Time-Frequency Analysis

Multi-Window Gabor Systems

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Abstract:

The main focus of this thesis is time-frequency analysis. In this perspective the current interest include a mathematical examination of multi-window Gabor systems, hence applications of the presented theory is not included in the present work.
The thesis has a hierarchical structure. We start by presenting the most important results regarding frames and operators on the Hilbert space $L^2(\mathbb{R})$. Secondly, in Chapter 3, we will present the Zak transform and the piecewise Zak transform, and show their main properties. This then leads to Chapter 4, where an examination of the multi-window Gabor system will be given in the Zak transform domain. Issues regarding undersampling, critical sampling and oversampling will be covered and calculations of the frame bounds and the dual frame is considered as well. Chapter 5 deals with a generalization of the multi-window Gabor system, where a countable set of generators is incorporated to the system. Chapter 6 presents a different approach for the analysis of multi-window Gabor systems. We introduce the concept of quilted Gabor frames and show frame properties of a specific situation. The thesis will finish with a conclusion and give a perspective to an application of the presented theory.

Contents

Preface

Aalborg University, January 6, 2017

This project has been prepared by Zeineb Al-Jawahri, as documentation for the master's thesis throughout the fall semester 2016 of the 11th semester of Mathematics at the Faculty of Engineering and Science of Aalborg University. The thesis is written in the period from September 1st 2016 to January 6th, 2017. The overall theme is time-frequency analysis. In this respect, the thesis will concern multi-window Gabor systems. The thesis will appeal first to mathematicians, who have a mathematical understanding corresponding to a bachelor's degree, but it is also accessible to anyone interested in time-frequency analysis, and whom have a basic knowledge of Hilbert space theory, Fourier theory and measure theory.
I would like to thank my supervisor professor Morten Nielsen for his guidance throughout the development of this thesis.

The thesis will use the Vancouver reference style to indicate the sources used. A proof will end with a black square (■), and a definition will end with a triangle (△). When the thesis refers to an equation used earlier in the project, the numbering of the equation will be written in parentheses. By reference without parentheses, it will always be mentioned whether the reference is to a theorem, a section etc. In the appendix some additional definitions and results are added.

The thesis has been compiled with the assistance of LaTeX.

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Danish abstract

Det primære fokus for dette speciale er tids-frekvens analyse. I den forbindelse tages der udgangspunkt i en matematisk beskrivelse af multible Gabor systemer. Specialet vil derfor ikke omhandle anvendelser af den præsenterede teori.

Indledningsvis introduceres motivet bag specialeafhandlingen. Dernæst præsenteres de vigtigste definitioner og resultater vedrørende frames og operatorer i Hilbertrummet $L^2(\mathbb{R})$. I Kapitel 3 introduceres Zak transformationen og den stykvise Zak transformation, hvortil de vigtigste egenskaber præsenteres. Særligt anvendeligt i det videre arbejde er, at Zak transformationen er en unitær afbildning. Det fører os naturligt videre til Kapitel 4, hvor frame egenskaber for det multible Gabor system studeres i Zak transformations domænet. Ved at betragte tilfældet hvor produktet ab af tids- og frekvensparametrene er et rationelt tal vises det, at den tilhørende frame operator kan repræsenteres ved en funktion med matrix værdier. Områder vedrørende undersampling, kritisk sampling og oversampling vil blive behandlet, og beregninger af frame grænserne samt dual framen vil ligeledes blive dækket. Kapitel 5 omhandler en generalisering af det multible Gabor system, præsenteret i forrige kapitel. Systemet udvides til at inkorporere en højst tællelig mængde frembringere. I Kapitel 6 præsenteres en alternativ tilgang til konstruktion af multible Gabor systemer. Vi introducerer begrebet quilted Gabor frames og viser frame egenskaber for en specifik situation. Specialet afsluttes med en konklusion samt en perspektivering til anvendelsen af den gennemgående teori om multible Gabor systemer.

1. Introduction

We will begin this master's thesis with an introduction of the problem that will be studied. This will serve as a motivator and set the framework for the following work.

The thesis will be concerned with some aspects of time-frequency analysis. The main motivation for this study comes from considering the analysis of one-dimensional signals, which may be classified into two different types: 1) Spectral analysis of stationary sinusoidal signals and 2) spectral analysis of non-stationary signals with time-varying parameters. For the former case it is assumed that the parameters, which characterize the sinusoidal signals, such as frequency, do not change with time. In this case, the Fourier transform is a tool which provides global spectral information about the signal. However, many physical processes and signals are non-stationary, meaning they evolve with time, for example speech or music. Since the Fourier system is not well adapted to represent local information in time of a function, i.e., it does not give information about which frequencies that occur at a given time, we search for another method with the ability to provide information about the frequencies occurring at any given time, that is, we seek for a joint time-frequency representation of signals. However, due to the so-called uncertainty principles an instantaneous time-frequency description of signals is impossible. Nevertheless, there exists several ideas for joint time-frequency representations of signals, which are both possible and realizable.

One attempt to address the problem of a joint time-frequency representation of signals is the short-time Fourier transform (STFT). The idea of the STFT is to restrict f to an interval, which is accomplished by taking the inner product of the signal f with a window function g, which is chosen to be smooth to avoid artificial discontinuities, and nonzero for only a short period of time. Then one takes the Fourier transform of this restriction. By letting the window function g slide along the time axis one obtains a description of the signal's frequency spectrum as a function of time. Formally, the STFT of a function $f \in L^2(\mathbb{R})$ with respect to a given window function g is defined as

$$\mathcal{V}_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}\,\mathrm{e}^{-2\pi i t\omega}\,\mathrm{d}t, \quad \text{for } x,\omega \in \mathbb{R}.$$

The function f can then be recovered from its STFT via the inversion formula of the STFT [10]. However, whilst the STFT is a satisfactory method in describing the behaviour of a signal f in time and frequency simultaneously, it is not an ideal tool in many practical purposes, since we are generally not interested in calculating the inner product of the function f with the window function g in every point of the time-frequency plane. This fact suggest to ask for a discretization of the STFT. An attempt to a discretization is to sample the STFT over a time-frequency lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$ are fixed time- and frequency parameters respectively. In this perspective D. Gabor [9] proposed a discrete and linear time-frequency representation of signals. He suggested to expand a function f into a series of elementary functions, which are generated by a single modulated and translated window function g. This leads to a series expansion of the function f of the form

$$f = \sum_{m,n\in\mathbb{Z}} c_{m,n} M_{mb} T_{na} g, \quad m,n \in \mathbb{Z} \quad a,b > 0. \tag{1.1}$$

The Gabor elementary functions $g_{m,n} := M_{mb}T_{na}g$ essentially occupies a certain area in the time-frequency plane and thus each of the Gabor coefficients $c_{m,n}$, associated to a certain area of the time-frequency plane via $g_{m,n}$, are a measure of the time-frequency content of f at time na and frequency mb. The set of the functions $g_{m,n}$ is called a Gabor system. These systems are useful in analysing some classes of signals, however, the single-window Gabor system come with some limitations. They are designed such that they only feature one resolution over the whole time-frequency plane, determined by the time and frequency parameters a and b. This is not ideal for certain type of signals, since different resolutions might be necessary in order to obtain a precise, and at the same time, a sparse representation of the signal. In analysing music signals one might have an interest in using both wider windows with good frequency concentration in the low-frequency regions and short windows, which do not have to be very localized in frequency, in the high-frequency regions [6].

This thesis will study the theory of a generalization of the single-window Gabor system, where several window functions of different shape are incorporated to the system. The new system will be called a *multi-window Gabor system*. The current interest will include a theoretical examination of multi-window Gabor systems, hence applications of the presented theory will not be included. We will study two different approaches for constructing and analysing such systems. The first approach utilizes the Zak transform and uses matrix algebra for the analysis of frame properties of the multi-window Gabor system. The second approach uses an admissible covering and an associated, so-called BAPU, to partition the time-frequency plane into different strips. Then a suitable Gabor frame, from a collection of given Gabor frames, is assigned to each of these strips.

The organization of the thesis is as follows. In Chapter 2 we will review some basic theory of frames in Hilbert spaces and introduce Gabor frames in $L^2(\mathbb{R})$ together with some of their fundamental properties. The chapter will recall the main results concerning frames and frame operators, however, the proofs of the presented results in this chapter have been omitted. They can be found in a previous project [1].

In Chapter 3 we will introduce the Zak transform (ZT) and study its connection to Gabor systems. The chapter will include several properties of the ZT and the piecewise Zak transform (PZT), where especially their unitary properties will be of significance importance in the following two chapters. In Chapter 4 we will examine one approach in constructing and characterizing multi-window Gabor systems. This approach is developed by Meir Zibulski and Yehoshua Y. Zeevi. We combine the ZT with the concept of frames and show how the frame operator associated with the multi-window Gabor frame can be represented as a finite order matrix-valued function. Chapter 5 will cover a generalization of the multi-window Gabor system presented in the previous chapter, where the system is extended to incorporate an at most countable set of window functions. Chapter 6 treats another approach in constructing and characterizing multi-window Gabor systems. This approach, which goes under the name *quilted Gabor frames*, is developed by Monika Dörfler. The thesis will finish with a conclusion in Chapter 7 and present a perspective to where the presented theory may be used in practice.

2. Frames in Hilbert spaces

This chapter contains a brief review of some definitions and results regarding frames and frame operators in Hilbert spaces. Since the presented theory in this chapter have been addressed in a previous project [1], the proofs will be omitted. The proofs and other properties of frames and frame operators can be found in [5]. Section 2.1 will concern a more specific Hilbert space, namely the $L^2(\mathbb{R})$-space, where the so-called Gabor system arises naturally. Studying this particularly space follows from the fact that we are mainly interested in time-frequency analysis of signals, and since signals are continuous and cannot have infinite energy, we may assume them to be members of the $L^2(\mathbb{R})$-space from $\mathbb{R} \to \mathbb{C}$.

Letting $\mathcal{H}$ denote a separable Hilbert space, we define the concept of a frame in $\mathcal{H}$ as follows.

Definition 2.1: A sequence of elements $\{f_k\}_{k\geq 1}$ in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist positive constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$ we have

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 .$$

The constants A and B are called frame bounds and they are not unique. If $A = B$ we call the frame a *tight* frame. △

If $\{f_k\}_{k\geq 1} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ then the closed linear span of $\{f_k\}_{k\geq 1}$ is the entire space $\mathcal{H}$, thus $\{f_k\}_{k\geq 1}$ is complete. A frame $\{f_k\}_{k\geq 1}$, which is a frame for all of $\mathcal{H}$, allows every other element in $\mathcal{H}$ to be written as a linear combination of the elements f_k, i.e.

$$f = \sum_{k=1}^{\infty} c_k(f) f_k. \tag{2.1}$$

Though, unlike a basis in a Hilbert space, the corresponding coefficients $c_k(f)$ are not necessarily unique.

Processing further, we define the frame operator as follows.

Definition 2.2: Given a frame $\{f_k\}_{k\geq 1} \in \mathcal{H}$, the frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined as

$$Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$ △

Some important properties can be stated for the frame operator S.

Lemma 2.3: Let $\{f_k\}_{k\geq 1}$ be a frame with frame operator S and frame bounds A, B. Then

1) S is bounded, invertible, self-adjoint and positive.

Lemma 2.3 states that the frame operator S is invertible hence the inverse frame operator S^{-1} is well defined. By mapping each element of the frame with S^{-1}, we arrive at the canonical dual frame of $\{f_k\}_{k\geq 1}$.

Definition 2.4: Let $\{f_k\}_{k\geq 1}$ be a frame for a Hilbert space $\mathcal{H}$. The canonical dual frame, denoted $\{\tilde{f}_k\}_{k\geq 1}$, is defined as

$$\tilde{f}_k = S^{-1} f_k, \quad k \geq 1. \tag{2.2}$$

△

An important property of the dual frame relates to the decomposition of elements $f \in \mathcal{H}$.

Theorem 2.5: Let $\{f_k\}_{k\geq 1}$ be a frame with frame operator S. Then

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \left\langle f, S^{-1}f_k \right\rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}. \tag{2.3}$$

The series in Equation (2.3) converges unconditionally for all $f \in \mathcal{H}$.

Now that we have reviewed some of the basic theory of frames and frame operators in Hilbert spaces, we continue to consider Gabor frames in $L^2(\mathbb{R})$.

2.1 Gabor frames

The theory of Gabor analysis is based open two specific operators on $L^2(\mathbb{R})$, namely the translation operator T_a and the modulation operator M_b defined as

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (T_a g)(x) = g(x-a).$$
$$M_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (M_b g)(x) = \mathrm{e}^{2\pi i b x}\, g(x).$$

A Gabor system is a collection of functions which are built from a so-called window function $g : \mathbb{R} \to \mathbb{C}$ through shifts in time and frequency determined by a, the time (or shift) parameter and b, the modulation parameter. That is

$$\{g_{m,n}\} := \{(M_{mb}T_{na}g)(x)\}_{m,n\in\mathbb{Z}} = \{\mathrm{e}^{2\pi i m b x}\, g(x-na) \ : \ m,n \in \mathbb{Z}, \forall x \in \mathbb{R}\},$$

where $a, b > 0$ is a Gabor system. Recalling the definition of a frame in a Hilbert space $\mathcal{H}$ (see Definition 2.1) we arrive at the definition of a Gabor frame.

Definition 2.6: A Gabor system $\{g_{m,n}\}$ is a Gabor frame for $L^2(\mathbb{R})$ if there exists positive constants $0 < A \leq B < \infty$ such that for all $f \in L^2(\mathbb{R})$ we have

$$A \left\| f \right\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, g_{m,n} \rangle|^2 \leq B \left\| f \right\|^2. \qquad \triangle$$

An important result is that not for all choices of time and frequency parameters a and b will the Gabor system $\{g_{m,n}\}$ constitute a frame. This is stated in the following.

Theorem 2.7: Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given.

i) If $ab > 1$, then $\{g_{m,n}\}$ is not a Gabor frame for $L^2(\mathbb{R})$.

ii) If $\{g_{m,n}\}$ is a Gabor frame for $L^2(\mathbb{R})$, then $ab = 1$ if and only if $\{g_{m,n}\}$ is a Riesz basis.

Assuming that the Gabor system $\{g_{m,n}\}$ constitutes a frame, then the corresponding frame operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined as

$$Sf = \sum_{m,n\in\mathbb{Z}} \langle f, g_{m,n} \rangle g_{m,n}$$

is, due to Lemma 2.3, bounded and invertible, hence the inverse frame operator S^{-1} is well-defined. A dual frame $\tilde{g} = S^{-1}g$ for a Gabor frame then leads to the reconstruction formula

$$f = SS^{-1}f = \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{m,n} \rangle g_{m,n} = \sum_{m,n\in\mathbb{Z}} \langle f, g_{m,n} \rangle \tilde{g}_{m,n}, \quad \forall f \in L^2(\mathbb{R}),$$

where the series converges unconditionally for all $f \in L^2(\mathbb{R})$.

