \begin{bmatrix} \sigma[k] \in T, \text{ and} \\ \forall i < k : \sigma[k] \in S \end{bmatrix} \right\}.$$

$\triangle$

We will refer to the set S and T as the *branching variables*, as they describe the branching structure on which we are interested in the accumulated weights.

Example 2.3. Consider the WKS shown in Figure 2.2. If we divide the set of states into the following two branching variables: $S = \{m_0, m_2\}$, i.e. all the states assigned the label a; and $T = \{m_1, m_3, m_4\}$, i.e. all the states assigned the label b. Then the branching weight of going from m_0, through states in S, to a state in T, i.e. $\Theta(m_0)(S,T)$, is

$$\Theta(m_0)(S,T) = \{1, 5, 6\}.$$

As the only available paths satisfying the criteria of Definition 2.6 are m_0m_1, which accumulates a weight of 1; $m_0m_2m_3$, which accumulates a weight of 5; or $m_0m_2m_4$, which accumulates a weight of 6. $\triangle$

The following two lemmas states that union and intersection are invariant over our branching weight function.

Lemma 2.5. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, $m \in M$, $T \subseteq M$, and I a set of indexes. If for all $i \in I$, $S_i \subseteq M$, then*

$$\Theta(m)(\bigcup_{i \in I} S_i, T) = \bigcup_{i \in I} \Theta(m)(S_i, T).$$

Similarly, let $S \subseteq M$*. If for all* $i \in I$, $T_i \subseteq M$*, then*

$$\Theta(m)(S, \bigcup_{i\in I} T_i) = \bigcup_{i\in I} \Theta(m)(S, T_i)\,.$$

Lemma 2.6. *Let* $\mathcal{M} = \langle M, \theta, \ell \rangle$ *be a WKS,* $m \in M$*,* $T \subseteq M$*, and* I *a set of indexes. If for all* $i \in I$*,* $S_i \subseteq M$*, then*

$$\Theta(m)(\bigcap_{i\in I} S_i, T) = \bigcap_{i\in I} \Theta(m)(S_i, T)\,.$$

Similarly, let $S \subseteq M$*. If for all* $i \in I$*,* $T_i \subseteq M$*, then*

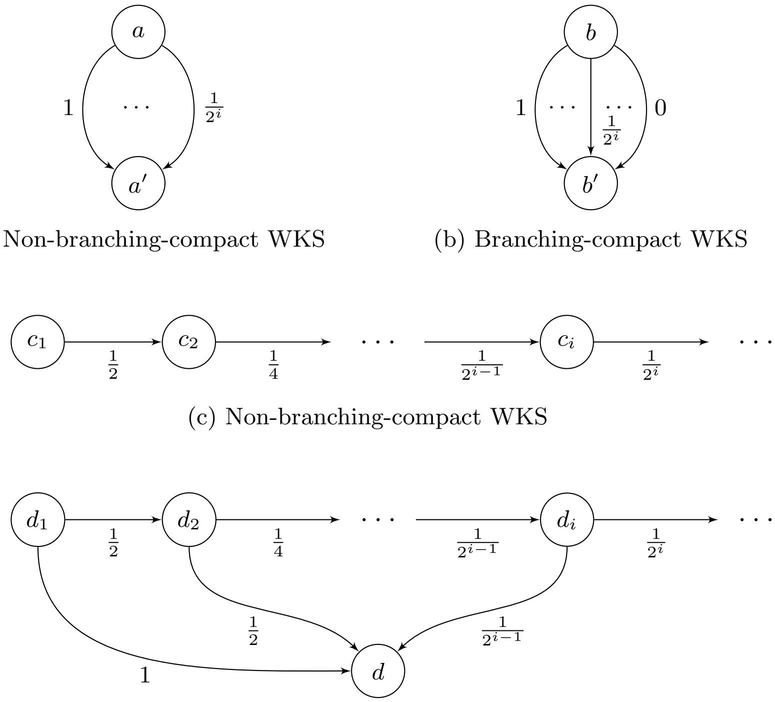
$$\Theta(m)(S, \bigcap_{i\in I} T_i) = \bigcap_{i\in I} \Theta(m)(S, T_i)\,.$$

Lastly, we conclude this chapter with our definition of branching-compact WKS. Intuitively, one can think of a branching-compact WKS as follows: If we can perform some behaviour with accumulated weight approaching some limit, then we can also perform said behaviour with an accumulated weight equal to the limit. Furthermore, the accumulated weight of all behaviour is bounded, meaning that there must exists some cheapest and most expensive way to perform said behaviour.

Definition 2.7 (Branching-Compact)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS. We say that $\mathcal{M}$ is <u>branching-compact</u> if and only if for all $m \in M$ and $S, T \subseteq M$, the set $\Theta(m)(S, T)$ is closed (in $\mathbb{R}$). △

Example 2.4. Consider the four WKS' shown in Figure 2.2. Here the WKS (a) is not branching-compact as we can perform a transition of weight approaching 0, but not actually a 0-transition. The WKS (b) is a branching-compact version of (a) as here we have added a 0-transition.

The WKS (c) is also not branching-compact. Here when we continue along the path we approach an accumulated weight of 1, but never reach it. The WKS (d) is a branching-compact version of (c), here we complete the WKS by adding an option to transition with the remaining needed weight to some final state. △



(a) Non-branching-compact WKS

(b) Branching-compact WKS

(c) Non-branching-compact WKS

(d) Branching-compact WKS

Fig. 2.3: Four WKS that illustrate some examples of branching-compactness.

3

Weighted Branching Behaviour

In this chapter we introduce what weighted behaviour in branching-time means. We create an equivalence relation and show that it is characterized by a logic similar to that of presented in [CHM$^+$15, JLSO16] and [FLM16, Jen18], which themselves are just variants of CTL with weights.

Like Giacalone et al. [GJS90] and Desharnais et al. we also develop a notion of relaxing our bisimulation and therefrom deriving a distance between different weighted systems.

3.1 Weighted Branching Bisimulation

In this section we introduce exactly what we mean by behaviour in weighted systems in branching time. We will introduce the concept of Weighted Branching Bisimulation.

When we talk about behaviour in branching-time, what we usually talk about are possible or inevitable futures, in which we behave a certain way. One can asks such questions as: Is it always the case that that there exists a way for me to return to my starting position? Will I inevitably end up in a deadlock? So in a sense, our behaviour is characterised by all of our possible futures.

It is however a little more stringent than that. In branching-time we cannot see individual moves, but rather only the overall aggregates of our behaviour. We abstract away from internal behaviour and only focus on observable outcomes in an abstract way. But what then is an internal action? For CCS and LTS they are the so called τ-transitions, transitions that typically produced as a communication between two systems. For LTS, branching bisimulations is defined by van Glabbeek and Weijland as follows:

Definition 3.1 (Branching Bisimulation [vGW89])**.** Two graphs, g and h, are branching bisimilar if there exists a symmetric relation $\mathcal{R}$ between the nodes of g and h, such that the roots of g and h are related by $\mathcal{R}$ and if $r\mathcal{R}s$ and $r \xrightarrow{a} r'$, then either:

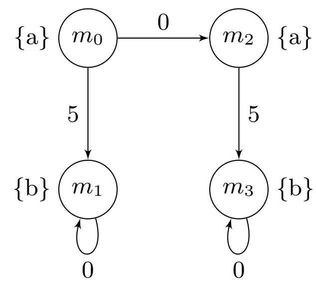


Fig. 3.1: A WKS illustrating the concept of an internal action (i.e. 0 transitions) for weighted systems.

1. $a = \tau$ and $r'\mathcal{R}s$, or
2. there exists a path $s \Rightarrow s_1 \xrightarrow{a} s_2 \Rightarrow s'$ such that $r\mathcal{R}s_1$, $r'\mathcal{R}s_2$, and $r'\mathcal{R}s'$.

△

Basically, this definition states that in order for you and I to be branching bisimilar, whenever you can perform an action then either:

1. You performed an internal action (i.e. τ), in which case we must remain branching bisimilar. So internal actions must have no effect on our external, observable behaviour.
2. You perform an external action (e.g. an action with the label a), in which case I must be able to match with a sequence of internal actions preserving our initial behaviour, then match your external action, ending up in a new state where our behaviour is again branching bisimilar.

In a sense, internal action can be percieved as *doing nothing*, at least for an external observer. So what is an analogue of (1) for weighted systems with state-based behaviour? How should we characterise internal behaviour? Well, as mentioned, an internal action should not be able to change our external behaviour. So we propose that for WKS, an internal action is a transition of weight 0 to an otherwise behaviourally equivalent state.

Example 3.1. Consider the WKS shown in Figure 3.1. Here we would like for the transition $m_0 \xrightarrow{0} m_2$ to be treated as an internal action. As to any external observer, the act of going from a state where a is satisfied to a state where b is satisfied, requires that we accumulate a weight of 5, whether we take the path $m_0 \xrightarrow{5} m_1$ or $m_0 \xrightarrow{0} m_2 \xrightarrow{5} m_3$. △

Finding an analogue for (2) is slightly more complicated. We would still like for behaviour to be preserved along the way and for end behaviour to be preserved. It would however make for a mundane bisimulation, if we required that weight must be matched by a sequence of zero-transitions and then exactly one transition with similar weight (as is the case when using

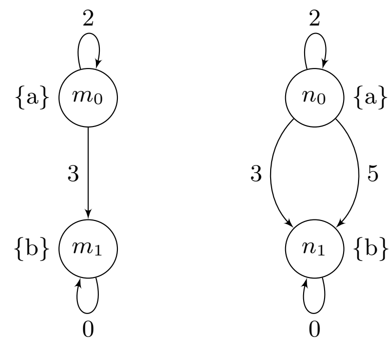


Fig. 3.2: A WKS illustrating the concept of matching transitions with paths for weighted systems.

labels). It would be nicer if we only required that the accumulated weight of the matching path is the same as that of the transition it is matching.

Example 3.2. Consider the WKS show in Figure 3.2 and the transition $n_0 \xrightarrow{5} n_1$. Can m_0 match this transition with a reasonable path? Consider the path $m_0 \xrightarrow{2} m_0 \xrightarrow{3} m_1$. Clearly the end behaviour is preserved as m_1 and n_1 both satisfy b and can only perform a 0-loop. We would however also argue that behaviour is preserved along the way, as m_0 and n_0 both satisfy a, can perform a 2-loop, transition with weight 3 and as shown, with the path $m_0 \xrightarrow{2} m_0 \xrightarrow{3} m_1$, with weight 5, to the equivalent states of m_1 and n_1 respectively. △

With these thoughts in mind, we now present our definition of Weighted Branching Bisimulation.

Definition 3.2 (Weighted Branching Bisimulation)**.** Given a WKS $\mathcal{M} = \langle M, \theta, \ell \rangle$, a relation $\mathcal{R} \subseteq M \times M$ is a Weighted Branching Bisimulation (WBB) if and only if whenever $m\mathcal{R}n$ then,

1. $\ell(m) = \ell(n)$, and
2. $\forall S, T \subseteq M : \Theta(m)(\mathcal{R}(S), \mathcal{R}(T)) = \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))$.

We use $\approx$ to denote the largest Weighted Branching Bisimulation. △

Condition (1) of WBB simply states that any pair of bisimilar states must satisfy the same atomic propositions.

Condition (2) of WBB states that in order for a pair of states to be bisimilar, they must be able to accumulate the same weights when transitioning through any arbitrary source behavioural class ($\mathcal{R}(S)$) to another arbitrary terminal behavioural class ($\mathcal{R}(T)$).

We concluded this section by stating that the largest weighted branching bisimulation, $\approx$, is an equivalence relation.

Proposition 3.1. *The relation $\approx$ is an equivalence relation.*

Proof. Trivial, as $\ell(m) = \ell(n)$ and $\forall S, T \subseteq M : \Theta(m)(\mathcal{R}(S), \mathcal{R}(T)) = \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))$ clearly implies that $\approx$ is reflexive, transitive, and symmetric. $\square$

3.2 Weighted Branching Logic

In this section we introduce the logic which we will use to inquire about whether a given model satisfies certain properties/specifications. As mentioned earlier, we take inspiration from [CHM+15, JLSO16], and therefore also develop a weighted analogue of CTL without the next-operator. We show that our Weighted Branching Bisimulation is in fact characterised by this logic, meaning that two systems are behaviourally equivalent if and only if they satisfy the exact same formulae.