2.1.1 Gabor frames and Gabor bases

An interesting question in relation to the study of Gabor frames for $L^2(\mathbb{R})$, is why there is a need to study frames over bases. We already know that any separable Hilbert space $\mathcal{H}$ contains an orthonormal basis and by [5, Example 3.5.3] $L^2(\mathbb{R})$ has an orthonormal basis given by

$$\{\mathrm{e}^{2\pi imx}\,\chi_{[0,1]}(x-n)\}_{m,n\in\mathbb{Z}} = \{M_m T_n g(x)\}_{m,n\in\mathbb{Z}}, \quad x \in \mathbb{R}, \tag{2.4}$$

where $g = \chi_{[0,1]}$. This is in fact a Gabor basis, which automatically satisfies the conditions of being a frame with optimal frame bounds $A = B = 1$. The main disadvantage of using Gabor bases over Gabor frames is stated in Theorem 2.8, the so-called Balian-Low Theorem, which shows that if $\{M_m T_n g(x)\}_{m,n\in\mathbb{Z}}$ constitute a Riesz basis for $L^2(\mathbb{R})$ then the window function g cannot be well localized in both time and frequency.

Theorem 2.8: Let $g \in L^2(\mathbb{R})$. If $\{M_m T_n g(x)\}_{m,n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$, then

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2 \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} |\xi\hat{g}(\xi)|^2 \mathrm{d}\xi\right) = \infty,$$

where $\hat{g}$ is the Fourier transform of g defined as $\hat{g}(\xi) = \int_{\mathbb{R}} g(x)\,\mathrm{e}^{-2\pi ix\xi}\,\mathrm{d}x$.

When using Gabor frames with an oversampling rate instead of a Gabor basis, we are able to overcome the problem addressed in the Balian-Low Theorem.

After having reviewed some of the important definitions and properties of frames in Hilbert spaces, we will move on to consider the Zak transform, as this is a useful tool in connection with analysing Gabor systems.

3. The Zak transform

This chapter is based on [5] and [11]. We will introduce the Zak transform as this will be relevant in the following sections, where we will study multi-window Gabor systems. The Zak transform have great application in signal processing, where the transform of a given signal will be a mixed time-frequency representation of the signal. We begin with the definition.

Definition 3.1: Let $f \in L^2(\mathbb{R})$. For a fixed parameter $\lambda > 0$, the Zak transform $Z_\lambda f$ of f is defined as a function of two real variables:

$$(Z_\lambda f)(x,\omega) = \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x-k))\, \mathrm{e}^{2\pi i k\omega}, \quad x,\omega \in \mathbb{R}. \tag{3.1}$$

△

We note that the superscript λ in the definition of the Zak transform, will sometimes be omitted when the value of λ is irrelevant.

In the following lemma we will prove that the Zak transform is a unitary operator, that is, it preserve the linear space structure, the inner product and hence the topology of the space on which it acts.

Lemma 3.2: Given $\lambda > 0$, the Zak transform Z_λ is a unitary operator of $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$.

Proof: Let $Q = [0,1)^2$ and consider first the case where $\lambda = 1$. Given $f \in L^2(\mathbb{R})$ we wish to show that $Z_1 f$ is well-defined as a function in $L^2(Q)$. Introduce the functions

$$F_k(x,\omega) = f(x-k)\,\mathrm{e}^{2\pi i k\omega}, \quad k\in\mathbb{Z}, \quad (x,\omega)\in Q.$$

These functions belong to $L^2(Q)$, since for all $k \in \mathbb{Z}$ we observe that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \|F_k\|^2_{L^2(Q)} &= \sum_{k\in\mathbb{Z}} \int_0^1\int_0^1 |F_k(x,\omega)|^2 \,\mathrm{d}\omega\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \int_0^1\int_0^1 \left| f(x-k)\,\mathrm{e}^{2\pi i k\omega}\right|^2 \,\mathrm{d}\omega\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \left(\int_0^1 \left|\mathrm{e}^{2\pi i k\omega}\right|^2 \mathrm{d}\omega \int_0^1 |f(x-k)|^2 \mathrm{d}x\right) \\
&= \sum_{k\in\mathbb{Z}} \left(\int_0^1 1\mathrm{d}\omega \int_0^1 |f(x-k)|^2 \mathrm{d}x\right) \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 |f(x-k)|^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x \\
&= \|f\|^2 < \infty.
\end{aligned} \tag{3.2}$$

Moreover, for $j \neq k$ we have

$$\begin{aligned}
\langle F_k, F_j\rangle_{L^2(Q)} &= \int_0^1\int_0^1 f(x-k)\,\mathrm{e}^{2\pi i k\omega}\, \overline{f(x-j)\,\mathrm{e}^{2\pi i j\omega}} \mathrm{d}\omega\mathrm{d}x \\
&= \int_0^1 f(x-k)\overline{f(x-j)} \left(\int_0^1 \mathrm{e}^{2\pi i (k-j)\omega}\,\mathrm{d}\omega\right) \mathrm{d}x \\
&= 0,
\end{aligned} \tag{3.3}$$

since, in the last term of Equation (3.3), we have

$$\int_0^1 \mathrm{e}^{2\pi i(k-j)\omega}\,\mathrm{d}\omega = \left[\frac{1}{2\pi i(k-j)}\,\mathrm{e}^{2\pi i(k-j)\omega}\right]_{\omega=0}^{1} = \frac{1}{2\pi i(k-j)}\left(\mathrm{e}^{2\pi i(k-j)} - \mathrm{e}^{0}\right) = 0.$$

Hence F_k and F_j are orthogonal, when $k \neq j$, and thus, by the Pythagorean theorem for infinite sums we have

$$\left\|\sum_{k\in\mathbb{Z}} F_k\right\|_{L^2(Q)}^2 = \sum_{k\in\mathbb{Z}} \|F_k\|_{L^2(Q)}^2 . \tag{3.4}$$

A combination of Equations (3.4) and (3.2) yields

$$\|Z_1 f\|_{L^2(Q)}^2 = \left\|\sum_{k\in\mathbb{Z}} F_k\right\|_{L^2(Q)}^2 = \sum_{k\in\mathbb{Z}} \|F_k\|_{L^2(Q)}^2 = \|f\|^2 .$$

Thus we have shown that the Zak transform $Z_1 f$ is a well defined isometry from $L^2(\mathbb{R})$ to $L^2(Q)$.

We recall by Equation (2.4) that $L^2(\mathbb{R})$ has an orthonormal basis given by

$$\{M_m T_n \chi_{[0,1]}\}_{m,n\in\mathbb{Z}} = \{\mathrm{e}^{2\pi i m x}\,\chi_{[0,1]}(x-n)\}_{m,n\in\mathbb{Z}}, \quad x \in \mathbb{R}.$$

For $(x,\omega) \in Q$ we find that

$$\begin{aligned}
(Z_1 M_m T_n \chi_{[0,1]})(x,\omega) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\pi i m(x-k)}\,\chi_{[0,1]}(x-n-k)\,\mathrm{e}^{2\pi i k\omega} \\
&= \sum_{k'\in\mathbb{Z}} \mathrm{e}^{2\pi i m(x-(k'-n))}\,\chi_{[0,1]}(x-k')\,\mathrm{e}^{2\pi i(k'-n)\omega} \\
&= \mathrm{e}^{2\pi i m x}\,\mathrm{e}^{2\pi i m n}\,\mathrm{e}^{-2\pi i n\omega} \sum_{k'\in\mathbb{Z}} \chi_{[0,1]}(x-k')\,\mathrm{e}^{2\pi i k'\omega}\,\mathrm{e}^{-2\pi i m k'} \\
&= \mathrm{e}^{2\pi i m x}\,\mathrm{e}^{-2\pi i n\omega} \sum_{k'\in\mathbb{Z}} \chi_{[0,1]}(x-k')\,\mathrm{e}^{2\pi i k'\omega} \\
&= \mathrm{e}^{2\pi i m x}\,\mathrm{e}^{-2\pi i n\omega},
\end{aligned}$$

where the last equality follows since $x \in [0,1)$, so the only part in the sum, which is different from zero, is when $k' = 0$. This means that Z_1 maps the orthonormal basis $\{M_n T_n \chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ to the orthonormal basis $\{\mathrm{e}^{-2\pi i n\omega}\,\mathrm{e}^{2\pi i m x}\}_{m,n\in\mathbb{Z}}$ for $L^2(Q)$. Finally, since Z_1 is a linear and bounded operator from $L^2(\mathbb{R})$ to $L^2(Q)$, we have, for an arbitrary $g \in L^2(Q)$ that

$$\begin{aligned}
g(x,\omega) &= \sum_{m,n\in\mathbb{Z}} c_{m,n}\,\mathrm{e}^{2\pi i m x}\,\mathrm{e}^{-2\pi i n\omega} \\
&= \sum_{m,n\in\mathbb{Z}} c_{m,n}\left(Z_1 M_m T_n \chi_{[0,1]}\right)(x,\omega) \\
&= \left(Z_1 \sum_{m,n\in\mathbb{Z}} c_{m,n} M_m T_n \chi_{[0,1]}\right)(x,\omega), \quad (x,\omega) \in \mathbb{R},
\end{aligned}$$

which shows that Z_1 is a surjective operator on $L^2(Q)$. Altogether this implies that Z_1 is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(Q)$.

Now, for the general case we will use that the Dilation operator (see Definition A.1) is a unitary operator by Lemma A.2. We observe that

$$
\begin{aligned}
(Z_\lambda f)(x,\omega) &= \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x-k))\, \mathrm{e}^{2\pi i k\omega} \\
&= \frac{1}{\sqrt{\lambda^{-1}}} \sum_{k\in\mathbb{Z}} f\left(\frac{x-k}{\lambda^{-1}}\right) \mathrm{e}^{2\pi i k\omega} \\
&= \sum_{k\in\mathbb{Z}} D_{\lambda^{-1}} f(x-k)\, \mathrm{e}^{2\pi i k\omega} \\
&= (Z_1 D_{\lambda^{-1}} f)(x,\omega), \quad (x,\omega) \in \mathbb{R}.
\end{aligned}
$$

That is, the Zak transform Z_λ is a composition of two unitary operators, hence it is itself unitary. ■

Some other relevant properties of the Zak transform are stated in the following lemma.

Lemma 3.3: Consider the Zak transform Z_λ with $\lambda > 0$, $f \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$.

i) The Zak transform satisfies the periodic relation in the frequency variable ω:

$$(Z_\lambda f)(x,\omega+n) = (Z_\lambda f)(x,\omega), \tag{3.5}$$

ii) and the quasi-periodic relation in the time variable x:

$$(Z_\lambda f)(x+n,\omega) = \mathrm{e}^{2\pi i\omega n}(Z_\lambda f)(x,\omega). \tag{3.6}$$

Proof: For the periodic relation we observe that

$$
\begin{aligned}
(Z_\lambda f)(x,\omega+n) &= \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x-k))\, \mathrm{e}^{2\pi i(\omega+n)k} \\
&= \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x-k))\, \mathrm{e}^{2\pi i\omega k}\, \mathrm{e}^{2\pi i n k} \\
&= \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x-k))\, \mathrm{e}^{2\pi i\omega k} \\
&= (Z_\lambda f)(x,\omega), \quad x,\omega \in \mathbb{R}.
\end{aligned}
$$

And for the quasi-periodic relation we have

$$
\begin{aligned}
(Z_\lambda f)(x+n,\omega) &= \lambda^{1/2} \sum_{k\in\mathbb{Z}} f(\lambda(x+n-k))\, \mathrm{e}^{2\pi i\omega k} \\
&= \lambda^{1/2} \sum_{j\in\mathbb{Z}} f(\lambda(x-j))\, \mathrm{e}^{2\pi i\omega(n+j)} \\
&= \mathrm{e}^{2\pi i\omega n}\, \lambda^{1/2} \sum_{j\in\mathbb{Z}} f(\lambda(x-j))\, \mathrm{e}^{2\pi i\omega j} \\
&= \mathrm{e}^{2\pi i\omega n}(Z_\lambda f)(x,\omega) \quad x,\omega \in \mathbb{R}.
\end{aligned}
$$

■

Lemma 3.3 shows that the Zak transform $Z_\lambda f$ is completely determined by its values on the unit square $(x,\omega) \in ([0,1)^2)$. Furthermore, in relation to translation and modulation, the Zak transform possesses the following properties:

$$(Z_\lambda T_a f)(x,\omega) = (Z_\lambda f)\left(x - \frac{a}{\lambda}, \omega\right), \quad x,\omega \in \mathbb{R} \tag{3.7}$$

$$(Z_\lambda M_b f)(x,\omega) = \mathrm{e}^{2\pi i b\lambda x}(Z_\lambda f)(x,\omega - b\lambda), \quad x,\omega \in \mathbb{R}. \tag{3.8}$$

The expressions in Equations (3.7) and (3.8) follows immediately by direct computation.

3.1 Piecewise Zak transform

Whereas the Zak transform is a useful tool for the analysis of Gabor systems in the case of critical sampling, i.e., when the product of time- and frequency parameters $ab = 1$, the piecewise Zak transform, introduced by Zibulski and Zeevi [11], generalizes the role of the Zak transform in the case of oversampling.

Definition 3.4: Let $\lambda > 0$ and $p \in \mathbb{N}$ be given. The piecewise Zak transform (PZT) of length p is a vector-valued function $F(x,\omega)$ defined as

$$F(x,\omega) = [F_{0,0}(x,\omega), F_{0,1}(x,\omega), \dots, F_{0,p-1}(x,\omega)]^T, \tag{3.9}$$

where

$$F_{0,j}(x,\omega) = (Z_\lambda f)\left(x, \omega + \frac{j}{p}\right), \quad 0 \le j \le p-1,\ j \in \mathbb{Z},\ (x,\omega) \in \mathbb{R}, \tag{3.10}$$

for all $f \in L^2(\mathbb{R})$. △

We point out to the reader that we use the notation $F_{0,j}(x,\omega)$, as in Equation (3.10), since this type of notation will be used in various other contexts throughout the project. However, it should be noted that in Chapter 5 we will write $G^f_{0,j}(x,\omega)$ for $F_{0,j}(x,\omega)$.

The vector-valued function $F(x,\omega)$ belongs to the Hilbert space $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$, consisting of vector-valued functions, with an inner product defined by

$$\langle F, G\rangle = \int_0^1 \int_0^{1/p} \sum_{j=0}^{p-1} F_{0,j}(x,\omega)\overline{G_{0,j}(x,\omega)}\mathrm{d}\omega\mathrm{d}x, \quad \forall\, F, G \in L^2([0,1)\times[0,1/p);\mathbb{C}^p).$$

As proven in Lemma 3.2, the Zak transform is a unitary operator. What we will see next is that the PZT is likewise a unitary operator from $L^2(\mathbb{R})$ onto $L^2(U;\mathbb{C}^p)$, where we set $U = [0,1)\times[0,1/p)$.

Theorem 3.5: Given $\lambda > 0$, the PZT is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(U;\mathbb{C}^p)$.

Proof: Let $f \in L^2(\mathbb{R})$ and recall by Lemma 3.2 that $Z_\lambda f \in L^2(Q)$. Then we have that $F \in L^2(U;\mathbb{C}^p)$ and F is an isometry from $L^2(\mathbb{R})$ to $L^2(U;\mathbb{C}^p)$, since

$$\begin{aligned}\|F\|^2_{L^2(U;\mathbb{C}^p)} &= \int_0^1 \int_0^{1/p} \sum_{j=0}^{p-1} |F_{0,j}(x,\omega)|^2 \mathrm{d}\omega\mathrm{d}x \\ &= \int_0^1 \int_0^{1/p} \sum_{j=0}^{p-1} \left| Z_\lambda f\left(x, \omega + \frac{j}{p}\right)\right|^2 \mathrm{d}\omega\mathrm{d}x\end{aligned}$$

Letting $u = \omega + \frac{j}{p}$ we find

$$\begin{aligned}\|F\|^2_{L^2(U;\mathbb{C}^p)} &= \int_0^1 \sum_{j=0}^{p-1} \int_{\frac{j}{p}}^{\frac{j+1}{p}} |Z_\lambda f(x,u)|^2 \mathrm{d}u\mathrm{d}x \\ &= \int_0^1 \int_0^1 |Z_\lambda(x,u)|^2 \mathrm{d}u\mathrm{d}x \\ &= \|Z_\lambda f\|^2_{L^2(Q)} = \|f\|^2 < \infty.\end{aligned}$$

Next we will show that the PZT is a surjective operator from $L^2(\mathbb{R})$ to $L^2(U;\mathbb{C}^p)$.

For $G(x,\omega) \in L^2(U;\mathbb{C}^p)$ we write

$$G(x,\omega) = \begin{bmatrix} G_{0,0}(x,\omega) \\ G_{0,1}(x,\omega) \\ \vdots \\ G_{0,p-1}(x,\omega) \end{bmatrix}, \quad (x,\omega) \in U,$$

where $G_{0,j} \in L^2(U)$ for all j, with $0 \le j \le p-1$. Since $G_{0,j} \in L^2(U)$ we know that

$$\sum_{j=0}^{p-1} \int_0^1 \int_0^{1/p} |G_{0,j}(x,\omega)|^2 \mathrm{d}\omega \mathrm{d}x < \infty. \tag{3.11}$$

Now define $\tilde{G}(x,\omega) : Q \to \mathbb{C}$ as:

$$\tilde{G}(x,\omega) = \begin{cases} G_{0,0}\left(x, \omega - \frac{0}{p}\right) & \text{if } \omega \in \left[0, \frac{1}{p}\right) \\ G_{0,1}\left(x, \omega - \frac{1}{p}\right) & \text{if } \omega \in \left[\frac{1}{p}, \frac{2}{p}\right) \\ \vdots & \\ G_{0,p-1}\left(x, \omega - \frac{p-1}{p}\right) & \text{if } \omega \in \left[\frac{p-1}{p}, 1\right). \end{cases}$$

Then $\tilde{G}(x,\omega) \in L^2(Q)$. By Lemma 3.2 there exists $g \in L^2(\mathbb{R})$ such that for all $(x,\omega) \in Q$ we have $Z_\lambda g(x,\omega) = \tilde{G}(x,\omega)$, i.e. we may write

$$G(x,\omega) = \begin{bmatrix} G_{0,0}(x,\omega) \\ G_{0,1}(x,\omega) \\ \vdots \\ G_{0,p-1}(x,\omega) \end{bmatrix} = \begin{bmatrix} \tilde{G}\left(x, \omega + \frac{0}{p}\right) \\ \tilde{G}\left(x, \omega + \frac{1}{p}\right) \\ \vdots \\ \tilde{G}\left(x, \omega + \frac{p-1}{p}\right) \end{bmatrix} = \begin{bmatrix} Z_\lambda g\left(x, \omega + \frac{0}{p}\right) \\ Z_\lambda g\left(x, \omega + \frac{1}{p}\right) \\ \vdots \\ Z_\lambda g\left(x, \omega + \frac{p-1}{p}\right) \end{bmatrix}, \quad (x,\omega) \in U.$$

Thus we have shown that the PZT is a surjective operator on $L^2(U;\mathbb{C}^p)$. Combining the obtained results shows that the PZT is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(U;\mathbb{C}^p)$. ■

The results of Lemma 3.2 and Theorem 3.5 imply that

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \langle Z_\lambda f, Z_\lambda g \rangle_{L^2(Q)} = \langle F, G \rangle_{L^2(U;\mathbb{C}^p)},$$

written explicitly as

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\overline{g(x)} \mathrm{d}x &= \int_0^1 \int_0^1 (Z_\lambda f)(x,\omega)\overline{(Z_\lambda g)(x,\omega)} \mathrm{d}\omega \mathrm{d}x \\ &= \int_0^1 \int_0^{1/p} \sum_{j=0}^{p-1} F_{0,j}(x,\omega)\overline{G_{0,j}(x,\omega)} \mathrm{d}\omega \mathrm{d}x. \end{aligned} \tag{3.12}$$

An important consequence of the unitary property of the PZT is the ability to represent the frame operator, associated to the multi-window Gabor frame, as a finite order matrix-valued function. This leads to the following chapter.

4. Multi-window Gabor systems

This chapter is highly inspired by [11]. We will investigate properties of multi-window Gabor systems of the form

$$\begin{aligned}\{g_{r,m,n}\} &:= \{(M_{mb_r}T_{na_r}g_r)(x)\}_{m,n\in\mathbb{Z}} \\ &= \{\mathrm{e}^{2\pi imb_rx}\, g_r(x-na_r)\ :\ m,n\in\mathbb{Z},\ r\in\mathcal{R},\ \forall x\in\mathbb{R}\},\end{aligned} \tag{4.1}$$

where $a_r, b_r > 0$ and $r \in \mathcal{R} = \{0, 1, \ldots, R-1\}$. That is, $\{g_r(x)\}$ is the set of R distinct window functions.

Definition 4.1: Assume that the time- and frequency parameters $a_r, b_r > 0$ and $g_r \in L^2(\mathbb{R})$ for $r \in \mathcal{R} = \{0, 1, \ldots, R-1\}$ with $R \in \mathbb{N}$. If the Gabor system $\{g_{r,m,n}\}$ constitute a frame for $L^2(\mathbb{R})$, we call the system a multi-window Gabor frame. △

In order to characterize the properties of the sequence $\{g_{r,m,n}\}$, we consider the operator

$$Sf = \sum_{r=0}^{R-1}\sum_{m,n\in\mathbb{Z}} \langle f, g_{r,m,n}\rangle g_{r,m,n}, \quad \forall f\in L^2(\mathbb{R}). \tag{4.2}$$

Recalling the definition of a frame operator (see Definition 2.2) then, under the assumption that $\{g_{r,m,n}\}$ constitute a frame, Sf is the associated frame operator. Though we will call Sf the frame operator regardless of $\{g_{r,m,n}\}$ being a frame.

From now on we will consider properties of the Gabor system in Equation (4.1) in the case where the product ab of time and frequency parameters is a rational number. This allows us to represent the frame operator S as a finite-order matrix-valued function in the PZT domain, as stated in Theorem 4.2. Moreover, we will assume that all the window functions g_r, for $r \in \mathcal{R}$, have the same time- and frequency parameters a and b, and that the Zak transform has parameter value $\lambda = \frac{1}{b}$.

Theorem 4.2: Let $ab = p/q$ with $p, q \in \mathbb{N}$, and let S_z be the operator that maps the PZT of $f \in L^2(\mathbb{R})$ to the PZT of Sf. Then S_z is given by the following matrix algebra:

$$(S_zF)(x,\omega) = S(x,\omega)F(x,\omega), \quad (x,\omega)\in U = [0,1)\times[0,1/p). \tag{4.3}$$

$S(x, \omega)$ is a matrix-valued function of size $p \times p$, whose (j, k)th entry is given by

$$S_{j,k}(x,\omega) = \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\cdot\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\cdot\frac{p}{q}, \omega + \frac{k}{p}\right)}, \tag{4.4}$$

with $j, k = 0, \ldots p-1$,

and the vector-valued function $F(x, \omega) \in L^2(U; \mathbb{C}^p)$ is given by Equations (3.9) and (3.10).

Proof: From Equations (3.7) and (3.8) we obtain

$$\begin{aligned}\left(Z_{\frac{1}{b}}g_{r,m,n}\right)(x,\omega) &= (Z_{\frac{1}{b}}M_{mb}T_{na}g_r)(x,\omega)\\ &= (Z_{\frac{1}{b}}M_{mb}g_r)\left(x-\frac{na}{1/b},\omega\right)\\ &= \mathrm{e}^{2\pi imb(1/b)x}(Z_{\frac{1}{b}}g_r)\left(x-nab,\omega-mb\cdot 1/b\right)\\ &= \mathrm{e}^{2\pi imx}(Z_{\frac{1}{b}}g_r)\left(x-n\cdot\frac{p}{q},\omega-m\right)\\ &= \mathrm{e}^{2\pi imx}(Z_{\frac{1}{b}}g_r)\left(x-n\cdot\frac{p}{q},\omega\right),\quad (x,\omega)\in Q. \end{aligned}\tag{4.5}$$

By Equation (4.5) and the quasi-periodic property of the Zak transform (see Equation (3.6) in Lemma 3.3), we make a change of variable $n=n'q+l,\ 0\le l\le q-1,\ n'\in\mathbb{Z}$ and find that

$$\begin{aligned}\left(Z_{\frac{1}{b}}g_{r,m,n'q+l}\right)(x,\omega) &= \mathrm{e}^{2\pi imx}(Z_{\frac{1}{b}}g_r)\left(x-(n'q+l)\cdot\frac{p}{q},\omega\right)\\ &= \mathrm{e}^{2\pi imx}(Z_{\frac{1}{b}}g_r)\left(x-n'p-\frac{lp}{q},\omega\right)\\ &= \mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega\right),\quad (x,\omega)\in Q. \end{aligned}\tag{4.6}$$

Now using that S_z maps the PZT of $f\in L^2(\mathbb{R})$ to the PZT of Sf we have

$$\begin{aligned}(S_zF)_{0,j}(x,\omega) &= \left(Z_{\frac{1}{b}}Sf\right)\left(x,\omega+\frac{j}{p}\right)\\ &= \left(Z_{\frac{1}{b}}\sum_{r=0}^{R-1}\sum_{m,n'\in\mathbb{Z}}\sum_{l=0}^{q-1}\langle f,g_{r,m,n'q+l}\rangle g_{r,m,n'q+l}\right)\left(x,\omega+\frac{j}{p}\right)\\ &= \sum_{r=0}^{R-1}\sum_{m,n'\in\mathbb{Z}}\sum_{l=0}^{q-1}\langle f,g_{r,m,n'q+l}\rangle\left(Z_{\frac{1}{b}}g_{r,m,n'q+l}\right)\left(x,\omega+\frac{j}{p}\right)\\ &= \sum_{r=0}^{R-1}\sum_{m,n'\in\mathbb{Z}}\sum_{l=0}^{q-1}\langle f,g_{r,m,n'q+l}\rangle\,\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right), \end{aligned}\tag{4.7}$$

where the last equality follows from Equation (4.6). Now look at the inner product in Equation (4.7) and recall that the PZT is a unitary operator of $L^2(\mathbb{R})$ onto $L^2(U;\mathbb{C}^p)$. Thus using Equation (3.12) we find

$$\begin{aligned}\langle f,g_{r,m,n'q+l}\rangle &= \int_{-\infty}^{\infty} f(x)\overline{g_{r,m,n'q+l}(x)}\mathrm{d}x\\ &= \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{\left(Z_{\frac{1}{b}}g_{r,m,n'q+l}\right)\left(x,\omega+\tfrac{k}{p}\right)}\mathrm{d}\omega\mathrm{d}x\\ &= \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)}\mathrm{d}\omega\mathrm{d}x, \end{aligned}\tag{4.8}$$

where the last equality follows by Equation (4.6). Combining Equations (4.7) and (4.8) we obtain

$$
\begin{aligned}
(S_zF)_{0,j}(x,\omega) &= \sum_{r=0}^{R-1}\sum_{m,n'\in\mathbb{Z}}\sum_{l=0}^{q-1} \mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right)\\
&\times \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)}\mathrm{d}\omega\mathrm{d}x\\
&= \sum_{r=0}^{R-1}\sum_{l=0}^{q-1}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right)\sum_{m,n'\in\mathbb{Z}}\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}\\
&\times \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}}\mathrm{d}\omega\mathrm{d}x\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right)\sum_{m,n'\in\mathbb{Z}}\sqrt{p}\,\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}\\
&\times \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)\sqrt{p}\,\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}}\mathrm{d}\omega\mathrm{d}x.
\end{aligned}
$$

According to [11, p. 194], $\{\sqrt{p}\,\mathrm{e}^{2\pi imx}\,\mathrm{e}^{-2\pi in'p\omega}\}_{m,n'\in\mathbb{Z}}$ constitutes an orthonormal basis for $L^2(U)$, hence we have, almost everywhere,

$$
\begin{aligned}
(S_zF)_{0,j}(x,\omega) &= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right)\sum_{k=0}^{p-1}F_{0,k}(x,\omega)\overline{(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)}\\
&= \sum_{k=0}^{p-1}F_{0,k}(x,\omega)\frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1}(Z_{\frac{1}{b}}g_r)\left(x-l\frac{p}{q},\omega+\frac{j}{p}\right)\overline{(Z_{\frac{1}{b}}g_r)\left(x-l\tfrac{p}{q},\omega+\tfrac{k}{p}\right)}\\
&= \sum_{k=0}^{p-1}F_{0,k}(x,\omega)S_{j,k}(x,\omega)\\
&= (S(x,\omega)F(x,\omega))_{0,j}.
\end{aligned}
$$

And the theorem is proved. ■

By Theorem 3.5 we recall that the PZT is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(U;\mathbb{C}^p)$, thus the representation of the frame operator in Equation (4.2) is isometrically isomorphic to the representation of the frame operator in Equation (4.3). Moreover, as a result of Theorem 4.2 we are able to interpret properties of the sequence $\{g_{r,m,n}\}$ by examining properties of the matrix-valued function $S(x,\omega)$. Using Equation (4.2) we have that

$$
\sum_{r=0}^{R-1}\sum_{m,n\in\mathbb{Z}}|\langle f,g_{r,m,n}\rangle|^2=\langle Sf,f\rangle, \tag{4.9}
$$

and by Equation (3.12) we observe that

$$\begin{aligned}\langle Sf, f\rangle &= \langle S_z F, F\rangle_{L^2(U;\mathbb{C}^p)}\\ &= \int_0^1\int_0^{1/p}\sum_{j=0}^{p-1}(S(x,\omega)F(x,\omega))_{0,j}\,\overline{F_{0,j}(x,\omega)}\mathrm{d}\omega\mathrm{d}x\\ &= \int_0^1\int_0^{1/p}\overline{F(x,\omega)^T}S(x,\omega)F(x,\omega)\mathrm{d}\omega\mathrm{d}x. \end{aligned}\tag{4.10}$$

A combination of Equations (4.9) and (4.10) yields

$$\sum_{r=0}^{R-1}\sum_{m,n\in\mathbb{Z}}|\langle f, g_{r,m,n}\rangle|^2 = \langle Sf, f\rangle = \int_0^1\int_0^{1/p}\overline{F(x,\omega)^T}S(x,\omega)F(x,\omega)\mathrm{d}\omega\mathrm{d}x. \tag{4.11}$$

Another important property of the matrix $S(x,\omega)$ is stated in the following lemma.