CTL is composed of two different kinds of logical expressions, state-formulae and path-formulae. State-formulae are formulae pertaining to properties regarding states, such as whether not a state satisfies an atomic propositions, or if all paths from here satisfies a certain path-formulae. Path-formulae are formulae pertaining to properties regarding paths, in the weighted analogue the only kind of path-formula we have are untils—formulae that required some initial condition to be satisfied up till some terminal condition is satisfied.

The weighted part of the logic presented here, Weighted Branching Logic, is that we add closed intervals to the existential and universal quantifiers of CTL, requiring that either there exists a path with accumulated weight within said interval that satisfies a given until-formula, or all paths satisfying a given until-formula accumulate a weight within this interval.

Definition 3.3 (Syntax)**.** Let AP be a set of atomic propositions, then the set of formulae of Weighted Branching Logic (WBL), denoted $\mathcal{L}$, are induced by the following context-free grammar:

$$\begin{aligned} \mathcal{L}: \quad & \phi ::= a \mid \neg\phi \mid \phi \wedge \phi \mid E_{[v,u]}\psi \mid A_{[v,u]}\psi \\ & \psi ::= \phi U \phi \end{aligned}$$

where and $a \in AP$, $v, u \in \mathbb{Q}$ and $\bowtie \in \{\leq, \geq\}$. $\triangle$

Definition 3.4 (Semantics)**.** The satisfiability relation for Weighted Branching Logic (WBL), $\models \subseteq \mathfrak{M} \times \mathcal{L}$, is given, for arbitrary $(\mathcal{M}, m) \in \mathfrak{M}$:

$$(\mathcal{M}, m) \models a \quad \text{iff} \quad a \in \ell(m) \tag{3.1}$$

$$(\mathcal{M}, m) \models \neg\phi \quad \text{iff} \quad \text{not } (\mathcal{M}, m) \models \phi \tag{3.2}$$

$$(\mathcal{M}, m) \models \phi_1 \wedge \phi_2 \quad \text{iff} \quad (\mathcal{M}, m) \models \phi_1 \text{ and } (\mathcal{M}, m) \models \phi_2 \tag{3.3}$$

$$(\mathcal{M}, m) \models E_{[v,u]}\phi_1 U \phi_2 \quad \text{iff} \quad [v, u] \cap \Theta(m)(\llbracket\phi_1\rrbracket, \llbracket\phi_2\rrbracket) \neq \emptyset \tag{3.4}$$

$$(\mathcal{M}, m) \models A_{[v,u]}\phi_1 U \phi_2 \quad \text{iff} \quad [v, u] \cup \Theta(m)(\llbracket\phi_1\rrbracket, \llbracket\phi_2\rrbracket) = [v, u] \tag{3.5}$$

where $a \in AP$, $v, u \in \mathbb{R}$, and $\bowtie \in \{\leq, \geq\}$; and $[\![\phi]\!]$ is the set of states satisfying ϕ, i.e. $[\![\phi]\!] = \{m \in M \mid (\mathcal{M}, m) \models \phi\}$. △

Often when $\mathcal{M}$ is clear from context, we will just use the shorthand $m \models \phi$.

Remark 3.1. Unlike the syntax, we define the semantics for WBL on arbitrary closed intervals real numbers. The reason for the restriction on the syntax is that we require that the logic be countable, so that we may perform induction upon it later. This is however not needed for the semantics, and infact we would later like to reason about when certain systems satisfy formulae with arbitrary closed intervals.

The problem of model checking the until-formulae of WBL have been shown to be NP-hard by Jensen et al. [JLSO16], and to be contained in P if we restrict ourselves to only have upper bounds (i.e. only intervals of the form $[0, u]$).

We now introduce some constructs that will be helpfull when we wish to prove that our bisimulation (Definition 3.2) is characterised by WBL. First, for a given WKS, $\mathcal{M} = \langle M, \theta, \ell \rangle$, and state $m \in M$, let

$$(\!|m|\!) = \{\phi \in \mathcal{L} \mid m \models \phi\}$$

be the set of formulae satisfied by m. Now, as our logic is countable, we can enumerate the elements of $(\!|m|\!)$, like so

$$\{\phi_1, \phi_2, ...\} = (\!|m|\!) \,.$$

For the following proof, we would like to take increasing finite bites of $(\!|m|\!)$, as follows:

$$(\!|m|\!)_k = \{\phi_1, ..., \phi_k\}$$

where $k \in \mathbb{N}$.

We now conclude this section with one of the main theorems of this Thesis, namely that our bisimulation, WBB, is characterised by our logic, WBL, on branching-compact, countable, and label-finite WKS.

Theorem 3.1. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a <u>branching-compact</u> and <u>countable</u> WKS, for arbitrary $m, n \in M$,*

$$m \approx n \text{ if and only if } \forall \phi \in \mathcal{L} : m \models \phi \Leftrightarrow n \models \phi \,.$$

Proof. **(⇒) :** We show that if $m \approx n$ then $\forall \phi \in \mathcal{L} : m \models \phi \Leftrightarrow n \models \phi$. Induction on the structure of $\phi \in \mathcal{L}$. Suppose $m \approx n$ and that $m \models \phi$, we show that this implies that $n \models \phi$.

Case $\phi = a$:

By Definition 3.4 we have that $m \models a$ if and only if $a \in \ell(m)$. Since $m \approx n$ we have by Definition 3.2 that $\ell(m) = \ell(n)$, and therefore $a \in \ell(n)$. So by Definition 3.4 we have that $n \models a$.

Case $\phi = \neg\phi_1$:
By Definition 3.4 we have that $m \models \neg\phi_1$ if and only if it is not the case that $m \models \phi_1$. By Proposition 3.1 we have that $\approx$ is symmetric, and since $m \approx n$ we have that $n \approx m$. Assume now towards a contradiction that $n \models \phi_1$, then by induction we have that $m \models \phi_1$, thereby contradicting our assumption that $m \models \neg\phi_1$. So it must not be the case that $n \models \phi_1$, and therefore, by Definition 3.4, we have that $n \models \neg\phi_1$.
Case $\phi = \phi_1 \wedge \phi_2$:
By Definition 3.4 we have that $m \models \phi_1 \wedge \phi_2$ if and only if $m \models \phi_1$ and $m \models \phi_2$. By induction we have that $n \models \phi_1$ and $n \models \phi_2$, and therefore, by Definition 3.4, we have that $n \models \phi_1 \wedge \phi_2$.
Case $\phi = E_{[v,u]}\, \phi_1\, U\, \phi_2$:
By Definition 3.4 we have that $m \models E_{[v,u]}\, \phi_1\, U\, \phi_2$ if and only if

$$[v,u] \cap \Theta(m)([\![\phi_1]\!],[\![\phi_2]\!]) \neq \emptyset\,.$$

Since $m \approx n$ we have by Definition 3.2 that $\Theta(m)([\![\phi_1]\!],[\![\phi_2]\!]) = \Theta(n)([\![\phi_1]\!],[\![\phi_2]\!])$, which in turn implies that

$$[v,u] \cap \Theta(n)([\![\phi_1]\!],[\![\phi_2]\!]) \neq \emptyset\,.$$

So, by Definition 3.4 we have that $n \models E_{[v,u]}\, \phi_1\, U\, \phi_2$.
Case $\phi = A_{[v,u]}\, \phi_1\, U\, \phi_2$:
By Definition 3.4 we have that $m \models A_{[v,u]}\, \phi_1\, U\, \phi_2$ if and only if

$$[v,u] \cup \Theta(m)([\![\phi_1]\!],[\![\phi_2]\!]) = [v,u]\,.$$

Since $m \approx n$ we have by Definition 3.2 that $\Theta(m)([\![\phi_1]\!],[\![\phi_2]\!]) = \Theta(n)([\![\phi_1]\!],[\![\phi_2]\!])$, which in turn implies that

$$[v,u] \cup \Theta(n)([\![\phi_1]\!],[\![\phi_2]\!]) = [v,u]\,.$$

So, by Definition 3.4 we have that $n \models A_{[v,u]}\, \phi_1\, U\, \phi_2$.

($\Leftarrow$) : We show that if $\forall\phi \in \mathcal{L} : m \models \phi \Leftrightarrow n \models \phi$ then $m \approx n$. It is sufficient to show that the relation

$$\mathcal{R} = \{(m,n) \in M \times M \mid \forall\phi \in \mathcal{L} : m \models \phi \Leftrightarrow \models \phi\}$$

is a Weighted Branching Bisimilation relation.

Suppose that $m, n \in M$ and $m\mathcal{R}n$.

Clearly, $m\mathcal{R}n$ implies $\ell(m) = \ell(n)$.

We now show that $m\mathcal{R}n$ implies $\Theta(m)(S^{\mathcal{R}}, T^{\mathcal{R}}) = \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})$, for all $S, T \subseteq M$. We handle the case of S and T being either finite or infinite separately.

Finite Case.

First, without a loss of generality, suppose that both S and T are finite. We can therefore construct a sequence of formulae, $\chi_0^S, \chi_1^S, ...$, defined for an arbitrary $k \in \mathbb{N}$ as follows:

$$\chi_k^S = \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_k} \phi \, .$$

Since S and $(\!|s|\!)_k$ are finite, we have that χ_k^S is well formed.

This allows us to create a decreasing sequence of sets of states, $[\![\chi_0^S]\!] \supseteq [\![\chi_1^S]\!] \supseteq ...$, for which we now show that

$$\bigcap_{k \in \mathbb{N}} [\![\chi_k^S]\!] = S^{\mathcal{R}} \, . \tag{3.6}$$

If $s' \in S^{\mathcal{R}}$ then there exists some $s \in S$ such that $(\!|s|\!) = (\!|s'|\!)$. This in turn implies that $s' \models \bigwedge_{\phi \in (\!|s|\!)_k} \phi$, and hence $s' \models \chi_k^S$, for any $k \in \mathbb{N}$. Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![\chi_k^S]\!] \supseteq S^{\mathcal{R}} \, . \tag{3.7}$$

If $s' \notin S^{\mathcal{R}}$ then for all $s \in S$ there exists some $\phi_s \in \mathcal{B}_I$ such that $s \models \phi_s$ and $s' \not\models \phi_s$. We can now create a formula, $\varPhi_S$, that distinguishes s' from any $s \in S$,

$$\varPhi_S = \bigvee_{s \in S} \phi_s , \text{ where } s \models \phi_s , \text{ and } s' \not\models \phi_s \, .$$

There must exist some $k \in \mathbb{N}$ such that for each $s \in S$ we have that $\phi_s \in (\!|s|\!)_k$. Therefore, we get that $s' \notin [\![\chi_k^S]\!]$, due to there existing some $\phi_s \in (\!|s|\!)_k$ such that $s' \not\models \phi_s$, for any $s \in S$, i.e.

$$s' \not\models \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_k} \phi \, ,$$

Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![\chi_k^S]\!] \subseteq S^{\mathcal{R}} \, . \tag{3.8}$$

By Equation (3.7) and 3.8 combined we have now shown that Equation (3.6) holds. By similar reasoning we can create a sequence of formulae $[\![\chi_0^T]\!] \supseteq [\![\chi_1^T]\!] \supseteq ...$ where

$$\bigcap_{h \in \mathbb{N}} [\![\chi_h^T]\!] = T^{\mathcal{R}} \, . \tag{3.9}$$

We now show that

$$\forall k, h \in \mathbb{N} : \Theta(m)([\![\chi_k^S]\!], [\![\chi_h^T]\!]) = \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])\,. \tag{3.10}$$

Assume towards a contradiction that

$$\exists k, h \in \mathbb{N} : \Theta(m)([\![\chi_k^S]\!], [\![\chi_h^T]\!]) \neq \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])\,.$$

which implies there exists some $w \in \mathbb{R}$ in one but not the other. Assume with a loss of generality that $w \in \Theta(m)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$ and that $w \notin \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$. Since $\mathcal{M}$ is branching-compact there must exists a pair of rational numbers, $v, u \notin \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$ such that $w \in [v, u]$.