Lemma 4.3: The matrix-valued function $S(x,\omega)$ is self-adjoint and positive semi-definite for each $(x,\omega)\in U = ([0,1)\times[0,1/p))$.

Proof: We start by showing that $S(x,\omega)$ is self-adjoint:

$$\begin{aligned}S_{j,k}(x,\omega) &= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\cdot\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\cdot\tfrac{p}{q}, \omega+\tfrac{k}{p}\right)}\\ &= \overline{\frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\cdot\tfrac{p}{q}, \omega + \tfrac{k}{p}\right)\overline{Zg_r\left(x - l\cdot\tfrac{p}{q}, \omega+\tfrac{j}{p}\right)}}\\ &= \overline{S_{k,j}}(x,\omega), \quad (x,\omega)\in U.\end{aligned}$$

Next, to show that $S(x,\omega)$ is positive semi-definite we define a $Rq\times p$ matrix $G(x,\omega)$ by

$$G(x,\omega) = \begin{bmatrix} G^0(x,\omega)\\ \vdots \\ G^{R-1}(x,\omega)\end{bmatrix}, \tag{4.12}$$

where each $G^r(x,\omega)$, for $r = \{0,1,\ldots,R-1\}$, is a $q\times p$ matrix-valued function with elements given by

$$G^r_{l,k}(x,\omega) = Zg_r\left(x - l\cdot\frac{p}{q}, \omega+\frac{k}{p}\right), \qquad l = 0,\ldots,q-1, \quad k = 0,\ldots,p-1. \tag{4.13}$$

By calculating $\frac{1}{p}G(x,\omega)^T\overline{G(x,\omega)}$ we find

$$\begin{aligned}\left(\frac{1}{p}G(x,\omega)^T\overline{G(x,\omega)}\right)_{j,k} &= \frac{1}{p}\sum_{l=0}^{Rq-1} G_{j,l}(x,\omega)^T\overline{G_{l,k}(x,\omega)}\\ &= \frac{1}{p}\sum_{l=0}^{Rq-1} G_{l,j}(x,\omega)\overline{G_{l,k}(x,\omega)}\\ &= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} G^r_{l,j}(x,\omega)\overline{G^r_{l,k}(x,\omega)}\\ &= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\cdot\frac{p}{q},\omega+\frac{j}{p}\right)\overline{Zg_r\left(x-l\cdot\tfrac{p}{q},\omega+\tfrac{k}{p}\right)}\\ &= S_{j,k}(x,\omega), \qquad j,k = 0,\ldots,p-1.\end{aligned}\tag{4.14}$$

Thus we have a factorization of the matrix $S(x, \omega)$ gives as

$$S(x, \omega) = \frac{1}{p} G(x, \omega)^T \overline{G(x, \omega)}. \tag{4.15}$$

For any $F \in \mathbb{C}^p$ we have

$$\begin{aligned}
\overline{F^T}(S(x, \omega))F &= \overline{F^T} \left(\frac{1}{p} G(x, \omega)^T \overline{G(x, \omega)} \right) F \\
&= \frac{1}{p} \left(G(x, \omega) \overline{F} \right)^T \left(\overline{G(x, \omega) F} \right) \\
&\geq 0, \qquad (x, \omega) \in U.
\end{aligned}$$

This shows that $S(x, \omega)$ is positive semi-definite for $(x, \omega) \in U$. ■

An examination of the sequence $\{g_{r,m,n}\}$ can be divided into three different cases. For this we define the redundancy factor $d := \frac{R}{ab}$, and consider the case of

- undersampling, where $d < 1$,
- critical sampling, where $d = 1$, and
- oversampling, where $d > 1$.

For the single-window Gabor system we noted in Theorem 2.7 that in the case of undersampling, $\{g_{m,n}\}$ cannot constitute a frame for $L^2(\mathbb{R})$. Also, in the case of critical sampling, the Balian-Low Theorem (see Theorem 2.8) states that one cannot, simultaneously, achieve stability (in the form of bases properties) and good time-frequency localization. However, in the case of oversampling we are able to avoid the problem imposed by the Balian-Low Theorem. In what follows we will consider the cases of undersampling, critical sampling and oversampling for the multi-window Gabor system $\{g_{r,m,n}\}$.

4.1 Undersampling

In the case of undersampling, the multi-window Gabor system $\{g_{r,m,n}\}$ cannot constitute a frame. Assuming that ab is a rational number we will use matrix algebra to prove the incompleteness of $\{g_{r,m,n}\}$. For this we need the following lemma.

Lemma 4.4: Assume we are given $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R - 1$, and a matrix-valued function $S(x, \omega), (x, \omega) \in U$, defined as in Equation (4.4). If $\det(S)(x, \omega) = 0$ for a measurable set in U with measure greater than zero, there exists a vector-valued function $F \in L^2(U; \mathbb{C}^p), F \neq 0$, which satisfies

$$S(x, \omega) F(x, \omega) = 0, \qquad \text{a.e. on} \quad U.$$

In order to prove Lemma 4.4 we need to first introduce the following lemma. We note that the proof of this lemma is an addition, prepared by the author, and which can not be found in the primary source used in this chapter.

Lemma 4.5: Let $A \in \mathbb{C}^{p \times p}$ be a fixed self-adjoint positive semidefinite singular matrix and let $B = \lim_{\tau \to \infty} \exp(-\tau A)$. Then $AB = 0$.

Proof: Since any self-adjoint matrix is diagonalizable, we may write the matrix A as

$$A = PDP^{-1}, \tag{4.16}$$

where D is a $p \times p$ diagonal matrix with the eigenvalues of A on its diagonal and the columns of P are the eigenvectors corresponding to distinct eigenvalues of A. Note that the columns of P form an orthonormal set since A is self-adjoint. Moreover, A is positive semidefinite and singular, thus the eigenvalues μ_i for $i = 1, \dots, p$ are non-negative and at least one of the eigenvalues is zero. Now we may assume D is given as

$$D = \begin{bmatrix} \mu_1 & & & & & \\ & \ddots & & & & \\ & & \mu_k & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

We note that the diagonal decomposition of A in Equation (4.16) allows us to easily compute the exponential of the matrix. Thus

$$\exp(A) = P \exp(D) P^{-1},$$

and for $\tau > 0$ we have

$$\exp(-\tau A) = P \exp(-\tau D) P^{-1}.$$

Now, we have defined the matrix $B = \lim_{\tau \to \infty} \exp(-\tau A)$, so letting $\tau \to \infty$ we find that

$$B = P\tilde{D}P^{-1}, \quad \text{where } \tilde{D} = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \tag{4.17}$$

and we obtain

$$AB = AP\tilde{D}P^{-1} = PD\tilde{D}P^{-1} = 0.$$

This proves the lemma. ∎

We note by Equation (4.17) that $B \neq 0_{p \times p}$. Thus for a vector $v = Bw$, where w is chosen arbitrarily, we have $Av = 0$. The important fact is that, because $B \neq 0_{p \times p}$ we may choose p linearly independent vectors w which will lead to at least one vector $v \neq 0$. Now we are ready to prove Lemma 4.4.

Proof: (Proof of Lemma 4.4) Assume $\det(S)(x, \omega) = 0$ for a measurable set J with measure greater than zero. For a fixed vector $v \in \mathbb{C}^p$ we define a vector-valued function

$$v(x, \omega; \tau) = \exp(-\tau S(x, \omega))v \quad \text{and} \quad F(x, \omega) = \lim_{\tau \to \infty} v(x, \omega; \tau), \qquad \text{for } \tau > 0.$$

We note that $S(x,\omega)$ is measurable for a.e. (x,ω) since each entry of $S(x,\omega)$ consists of a sum and product of functions in $L^2(\mathbb{R})$. This implies that $v(x,\omega;\tau)$ is measurable for a.e. (x,ω). We need to show that $v(x,\omega;\tau)$ is pointwise convergent, which we will do by showing that $v(x,\omega;\tau)$ is a Cauchy sequence for $\tau \in \mathbb{N}$. By the properties of $S(x,\omega)$ we can make a diagonalizable decomposition and write $v(x,\omega;\tau)$ as

$$v(x,\omega;\tau) = \mathrm{e}^{-\tau S(x,\omega)}\, v = P\,\mathrm{e}^{-\tau D}\, P^{-1} v = P\left(\sum_{k=0}^{\infty} \frac{(-\tau D)^k}{k!}\right) P^{-1} v,$$

where D is a diagonal matrix-valued function with the eigenvalues of $S(x,\omega)$ on its diagonal, and P is unitary (we note that such a decomposition is possible pointwise a.e. since $S(x,\omega)$ is self-adjoint). Now, for a given $\varepsilon > 0$ and for τ large we use the Frobenius norm and find that

$$\begin{aligned}\left\|\sum_{k=0}^{\tau+q} \frac{(-\tau D)^k}{k!} - \sum_{k=0}^{\tau} \frac{(-\tau D)^k}{k!}\right\|_F &= \left\|\sum_{k=\tau+1}^{\tau+q} \frac{(-\tau D)^k}{k!}\right\|_F \le \sum_{k=\tau+1}^{\tau+q} \left\|\frac{(-\tau D)^k}{k!}\right\|_F \\ &\le \sum_{k=\tau+1}^{\infty} \left\|\frac{(-\tau D)}{k!}\right\|_F^k \\ &< \varepsilon,\end{aligned}$$

where the last inequality follows since $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is a convergent series. This implies that

$$\operatorname*{ess\,sup}_{(x,\omega)\in U} |v(x,\omega;\tau+q) - v(x,\omega;\tau)| = \|v(x,\omega;\tau+q) - v(x,\omega;\tau)\|_\infty < \varepsilon, \tag{4.18}$$

which shows that $v(x,\omega;\tau)$ is uniformly Cauchy, implying that $v(x,\omega;\tau)$ is pointwise convergent. Then by [2, Korollar 2.8] the limit functions $F_j(x,\omega)$ are measurable as pointwise limits of measurable functions. Moreover, $\sum_{j=0}^{p-1} |F_j(x,\omega)|^2 \le \|v\|^2$, hence it follows that $F \in L^2(U;\mathbb{C}^p)$. While $F(x,\omega) = 0$ outside the set J, choosing p linearly independent fixed vectors v will lead to at least one $F(x,\omega)$ which is not zero a.e. on J. A combination of this and Lemma 4.5 completes the proof. ■

The proof of Lemma 4.4 is inspired by [11, p. 196], however, additional arguments have been provided by the author.

We recall that $ab = p/q$. Thus in the undersampling case we have

$$d = \frac{R}{ab} < 1 \Leftrightarrow \frac{R}{p/q} < 1 \Leftrightarrow Rq < p.$$

By Equation (4.15) we can write the $p \times p$ matrix $S(x,\omega)$ as

$$S(x,\omega) = \frac{1}{p} G(x,\omega)^T \overline{G(x,\omega)},$$

where $G(x,\omega)$ is a $Rq \times p$ matrix. Since $Rq < p$, the rank of $G(x,\omega)$ is less than p, which implies that the rank of $S(x,\omega)$ is less than p. That is, the reduced row echelon form of $S(x,\omega)$ contains some zero rows, which means that $S(x,\omega)$ is not invertible and therefore $\det(S(x,\omega)) = 0$ a.e. on U. Then, according to Lemma 4.4, there exists an $F \in L^2(U;\mathbb{C}^p)$, $F \neq 0$ which satisfies

$$S(x,\omega)F(x,\omega) = 0,$$

a.e. on U. By Equation (4.11) this implies that

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} |\langle f, g_{r,m,n}\rangle|^2 = 0,$$

which in turn means that the lower frame condition of the sequence $\{g_{r,m,n}\}$ is not satisfied, hence $\{g_{r,m,n}\}$ cannot constitute a frame for $L^2(\mathbb{R})$.

4.2 Critical sampling and oversampling

We will now examine properties of the sequence $\{g_{r,m,n}\}$ in the case of critical sampling and oversampling. The following theorem gives a relation between the determinant of the matrix $S(x,\omega)$ and the completeness of $\{g_{r,m,n}\}$. The proof of Theorem 4.6 is highly inspired by [11, p. 197].

Theorem 4.6: Assume we are given $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R-1$ and a matrix-valued function $S(x,\omega), (x,\omega) \in U$, defined as in Equation (4.4). The sequence $\{g_{r,m,n}\}$, with $ab = p/q, p, q \in \mathbb{N}$, is complete, if and only if $\det(S)(x,\omega) \neq 0$ a.e. on U.

Proof: We first show that if the sequence $\{g_{r,m,n}\}$ is complete then $\det(S)(x,\omega) \neq 0$ a.e. Assume to the contrary that $\det(S)(x,\omega) = 0$ on a measurable set with measure greater than zero. Then, by Lemma 4.4, there exists a vector-valued function $F \in L^2(U;\mathbb{C}^p), F \neq 0$ such that $S(x,\omega)F(x,\omega) = 0$ a.e. on U. Since this implies that $\{g_{r,m,n}\}$ is incomplete, we have a contradiction.
Next we will show that if $\det(S)(x,\omega) \neq 0$ a.e. then the sequence $\{g_{r,m,n}\}$ is complete. Assume to the contrary that $\{g_{r,m,n}\}$ is incomplete. Then there exists an $F \in L^2(U;\mathbb{C}^p)$ with $F \neq 0$ which satisfies $S(x,\omega)F(x,\omega) = 0$ a.e. Since $F \neq 0$, this implies that $\det(S)(x,\omega) = 0$, which is a contradiction. ■

Recalling that a square matrix is invertible if and only if its determinant is different from zero, Theorem 4.6 states that the sequence $\{g_{r,m,n}\}$ constitute a frame for $L^2(\mathbb{R})$ if and only if $S(x,\omega)$ is invertible a.e. on U.

Another interesting application in using the matrix-valued function $S(x,\omega)$ to examine properties of the sequence $\{g_{r,m,n}\}$ relates to the frame bounds of $\{g_{r,m,n}\}$. These can be derived by calculating the eigenvalues of $S(x,\omega)$. Assuming that $\{g_{r,m,n}\}$ constitute a frame, we recall that the associated frame operator S is self-adjoint, bounded, linear and positive definite.

We define

$$\begin{aligned}\mu_{\max}(S) &:= \operatorname*{ess\,sup}_{(x,\omega)\in U} \left(\max_{1\leq j\leq p} \mu_i(S)(x,\omega) \right) \\ &= \inf \left\{ a \in [0,\infty] \ : \ \max_{1\leq j\leq p} \mu_i(S)(x,\omega) \leq a \text{ for a.e. } (x,\omega) \in U \right\}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\mu_{\min}(S) &:= \operatorname*{ess\,inf}_{(x,\omega)\in U} \left(\min_{1\leq j\leq p} \mu_i(S)(x,\omega) \right) \\ &= \sup \left\{ b \in [0,\infty] \ : \ \min_{1\leq j\leq p} \mu_i(S)(x,\omega) \leq b \text{ for a.e. } (x,\omega) \in U \right\},\end{aligned}$$

where $\mu_i(S)(x,\omega)$ are the eigenvalues of $S(x,\omega)$. By the properties of the frame operator S we have:

$$\mu_{\min}(S)\overline{F(x,\omega)^T}F(x,\omega) \leq \overline{F(x,\omega)^T}S(x,\omega)F(x,\omega) \leq \mu_{\max}(S)\overline{F(x,\omega)^T}F(x,\omega)$$

a.e., and by Equation (4.11) we obtain:

$$\mu_{\min}(S)\left\|f\right\|^2 \leq \langle Sf,f\rangle \leq \mu_{\max}(S)\left\|f\right\|^2.$$

So $\mu_{\min}(S)$ constitutes a lower frame bound and $\mu_{\max}(S)$ an upper frame bound. Moreover, under the assumption that $\{g_{r,m,n}\}$ constitutes frame, the associated optimal frame bounds satisfy

$$A = \inf_{f\neq 0}\frac{\langle Sf,f\rangle}{\langle f,f\rangle} \quad \text{and} \quad B = \sup_{f\neq 0}\frac{\langle Sf,f\rangle}{\langle f,f\rangle},$$

therefore

$$A\left\|f\right\|^2 \leq \langle Sf,f\rangle \leq B\left\|f\right\|^2.$$

Now we claim that the minimal eigenvalue for S is the optimal lower frame bound, and similarly that the maximal eigenvalue of S is the optimal upper frame bound, that is

$$A = \mu_{\min}(S) \quad \text{and} \quad B = \mu_{\max}(S). \tag{4.20}$$

To see this we must be able to choose a measurable function $f(x,\omega)$ such that $f(x,\omega)_{\max}$ is the corresponding normalized eigenvector of the maximal eigenvalue and zero for other values of (x,ω). Hence we wish to show the existence of such a function. We note that the following contribution is developed by the author and cannot be found in the primary source used in this chapter.

Define the $p\times p$ matrix $C(x,\omega)$ as

$$C(x,\omega) = S(x,\omega) - \mu_{\max}(x,\omega)I_{p\times p}, \tag{4.21}$$

where $S(x,\omega),(x,\omega)\in U$ and $\mu_{\max}(x,\omega)$ are defined as in Equations (4.4) and (4.19) respectively. Then $C(x,\omega)$ is clearly singular. The main problem is to show that $C(x,\omega)$ is measurable. Now consider

$$S(x,\omega) = \begin{bmatrix} S_{0,0}(x,\omega) & \dots & S_{0,p-1}(x,\omega) \\ \vdots & \ddots & \vdots \\ S_{p-1,0}(x,\omega) & \dots & S_{p-1,p-1}(x,\omega) \end{bmatrix}.$$

We know that the entries of $S(x,\omega)$ are measurable functions. Assuming that the entries of $S(x,\omega)$ are complex-valued, we can write the functions in the form

$$S_{j,k}(x,\omega) = S'_{j,k}(x,\omega) + iS''_{j,k}(x,\omega), \quad j,k = 0,\dots p-1, \tag{4.22}$$

where $S'_{j,k}(x,\omega)$ and $S''_{j,k}(x,\omega)$ are real-valued. Then each of the real-valued functions can be split into their positive and negative part, i.e.

$$S'_{j,k}(x,\omega) = S^{+'}_{j,k}(x,\omega) - S^{-'}_{j,k}(x,\omega)$$
$$S''_{j,k}(x,\omega) = S^{+''}_{j,k}(x,\omega) - S^{-''}_{j,k}(x,\omega),$$

for $j,k = 0,\dots p-1$ and $S'_{j,k}(x,\omega), S''_{j,k}(x,\omega) \geq 0$. Thus the entries of $S(x,\omega)$ consists of a finite linear combination of measurable positive functions, hence it follows from [2, Theorem

4.1] that for each entry of $S(x,\omega)$ there exists an increasing sequence $s_1 \leq s_2 \leq \cdots$ of simple measurable functions s_n, with $S_{j,k}(x,\omega) = \lim_{n\to\infty} s_n$ for $j,k = 0,\dots p-1$. That is, each entry of the matrix $S(x,\omega)$ can be approximated by simple measurable functions, hence we arrive at an approximation of the matrix $S(x,\omega)$. Call this matrix $\tilde{S}(x,\omega)$. Then $\max_{1\leq j\leq p} \mu_i((\tilde{S}))(x,\omega)$ is a new simple measurable function. By taking the limit of these functions it follows that $\mu_{\max}(S)(x,\omega)$ is measurable as a pointwise limit of simple measurable functions. All together this implies that the matrix $C(x,\omega)$, defined in Equation (4.21), is measurable. Now, imposing Lemma 4.4 we know there exists a matrix $B \neq 0_{p\times p}$ such that $CB = 0$. If we choose p linearly independent vectors w and set $v = Bw$, then at least one vector $v \neq 0$, and $Cv = 0$. But this vector v is exactly the eigenvector $f(x,\omega)_{\max}$ corresponding to the eigenvalue $\mu_{\max}(x,\omega)$ for $S(x,\omega)$. Hence it follows that $f(x,\omega)_{\max}$ is a measurable function, and we finally obtain

$$B = \sup_{\|f\|=1} \langle Sf, f\rangle = \sup_{\|f\|=1} \langle \mu_{\max} f, f\rangle = \mu_{\max},$$

which proves our claim that the maximum eigenvalue of S is exactly the optimal upper frame bound B. The claim concerning the lower frame bound can be proven in a similar manner.

We summarize the stated facts about the frame bounds of the sequence $\{g_{r,m,n}\}$ and the eigenvalues of $S(x,\omega)$ in the following theorem.

Theorem 4.7: Assume we are given $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R-1$ and a matrix-valued function $S(x,\omega), (x,\omega) \in U$ defined as in Equation (4.4). The sequence $\{g_{r,m,n}\}$, where $ab = p/q, p, q \in \mathbb{N}$ constitute a frame for $L^2(\mathbb{R})$, if and only if

$$0 < \mu_{\min}(S) \leq \mu_{\max}(S) < \infty.$$

In practical terms the task of finding the eigenvalues of the matrix $S(x,\omega)$ can be relative time consuming, as the dimension of the $p \times p$ matrix $S(x,\omega)$ increases. Therefore we state, in Theorem 4.9, an alternative way of determining whether $\{g_{r,m,n}\}$ constitute a frame for $L^2(\mathbb{R})$, where we avoid calculations of the eigenvalues. Before presenting the theorem, we introduce the following lemma.

Lemma 4.8: Assume we are given $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R-1$. The sequence $\{g_{r,m,n}\}$, with $ab = p/q, p, q \in \mathbb{N}$ has an upper frame bound $B < \infty$, if and only if $(Zg_r)(x,\omega)$ are all bounded a.e. on $(0,1]^2$ ($Zg_r \in L^\infty((0,1]^2)$).

Proof: Recalling the periodic and quasi-periodic properties of the Zak transform (see Lemma 3.3) we note that $Zg_r \in L^\infty((0,1]^2)$ is equivalent to $Zg_r \in L^\infty(\mathbb{R})$.
First assume $Zg_r \in L^\infty((0,1]^2)$. Then $(Zg_r)(x,\omega)$ are bounded a.e, which implies that the entries $S_{j,k}(x,\omega)$ of the matrix $S(x,\omega)$, defined in Equation (4.4), are bounded a.e. for all $0 \leq j,k \leq p-1$. Hence there exists an $M < \infty$ such that for any $F \in \mathbb{C}^p$ we have

$$\overline{F^T} S(x,\omega) F \leq M \overline{F^T} F$$

a.e., and by Equation (4.11) we obtain

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} |\langle f, g_{r,m,n}\rangle|^2 = \langle Sf, f\rangle \leq M \|f\|^2 .$$

That is, the optimal upper frame bound $B \leq M < \infty$.
Now suppose that $B < \infty$. We consider the trace of the matrix-valued function $S(x,\omega)$.

Thus by Proposition A.3 we find

$$\sum_{j=0}^{R-1} \mu_j(x,\omega) = \frac{1}{p} \sum_{r=0}^{R-1} \sum_{l=0}^{q-1} \left| Zg_r \left(x - l \cdot \frac{p}{q}, \omega + \frac{j}{p} \right) \right|^2, \quad \text{for } j = 0, \dots p-1, \tag{4.23}$$

with μ_j the eigenvalues of S. By Equation (4.20) $\mu_{\max} = B < \infty$, that is, the left- and right-hand side of Equation (4.23) are bounded a.e. This implies that $\left| Zg_r \left(x - l \cdot \frac{p}{q}, \omega + \frac{j}{p} \right) \right|$ are bounded a.e. where $0 \leq r \leq R-1, 0 \leq l \leq q-1$ and $(x,\omega) \in U$. That is, $(Zg_r)(x,\omega)$ are all bounded a.e. on $(0,1]^2$. ■

Lemma 4.8 gives an equivalent condition for the existence of an upper frame bound $B < \infty$, in terms of the Zak transform of g_r. Assuming now that $\{g_{r,m,n}\}$ has an upper frame bound $B < \infty$ we state the following result.

Theorem 4.9: Let $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R-1$ be given, such that the sequence $\{g_{r,m,n}\}$, with $ab = p/q, p, q \in \mathbb{N}$ has an upper frame bound $B < \infty$. Then $\{g_{r,m,n}\}$ constitute a frame for $L^2(\mathbb{R})$, if and only if $0 < K \leq \det(S)(x,\omega)$ a.e. on U, where the matrix-valued function $S(x,\omega), (x,\omega) \in U$ is given by Equation (4.4).

Proof: First assume that $\{g_{r,m,n}\}$ constitute a frame for $L^2(\mathbb{R})$, then $\mu_{\min} > 0$ and $\det(S)(x,\omega) = \prod_{j=1}^{p} \mu_j(x,\omega)$ imply that $\det(S)(x,\omega) \geq K > 0$ a.e.
Now assume that $\det(S)(x,\omega) \geq K > 0$. Since the sequence $\{g_{r,m,n}\}$ has an upper frame bound $B < \infty$ it follows from Lemma 4.8 that the entries of $S(x,\omega)$ are bounded a.e. This further implies that $\mu_j(x,\omega) \leq B$ for all $1 \leq j \leq p$. Now using our assumption we have

$$0 < K \leq \det(S)(x,\omega) = \prod_{j=1}^{p} \mu_j(x,\omega).$$

That is, each eigenvalue of the matrix $S(x,\omega)$ is bounded from below a.e., hence there exists a lower frame bound $A = \mu_{\min} > 0$, and thus $\{g_{r,m,n}\}$ constitutes a frame. ■

We note that a combination of Lemma 4.8 and Theorem 4.9 yields a necessary and sufficient condition for the sequence $\{g_{r,m,n}\}$ to constitute a frame.

In the following example we consider the case of a single-window Gabor frame and show how to calculate the frame bounds in terms of calculating the eigenvalues of the matrix-valued function $S(x,\omega)$.

Example 4.10: We first consider the case where $p = 1$.

i) Let $p = 1$. Then $S(x,\omega)$ reduces to a scalar-valued function given as

$$S(x,\omega) = \sum_{l=0}^{q-1} \left| Zg \left(x - \frac{l}{q}, \omega \right) \right|^2, \tag{4.24}$$

and thus there is a single eigenvalue, which is the scalar itself. Hence by Equation (4.20) the frame bounds are

$$A = \operatorname*{ess\,inf}_{(x,\omega)\in U} S(x,\omega), \qquad B = \operatorname*{ess\,sup}_{(x,\omega)\in U} S(x,\omega).$$

ii) Now let $p = 2$. Then

$$S(x,\omega) = \begin{bmatrix} S_{0,0} & S_{0,1} \\ S_{1,0} & S_{1,1} \end{bmatrix}, \tag{4.25}$$

where $S_{j,k}$ are all functions of $(x,\omega) \in ([0,1) \times [0,1/2))$ and are given in Equation (4.4). We recall that $\mu \in \mathbb{C}$ is an eigenvalue of $S(x,\omega)$, if and only if

$$\det(S(x,\omega) - \mu I_{2\times 2}) = 0. \tag{4.26}$$

Thus using Equation (4.26) we obtain:

$$\begin{aligned} \det\left(\begin{bmatrix} S_{0,0}-\mu & S_{0,1} \\ S_{1,0} & S_{1,1}-\mu \end{bmatrix}\right) &= 0 && \Leftrightarrow \\ (S_{0,0}-\mu)(S_{1,1}-\mu) - (S_{1,0}S_{0,1}) &= 0 && \Leftrightarrow \\ \mu^2 - \mu(S_{0,0}+S_{1,1}) + (S_{0,0}S_{1,1} - S_{1,0}S_{0,1}) &= 0 \end{aligned}$$

This turns into solving a quadratic equation, hence

$$\begin{aligned} \mu_{1,2} &= \frac{1}{2}\left(S_{0,0}+S_{1,1} \pm \sqrt{(-S_{0,0}-S_{1,1})^2 - 4(S_{0,0}S_{1,1} - |S_{0,1}|^2)}\right) \\ &= \frac{1}{2}\left(S_{0,0}+S_{1,1} \pm \sqrt{(S_{0,0}-S_{1,1})^2 - 4|S_{0,1}|^2}\right). \end{aligned}$$

Now, by Equation (4.20) the frame bounds are

$$A = \operatorname*{ess\,inf}_{(x,\omega)\in U} \frac{1}{2}\left(S_{0,0}+S_{1,1} - \sqrt{(S_{0,0}-S_{1,1})^2 - 4|S_{0,1}|^2}\right)$$

$$B = \operatorname*{ess\,sup}_{(x,\omega)\in U} \frac{1}{2}\left(S_{0,0}+S_{1,1} + \sqrt{(S_{0,0}-S_{1,1})^2 - 4|S_{0,1}|^2}\right).$$

In the following section we will characterize a dual frame for the multi-window Gabor frame in the case where ab is a rational number.

4.3 The dual frame

In order to use the frame decomposition, such as the one represented in Theorem 2.5, Equation (2.3), one needs to be able to calculate a dual frame for the multi-window Gabor frame. In the case of a single-window Gabor frame, we noted, in Definition 2.4, that the canonical dual frame could be obtained by mapping each element of the frame with the associated inverse frame operator. This is also seen to be the case for the multi-window Gabor system. We state the following theorem.

Theorem 4.11: Assume $\{g_{r,m,n}\}$ is a frame for $L^2(\mathbb{R})$ with $ab = p/q, p, q \in \mathbb{N}$ and let $\{\gamma_{r,m,n}\}$ denote a dual frame of $\{g_{r,m,n}\}$. Then $\{\gamma_{r,m,n}\}$ is given by

$$\gamma_{r,m,n} = M_{mb}T_{na}\gamma_r = \gamma_r(x-na)\,\mathrm{e}^{2\pi imbx}, \quad 0 \le r \le R-1,\ m,n \in \mathbb{Z}, x \in \mathbb{R}, \tag{4.27}$$

where $\gamma_r = S^{-1}g_r$, and S is the frame operator for $\{g_{r,m,n}\}$.