We can now create a distinguishing formula

$$\chi = E_{[v,u]}\ \chi_k^S\ U\ \chi_h^T$$

for which $m \models \chi$ and $n \not\models \chi$, thereby contradicting that $m\mathcal{R}n$, so Equation (4.5) must hold.

By Lemma 2.6 we have that

$$\Theta(m)(\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!], \bigcap_{h\in\mathbb{N}} [\![\chi_h^T]\!]) = \bigcap_{k,h\in\mathbb{N}} \Theta(m)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$$
$$\Theta(n)(\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!], \bigcap_{h\in\mathbb{N}} [\![\chi_h^T]\!]) = \bigcap_{k,h\in\mathbb{N}} \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$$

By Equation (3.10), (3.6), and (3.9), we get that

$$\Theta(m)(S^{\mathcal{R}}, T^{\mathcal{R}}) = \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})\,.$$

Infinite Case.

Suppose that S and T are infinite. Since $\mathcal{M}$ is *countable* we have by Lemma 2.1 that there exists two sequences of finite sets of states, $S_0 \subseteq S_1 \subseteq ...$ and $T_0 \subseteq T_1 \subseteq ...$, such that $\bigcup_{k\in\mathbb{N}} S_k = S$ and $\bigcup_{h\in\mathbb{N}} T_h = T$.

Since S_k and T_h are finite for any $k, h \in \mathbb{N}$ we have by what we previously showed in the finite case , that

$$\forall k, h \in \mathbb{N} : \Theta(m)(S_k^{\mathcal{R}}, T_h^{\mathcal{R}}) = \Theta(n)(S_k^{\mathcal{R}}, T_h^{\mathcal{R}})\,.$$

We therefore have that

$$\bigcup_{k,h\in\mathbb{N}} \Theta(m)(S_k^{\mathcal{R}}, T_h^{\mathcal{R}}) = \bigcup_{k,h\in\mathbb{N}} \Theta(n)(S_k^{\mathcal{R}}, T_h^{\mathcal{R}})\,.$$

which is equivalent to

$$\Theta(m)(S^{\mathcal{R}}, T^{\mathcal{R}}) = \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})\,.$$

□

3.3 Behavioural Distance

Often we will base our quantitative models upon real-world empirical data that is measured with some degree of error, however our bisimulation only relate systems with exactly equal weighted branching behaviour. This restriction often, especially in the case of larger models where uncertainty can accumulate, renders the notion of an exact bisimulation useless.

In this section produce a relaxed bisimulation in which we allow for the weight of branching-behaviour to be matched with some degree of error. We show that this relaxation is characterised by a similar and intuitive relaxation of our WBL formulae. From this notion of relaxing our bisimulation we derive a distance between weighted systems, namely the greatest lower bounds of errors required for two systems to be relaxed bisimilar. We show that systems that are close together (in accordance with the derived distance) also satisfy similar formulae.

Definition 3.5 (Relaxed Bisimulation)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, $\mathcal{R} \subseteq M \times M$ and $\varepsilon \in \mathbb{R}_{\geq 0}$. We say that $\mathcal{R}$ is an ε-Weighted Branching Bisimulation (εWBB) if and only if whenever $m\mathcal{R}n$ then,

1. $\ell(m) = \ell(n)$, and
2. $\forall S, T \subseteq M : H\big(\Theta(m)(\mathcal{R}(S), \mathcal{R}(T)), \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))\big) \leq \varepsilon$.

We use $\overset{\varepsilon}{\approx}$ to denote the largest ε-Weighted Branching Bisimulation (εWBB). △

Intuitively condition (*1*) is the requirement of state-wise equal behaviour and condition (*2*) is that the accumulated weights of any branching-behaviour does not differ more than some given margin of error, ε.

If we were to restrict ourselves to a margin of error of 0, then for branching-compact WKS we would simply end up with WBB.

Proposition 3.2. *For <u>branching-compact</u> WKS we have that* $\approx = \overset{0}{\approx}$.

Proof. Consequence of the Hausdorff distance being a metric on closed sets and therefore satisfying the identity of indiscernibles axiom. □

We also have that as we increase our margin of error, we get a coarser and coarser relation.

Proposition 3.3. *Let* $\mathcal{M} = \langle M, \theta, \ell \rangle$ *be a <u>branching-compact</u> WKS and* $\varepsilon, \gamma \in \mathbb{R}_{\geq 0}$. *If* $\varepsilon \leq \gamma$ *then* $\overset{\varepsilon}{\approx} \subseteq \overset{\gamma}{\approx}$.

Proof. Left up to the reader. □

Remark 3.2. With the exception of $\varepsilon = 0$ on branching-compact WKS, the family of relations, $\overset{\varepsilon}{\approx}$, are not equivalence relations. While they are reflexive and symmetric, they are not transitive.

While our relaxed bisimulation does not satisfy transitivity, it does satisfy triangular inequality on branching-compact WKS.

Proposition 3.4. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a <u>branching-compact</u> WKS, $m, n, o \in M$, and $\varepsilon, \gamma \in \mathbb{R}_{\geq 0}$. If $m \overset{\varepsilon}{\approx} n$ and $n \overset{\varepsilon}{\approx} o$ then $m \overset{\varepsilon+\gamma}{\approx} o$.*

Proof. Consequence of the Hausdorff distance being a metric on closed sets and therefore satisfying the triangular inequality axiom. □

We now introduce the way in which we relax our WBL formulae such that we may later use this to characterise our relaxed bisimulations. The relaxation of formulae leaves most untouched, but increases the interval on the existential and universal quantifiers for until-formulae. If before a systems satisfied the formula $E_{[v,u]} a \, U \, b$, then it would make sense that any other systems that is bisimilar with a margin of $\varepsilon \in \mathbb{R}_{\geq 0}$ should at least satisfy $E_{[v-\varepsilon,u+\varepsilon]} a \, U \, b$

Definition 3.6 (Relaxed Formulae)**.** Given an $\varepsilon \in \mathbb{R}$, we define the ε-relaxation of WBL formulae, $\_^{\varepsilon} : \mathcal{L} \to \mathcal{L}$, inductively for arbitrary $\phi \in \mathcal{L}$ as follows:

$$\phi^{\varepsilon} = \begin{cases} a & \text{if } \phi = a \\ \neg \phi_1^{\varepsilon} & \text{if } \phi = \neg \phi_1 \\ \phi_1^{\varepsilon} \wedge \phi_2^{\varepsilon} & \text{if } \phi = \phi_1 \wedge \phi_2 \\ E_{[v-\varepsilon,u+\varepsilon]} \phi_1^{\varepsilon} U \phi_2^{\varepsilon} & \text{if } \phi = E_{[v,u]} \phi_1 U \phi_2 \\ A_{[v-\varepsilon,u+\varepsilon]} \phi_1^{\varepsilon} U \phi_2^{\varepsilon} & \text{if } \phi = A_{[v,u]} \phi_1 U \phi_2 \end{cases}$$

where $a \in AP$ and $v, u \in \mathbb{Q}$. △

The implication that we wish however only works for positive formulae. When we expand a positive formula the number of systems satisfying said formula increases, as we increase the likelihood of its behaviour accumulating weights within the given interval. However, in the case of negated formula the inverse is the true, as we relax our formulae, and thereby increasing our intervals, we decrease the likelyhood of our behaviour not being in that interval.

Definition 3.7 (Positive Formulae)**.** Let AP be a set of atomic propositions, the positive subset of WBL is induced by the following context-free grammar:

$$\mathcal{L}^{+}: \quad \phi ::= a \mid \neg a \mid \phi \wedge \phi \mid \phi \vee \phi \mid E_{[v,u]} \, \phi_1 \, U \, \phi_2 \mid A_{[v,u]} \, \phi_1 \, U \, \phi_2$$

where $a \in AP$ and $v, u \in \mathbb{Q}$. △

Theorem 3.2. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* branching-compact*,* countable*, and* label-finite *WKS, for arbitrary $m, n \in M$,*

$$m \overset{\varepsilon}{\approx} n \text{ if and only if } \forall \phi \in \mathcal{L}^{+} : \begin{array}{l} m \models \phi \implies n \models \phi^{\varepsilon} \\ n \models \phi \implies m \models \phi^{\varepsilon} \end{array}.$$

The proof is very similar to that of Theorem 3.1. For a full version, see Appendix A.

By taking the greatest lower bounds of margin of errors required for two systems to be relaxed bisimilar, we can induce a distance between weighted systems.

Definition 3.8 (Distance)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, we define the distance, $d : M \times M \to \mathbb{R}_{\geq 1}$, between two arbitrary states $m, n \in M$ as follows:

$$d(m,n) = \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid m \overset{\varepsilon}{\approx} n\} \tag{3.11}$$

where $\inf \emptyset = \infty$. △

And in fact, it is the case that this distance nicely forms a pseudometric.

Theorem 3.3. *Given a* <u>*branching-compact*</u> *WKS $\mathcal{M} = \langle M, \theta, \ell \rangle$, the distance function d is a relative-pseudometric, i.e. for arbitrary $m, n, o \in M$ we have that*

1. *$d(m,m) = 0$ (Identity)*
2. *$d(m,n) = d(n,m)$ (Symmetry), and*
3. *$d(m,o) \leq d(m,n) \cdot d(n,o)$ (Relative Triangular Inequality).*

Proof. (*1*) is a consequence of Proposition 3.2, (*2*) is a consequence of the Hausdorff distance being a metric on closed sets, and (*3*) is a consequence of Proposition 3.4. □

But more importantly, we have that our distance in a sense epitomizes our idea of branching-behaviour. This is shown in the following two theorems.

The first states that systems at distance zero are behaviourally equivalent and that our distance and relaxed bisimulation agree.

Theorem 3.4 (Behavioural Distance)**.** *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* branching-compact *WKS, $m, n \in M$, and $\varepsilon \in \mathbb{R}_{\geq 1}$, then $d(m,n) = \varepsilon$ if and only if $m \overset{\varepsilon}{\approx} n$.*

The second theorem states that state that are close together, also satisfy similar formulae. This theorem is of great importance, as it states that given a model of a weighted system with some known error, we can still infer properties on said system.

Theorem 3.5 (Robustness)**.** *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* branching-compact *WKS, $m, n \in M$, and $\varepsilon \in \mathbb{R}_{\geq 1}$. If $d(m,n) \leq \varepsilon$ then $\forall \phi \in \mathcal{L} : m \models \phi \implies n \models \phi^{\varepsilon}$.*

4

Cheaper Than

Often we are not interested in whether two weighted systems are behaviourally equivalent, usually it suffices to reason about whether one system is cheaper than another. But what exactly would it mean for a weighted system to be cheaper than another? And how does this reflect itself in branching-time? In this chapter we attempt to answer these questions.

We produce two classifications of what we mean by systems being cheaper than one another. The first is the Possibly Cheaper Than relations. Here we say that *you* are possibly cheaper than I, if whenever I can perform some behaviour, then there exists a cheaper way (i.e. it is possible) for *you* to do the same behaviour. The second is the Always Cheaper Than relations. Here we say *you* are always cheaper than I, if whenever I can perform some behaviour, then *you* always perform said behaviour in a cheaper manner.

Like in the previous chapter, we also here produce a notion of distance between systems based on our notion of Possibly Cheaper Than, thus producing a behavioural hemi-metric.

4.1 Possibly Cheaper Than

Like Hansen et al. [HLM$^+$17], when reasoning about whether a systems is cheaper than another, we are only really interested in the extremities of the weights in questions, the bounds.

In this section we introduce our Possibly Cheaper Than (PCT) relation, in which we compare the cheapest paths of differing systems characterised by some behaviour with one another. As mentioned, for a system to be possibly cheaper than another, it is only required for there to exists at least one cheaper than path for each behaviourally equivalent path in the systems we are comparing it to. This means that it is possible for a possibly cheaper than systems to have other, more expensive, paths as well.

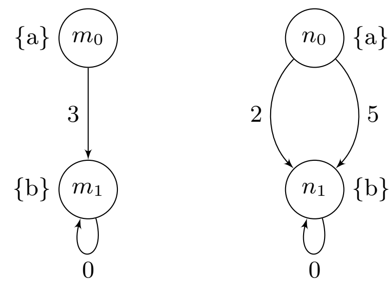


Fig. 4.1: A WKS illustrating the concept of Possibly Cheaper Than, where $m_0 \triangleright n_0$.