We see that a dual frame for the multi-window Gabor frame is generated by R window functions γ_r, and the structure of γ_r is identical to the structure of the frame $\{g_{r,m,n}\}$. To prove Theorem 4.11 we need the following lemma.

Lemma 4.12: Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Assume that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ and let S be the associated frame operator. Then the following holds:

(i)
$$SM_{mb}T_{na} = M_{mb}T_{na}S, \quad \forall\, m,n \in \mathbb{Z}. \tag{4.28}$$

(ii)
$$S^{-1}M_{mb}T_{na} = M_{mb}T_{na}S^{-1}, \quad \forall\, m,n \in \mathbb{Z}. \tag{4.29}$$

Proof: Let $f \in L^2(\mathbb{R})$. We prove the commutation relation in Equation (4.28) in the PZT domain. Let $ab = p/q, p, q \in \mathbb{N}$, then by Equation (4.5) we have that

$$F_{m,n}(x,\omega) = \mathrm{e}^{2\pi i m x}\, F\left(x - n\frac{p}{q}, \omega\right), \quad (x,\omega) \in U, \tag{4.30}$$

where $F_{m,n}$ is the PZT of $f_{m,n}$ and F is the PZT of f. Furthermore, by the definition of the frame operator S in the PZT domain (see Theorem 4.2) we have, for $j,k = 0,\dots p-1$,

$$\begin{aligned}
S_{j,k}\left(x - \frac{p}{q}, \omega\right) &= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\frac{p}{q} - \frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q} - \frac{p}{q}, \omega + \frac{k}{p}\right)}\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - (l+1)\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - (l+1)\frac{p}{q}, \omega + \frac{k}{p}\right)}\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=1}^{q} Zg_r\left(x - l\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q}, \omega + \frac{k}{p}\right)}\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\left(\sum_{l=1}^{q-1} Zg_r\left(x - l\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q}, \omega + \frac{k}{p}\right)}\right.\\
&\quad \left. + Zg_r\left(x - p, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - p, \omega + \frac{k}{p}\right)}\right).
\end{aligned}$$

Now, using the quasi-periodic relation of the Zak transform (see Lemma 3.3, Equation (3.6)) we find

$$\begin{aligned}
S_{j,k}\left(x - \frac{p}{q}, \omega\right) &= \frac{1}{p}\sum_{r=0}^{R-1}\left(\sum_{l=1}^{q-1} Zg_r\left(x - l\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q}, \omega + \frac{k}{p}\right)}\right.\\
&\quad \left. + \mathrm{e}^{-2\pi i\omega p}\, Zg_r\left(x, \omega + \frac{j}{p}\right)\overline{\mathrm{e}^{-2\pi i\omega p}\, Zg_r\left(x, \omega + \frac{k}{p}\right)}\right)\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\left(\sum_{l=1}^{q-1} Zg_r\left(x - l\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q}, \omega + \frac{k}{p}\right)}\right.\\
&\quad \left. + Zg_r\left(x, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x, \omega + \frac{k}{p}\right)}\right)\\
&= \frac{1}{p}\sum_{r=0}^{R-1}\sum_{l=0}^{q-1} Zg_r\left(x - l\frac{p}{q}, \omega + \frac{j}{p}\right)\overline{Zg_r\left(x - l\frac{p}{q}, \omega + \frac{k}{p}\right)}\\
&= S_{j,k}(x,\omega).
\end{aligned}$$

This shows that $S_{j,k}(x - p/q, \omega) = S_{j,k}(x, \omega)$. Using this together with Equation (4.30) we obtain

$$\begin{aligned} S(x,\omega)F_{m,n}(x,\omega) &= S(x,\omega)\,\mathrm{e}^{2\pi imx}\,F\left(x - n\frac{p}{q},\omega\right) \\ &= \mathrm{e}^{2\pi imx}\,S\left(x - n\frac{p}{q},\omega\right)F\left(x - n\frac{p}{q},\omega\right) \\ &= \mathrm{e}^{2\pi imx}\,S\left(x - nab,\omega\right)F\left(x - nab,\omega\right) \\ &= \mathrm{e}^{2\pi imx}\,S\left(x - \frac{na}{\lambda},\omega\right)F\left(x - \frac{na}{\lambda},\omega\right), \end{aligned} \tag{4.31}$$

where the last equality follows since the Zak parameter $\lambda = \frac{1}{b}$. We notice that the right-hand side of Equation (4.31) corresponds exactly to a translation and modulation in the Zak domain, and since the PZT is a unitary operator we have proven that the frame operator S commutes with the inherent modulation and translation operators.
To prove (ii) we first note that since both the operators M_{mb} and T_{na} are invertible, the product of the two operators is invertible. Thus using the result of part (i) we obtain:

$$\begin{aligned} SM_{mb}T_{na} &= M_{mb}T_{na}S & \Leftrightarrow \\ (M_{mb}T_{na})^{-1}S^{-1} &= S^{-1}(M_{mb}T_{na})^{-1} & \Leftrightarrow \\ S^{-1} &= M_{mb}T_{na}S^{-1}(M_{mb}T_{na})^{-1} & \Leftrightarrow \\ S^{-1}M_{mb}T_{na} &= M_{mb}T_{na}S^{-1}. \end{aligned}$$ ■

Now we are ready to prove Theorem 4.11.

Proof: (Proof of Theorem 4.11) As $\{g_{r,m,n}\}$ is assumed to be a frame, the frame operator S is invertible a.e. Thus using Equation (4.29) we can write a dual frame of $\{g_{r,m,n}\}$ as

$$\begin{aligned} S^{-1}g_{r,m,n} &= S^{-1}M_{mb}T_{na}g_r \\ &= M_{mb}T_{na}S^{-1}g_r \\ &= M_m T_{na}\gamma_r, \end{aligned}$$

where $\gamma_r = S^{-1}g_r$ for $0 \le r \le R-1$ and $m, n \in \mathbb{Z}$. This proves the theorem. ■

Using the matrix representation of the frame operator, and since $\gamma_r = S^{-1}g_r$, the PZT of γ_r becomes

$$\Gamma_r(x,\omega) = S^{-1}(x,\omega)G_r(x,\omega), \tag{4.32}$$

where $\Gamma_r(x,\omega)$ and $G_r(x,\omega)$ are vector-valued functions in $L^2(U;\mathbb{C}^p)$ and $S^{-1}(x,\omega)$ is the inverse of the matrix-valued function $S(x,\omega)$.

In the following example we consider the case of a single-window Gabor frame and show how to find an expression of a dual frame by using Equation (4.32) for the case of $p = 1$ and $p = 2$. We note that with an expression of a dual frame and by exploiting the unitary nature of the Zak transform, the expansion coefficients $c_{m,n}$ can be computed as the inner product between Zf and $Z\gamma$ in the Zak transform domain.

Example 4.13: We first consider the case where $p = 1$.

i) For $p = 1$ the scalar-valued function $S(x, \omega)$ is given by Equation (4.24). Thus by Equation (4.32) we have

$$(Z\gamma)(x, \omega) = \frac{Zg(x, \omega)}{\sum_{l=0}^{q-1} |Zg(x - l/q, \omega)|^2}. \tag{4.33}$$

By Equation (4.33) we observe that if the denominator is close to be a constant, then $Z\gamma$ and Zg will differ by a constant factor. Hence we can make $Z\gamma$ and Zg look alike by constructing the window function g in such a way the denominator is approximately constant.

ii) Let $p = 2$ and

$$S(x, \omega) = \begin{bmatrix} S_{0,0} & S_{0,1} \\ S_{1,0} & S_{1,1} \end{bmatrix},$$

where $S_{j,k}$ are all functions of $(x, \omega) \in ([0, 1) \times [0, 1/2))$ and are given in Equation (4.4). The inverse of the matrix-valued function $S(x, \omega)$ is given as

$$S^{-1}(x, \omega) = \frac{1}{\det(S)} \begin{bmatrix} S_{1,1} & -S_{0,1} \\ -S_{1,0} & S_{0,0} \end{bmatrix}.$$

Thus

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \frac{1}{\det(S)} \begin{bmatrix} S_{1,1} & -S_{0,1} \\ -S_{1,0} & S_{0,0} \end{bmatrix} \begin{pmatrix} G_0 \\ G_1 \end{pmatrix}$$

and

$$\Gamma_0 = \frac{S_{1,1}G_0 - S_{0,1}G_1}{S_{1,1}S_{0,0} - S_{1,0}S_{0,1}}, \quad \Gamma_1 = \frac{S_{0,0}G_1 - S_{1,0}G_0}{S_{1,1}S_{0,0} - S_{1,0}S_{0,1}}.$$

Based on Equation (3.10) we have

$$(Z\gamma)(x, \omega) = \begin{cases} \Gamma_0(x, \omega) & \text{if } \omega \in \left[0, \frac{1}{2}\right) \\ & \qquad\qquad x \in [0, 1) \\ \Gamma_1(x, \omega) & \text{if } \omega \in \left[\frac{1}{2}, 1\right). \end{cases}$$

And thus we have obtained an expression for a dual frame.

4.4 Extension of the Balian-Low Theorem

In this section we exploit some further properties of the multi-window Gabor system. We recall Theorem 2.8, by Balian and Low, which states that, for the single-window Gabor system, in the case where $\{g_{m,n}\}$ constitutes a Riesz basis, one cannot choose a "well-behaved", rapidly decaying and smooth function as generator for the frame. Now the question is whether this theorem still holds true in the case of multi-window Gabor systems. The first result we present is an extension of the Balian-Low Theorem, which states that, in the case of critical sampling, by choosing R well-behaved window functions, the Balian-Low condition still holds true.

Theorem 4.14: Let $g_r \in L^2(\mathbb{R})$ for $0 \leq r \leq R - 1$ be given and assume we are in the case of critical sampling, i.e. $\frac{R}{ab} = 1$ for $a, b > 0$. If the sequence $\{g_{r,m,n}\}$, given in Equation (4.1), constitutes a frame, then

$$\left(\int_{-\infty}^{\infty} |xg_r(x)|^2 \mathrm{d}x\right) \left(\int_{-\infty}^{\infty} |\xi\hat{g}_r(\xi)|^2 \mathrm{d}\xi\right) = \infty, \quad \text{for some } 0 \leq r \leq R - 1.$$

Despite the fact that the Balian-Low condition still holds for the multi-window Gabor system, there are still some advantages of using several window functions as opposed to using a single one. If one can choose a non-well-behaved window function and add this function to a set of well-behaved functions, such that the inclusive set of these functions generates a frame for critical sampling, then one can, in a way, overcome the constraints imposed by the Balian-Low Theorem on the choice of window functions. The following proposition, inspired by [11, Proposition 1], gives a necessary condition to create such a set.

Proposition 4.15: Assume we are given a set $\{g_r\}, 0 \leq r \leq R-2$ of $R-1$ window functions. Let $G_0(x,\omega)$ be a $R-1 \times R$ matrix-valued function with entries given by

$$G^r_{0,k}(x,\omega) = Zg_r(x,\omega + k/R) \tag{4.34}$$

and let

$$P(x,\omega) = \overline{G_0(x,\omega)} G_0(x,\omega)^T. \tag{4.35}$$

Then there exists a window function $g_{R-1}(x)$ such that the inclusive set $\{g_r\}, 0 \leq r \leq R-1$, generates a frame for the critical sampling case, if and only if $0 < K \leq \det(P)(x,\omega)$ a.e. on $[0,1) \times [0,1/R)$.

Proof: We first recall that in the critical sampling case,

$$\frac{R}{ab} = 1 \Leftrightarrow R = ab \Leftrightarrow R = \frac{p}{q},$$

so we may assume $p = R$ and $q = 1$. By adding a row vector, whose entries depends on Zg_{R-1}, to the matrix-valued function $G_0(x,\omega)$ we obtain an $R \times R$ matrix-valued function $G(x,\omega)$, which corresponds to the inclusive set $\{g_r\}$ for $0 \leq r \leq R-1$. We will show, by contradiction, that if the inclusive set generates a frame for the critical sampling case, then $0 < K \leq \det(P)(x,\omega)$ a.e. on $[0,1) \times [0,1/R)$.
Assume $\operatorname{ess\,inf} \det(P)(x,\omega) = 0$, then, by the factorization of the matrix-valued function $S(x,\omega)$ (See Equation (4.15)), we have $\operatorname{ess\,inf} \det(S)(x,\omega) = 0$ for any choice of Zg_{R-1}. Thus Theorem 4.6 imply that the sequence $\{g_{r,m,n}\}$ is incomplete, hence it cannot constitute a frame.
Next, assume that $\operatorname{ess\,inf} \det(P)(x,\omega) > 0$. Define $S_0(x,\omega) = G_0(x,\omega)^T \overline{G_0(x,\omega)}$, where $G_0(x,\omega)$ is a $R-1 \times R$ matrix. Since $R-1 < R$ the rank of $G_0(x,\omega)$ is less than R, and so $\operatorname{rank}(S_0(x,\omega)) < R$, which means that $S_0(x,\omega)$ is not invertible and therefore $\det(S_0)(x,\omega) = 0$ a.e. Then, by Lemma 4.4, there exists a vector-valued function $F \in L^2(U;\mathbb{C}^p), F \neq 0$ which satisfies $S_0(x,\omega)F(x,\omega) = 0$. Hence F is orthogonal to all the rows of G_0. By adding the row vector $F^T(x,\omega)$ to $G_0(x,\omega)$ we have constructed an $R \times R$ matrix $G(x,\omega)$ whose rows are "linearly independent". Thus we obtain $0 < K \leq \det(S)(x,\omega)$ a.e. on $[0,1) \times [0,1/R)$. That is, for $(Zg_{R-1})(x,\omega + j/R) = F_{0,j}(x,\omega), (x,\omega) \in ([0,1) \times [0,1/R))$, the sequence $\{g_{r,m,n}\}$ constitutes a frame. ∎

In the following example we will consider one well-behaved window function and one non-well-behaved window function, and see that the set of these two windows constitute a frame in the case of critical sampling. We note that Example 4.16 is inspired by [11, Example 1], however, several intermediate calculations, including calculations of the Zak transform of the two window functions, are added by the author.