Example 4.1. Consider the WKS shown in Figure 4.1. Here n_0 is possibly cheaper than m_0, as if m_0 wished to move to a state where b is satisfied, it would cost 3, but n_0 is capable of doing a similar move with only a cost of 2.

Note however, that while it is possibly for n_0 to be cheaper than m_0, it is not always the case, as n_0 can also move to a state where b is satisfied with a more expensive cost of 5. △

The PCT relation is well suited for systems in which we are only interested in the best outcomes and deterministically can choose the cheapest path. E.g. when planning a route, we might only be interested in the shortest or fastest route, thereby ignoring all other paths.

So what we want is a relation that compares the cheapest ways for two systems to perform some branching behaviour.

Definition 4.1 (Possibly Cheaper Than)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, and $\mathcal{R} \subseteq M \times M$. We say $\mathcal{R}$ is a Possibly Cheaper Than relation if and only if whenever $m\mathcal{R}n$ then,

1. $\ell(m) \subseteq \ell(n)$, and
2. $\forall S, T \subseteq M : \inf \Theta(m)(S,T) \geq \inf \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))$.

We use $\trianglerighteq$ to denote the largest Possibly Cheaper Than (PCT) relation. △

Intuitively, condition (*1*) can be read as: for any state n that we consider possibly cheaper than a state m, must be able to at least simulate the (state-wise) behaviour of m. Condition (*2*) can be read as: for whatever behaviour m can perform, the cheapest way for n to perform the same behaviour is equal to or less than m.

As with WBB, we now show that our PCT relation adheres to an underlining structure, namely that it is a preorder.

Proposition 4.1. *The relation* $\trianglerighteq$ *is a preorder.*

The proof is left up to the reader.

Unlike $\approx$, the relation $\trianglerighteq$ is not an equivalence relation, as it is not symmetric. E.g. consider the WKS shown in Figure 4.1, as mentioned, n_0 is cheaper

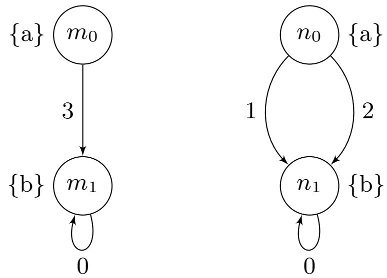


Fig. 4.2: A WKS illustrating the concept of Always Cheaper Than, where $m_0 \triangleright n_0$.

than m_0. The converse is however not the case, as m_0 cannot perform a move cheaper than 3, while n_0 can perform one of cost 2.

4.2 Always Cheaper Than

As mentioned, PCT is of use when we somehow can deterministically choose the cheapest path through a system. But what if we cannot? Then for one weighted system to be cheaper than another, we must ensure that no matter what move we perform, it is always cheaper than the other. We therefore in this section introduce the concept of Always Cheaper Than (ACT).

Example 4.2. Consider the WKS illustrated in Figure 4.2. Here n_0 is always cheaper than m_0, as the cost for m_0 to move to a state where b is satisfied is at least 3, while for n_0 to move to a state where b is satisfied, the cost is at most 2.

△

So in this case, we want a relation that compares the cheapest way for one system to perform some branching behaviour, with the most expensive way for another. This way we ensure that the other system is always cheaper than the first.

Definition 4.2 (Always Cheaper Than)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, and $\mathcal{R} \subseteq M \times M$. We say $\mathcal{R}$ is an Always Cheaper Than relation if and only if whenever $m\mathcal{R}n$ then,

1. $\ell(m) \subseteq \ell(n)$, and
2. $\forall S, T \subseteq M : \inf \Theta(m)(S,T) \geq \sup \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))$.

We use $\triangleright$ to denote the largest Always Cheaper Than (ACT) relation. △

Again, we can intuitively read condition (*1*) as: for any state n that we consider possibly cheaper than a state m, n must be able to at least simulate the (state-wise) behaviour of m. Condition (*2*) can be read as: for whatever branching behaviour m can perform, the most expensive way for n to perform the same behaviour is equal to or less than m.

Again we have that $\rhd_{\mathsf{A}}$ forms a preorder, but not an equivalence relation for similar reasons as that for $\rhd_{\mathsf{E}}$.

Proposition 4.2. *The relation $\rhd_{\mathsf{A}}$ is a preorder.*

The proof is left up for the reader.

4.3 Bounded Branching Logic

Inspired by the Markovian-like modal logic introduced by Hansen et al. in [HLM+17], we introduce a similar branching logic, Bounded Branching Logic (BBL), in this section. With BBL we wish to reason about the extremities of our weighted systems branching behaviour, namely the cheapest and most expensive ways for said systems to perform some behaviour. We say that our logic is Markovian-like, as it takes base in the Markov operators, *most* (M_r) and *least* (L_r), for stochastic systems [CLM11].

Like with WBL we still use until-formulae to express behaviour in branching time, we however replace the existential and universal quantifiers with an *infimum* ($I_{\leq u}$) quantifier and a *supremum* ($S_{\leq u}$) quantifier.

Definition 4.3 (Syntax)**.** Let AP be a set of atomic propositions, then the set of formulae of Bounded Branching Logic (BBL), denoted $\mathcal{B}$, are induced by the following context-free grammar:

$$\begin{aligned} \mathcal{B}: \quad & \phi ::= a \mid \neg\phi \mid \phi \wedge \phi \mid I_{\leq u}\psi \mid S_{\leq u}\psi \\ & \psi ::= \phi U \phi \end{aligned}$$

where $a \in AP$, and $u \in \mathbb{Q}$. △

Definition 4.4 (Semantics)**.** The satisfiability relation for BBL, $\models \subseteq \mathfrak{M} \times \mathcal{B}$, is given for arbitrary $(\mathcal{M}, m) \in \mathfrak{M}$:

$$\begin{aligned} (\mathcal{M}, m) &\models a && \text{iff} \quad a \in \ell(m) \\ (\mathcal{M}, m) &\models \neg\phi && \text{iff} \quad \text{not } (\mathcal{M}, m) \models \phi \\ (\mathcal{M}, m) &\models \phi_1 \wedge \phi_2 && \text{iff} \quad (\mathcal{M}, m) \models \phi_1 \text{ and } (\mathcal{M}, m) \models \phi_2 \\ (\mathcal{M}, m) &\models I_{\bowtie u}\phi_1 U \phi_2 && \text{iff} \quad \inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \leq u \\ (\mathcal{M}, m) &\models S_{\bowtie u}\phi_1 U \phi_2 && \text{iff} \quad \sup \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \leq u \end{aligned}$$

where $a \in AP$, and $u \in \mathbb{R}$. △

We conclude this section we two characterisation proofs of both PCT and ACT with sublogics of BBL.

4.3.1 Characterisation of Possibly Cheaper Than

As we are only concerned about the cheapest paths when considering PCT, we restrict our logic to one where we only have the basic boolean operators and the infimum operator.

Definition 4.5 (Possibly-Sublogic)**.** Let AP be a set of atomic propositions, we define the following sublogic, $\mathcal{B}_I$, of BBL as follows;

$$\mathcal{B}_I : \quad \phi ::= \top \mid \bot \mid a \mid \phi \wedge \phi \mid \phi \vee \phi \mid I_{\leq u}\, \phi \, U \, \phi$$

where $a \in AP$ and $u \in \mathbb{Q}$. $\triangle$

The semantics are still given by Definition 4.4.

In order to show that this PCT is characterised by this sublogic, we require a few additional constructs. First, for a given WKS, $\mathcal{M} = \langle M, \theta, \ell \rangle$, and state $m \in M$, let

$$(\!|m|\!) = \{\phi \in \mathcal{B}_I \mid m \models \phi\}$$

be the set of formulae satisfied by m. Now, as our logic is countable, we can enumerate the elements of $(\!|m|\!)$, like so

$$\{\phi_1, \phi_2, ...\} = (\!|m|\!)\,.$$

For the following proof, we would like to take increasing finite bites of $(\!|m|\!)$, as follows:

$$(\!|m|\!)_k = \{\phi_1, ..., \phi_k\}$$

where $k \in \mathbb{N}$.

With this in hand, we are now ready to show that we can characterise our Possibly Cheaper Than relation with the given sublogic, i.e. a system is cheaper than another if and only if it satisfies the same formulae as the other.

There is the caveat that we require for the WKS in question to be both *branching-compact* and *countable*.

Theorem 4.1. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* <u>*branching-compact*</u> *and* <u>*countable*</u> *WKS, for arbitrary $m, n \in M$:*

$$m \trianglerighteq n \textit{ if and only if } \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi$$

Proof. **($\Rightarrow$) :** We show that if $m \trianglerighteq n$ then $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi$. Induction on the structure of $\phi \in \mathcal{B}_I$. Suppose that $m \trianglerighteq n$ and that $m \models \phi$, we show that $n \models \phi$.
Case $\phi = a$:
By Definition 4.4 we have that $m \models a$ if and only if $a \in \ell(m)$. Since $m \trianglerighteq n$ we have by Definition 4.1 that $\ell(m) \subseteq \ell(n)$, and therefore $a \in \ell(n)$. So by Definition 4.4 we have that $n \models a$.

Case $\phi = \phi_1 \wedge \phi_2$:
By Definition 4.4 we have that $m \models \phi_1 \wedge \phi_2$ if and only if $m \models \phi_1$ and $m \models \phi_2$. By induction we have that $n \models \phi_1$ and $n \models \phi_2$, and therefore, by Definition 4.4, we have that $n \models \phi_1 \wedge \phi_2$.
Case $\phi = \phi_1 \vee \phi_2$:
Derived from Definition 4.4 we have that $m \models \phi_1 \vee \phi_2$ if and only if $m \models \phi_1$ or $m \models \phi_2$. By induction we have that either $n \models \phi_1$ or $n \models \phi_2$, and therefore, by Definition 4.4, we have that $n \models \phi_1 \vee \phi_2$.
Case $\phi = I_{\leq u} \phi_1 U \phi_2$:
By Definition 4.4 we have that $m \models I_{\leq u} \phi_1 U \phi_2$ if and only if

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \leq u\,.$$

Since $m \trianglerighteq n$, we have by Definition 4.1 that

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \inf \Theta(n)(\trianglerighteq([\![\phi_1]\!]), \trianglerighteq([\![\phi_2]\!]))\,.$$

By induction we have that $\trianglerighteq([\![\phi_1]\!]) = [\![\phi_1]\!]$ and that $\trianglerighteq([\![\phi_2]\!]) = [\![\phi_2]\!]$, we therefore get that

$$u \geq \inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \inf \Theta(n)([\![\phi_1]\!], [\![\phi_2]\!])\,,$$

and therefore by Definition 4.4 we have that $n \models I_{\leq u} \phi_1 U \phi_2$.

($\Leftarrow$) : We show that if $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi$ then $m \trianglerighteq n$. It is sufficient to show that the relation

$$\mathcal{R} = \{(m, n) \in M \times M \mid \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi\}$$

is a Possibly Cheaper Than relation. Suppose that $m, n \in M$ and $m \mathcal{R} n$.

Clearly, $m \mathcal{R} n$ implies $\ell(m) \subseteq \ell(n)$.

We now show that $m \mathcal{R} n$ implies $\inf \Theta(m)(S, T) \geq \inf \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))$, for all $S, T \subseteq M$. We handle the case of S and T being either finite or infinite separately.

Finite Case.

First, without a loss of generality, suppose that both S and T are finite. We can therefore construct a sequence of formulae, $\chi^S_0, \chi^S_1, ...$, defined for an arbitrary $k \in \mathbb{N}$ as follows:

$$\chi^S_k = \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)^I_k} .$$

Since S and $(\!|s|\!)^I_k$ are finite, we have that χ^S_k is well formed.