Example 4.16: Assume we are in the case of critical sampling, that is $\frac{R}{ab} = 1$ and let $a = R = 2$ and $b = 1$. Let one of the two windows be given as $g_0(x) = \mathrm{e}^{-\beta|x|}, x \in \mathbb{R}$, and the other be given as $g_1(x) = g_0(x)$ for $x \geq 0$ and $g_1(x) = 0$ for $x < 0$. We first notice that, since $g_0(x)$ is a smooth function, it is well-behaved, i.e.

$$\int_{-\infty}^{\infty} |xg_0(x)|^2 \mathrm{d}x < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\xi \hat{g_0}(\xi)|^2 \mathrm{d}\xi < \infty.$$

It is also easily seen, by the definition of $g_1(x)$ that $g_1(x) \in L^2(\mathbb{R})$. However, $g_1(x)$ is not a continuous function, so $\hat{g}_1(\xi) \notin L^1(\mathbb{R})$. This can be seen as follows: We assume that $\hat{g}_1(\xi) \in L^1(\mathbb{R})$. Since the Fourier transform maps $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$, the space of continuous functions vanishing at infinity, then, under the assumption that the inversion formula holds, $g_1(x) \in C_0(\mathbb{R})$, but this is a contradiction. Using this fact and by imposing the Cauchy-Schwarz inequality we find that

$$\begin{aligned}\infty = \int_{-\infty}^{\infty} |\hat{g}_1(\xi)| \mathrm{d}\xi &= \int_{-\infty}^{\infty} |\hat{g}_1(\xi)|(1+|\xi|)\frac{1}{1+|\xi|}\mathrm{d}\xi \\ &\leq \left(\int_{-\infty}^{\infty} |\hat{g}_1(\xi)|^2(1+|\xi|)^2 \mathrm{d}\xi\right)^{1/2} \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\xi|)^2}\mathrm{d}\xi\right)^{1/2}.\end{aligned} \tag{4.36}$$

We recall that on a finite dimensional vector space all norms are equivalent, that is $(1+|\xi|)^2 \sim 1+|\xi|^2$. Thus the only way that Equation (4.36) holds is if $\xi\hat{g}_1(\xi) \notin L^2(\mathbb{R})$.

By Definition 3.1 with $\lambda = 1$ the Zak transform of $g_0(x)$ is

$$\begin{aligned}(Zg_0)(x,\omega) &= \sum_{k\in\mathbb{Z}} g_0(x-k)\,\mathrm{e}^{2\pi ik\omega} \\ &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\beta|x-k|}\,\mathrm{e}^{2\pi ik\omega}\end{aligned} \tag{4.37}$$

Let $0 \leq x < 1$. Then we can split the sum in Equation (4.37) and write

$$\begin{aligned}(Zg_0)(x,\omega) &= \sum_{k=-\infty}^{0} \mathrm{e}^{-\beta(x-k)}\,\mathrm{e}^{2\pi ik\omega} + \sum_{k=1}^{\infty} \mathrm{e}^{-\beta|x-k|}\,\mathrm{e}^{2\pi ik\omega} \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{-\beta(x+k)}\,\mathrm{e}^{-2\pi ik\omega} + \sum_{k=1}^{\infty} \mathrm{e}^{-\beta(k-x)}\,\mathrm{e}^{2\pi ik\omega}.\end{aligned} \tag{4.38}$$

Now look at the first term of Equation (4.38) and set $a_k = \mathrm{e}^{-\beta k}\,\mathrm{e}^{-2\pi ik\omega}$. Then the ratio $r = a_{k+1}/a_k = \mathrm{e}^{-\beta}\,\mathrm{e}^{-2\pi i\omega}$. By defining

$$z = \mathrm{e}^{2\pi i\omega}, \quad u = \mathrm{e}^{-\beta}, \tag{4.39}$$

for $0 \leq \omega \leq 1$ we obtain

$$\sum_{k=0}^{\infty} \mathrm{e}^{-\beta(x+k)}\,\mathrm{e}^{-2\pi ik\omega} = \mathrm{e}^{-\beta x}\sum_{k=0}^{\infty} r^k = u^x\frac{1}{1-\frac{u}{z}} = \frac{zu^x}{z-u}. \tag{4.40}$$

For the second term of Equation (4.38) we set $\tilde{a}_k = \mathrm{e}^{-\beta k}\,\mathrm{e}^{2\pi ik\omega}$. Then we have that the ratio $\tilde{r} = \tilde{a}_{k+1}/\tilde{a}_k = \mathrm{e}^{-\beta}\,\mathrm{e}^{2\pi ik\omega}$, and with z and u given as in Equation (4.39) we have

$$\sum_{k=1}^{\infty} \mathrm{e}^{-\beta(k-x)}\,\mathrm{e}^{2\pi ik\omega} = \mathrm{e}^{\beta x}\sum_{k=1}^{\infty} \tilde{r}^k = u^{-x}\frac{uz}{1-uz} = \frac{zu^{1-x}}{1-uz}, \tag{4.41}$$

By combining Equations (4.40) and (4.41) we obtain:

$$(Zg_0)(x,\omega) = \frac{zu^x}{z-u} + \frac{zu^{1-x}}{1-uz}, \quad 0 \leq x,\ \omega \leq 1.$$

With $(x, \omega) = \left(\frac{1}{2}, \frac{1}{2}\right)$ we see that

$$(Zg_0)(1/2, 1/2) = \frac{\mathrm{e}^{\pi i}\,\mathrm{e}^{-\beta/2}}{\mathrm{e}^{\pi i} - \mathrm{e}^{-\beta}} + \frac{\mathrm{e}^{\pi i}\,\mathrm{e}^{-\beta/2}}{1 - \mathrm{e}^{-\beta}\,\mathrm{e}^{\pi i}} = \frac{-\,\mathrm{e}^{-\beta/2}}{-(1+\mathrm{e}^{-\beta})} - \frac{\mathrm{e}^{-\beta/2}}{1+\mathrm{e}^{-\beta}} = 0.$$

That is, within the unit square $[0, 1) \times [0, 1)$, at the point $(x, \omega) = (1/2, 1/2)$, Zg_0 has a zero. This means that the function g_0 cannot constitute a frame for the critical sampling case. Now consider $g_1(x)$. Then the Zak transform is given as

$$(Zg_1)(x, \omega) = \frac{zu^x}{z - u}, \quad 0 \leq x < 1, 0 \leq \omega \leq 1,$$

with z and u given by Equation (4.39). In this case Zg_1 has no zeros in the unit square. By defining

$$D(x, \omega) := \det\left(G(x, \omega)^T\right), \tag{4.42}$$

we notice that $\det(P)(x, \omega)$, where $P(x, \omega)$ is defined as in Equation (4.35), is essentially the square root of $D(x, \omega)$. Hence, by Proposition 4.15 it follows that the two windows $g_0(x)$ and $g_1(x)$ generates a frame for the critical sampling case if and only if $D(x, \omega) \neq 0$ a.e.
By Equation (4.34) the 2×2 matrix $G(x, \omega)^T$ is given as

$$G(x, \omega)^T = \begin{bmatrix} (Zg_0)(x, \omega) & (Zg_1)(x, \omega) \\ (Zg_0)(x, \omega + 1/2) & (Zg_1)(x, \omega + 1/2) \end{bmatrix},$$

where

$$(Zg_0)(x, \omega + 1/2) = \frac{zu^x}{z + u} - \frac{zu^{1-x}}{1 + uz} \quad \text{and} \quad (Zg_1)(x, \omega + 1/2) = \frac{zu^x}{z + u}.$$

Thus by Equation (4.42) we have

$$\begin{aligned} D(x, \omega) &= (Zg_0)(x, \omega)(Zg_1)(x, \omega + 1/2) - (Zg_0)(x, \omega + 1/2)(Zg_1)(x, \omega) \\ &= \left(\frac{zu^x}{z - u} + \frac{zu^{1-x}}{1 - uz}\right)\left(\frac{zu^x}{z + u}\right) - \left(\frac{zu^x}{z + u} - \frac{zu^{1-x}}{1 + uz}\right)\left(\frac{zr^u}{z - u}\right) \\ &= \frac{z^2 u}{(z + u)(1 - uz)} + \frac{z^2 u}{(z - u)(1 + uz)} \\ &= \frac{2z^3 u - 2z^3 u^3}{(z^2 - u^2)(1 - u^2 z^2)}, \end{aligned}$$

where $0 \leq x < 1, 0 \leq \omega \leq 1$. Hence we see that $D(x, \omega)$ does not vanish, therefore the set of the two windows $g_0(x)$ and $g_1(x)$ constitute a frame for the critical sampling case.

4.5 A different sampling density for each window

Up until now the characterization of the multi-window Gabor system in Equation (4.1) have been considered under the assumption that each window function $g_r(x)$ have the same time- and frequency parameters a and b. In this section we generalize the Gabor system and consider the case, where for each window function $g_r(x)$ there is a different set of parameters $a_r, b_r \in \mathbb{Q}$. That is, we consider

$$g_{r,m,n}(x) = \mathrm{e}^{2\pi i m b_r x}\, g_r(x - n a_r), \quad m, n \in \mathbb{Z}, \quad \forall\, x \in \mathbb{R}, \quad a_r, b_r > 0, \tag{4.43}$$

where the redundancy factor of the combined space is given as

$$d := \sum_{r=0}^{R-1} \frac{1}{a_r b_r}.$$

For the characterization of the sequence $\{g_{r,m,n}\}$, we consider the three different cases of sampling separately, i.e. the case of undersampling ($d < 1$), the case of critical sampling ($d = 1$) and the case of oversampling ($d > 1$). To ease the analysis of this generalized system, we change it into an equivalent one where $a_r = a$ and $b_r = b$ for all $0 \le r \le R-1$. This enables us to use the methods, which we are already familiar with from the previous sections, for the analysis. Let

$$a_r = \frac{p_r}{q_r}, \quad \text{with } \gcd(p_r, q_r) = 1.$$

Introduce the sizes Q and P, which represents the greatest common divisor of all q_r and p_r respectively. That is

$$Q = \gcd(q_0, q_1, \ldots, q_{R-1}) \quad \text{and} \quad P = \gcd(p_0, p_1, \ldots, p_{R-1}).$$

Moreover, let

$$a = \frac{1}{QP} \prod_{r=0}^{R-1} p_r.$$

Then the quotient $a/a_r \in \mathbb{Z}$ for all $0 \le r \le R-1$, which can be seen as follows:

$$\begin{aligned}
\frac{a}{a_r} &= \frac{1}{a_r} \frac{1}{QP} \prod_{n=0}^{R-1} p_n \\
&= \frac{q_r}{p_r} \frac{1}{QP} \prod_{n=0}^{R-1} p_n \\
&= \frac{p_r}{p_r} q_r \frac{1}{QP} \prod_{n=0, n\neq r}^{R-1} p_n \\
&= \frac{q_r}{Q} \frac{1}{P} \prod_{n\in\mathcal{R}, n\neq r}^{R-1} p_n, \quad r \in \mathcal{R} = \{0, 1, \ldots, R-1\}.
\end{aligned}$$

According to [11, p. 211], the number a is the smallest number which fulfils that a/a_r is an integer for all r. The calculation of b in terms of the values of b_r can be found in a similar manner. In the new system we replace each window function $g_r(x)$ with $R_r^a R_r^b$ new window functions, where $R_r^a = a/a_r$ and $R_r^b = b/b_r$. Thus we obtain new window functions given as

$$\tilde{g}_{r,k,j}(x) = M_{kb_r} T_{ja_r} g_r(x) = \mathrm{e}^{2\pi i k b_r x}\, g_r(x - ja_r), \quad 0 \le j \le R_r^a - 1, \quad 0 \le k \le R_r^b - 1.$$

Now the new system becomes

$$\{(M_{mb} T_{na} \tilde{g}_{r,k,j})(x) \ : m, n \in \mathbb{Z},\ r \in \mathcal{R},\ 0 \le j \le R_r^a - 1,\ 0 \le k \le R_r^b - 1,\ \forall x \in \mathbb{R}\}.$$

By using the commutator relations of the translation and modulation operators [see 5, p. 48] we have

$$\begin{aligned}
\{M_{mb} T_{na} \tilde{g}_{r,k,j}\} &= \{M_{mb} T_{na} M_{kb_r} T_{ja_r} g_r\} \\
&= \{M_{mb}\, \mathrm{e}^{-2\pi i k b_r n a}\, M_{kb_r} T_{na} T_{ja_r} g_r\} \\
&= \{\mathrm{e}^{-2\pi i k b_r n a}\, M_{mb + kb_r} T_{na + ja_r} g_r\} \\
&= \{\mathrm{e}^{-2\pi i k b_r n a}\, M_{(mR_r^b + k) b_r} T_{(nR_r^a + j) a_r} g_r\} \\
&= \{\mathrm{e}^{-2\pi i k b_r \tilde{n} a}\, M_{m b_r} T_{n a_r} g_r\},
\end{aligned} \tag{4.44}$$

where the last equality follows since $mR_r^b+k \in \mathbb{Z}$ when $m \in \mathbb{Z}$ and $0 \leq k < R_r^b-1$, and likewise $nR_r^a + j \in \mathbb{Z}$ since $n \in \mathbb{Z}$ and $0 \leq j < R_r^a - 1$. Hence we see that the new system and the original system, given in Equation (4.43), differ by only a complex factor $\mathrm{e}^{-2\pi i k b_r \tilde{n} a}$. Moreover, the frame operators S and $\tilde{S}$ associated to $\{M_{mb}T_{na}g_r\}$ and $\{M_{mb}T_{na}\tilde{g}_{r,j,k}\}$, respectively, are identical. We show this by considering the definition of the frame operator and by using Equation (4.44). Thus for $f \in L^2(\mathbb{R})$ we have

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} \sum_{j=0}^{R_r^a-1} \sum_{k=0}^{R_r^b-1} \langle f, M_{mb}T_{na}\tilde{g}_{r,j,k}\rangle M_{mb}T_{na}\tilde{g}_{r,j,k} =$$

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} \sum_{j=0}^{\frac{a}{a_r}-1} \sum_{k=0}^{\frac{b}{b_r}-1} \left\langle f, \mathrm{e}^{-2\pi i k b_r n a}\, M_{mb+kb_r}T_{na+ja_r}g_r \right\rangle \mathrm{e}^{-2\pi i k b_r n a}\, M_{mb+kb_r}T_{na+ja_r}g_r =$$

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} \sum_{j=0}^{\frac{a}{a_r}-1} \sum_{k=0}^{\frac{b}{b_r}-1} \mathrm{e}^{2\pi i k b_r n a} \langle f, M_{mb+kb_r}T_{na+ja_r}g_r \rangle\, \mathrm{e}^{-2\pi i k b_r n a}\, M_{mb+kb_r}T_{na+ja_r}g_r =$$

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} \sum_{j=0}^{\frac{a}{a_r}-1} \sum_{k=0}^{\frac{b}{b_r}-1} \langle f, M_{mb+kb_r}T_{na+ja_r}g_r\rangle M_{mb+kb_r}T_{na+ja_r}g_r =$$

$$\sum_{r=0}^{R-1} \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb_r}T_{na_r}g_r\rangle M_{mb_r}T_{na_r}g_r.$$

Then, by using Theorem 4.7, we see that the original system is a frame, if and only if the new system is a frame. Moreover, since the frame operators associated to $\{M_{mb}T_{na}g_r\}$ and $\{M_{mb}T_{na}\tilde{g}_{r,j,k}\}$ are identical, it follows that the dual frame of the new system is identical to the dual frame of the original system (up to a complex factor). Hence using Equation (4.44), we can write a dual frame for the new system as

$$\begin{aligned}\{M_{mb}T_{na}\tilde{\gamma}_{r,j,k}\} &= \{M_{mb}T_{na}M_{kb_r}T_{ja_r}S^{-1}g_r\} \\ &= \{\mathrm{e}^{-2\pi i k b_r n a}\, M_{mb_r}T_{na_r}S^{-1}g_r\}.\end{aligned}$$

In the following example we will consider the case where the new system created is identical to the original system.