This allows us to create a decreasing sequence of sets of states, $[\![\chi^S_0]\!] \supseteq [\![\chi^S_1]\!] \supseteq ...$, for which we now show that

$$\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!] = \mathcal{R}(S)\,. \tag{4.1}$$

If $s' \in \mathcal{R}(S)$ then there exists some $s \in S$ such that $(\!|s|\!) \subseteq (\!|s'|\!)$. This in turn implies that $s' \models \bigwedge_{\phi\in(\!|s|\!)_k^I} \phi$, and hence $s' \models \chi_k^S$, for any $k \in \mathbb{N}$. Ergo,

$$\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!] \supseteq \mathcal{R}(S)\,. \tag{4.2}$$

If $s' \notin \mathcal{R}(S)$ then for all $s \in S$ there exists some $\phi_s \in \mathcal{B}_I$ such that $s \models \phi_s$ and $s' \not\models \phi_s$. We can now create a formula, Φ_S, that distinguishes s' from any $s \in S$,

$$\Phi_S = \bigvee_{s\in S} \phi_s,\ \text{where } s \models \phi_s,\ \text{and } s' \not\models \phi_s\,.$$

There must exist some $k \in \mathbb{N}$ such that for each $s \in S$ we have that $\phi_s \in (\!|s|\!)_k^I$. Therefore, we get that $s' \notin [\![\chi_k^S]\!]$, due to there existing some $\phi_s \in (\!|s|\!)_k$ such that $s' \not\models \phi_s$, for any $s \in S$, i.e.

$$s' \not\models \bigvee_{s\in S} \bigwedge_{\phi\in(\!|s|\!)_h} \phi\,,$$

Ergo,

$$\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!] \subseteq \mathcal{R}(S)\,. \tag{4.3}$$

By Equation (4.2) and 4.3 combined we have now shown that Equation (4.1) holds. By similar reasoning we can create a sequence of formulae $[\![\chi_0^T]\!] \supseteq [\![\chi_1^T]\!] \supseteq ...$ where

$$\bigcap_{h\in\mathbb{N}} [\![\chi_h^T]\!] = \mathcal{R}(T)\,. \tag{4.4}$$

We now show that

$$\forall k, h \in \mathbb{N} : \inf \Theta(m)(S,T) \geq \inf \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])\,. \tag{4.5}$$

Assume towards a contradiction that

$$\exists k, h \in \mathbb{N} : \inf \Theta(m)(S,T) < q < \inf \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])\,.$$

where $q \in \mathbb{Q}$. Since $\inf \Theta(m)$ is monotonic and since $S \subseteq [\![\chi_k^S]\!]$ and $T \subseteq [\![\chi_h^T]\!]$ we have that

$$\inf \Theta(m)([\![\chi_k^S]\!], [\![\chi_h^T]\!]) \leq \inf \Theta(m)(S,T) < q\,.$$

We can now create a distinguishing formula

$$\chi = I_{\leq u}\ \chi_k^S\ U\ \chi_h^T$$

for which $m \models \chi$ and $n \not\models \chi$, thereby contradicting that $m\mathcal{R}n$, so Equation (4.5) must hold.

By Lemma 2.6 we have that

$$\Theta(n)(\bigcap_{k\in\mathbb{N}} [\![\chi_k^S]\!], \bigcap_{h\in\mathbb{N}} [\![\chi_h^T]\!]) = \bigcap_{k,h\in\mathbb{N}} \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$$

Since $\mathcal{M}$ is branching-compact and due to Equation (4.5), we get by Lemma 2.3 that

$$\inf \Theta(m)(S,T) \geq \inf \bigcap_{k,h\in\mathbb{N}} \Theta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$$

and by Equation (4.1) and (4.4) we get that

$$\inf \Theta(m)(S,T) \geq \inf \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))\,. \tag{4.6}$$

Infinite Case.

Suppose that S and T are infinite. Since $\mathcal{M}$ is *countable* we have by Lemma 2.1 that there exists two sequences of finite sets of states, $S_0 \subseteq S_1 \subseteq ...$ and $T_0 \subseteq T_1 \subseteq ...$, such that $\bigcup_{k\in\mathbb{N}} S_k = S$ and $\bigcup_{h\in\mathbb{N}} T_h = T$.

Since S_k and T_h are finite for any $k, h \in \mathbb{N}$ we have by what we previously showed in the finite case (Equation (4.6)), that

$$\forall k, h \in \mathbb{N} : \inf \Theta(m)(S_k, T_h) \geq \inf \Theta(n)(\mathcal{R}(S_k), \mathcal{R}(T_h))$$

Clearly, we have that

$$\mathcal{R}(S) = \mathcal{R}(\bigcup_{k\in\mathbb{N}} S_k) = \bigcup_{k\in\mathbb{N}} \mathcal{R}(S_k)$$

and similarly

$$\mathcal{R}(T) = \mathcal{R}(\bigcup_{h\in\mathbb{N}} T_h) = \bigcup_{h\in\mathbb{N}} \mathcal{R}(T_h)\,.$$

Therefore, due to , we have by Lemma 2.2 that

$$\inf \bigcup_{k,h\in\mathbb{N}} \Theta(m)(S_k, T_h) \geq \inf \bigcup_{k,h\in\mathbb{N}} \Theta(n)(\mathcal{R}(S_k), \mathcal{R}(T_h))$$

which is equivalent to

$$\inf \Theta(m)(S,T) \geq \inf \Theta(n)(\mathcal{R}(S), \mathcal{R}(T))\,.$$

$\square$

4.3.2 Characterisation of Always Cheaper Than

Much in the same way as with PCT, we here also require a different sublogic. However, as we this time wish to reason about the lower bounds in one and the upper bounds in another, two different sublogics are required for the two systems we wish to relate.

We still make use of the previously defined logic $\mathcal{B}_I$ given in Definition 4.5 for the systems we wish to relate to. For the system we wish to always be cheaper, the following sublogic is used:

Definition 4.6 (Always-Sublogic)**.** Let AP be a set of atomic propositions, we define the following sublogic, $\mathcal{B}_S$, of BBL as follows;

$$\mathcal{B}_S : \quad \phi ::= \top \mid \bot \mid a \mid \phi \wedge \phi \mid \phi \vee \phi \mid S_{\leq u}\, \phi\, U\, \phi$$

where $a \in AP$ and $u \in \mathbb{Q}$. △

The semantics are still given by Definition 4.4.

We also require a way to map between the two logics. This is done in a straight forward manner where we turn infima expressions ($I_{\leq u}$) into corresponding suprema expressions ($S_{\leq_u}$), leaving anything else untouched.

Definition 4.7. We define the following mapping $\_^{\sqcap} : \mathcal{B}_I \to \mathcal{B}_S$, for arbitrary $\phi \in \mathcal{B}_I$ as follows:

$$\phi^{\sqcap} = \begin{cases} \top & \text{if } \phi = \top \\ \bot & \text{if } \phi = \bot \\ a & \text{if } \phi = a, \text{ and } a \in AP \\ \phi_1^{\sqcap} \wedge \phi_2^{\sqcap} & \text{if } \phi = \phi_1 \wedge \phi_2 \\ \phi_1^{\sqcap} \vee \phi_2^{\sqcap} & \text{if } \phi = \phi_1 \vee \phi_2 \\ S_{\leq u}\, \phi_1^{\sqcap}\, U\, \phi_2^{\sqcap} & \text{if } \phi = I_{\leq u}\, \phi_1\, U\, \phi_2, \text{ and } u \in \mathbb{Q} \end{cases}$$

where $\phi \in \mathcal{L}_I$. △

We also need an additional construct. For a given WKS $\mathcal{M} = \langle M, \theta, \ell \rangle$, let

$$(\!| m |\!)^{\sqcap} = \{\phi \in \mathcal{B}_I \mid s \models \phi^{\sqcap}\}$$

be the inverse image of $\_^{\sqcap}$ on the set of $\mathcal{B}_S$ formulae satisfied by a state $m \in M$.

We are now ready to give a logical characterisation of ACT, again restricting ourselves to *branching-compact* and *countable* WKS.

Theorem 4.2. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* <u>*branching-compact*</u> *and* <u>*countable*</u> *WKS, for arbitrary $m, n \in M$, then*

$$m \trianglerighteq n \text{ if and only if } \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^{\sqcap}$$

The proof is very similar to that of Theorem 4.1. For a full version, see Appendix A.

4.4 Cheaper Than Distance

We conclude this chapter by introducing similar ideas regarding distance between weighted systems as we did for WBB, only this time for PCT. Many of the results are very similar, the only notable difference being that the distance now only forms a behavioural hemimetric.

First, we introduce the idea of relaxing our PCT relation. Here we simply subtract the margin of error from the greatest lower bound of accumulated weight of whatever branching behaviour in question.

Definition 4.8 (ε-Possibly Cheaper Than)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS, $\mathcal{R} \subseteq M \times M$ and $\varepsilon \in \mathbb{R}_{\geq 0}$. We say $\mathcal{R}$ is a ε-Possibly Cheaper Than relation if and only if whenever $m \mathcal{R} n$ then

1. $\ell(m) \subseteq \ell(n)$, and
2. $\forall S, T \subseteq M : \inf \Theta(m)(S,T) \geq \inf \Theta(n)(\mathcal{R}(S), \mathcal{R}(T)) - \varepsilon$.

We use $\rhd_{\varepsilon}$ to denote the largest ε-PCT relation. △

Here condition *(2)* can be read intuitively as, whatever m does, n can do cheaper if we subtract ε.

Again we introduce a similar notion of relaxing our logic, so that we may characterise our relaxed PCT.

Definition 4.9. We define the family of functions $\_^{\varepsilon} : \mathcal{L}_I \to \mathcal{L}_I$, for $\varepsilon \in \mathbb{R}_{\geq 1}$, as follows:

$$\phi^{\varepsilon} = \begin{cases} \top & \text{if } \phi = \top \\ \bot & \text{if } \phi = \bot \\ a & \text{if } \phi = a, \text{ and } a \in AP \\ \phi_1^{\varepsilon} \wedge \phi_2^{\varepsilon} & \text{if } \phi = \phi_1 \wedge \phi_2 \\ \phi_1^{\varepsilon} \vee \phi_2^{\varepsilon} & \text{if } \phi = \phi_1 \vee \phi_2 \\ I_{\leq u+\varepsilon}\, \phi_1^{\varepsilon}\, U\, \phi_2^{\varepsilon} & \text{if } \phi = I_{\leq u}\, \phi_1\, U\, \phi_2, \text{ and } u \in \mathbb{Q} \end{cases}$$

where $\phi \in \mathcal{L}_I$. △

Theorem 4.3. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a <u>branching-compact</u> and <u>countable</u> WKS. For arbitrary $m, n \in M$ we have that*

$$m \rhd_{\varepsilon} n \text{ if and only if } \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^{\varepsilon}$$

The proof is very similar to that of Theorem 4.1. For a full version, see Appendix A.

As with WBB, we can induce a distance from the infimum of relaxed PCTs.

Definition 4.10 (Distance)**.** Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a WKS. The distance between arbitrary states $m, n \in M$ is given by the function $d : M \times M \to \mathbb{R}_{\geq 0}$, defined as follows:

$$d(m,n) = \inf\{\varepsilon \in \mathbb{R} \mid m \trianglerighteq_{\varepsilon} n\}$$

where $\inf \emptyset = \infty$. △

As with before, this distance adheres to our idea of relaxing our PCTs.

Theorem 4.4. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* branching-compact *WKS, $m, n \in M$ and $\varepsilon \in \mathbb{R}$.*

If $d(m,n) = \varepsilon$ then $m \trianglerighteq_{\varepsilon} n$.

Proof. Clearly, $d(m,n) = \varepsilon$ implies that $\ell(m) \subseteq \ell(n)$, as there must exists some $\gamma \geq \varepsilon$ such that $m \trianglerighteq_{\gamma} n$. Assume now towards a contradiction, that there exists some pair $S, T \subseteq M$ such that

$$\inf \Theta(m)(S,T) < \inf \Theta(n)(S^{\trianglerighteq_{\varepsilon}}, T^{\trianglerighteq_{\varepsilon}}) - \varepsilon\,. \tag{4.7}$$

However, since $d(m,n) = \inf\{\gamma \in \mathbb{R} | m \trianglerighteq_{\gamma} n\} = \varepsilon$ we have that there exists a decreasing sequence $\gamma_0 \geq \gamma_1 \geq, ..., \geq \varepsilon$ such that $\lim\limits_{k \to \infty} \gamma_k = \varepsilon$, for all $k \in \mathbb{N}$ we have that $m \trianglerighteq_{\gamma_k} n$, and

$$\forall n \in \mathbb{N} : \inf \Theta(m)(S,T) \geq \inf \Theta(n)(S^{\trianglerighteq_{\gamma_n}}, T^{\trianglerighteq_{\gamma_n}}) - \gamma_n\,. \tag{4.8}$$

Note that $S^{\trianglerighteq_{\varepsilon}} = \bigcap_{k \in \mathbb{N}} S^{\trianglerighteq_{\gamma_k}}$ and that $T^{\trianglerighteq_{\varepsilon}} = \bigcap_{k \in \mathbb{N}} T^{\trianglerighteq_{\gamma_k}}$. Since $\mathcal{M}$ is *branching-compact* we get by Lemma 2.3 that

$$\inf \Theta(m)(S,T) \geq \inf \Theta(n)(S^{\trianglerighteq_{\varepsilon}}, T^{\trianglerighteq_{\varepsilon}}) - \varepsilon\,. \tag{4.9}$$

□

As mentioned, this distance forms a hemimetric.

Theorem 4.5. *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* <u>branching-compact</u> *WKS, d is an extended hemi-metric, i.e. for $m, n, o \in M$:*

1. $d(m,m) = 0$
2. $d(m,o) \leq d(m,n) + d(n,o)$.

Proof. (*1*): Clearly $d(m,m) = 0$ for all $m \in M$, as $\inf \Theta(m)(S,T) \geq \inf \Theta(m)(S^{\trianglerighteq}, T^{\trianglerighteq}) - 0$, due to the monotonicity of $\inf \Theta(m)$ and that $S \subseteq S^{\trianglerighteq}$ and $T \subseteq T^{\trianglerighteq}$. □

Lastly, we conclude this section, by producing a robustness results stating that systems that are close together also satisfy similar formula.

Theorem 4.6 (Robustness)**.** *Let $\mathcal{M} = \langle M, \theta, \ell \rangle$ be a* <u>branching-compact</u> *WKS, $m, n \in M$, $\varepsilon \in \mathbb{R}$ and $\phi \in \mathcal{L}_I$. If $d(m,n) \leq \varepsilon$ and $m \models \phi$, then $n \models \phi^{\varepsilon}$.*

Proof. Follows from Theorem 4.3 and Theorem 4.4. □

5

Conclusion

In this Thesis we delved into world of weighted systems and perceived them through the temporal lense of branching-time. We developed a mathematically concise way of reasoning about weighted behaviour in branching-time, resulting in a behavioural relation, namely Weighted Branching Bisimulation.

We developed a temporal logic which can be used to reason about weighted behaviour in branching-time and showed that for a class of branching-compact systems, that this logic completely characterises our Weighted Branching Bisimulation.

Due to the restrictive nature of exact quantitative behavioural relations, we developed a notion of relaxing our bisimulation. In parallel with this we were also capable of relaxing our logic in a intuitive way that preserved our characterisation result. From this notion of relaxtion we induced a behavioural distance from the infimum of margin of errors required for two systems to be relaxed-Weighted Branching Bisimilar. We showed that this distance is a pseudometric on branching-compact systems, and that behave in accordance with our ideas of weighted branching behaviour. We also produced robustness results, stating that weighted systems close together (small distance from each other) satisfy similar formulae.

As it is often not required to reason about weighted systems as a whole, we produced two cheaper than relations. One in which it is only required for a cheaper than system to be possibly cheaper than the other, and another in which it must always be cheaper than the other.

We developed another temporal logic to reason about the bounds of weighted systems in branching time and show that we can characterise our cheaper than relations using fragments of this logic.

Lastly, we extended the notion of relaxing our behavioural relations to our cheaper than relations, showing similar results for them as we did for our bisimulation. We induced a distance and showed it is a hemimetric and similar robustness results as before.

5.1 Future Work

Decidability and complexity results for the problems of equivalence checking any of our behavioural relations, a long with the problems of model checking any of our logics on our specified models, are all open problems.

One could also be interested in further extending the work of Hansen et al. [HLM+17], and develop a notion of General Weighted Branching Bisimulation, in which one only require the bounds of weighted systems to agree.

Lastly, one could look into compositionality with regards to weighted systems. What does it mean to run two weighted systems in parallel, and can we communicate somehow?

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A

Proofs

Theorem 3.2

Proof. **(⇒) :** We show that if $m \overset{\varepsilon}{\approx} n$ then $\forall \phi \in \mathcal{L}^+ : \begin{array}{l} m \models \phi \implies n \models \phi^\varepsilon \\ n \models \phi \implies m \models \phi^\varepsilon \end{array}$.

Induction on the structure of $\phi \in \mathcal{L}$. Suppose $m \overset{\varepsilon}{\approx} n$ and that $m \models \phi$, we show that this implies that $n \models \phi^\varepsilon$.

Case $\phi = a$**:**

By Definition 3.4 we have that $m \models a$ if and only if $a \in \ell(m)$. Since $m \overset{\varepsilon}{\approx} n$ we have by Definition 3.5 that $\ell(m) = \ell(n)$, and therefore $a \in \ell(n)$. So by Definition 3.4 we have that $n \models a^\varepsilon = a$.

Case $\phi = a$**:**

By Definition 3.4 we have that $m \models \neg a$ if and only if $a \notin \ell(m)$. Since $m \overset{\varepsilon}{\approx} n$ we have by Definition 3.5 that $\ell(m) = \ell(n)$, and therefore $a \notin \ell(n)$. So by Definition 3.4 we have that $n \models \neg a^\varepsilon = \neg a$.

Case $\phi = \phi_1 \wedge \phi_2$**:**

By Definition 3.4 we have that $m \models \phi_1 \wedge \phi_2$ if and only if $m \models \phi_1$ and $m \models \phi_2$. By induction we have that $n \models \phi_1^\varepsilon$ and $n \models \phi_2^\varepsilon$, and therefore, by Definition 3.4, we have that $n \models \phi_1^\varepsilon \wedge \phi_2^\varepsilon$.

Case $\phi = E_{[v,u]}\ \phi_1\ U\ \phi_2$**:**

By Definition 3.4 we have that $m \models E_{[v,u]}\ \phi_1\ U\ \phi_2$ if and only if

$$[v, u] \cap \Theta(m)(\llbracket \phi_1 \rrbracket, \llbracket \phi_2 \rrbracket) \neq \emptyset\,.$$

Since $m \overset{\varepsilon}{\approx} n$ we have by Definition 3.5 that

$$H\left(\Theta(m)(\llbracket \phi_1 \rrbracket^{\overset{\varepsilon}{\approx}}, \llbracket \phi_2 \rrbracket^{\overset{\varepsilon}{\approx}}), \Theta(n)(\llbracket \phi_1 \rrbracket^{\overset{\varepsilon}{\approx}}, \llbracket \phi_2 \rrbracket^{\overset{\varepsilon}{\approx}})\right) \leq \varepsilon\,.$$

By induction we get that $\llbracket \phi_1 \rrbracket^{\overset{\varepsilon}{\approx}} = \llbracket \phi_1^\varepsilon \rrbracket$ and $\llbracket \phi_2 \rrbracket^{\overset{\varepsilon}{\approx}} = \llbracket \phi_2^\varepsilon \rrbracket$. This in turn gives us

$$[v - \varepsilon, u + \varepsilon] \cap \Theta(n)(\llbracket \phi_1^\varepsilon \rrbracket, \llbracket \phi_2^\varepsilon \rrbracket) \neq \emptyset\,.$$

So, by Definition 3.4 we have that $n \models E\ \phi_1^\varepsilon\ U_{[v-\varepsilon,u+\varepsilon]}\ \phi_2^\varepsilon$.
Case $\phi = A_{[v,u]}\ \phi_1\ U\ \phi_2$:
By Definition 3.4 we have that $m \models A_{[v,u]}\ \phi_1\ U\ \phi_2$ if and only if

$$[v,u] \cup \Theta(m)([\![\phi_1]\!],[\![\phi_2]\!]) = [v,u]\,.$$

Since $m \overset{\varepsilon}{\approx} n$ we have by Definition 3.5 that

$$H\left(\Theta(m)([\![\phi_1]\!]^{\overset{\varepsilon}{\approx}},[\![\phi_2]\!]^{\overset{\varepsilon}{\approx}}),\Theta(n)([\![\phi_1]\!]^{\overset{\varepsilon}{\approx}},[\![\phi_2]\!]^{\overset{\varepsilon}{\approx}})\right) \le \varepsilon\,.$$

By induction we get that $[\![\phi_1]\!]^{\overset{\varepsilon}{\approx}} = [\![\phi_1^\varepsilon]\!]$ and $[\![\phi_2]\!]^{\overset{\varepsilon}{\approx}} = [\![\phi_2^\varepsilon]\!]$. This in turn gives us

$$[v-\varepsilon,u+\varepsilon] \cup \Theta(n)([\![\phi_1^\varepsilon]\!],[\![\phi_2^\varepsilon]\!]) = [v-\varepsilon,u+\varepsilon]\,.$$

So, by Definition 3.4 we have that $n \models A\ \phi_1^\varepsilon\ U_{[v-\varepsilon,u+\varepsilon]}\ \phi_2^\varepsilon$.

($\Leftarrow$) : We show that the relation

$$\mathcal{R} = \left\{(m,n) \in M \times M \mid \forall \phi \in \mathcal{L} : \begin{matrix} m \models \phi \implies n \models \phi^\varepsilon \\ n \models \phi \implies m \models \phi^\varepsilon \end{matrix}\right\}$$

is a ε-Weighted Branching Bisimilation relation.

Suppose that $m, n \in M$ and $m\mathcal{R}n$.

Clearly, $m\mathcal{R}n$ implies $\ell(m) = \ell(n)$.

We now show that $m\mathcal{R}n$ implies $H(\Theta(m)(S^\mathcal{R},T^\mathcal{R}),\Theta(n)(S^\mathcal{R},T^\mathcal{R})) \le \varepsilon$, for all $S,T \subseteq M$. We handle the case of S and T being either finite or infinite separately.

Finite Case.

First, without a loss of generality, suppose that both S and T are finite. We can therefore construct a sequence of formulae, $\chi_0^S, \chi_1^S, \ldots$, defined for an arbitrary $k \in \mathbb{N}$ as follows:

$$\chi_k^S = \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_k} \phi\,.$$

Since S and $(\!|s|\!)_k$ are finite, we have that χ_k^S is well formed.

This allows us to create a decreasing sequence of sets of states, $[\![\chi_0^S]\!] \supseteq [\![\chi_1^S]\!] \supseteq \ldots$, for which we now show that

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] = S^\mathcal{R}\,. \tag{A.1}$$

If $s' \in S^\mathcal{R}$ then there exists some $s \in S$ such that $(\!|s|\!) \subseteq (\!|s'|\!)^\varepsilon$. This in turn implies that $s' \models \bigwedge_{\phi \in (\!|s|\!)_k} \phi^\varepsilon$, and hence $s' \models (\chi_k^S)^\varepsilon$, for any $k \in \mathbb{N}$. Ergo,

$$\bigcap_{k\in\mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] \supseteq S^{\mathcal{R}} . \tag{A.2}$$

If $s' \notin S^{\mathcal{R}}$ then for all $s \in S$ there exists some $\phi_s \in \mathcal{B}_I$ such that $s \models \phi_s$ and $s' \not\models \phi_s^\varepsilon$. We can now create a formula, $\varPhi_S$, that distinguishes s' from any $s \in S$,

$$\varPhi_S = \bigvee_{s\in S} \phi_s, \text{ where } s \models \phi_s, \text{ and } s' \not\models \phi_s^\varepsilon .$$

There must exist some $k \in \mathbb{N}$ such that for each $s \in S$ we have that $\phi_s \in (\!|s|\!)_k$. Therefore, we get that $s' \notin [\![(\chi_k^S)^\varepsilon]\!]$, due to there existing some $\phi_s \in (\!|s|\!)_k$ such that $s' \not\models \phi_s^\varepsilon$, for any $s \in S$, i.e.

$$s' \not\models \bigvee_{s\in S} \bigwedge_{\phi\in(\!|s|\!)_k} \phi^\varepsilon ,$$

Ergo,

$$\bigcap_{k\in\mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] \subseteq S^{\mathcal{R}} . \tag{A.3}$$

By Equation (3.7) and A.3 combined we have now shown that Equation (A.1) holds. By similar reasoning we can create a sequence of formulae $[\![\chi_0^T]\!] \supseteq [\![\chi_1^T]\!] \supseteq ...$ where

$$\bigcap_{h\in\mathbb{N}} [\![(\chi_h^T)^\varepsilon]\!] = T^{\mathcal{R}} . \tag{A.4}$$

We now show that

$$\forall k, h \in \mathbb{N} : H(\varTheta(m)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!]), \varTheta(n)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!])) \leq \varepsilon . \tag{A.5}$$

Assume towards a contradiction that

$$\exists k, h \in \mathbb{N} : H(\varTheta(m)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!]), \varTheta(n)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!])) > \varepsilon .$$

Assume without a loss of generality that there exists $w \in \varTheta(m)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!])$ and that for all $v \in \varTheta(n)([\![(\chi_k^S)^\varepsilon]\!], [\![(\chi_h^T)^\varepsilon]\!])$, $|w-v| > \varepsilon$. Since $\mathcal{M}$ is branching-compact there must exists a pair of rational numbers, $v, u \notin \varTheta(n)([\![\chi_k^S]\!], [\![\chi_h^T]\!])$ such that $w \in [v, u]$.

We can now create a distinguishing formula

$$\chi = E_{[v,u]}\ \chi_k^S\ U\ \chi_h^T$$

for which $m \models \chi$ and $n \not\models \chi^\varepsilon$, thereby contradicting that $m\mathcal{R}n$, so Equation (A.5) must hold.

By Lemma 2.6 we have that

$$\Theta(m)(\bigcap_{k\in\mathbb{N}}[\![(\chi_k^S)^\varepsilon]\!],\bigcap_{h\in\mathbb{N}}[\![(\chi_h^T)^\varepsilon]\!])=\bigcap_{k,h\in\mathbb{N}}\Theta(m)([\![(\chi_k^S)^\varepsilon]\!],[\![(\chi_h^T)^\varepsilon]\!])$$
$$\Theta(n)(\bigcap_{k\in\mathbb{N}}[\![(\chi_k^S)^\varepsilon]\!],\bigcap_{h\in\mathbb{N}}[\![(\chi_h^T)^\varepsilon]\!])=\bigcap_{k,h\in\mathbb{N}}\Theta(n)([\![(\chi_k^S)^\varepsilon]\!],[\![(\chi_h^T)^\varepsilon]\!])$$

By Equation (3.10), (3.6), and (3.9), we get that

$$H(\Theta(m)(S^{\mathcal{R}},T^{\mathcal{R}}),\Theta(n)(S^{\mathcal{R}},T^{\mathcal{R}}))\leq\varepsilon\,.$$

Infinite Case.

Suppose that S and T are infinite. Since $\mathcal{M}$ is *countable* we have by Lemma 2.1 that there exists two sequences of finite sets of states, $S_0\subseteq S_1\subseteq\ldots$ and $T_0\subseteq T_1\subseteq\ldots$, such that $\bigcup_{k\in\mathbb{N}}S_k=S$ and $\bigcup_{h\in\mathbb{N}}T_h=T$.

Since S_k and T_h are finite for any $k,h\in\mathbb{N}$ we have by what we previously showed in the finite case , that

$$\forall k,h\in\mathbb{N}:H(\Theta(m)(S_k^{\mathcal{R}},T_h^{\mathcal{R}}),\Theta(n)(S_k^{\mathcal{R}},T_h^{\mathcal{R}}))\leq\varepsilon\,.$$

We therefore have that

$$H(\bigcup_{k,h\in\mathbb{N}}\Theta(m)(S_k^{\mathcal{R}},T_h^{\mathcal{R}}),\bigcup_{k,h\in\mathbb{N}}\Theta(n)(S_k^{\mathcal{R}},T_h^{\mathcal{R}}))\leq\varepsilon\,.$$

which is equivalent to

$$H(\Theta(m)(S^{\mathcal{R}},T^{\mathcal{R}}),\Theta(n)(S^{\mathcal{R}},T^{\mathcal{R}}))\leq\varepsilon\,.$$

$\square$

Theorem 4.2

Proof. **(⇒) :** We show that if $m \triangleright n$ then $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^{\sqcap}$. Induction on the structure of $\phi \in \mathcal{B}_I$. Suppose that $m \triangleright n$ and that $m \models \phi$, we show that $n \models \phi^{\sqcap}$.

Case $\phi = a$:
By Definition 4.4 we have that $m \models a$ if and only if $a \in \ell(m)$. Since $m \triangleright n$ we have by Definition 4.2 that $\ell(m) \subseteq \ell(n)$, and therefore $a \in \ell(n)$. So by Definition 4.4 we have that $n \models a$ and by Definition 4.7 we have that $a = a^{\sqcap}$.

Case $\phi = \phi_1 \wedge \phi_2$:
By Definition 4.4 we have that $m \models \phi_1 \wedge \phi_2$ if and only if $m \models \phi_1$ and $m \models \phi_2$. By induction, we have that $n \models \phi_1^{\sqcap}$ and $n \models \phi_2^{\sqcap}$, and therefore, by Definition 4.4, we have that $n \models \phi_1^{\sqcap} \wedge \phi_2^{\sqcap}$. Lastly, by Definition 4.7 we have that $\phi_1^{\sqcap} \wedge \phi_2^{\sqcap} = (\phi_1 \wedge \phi_2)^{\sqcap}$.

Case $\phi = \phi_1 \wedge \phi_2$:
Derived from Definition 4.4 we have that $m \models \phi_1 \vee \phi_2$ if and only if $m \models \phi_1$ or $m \models \phi_2$. By induction, we have that $n \models \phi_1^{\sqcap}$ or $n \models \phi_2^{\sqcap}$, and therefore, by Definition 4.4, we have that $n \models \phi_1^{\sqcap} \vee \phi_2^{\sqcap}$. Lastly, by Definition 4.7 we have that $\phi_1^{\sqcap} \vee \phi_2^{\sqcap} = (\phi_1 \vee \phi_2)^{\sqcap}$.

Case $\phi = I_{\leq u}\phi_1 U \phi_2$:
By Definition 4.4 we have that $m \models I_{\leq u}\phi_1 U \phi_2$ if and only if

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \leq u\,.$$

Since $m \triangleright n$, we have by Definition 4.2 that

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \sup \Theta(n)([\![\phi_1]\!]^{\triangleright}, [\![\phi_2]\!]^{\triangleright})\,.$$

By induction, we have that $[\![\phi_1]\!]^{\triangleright} = [\![\phi_1^{\sqcap}]\!]$ and that $[\![\phi_2]\!]^{\triangleright} = [\![\phi_2^{\sqcap}]\!]$. We therefore get that

$$u \geq \inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \sup \Theta(n)([\![\phi_1^{\sqcap}]\!], [\![\phi_2^{\sqcap}]\!])\,.$$

Hence, by Definition 4.4, we have that $n \models S_{\leq u}\phi_1^{\sqcap}\, U\, \phi_2^{\sqcap}$, for which, by Definition 4.7, we have that $S_{\leq u}\phi_1^{\sqcap}\, U\, \phi_2^{\sqcap} = (I_{\leq u}\phi_1 U \phi_2)^{\sqcap}$.

(⇐) : We show that if $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^{\sqcap}$ then $m \triangleright n$. It is sufficient to show that the relation

$$\mathcal{R} = \{(m, n) \in M \times M \mid \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^{\sqcap}\}$$

is a Always Cheaper Than relation.

Suppose that $m, n \in M$ and $m\mathcal{R}n$.

Clearly, $m\mathcal{R}n$ implies $\ell(m) \subseteq \ell(n)$.

We now show that $m\mathcal{R}n$ implies $\inf \Theta(m)(S, T) \geq \sup \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})$, for all $S, T \subseteq M$. We handle the case of S and T being either finite or infinite separately.

Finite Case.

First, without a loss of generality, suppose that both S and T are finite. We can therefore construct a sequence of formulae, $\chi_0^S, \chi_1^S, ...$, defined for an arbitrary $k \in \mathbb{N}$ as follows:

$$\chi_k^S = \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_k} \phi .$$

Since S and $(\!|s|\!)_k$ are finite, we have that χ_k^S is well formed.

This allows us to create a decreasing sequence of sets of states, $[\![\chi_0^S]\!] \supseteq [\![\chi_1^S]\!] \supseteq ...$, for which we now show that

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\sqcap]\!] = S^\mathcal{R} . \tag{A.6}$$

If $s' \in S^\mathcal{R}$ then there exists some $s \in S$ such that $(\!|s|\!) \subseteq (\!|s'|\!)^\sqcap$. This in turn implies that $s' \models \bigwedge_{\phi \in (\!|s|\!)_k} \phi^\sqcap$, and hence $s' \models (\chi_k^S)^\sqcap$, for any $k \in \mathbb{N}$. Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\sqcap]\!] \supseteq S^\mathcal{R} . \tag{A.7}$$

If $s' \notin S^\mathcal{R}$ then for all $s \in S$ there exists some $\phi_s \in \mathcal{B}_I$ such that $s \models \phi_s$ and $s' \not\models \phi_s^\sqcap$. We can now create a formula, Φ_S, that distinguishes s' from any $s \in S$,

$$\Phi_S = \bigvee_{s \in S} \phi_s , \text{ where } s \models \phi_s , \text{ and } s' \not\models \phi_s^\sqcap .$$

There must exist some $k \in \mathbb{N}$ such that for each $s \in S$ we have that $\phi_s \in (\!|s|\!)_k$. Therefore, we get that $s' \notin [\![\chi_k^S]\!]$, due to there existing some $\phi_s \in (\!|s|\!)_k$ such that $s' \not\models \phi_s^\sqcap$, for any $s \in S$, i.e.

$$s' \not\models \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_h} \phi^\sqcap ,$$

Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\sqcap]\!] \subseteq S^\mathcal{R} . \tag{A.8}$$

By Equation (A.7) and A.8 combined we have now shown that Equation (A.6) holds. By similar reasoning we can create a sequence of formulae $[\![\chi_0^T]\!] \supseteq [\![\chi_1^T]\!] \supseteq ...$ where

$$\bigcap_{h \in \mathbb{N}} [\![(\chi_h^T)^\sqcap]\!] = T^\mathcal{R} . \tag{A.9}$$

We now show that

$$\forall k, h \in \mathbb{N} : \inf \Theta(m)(S,T) \geq \sup \Theta(n)(\llbracket (\chi_k^S)^\sqcap \rrbracket, \llbracket (\chi_h^T)^\sqcap \rrbracket)\,. \tag{A.10}$$

Assume towards a contradiction that

$$\exists k, h \in \mathbb{N} : \inf \Theta(m)(S,T) < u < \sup \Theta(n)(\llbracket (\chi_k^S)^\sqcap \rrbracket, \llbracket (\chi_h^T)^\sqcap \rrbracket)\,.$$

where $u \in \mathbb{Q}$. Since $\inf \Theta(m)$ is monotonic and since $S \subseteq \llbracket \chi_k^S \rrbracket$ and $T \subseteq \llbracket \chi_h^T \rrbracket$ we have that

$$\inf \Theta(m)(\llbracket \chi_k^S \rrbracket, \llbracket \chi_h^T \rrbracket) \leq \inf \Theta(m)(S,T) < q\,.$$

We can now create a distinguishing formula

$$\chi = I_{\leq u}\ \chi_k^S\ U\ \chi_h^T$$

for which $m \models \chi$ and $n \not\models \chi^\sqcap$, thereby contradicting that $m\mathcal{R}n$, so Equation (4.5) must hold.

By Lemma 2.6 we have that

$$\Theta(n)(\bigcap_{k\in\mathbb{N}} \llbracket (\chi_k^S)^\sqcap \rrbracket, \bigcap_{h\in\mathbb{N}} \llbracket (\chi_h^T)^\sqcap \rrbracket) = \bigcap_{k,h\in\mathbb{N}} \Theta(n)(\llbracket (\chi_k^S)^\sqcap \rrbracket, \llbracket (\chi_h^T)^\sqcap \rrbracket)$$

Since $\mathcal{M}$ is branching-compact and due to Equation (A.10), we get by Lemma 2.3 that

$$\inf \Theta(m)(S,T) \geq \sup \bigcap_{k,h\in\mathbb{N}} \Theta(n)(\llbracket \chi_k^S \rrbracket, \llbracket \chi_h^T \rrbracket)$$

and by Equation (A.6) and (A.9) we get that

$$\inf \Theta(m)(S,T) \geq \sup \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})\,. \tag{A.11}$$

Infinite Case.

Suppose that S and T are infinite. Since $\mathcal{M}$ is *countable* we have by Lemma 2.1 that there exists two sequences of finite sets of states, $S_0 \subseteq S_1 \subseteq ...$ and $T_0 \subseteq T_1 \subseteq ...$, such that $\bigcup_{k\in\mathbb{N}} S_k = S$ and $\bigcup_{h\in\mathbb{N}} T_h = T$.

Since S_k and T_h are finite for any $k, h \in \mathbb{N}$ we have by what we previously showed in the finite case (Equation (A.11)), that

$$\forall k, h \in \mathbb{N} : \inf \Theta(m)(S_k, T_h) \geq \sup \Theta(n)(S_k^{\mathcal{R}}, T_k^{\mathcal{R}})\,.$$

Therefore, by Lemma 2.2, we have that

$$\inf \bigcup_{k,h\in\mathbb{N}} \Theta(m)(S_k, T_h) \geq \sup \bigcup_{k,h\in\mathbb{N}} \Theta(n)(S_k^{\mathcal{R}}, T_h^{\mathcal{R}})\,.$$

which is equivalent to

$$\inf \Theta(m)(S,T) \geq \sup \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})\,.$$

$\square$

Theorem 4.3

Proof. **($\Rightarrow$) :** We show that if $m \trianglerighteq_\varepsilon n$ then $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^\varepsilon$. Induction on the structure of $\phi \in \mathcal{B}_I$. Suppose that $m \trianglerighteq_\varepsilon n$ and that $m \models \phi$, we show that $n \models \phi^\varepsilon$.
Case $\phi = a$:
By Definition 4.4 we have that $m \models a$ if and only if $a \in \ell(m)$. Since $m \trianglerighteq_\varepsilon n$ we have by Definition 4.8 that $\ell(m) \subseteq \ell(n)$, and therefore $a \in \ell(n)$. So by Definition 4.4 we have that $n \models a$ and by Definition 4.9 we have that $a = a^\varepsilon$.
Case $\phi = \phi_1 \wedge \phi_2$:
By Definition 4.4 we have that $m \models \phi_1 \wedge \phi_2$ if and only if $m \models \phi_1$ and $m \models \phi_2$. By induction, we have that $n \models \phi_1^\varepsilon$ and $n \models \phi_2^\varepsilon$, and therefore, by Definition 4.4, we have that $n \models \phi_1^\varepsilon \wedge \phi_2^\varepsilon$. Lastly, by Definition 4.9 we have that $\phi_1^\varepsilon \wedge \phi_2^\varepsilon = (\phi_1 \wedge \phi_2)^\varepsilon$.
Case $\phi = \phi_1 \wedge \phi_2$:
Derived from Definition 4.4 we have that $m \models \phi_1 \vee \phi_2$ if and only if $m \models \phi_1$ or $m \models \phi_2$. By induction, we have that $n \models \phi_1^\varepsilon$ or $n \models \phi_2^\varepsilon$, and therefore, by Definition 4.4, we have that $n \models \phi_1^\varepsilon \vee \phi_2^\varepsilon$. Lastly, by Definition 4.9 we have that $\phi_1^\varepsilon \vee \phi_2^\varepsilon = (\phi_1 \vee \phi_2)^\varepsilon$.
Case $\phi = I_{\leq u} \phi_1 U \phi_2$:
By Definition 4.4 we have that $m \models I_{\leq u} \phi_1 U \phi_2$ if and only if

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \leq u \,.$$

Since $m \trianglerighteq_\varepsilon n$, we have by Definition 4.8 that

$$\inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \inf \Theta(n)([\![\phi_1]\!]^{\trianglerighteq_\varepsilon}, [\![\phi_2]\!]^{\trianglerighteq_\varepsilon}) - \varepsilon \,.$$

By induction we have that $[\![\phi_1]\!]^{\trianglerighteq_\varepsilon} = [\![\phi_1^\varepsilon]\!]$ and that $[\![\phi_2]\!]^{\trianglerighteq_\varepsilon} = [\![\phi_2^\varepsilon]\!]$. We therefore get that

$$u \geq \inf \Theta(m)([\![\phi_1]\!], [\![\phi_2]\!]) \geq \inf \Theta(n)([\![\phi_1^\varepsilon]\!], [\![\phi_2^\varepsilon]\!]) - \varepsilon \,.$$

Hence, we get that

$$u + \varepsilon \geq \inf \Theta(n)([\![\phi_1^\varepsilon]\!], [\![\phi_2^\varepsilon]\!]) \,,$$

and therefore by Definition 4.4 we have that $n \models I_{\leq u+\varepsilon} \phi_1^\varepsilon U \phi_2^\varepsilon$.

($\Leftarrow$) : We show that if $\forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^\varepsilon$ then $m \trianglerighteq_\varepsilon n$. It is sufficient to show that the relation

$$\mathcal{R} = \{(m,n) \in M \times M \mid \forall \phi \in \mathcal{B}_I : m \models \phi \implies n \models \phi^\varepsilon\}$$

is a Possibly Cheaper Than relation. Suppose that $m, n \in M$ and $m \mathcal{R} n$.

Clearly, $m \mathcal{R} n$ implies $\ell(m) \subseteq \ell(n)$.

We now show that $m \mathcal{R} n$ implies $\inf \Theta(m)(S,T) \geq \inf \Theta(n)(S^{\mathcal{R}}, T^{\mathcal{R}})$, for all $S, T \subseteq M$. We handle the case of S and T being either finite or infinite separately.

Finite Case.

First, without a loss of generality, suppose that both S and T are finite. We can therefore construct a sequence of formulae, $\chi_0^S, \chi_1^S, ...$, defined for an arbitrary $k \in \mathbb{N}$ as follows:

$$\chi_k^S = \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_k} .$$

Since S and $(\!|s|\!)_k$ are finite, we have that χ_k^S is well formed.

This allows us to create a decreasing sequence of sets of states, $[\![\chi_0^S]\!] \supseteq [\![\chi_1^S]\!] \supseteq ...$, for which we now show that

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] = S^{\mathcal{R}} . \tag{A.12}$$

If $s' \in S^{\mathcal{R}}$ then there exists some $s \in S$ such that $(\!|s|\!) \subseteq (\!|s'|\!)^\varepsilon$. This in turn implies that $s' \models \bigwedge_{\phi \in (\!|s|\!)_k} \phi^\varepsilon$, and hence $s' \models (\chi_k^S)^\varepsilon$, for any $k \in \mathbb{N}$. Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] \supseteq S^{\mathcal{R}} . \tag{A.13}$$

If $s' \notin S^{\mathcal{R}}$ then for all $s \in S$ there exists some $\phi_s \in \mathcal{B}_I$ such that $s \models \phi_s$ and $s' \not\models \phi_s^\varepsilon$. We can now create a formula, Φ_S, that distinguishes s' from any $s \in S$,

$$\Phi_S = \bigvee_{s \in S} \phi_s, \text{ where } s \models \phi_s, \text{ and } s' \not\models \phi_s^\varepsilon .$$

There must exist some $k \in \mathbb{N}$ such that for each $s \in S$ we have that $\phi_s \in (\!|s|\!)_k$. Therefore, we get that $s' \notin [\![\chi_k^S]\!]$, due to there existing some $\phi_s \in (\!|s|\!)_k$ such that $s' \not\models \phi_s^\varepsilon$, for any $s \in S$, i.e.

$$s' \not\models \bigvee_{s \in S} \bigwedge_{\phi \in (\!|s|\!)_h} \phi^\varepsilon ,$$

Ergo,

$$\bigcap_{k \in \mathbb{N}} [\![(\chi_k^S)^\varepsilon]\!] \subseteq S^{\mathcal{R}} . \tag{A.14}$$

By Equation (A.13) and A.14 combined we have now shown that Equation (A.12) holds. By similar reasoning we can create a sequence of formulae $[\![\chi_0^T]\!] \supseteq [\![\chi_1^T]\!] \supseteq ...$ where

$$\bigcap_{h \in \mathbb{N}} [\![(\chi_h^T)^\varepsilon]\!] = T^{\mathcal{R}} . \tag{A.15}$$

We now show that

$$\forall k,h \in \mathbb{N} : \inf \Theta(m)(S,T) \geq \inf \Theta(n)([\![(\chi_k^S)^\varepsilon]\!],[\![(\chi_h^T)^\varepsilon]\!]) - \varepsilon\,. \tag{A.16}$$

Assume towards a contradiction that

$$\exists k,h \in \mathbb{N} : \inf \Theta(m)(S,T) < u < \inf \Theta(n)([\![(\chi_k^S)^\varepsilon]\!],[\![(\chi_h^T)^\varepsilon]\!]) - \varepsilon\,.$$

where $u \in \mathbb{Q}$. Since $\inf \Theta(m)$ is monotonic and since $S \subseteq [\![\chi_k^S]\!]$ and $T \subseteq [\![\chi_h^T]\!]$ we have that

$$\inf \Theta(m)([\![\chi_k^S]\!],[\![\chi_h^T]\!]) \leq \inf \Theta(m)(S,T) < q\,.$$

We can now create a distinguishing formula

$$\chi = I_{\leq u}\ \chi_k^S\ U\ \chi_h^T$$

for which $m \models \chi$ and $n \not\models \chi^\varepsilon$, thereby contradicting that $m\mathcal{R}n$, so Equation (4.5) must hold.

By Lemma 2.6 we have that

$$\Theta(n)(\bigcap_{k\in\mathbb{N}}[\![(\chi_k^S)^\varepsilon]\!],\bigcap_{h\in\mathbb{N}}[\![(\chi_h^T)^\varepsilon]\!]) = \bigcap_{k,h\in\mathbb{N}} \Theta(n)([\![(\chi_k^S)^\varepsilon]\!],[\![(\chi_h^T)^\varepsilon]\!])$$

Since $\mathcal{M}$ is <u>branching-compact</u> and due to Equation (A.16), we get by Lemma 2.3 that

$$\inf \Theta(m)(S,T) \geq \inf \bigcap_{k,h\in\mathbb{N}} \Theta(n)([\![\chi_k^S]\!],[\![\chi_h^T]\!]) - \varepsilon$$

and by Equation (A.12) and (A.15) we get that

$$\inf \Theta(m)(S,T) \geq \inf \Theta(n)(S^{\mathcal{R}},T^{\mathcal{R}}) - \varepsilon\,. \tag{A.17}$$

Infinite Case.

Suppose that S and T are infinite. Since $\mathcal{M}$ is *countable* we have by Lemma 2.1 that there exists two sequences of finite sets of states, $S_0 \subseteq S_1 \subseteq \ldots$ and $T_0 \subseteq T_1 \subseteq \ldots$, such that $\bigcup_{k\in\mathbb{N}} S_k = S$ and $\bigcup_{h\in\mathbb{N}} T_h = T$.

Since S_k and T_h are finite for any $k,h \in \mathbb{N}$ we have by what we previously showed in the finite case (Equation (A.17)), that

$$\forall k,h \in \mathbb{N} : \inf \Theta(m)(S_k,T_h) \geq \inf \Theta(n)(S_k^{\mathcal{R}},T_k^{\mathcal{R}}) - \varepsilon$$

Therefore, by Lemma 2.2, we have that

$$\inf \bigcup_{k,h\in\mathbb{N}} \Theta(m)(S_k,T_h) \geq \inf \bigcup_{k,h\in\mathbb{N}} \Theta(n)(S_k^{\mathcal{R}},T_h^{\mathcal{R}}) - \varepsilon$$

which is equivalent to

$$\inf \Theta(m)(S,T) \geq \inf \Theta(n)(S^{\mathcal{R}},T^{\mathcal{R}}) - \varepsilon\,.$$

$\square$