Example 4.17: Consider the case where $R = 2$, $b_r = 1$ for all r, $a_0 = p_0/q_0 = 5/1$ and $a_1 = p_1/q_1 = 5/4$. Then we are in the case of critical sampling, since

$$d := \sum_{r=0}^{R-1} \frac{1}{a_r b_r} = \frac{1}{a_0 b_0} + \frac{1}{a_1 b_1} = \frac{1}{5} + \frac{1}{5/4} = 1.$$

We have

$$\begin{aligned}Q &= \gcd(1,4) = 1 \\ P &= \gcd(5,5) = 5,\end{aligned}$$

so

$$a = \frac{1}{QP}(p_0 \cdot p_1) = \frac{1}{5}(5 \cdot 5) = 5.$$

Now, the original system consisting of two window functions $g_{0,m,n}$ and $g_{1,m,n}$ given as

$$\begin{aligned} g_{0,m,n} &= \mathrm{e}^{2\pi imx}\, g_0(x - 5n) \\ g_{1,m,n} &= \mathrm{e}^{2\pi imx}\, g_1\left(x - \frac{5}{4}n\right) \end{aligned}$$

will be replaced with

$$R_r^a R_r^b = \frac{a}{a_0}\frac{b}{b_0} \cdot \frac{a}{a_1}\frac{b}{b_1} = \frac{5}{5} \cdot \frac{5}{5/4} = 4$$

new window functions. The new window functions for the new system becomes

$$\begin{aligned} \tilde{g}_0(x) &= g_0(x) \\ \tilde{g}_1(x) &= g_1(x) \\ \tilde{g}_2(x) &= g_1\left(x - \frac{5}{4}\right) \\ \tilde{g}_3(x) &= g_1\left(x - \frac{5}{2}\right) \\ \tilde{g}_4(x) &= g_1\left(x - \frac{15}{4}\right), \end{aligned}$$

so, with $a_r = a = 5$ and $b_r = b = 1$ we have

$$\begin{aligned} \tilde{g}_{0,m,n} &= \mathrm{e}^{2\pi imx}\, g_0(x - 5n) \\ \tilde{g}_{1,m,n} &= \mathrm{e}^{2\pi imx}\, g_1(x - 5n) \\ \tilde{g}_{2,m,n} &= \mathrm{e}^{2\pi imx}\, g_1\left(x - \frac{5}{4} - 5n\right) \\ \tilde{g}_{3,m,n} &= \mathrm{e}^{2\pi imx}\, g_1\left(x - \frac{5}{2} - 5n\right) \\ \tilde{g}_{4,m,n} &= \mathrm{e}^{2\pi imx}\, g_1\left(x - \frac{15}{4} - 5n\right). \end{aligned}$$

In the new system $\tilde{g}_0$ replaced g_0 of the original system and $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4$ replaced g_1. Since the original system and the new system represent the same sequence $\{g_{r,m,n}\}$, they are identical.

Up until now we have examined the multi-window Gabor system in the case where the set of generators g of the system is finite. In the following chapter we will go a step further in the generalization and expand the system to incorporate an at most countable set of generators.

5. Generalization of the multi-window Gabor system

This chapter is highly inspired by [4]. We will examine properties of Gabor systems with at most countable set of generators g. We assume that $ab \in \mathbb{Q}$, and without loss of generality we take $a = p/q, p, q \in \mathbb{N}$ and $b = 1$. Let $\mathcal{A} \subset L^2(\mathbb{R})$ be at most countable, then the generalized multi-window Gabor system takes the following form:

$$\mathcal{G}(p/q, 1, \mathcal{A}) = \{e^{2\pi imx}\, g(x - np/q)\ : m, n \in \mathbb{Z},\ g \in \mathcal{A}\}. \tag{5.1}$$

By considering the case where the product ab is a rational number, we are able to use matrix algebra to analyse properties of the sequence $\mathcal{G}(p/q, 1, \mathcal{A})$, just like we did in the previous chapter. Let the Zak transform of $f \in L^2(\mathbb{R})$ be given as in Definition 3.1 with Zak parameter $\lambda = 1$, then we have the following definition of the Zibulski-Zeevi transform.

Definition 5.1: The Zibulski-Zeevi transform is a map

$$G : L^2(\mathbb{R}) \to L^2([0, 1/q] \times [0, 1/p], M_{q,p}(\mathbb{C})).$$

When G acts on a function $g \in L^2(\mathbb{R})$, we obtain a matrix-valued function, defined on $\mathbb{R}^2$, which we denote by G^g and whose elements are given as

$$G^g_{l,k}(x, \omega) = Zg\left(x - l \cdot \frac{p}{q}, \omega + \frac{k}{p}\right), \quad 0 \leq l \leq q-1,\ 0 \leq k \leq p-1, (x, \omega) \in \mathbb{R} \times \mathbb{R}.$$

The matrix G^g is called the Zibulski-Zeevi matrix. △

We notice that the matrix $G^g(x, \omega)$ corresponds to the matrix $G^r(x, \omega)$, which we defined in Chapter 4, Equation (4.13), that is, both matrices are of size $q \times p$. Though, as opposed to the matrix $G(x, \omega)$, defined in Equation (4.12), which consist of finitely many rows, the collection of the matrices $G^g(x, \omega)$ forms a new matrix, which may consist of infinitely many rows. Nevertheless, the row vectors in both matrices have finite length. As a consequence of this fact, the examination of frame properties for the sequence in Equation (5.1) will be carried out by considering the row vectors of $G^g(x, \omega)$, where the lth row of $G^g(x, \omega), 0 \leq l \leq q-1$ is denoted by $G^g_l(x, \omega)$. This leads to the following theorem.

Theorem 5.2: Let $g \in L^2(\mathbb{R})$ and $0 < a \leq b < \infty$ be given. Take $\mathcal{A} \subset L^2(\mathbb{R})$ to be at most countable. Then the following holds:

(i) $\mathcal{G}(p/q, 1, \mathcal{A})$ is a Bessel sequence with bound b, if and only if the system

$$\{G^g_l(x, \omega)\ :\ 0 \leq l \leq q-1,\ g \in \mathcal{A}\} \tag{5.2}$$

is a Bessel sequence with uniform bound pb for a.e. $(x, \omega) \in [0, 1/q] \times [0, 1/p]$.

(ii) $\mathcal{G}(p/q, 1, \mathcal{A})$ is a frame (respectively frame sequence) with bounds a, b, if and only if the system in Equation (5.2) is a frame (respectively frame sequence) with uniform bounds pa, pb for a.e. $(x, \omega) \in [0, 1/q] \times [0, 1/p]$.

In order to prove Theorem 5.2, we need to define shift-modulation invariant spaces and range functions, and present a result, Theorem 5.4, which characterizes the relation between shift-modulation invariant spaces and range functions. However, the proof of this result is beyond the scope of this project, so we will simply state the theorem and omit the proof. The reader is referred to [3] for a proof of Theorem 5.4.

Definition 5.3: A closed subspace $V \subset L^2(\mathbb{R})$ is shift-modulation invariant (SMI) if it is invariant under modulations and shifts, i.e.

$$M_m T_{np/q} V = V, \quad \forall\, n, m \in \mathbb{Z}. \qquad \triangle$$

By definition we notice that SMI spaces are exactly the subspaces of $L^2(\mathbb{R})$ which are generated by Gabor systems. The relation between SMI spaces in terms of appropriate range functions is given in the following.

Theorem 5.4: There is a one-to-one correspondence between SMI spaces V and measurable range functions

$$J : [0, 1/q] \times [0, 1/p] \to \{E : E \text{ is a subspace of } \mathbb{C}^p\}.$$

Given a measurable range function J, the associated SMI space is

$$V = \{f \in L^2(\mathbb{R}) \;:\; G_l^f(x, \omega) \in J(x, \omega),\ \forall\, 0 \le l \le q-1 \text{ and for a.e. } (x, \omega)\}. \tag{5.3}$$

If $V = \overline{\text{span}}\mathcal{G}(p/q, 1, \mathcal{A})$, the associated range function is

$$J(x, \omega) = \text{span}\{G_l^g(x, \omega) \;: 0 \le l \le q-1,\ g \in \mathcal{A}\}. \tag{5.4}$$

In terms of translation and modulation, we recall that the Zak transform satisfies the relations given in Equations (3.7) and (3.8). Thus by modulating the space V, defined in Equation (5.3), with $b = 1$ and translating it with $a = p/q$, we are still within the space V. This explains the claim that V is in fact an SMI space. Now we are ready to prove Theorem 5.2.

Proof: (Proof of Theorem 5.2) By the expression in Equation (4.8) we have

$$\langle f, g_{m,n'q+l}\rangle = \int_0^1 \int_0^{1/p} \left\langle G_0^f(x, \omega), G_l^g(x, \omega)\right\rangle \mathrm{e}^{2\pi i m x}\, \mathrm{e}^{-2\pi i n' p \omega}\, \mathrm{d}\omega \mathrm{d}x.$$

Since $\{\sqrt{p}\, \mathrm{e}^{2\pi i m x}\, \mathrm{e}^{-2\pi i n' p\omega}\}_{m,n' \in \mathbb{Z}}$ constitutes an orthonormal basis for $L^2([0,1] \times [0, 1/p])$, we use Parseval's identity [see 5, Theorem 3.2.2] and find that

$$\begin{aligned}
&\sum_{g \in \mathcal{A}} \sum_{m,n' \in \mathbb{Z}} \sum_{l=0}^{q-1} |\langle f, g_{m,n'q+l}\rangle|^2 \\
&= \frac{1}{p} \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \int_0^1 \int_0^{1/p} \left|\left\langle G_0^f(x, \omega), G_l^g(x, \omega)\right\rangle\right|^2 \mathrm{d}\omega \mathrm{d}x \\
&= \frac{1}{p} \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \sum_{l'=0}^{q-1} \int_0^{1/q} \int_0^{1/p} \left|\left\langle G_{l'}^f(x, \omega), G_{(l+l') \bmod q}^g(x, \omega)\right\rangle\right|^2 \mathrm{d}\omega \mathrm{d}x \\
&= \frac{1}{p} \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \sum_{l'=0}^{q-1} \int_0^{1/q} \int_0^{1/p} \left|\left\langle G_{l'}^f(x, \omega), G_l^g(x, \omega)\right\rangle\right|^2 \mathrm{d}\omega \mathrm{d}x.
\end{aligned} \tag{5.5}$$

We first assume that the system in Equation (5.2) is a frame sequence with uniform bounds pa, pb for a.e. $(x, \omega) \in [0, 1/q] \times [0, 1/p]$, that is

$$pa \|v\|^2 \leq \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \left| \langle v, G_l^g(x,\omega) \rangle \right|^2 \leq pb \|v\|^2 , \quad \forall\, v \in J(x,\omega) \text{ and a.e. } (x,\omega), \tag{5.6}$$

where $J(x,\omega)$ is given by Equation (5.4). Note that Theorem 5.4 guarantees that if $f \in \overline{\text{span}}\mathcal{G}(p/q, 1, \mathcal{A})$, then the object $v = G_{l'}^f(x,\omega)$, which we are interested in expanding, lies entirely within the space, which we have a frame for, that is $G_{l'}^f(x,\omega) \in J(x,\omega)$ for all $0 \leq l' \leq q-1$ and a.e. (x,ω). By inserting $v = G_{l'}^f(x,\omega)$ in Equation (5.6), integrating over $[0, 1/q] \times [0, 1/p]$ and summing over $0 \leq l' \leq q-1$ we obtain:

$$pa \|f\|^2 \leq \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \sum_{l'=0}^{q-1} \int_0^{1/q} \int_0^{1/p} \left| \left\langle G_{l'}^f(x,\omega), G_l^g(x,\omega) \right\rangle \right|^2 \mathrm{d}\omega \mathrm{d}x \leq pb \|f\|^2 \Leftrightarrow$$

$$a \|f\|^2 \leq \frac{1}{p} \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \sum_{l'=0}^{q-1} \int_0^{1/q} \int_0^{1/p} \left| \left\langle G_{l'}^f(x,\omega), G_l^g(x,\omega) \right\rangle \right|^2 \mathrm{d}\omega \mathrm{d}x \leq b \|f\|^2 \tag{5.7}$$

Combining Equation (5.7) with Equation (5.5) shows that $\mathcal{G}(p/q, 1, \mathcal{A})$ is a frame sequence with bounds a, b.
Conversely, suppose that $\mathcal{G}(p/q, 1, \mathcal{A})$ is a Bessel sequence with bound b. We will show that this implies that the system in Equation (5.2) is a Bessel sequence with uniform bound pb. Letting $D \subset \mathbb{C}^p$ be a countable dense subset, we show that for any $v \in D$

$$\sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} \left| \langle v, G_l^g(x,\omega) \rangle \right|^2 \leq pb \|P(x,\omega)v\|^2 , \quad \text{for a.e. } (x,\omega) \in [0, 1/q] \times [0, 1/p], \tag{5.8}$$

where $P(x,\omega)$ is the orthogonal projection of $\mathbb{C}^p$ onto $J(x,\omega)$. It is enough to check the inequality in Equation (5.8) on a dense subset of $\mathbb{C}^p$ since we are working with finite sums, so if the inequality holds for $v \in D$ then, in the limit, it holds for any element in $\mathbb{C}^p$. We will make a proof by contradiction, thus suppose the upper condition in Equation (5.8) is violated. Since D is countable we examine a countable set of candidate vectors $\{v_k\}_{k \in D}$. Recalling that a countable union of sets with measure zero has measure zero, there must exists at least one element in D, say v_0, such that Equation (5.8) fails on a measurable set $E \subset [0, 1/q] \times [0, 1/p]$ with positive measure. It then follows by standard arguments that for a given $\varepsilon > 0$ and $v_0 \in D$, there exists a set $E' \subseteq E$ with positive measure, such that

$$\sum_{g \in A} \sum_{l=0}^{q-1} \left| \langle P(x,\omega)v_0, G_l^g(x,\omega) \rangle \right|^2 \geq (pb + \varepsilon) \|P(x,\omega)v_0\|^2 , \quad \text{for a.e. } (x,\omega) \in E'. \tag{5.9}$$

Let $M \in L^2([0, 1/q] \times [0, 1/p], M_{q,p}(\mathbb{C}))$ and define its rows $M_{l'}(x,\omega)$ for $0 \leq l' \leq q-1$ by

$$M_{l'}(x,\omega) = \begin{cases} P(x,\omega)v_0 & \text{for } l' = 0 \text{ and } (x,\omega) \in E', \\ 0 & \text{otherwise.} \end{cases}$$

We recall by Lemma 3.2 that the Zak transform is a unitary operator, hence the Zibulski-Zeevi transform is unitary. Therefore there exists a unique $f \in L^2(\mathbb{R})$ such that $G^f(x,\omega) = M(x,\omega)$ for a.e. $(x,\omega) \in [0, 1/q] \times [0, 1/p]$, and by Theorem 5.4 $f \in \mathcal{G}(p/q, 1, \mathcal{A})$. Using

Equations (5.5) and (5.9) we obtain

$$\begin{aligned}\sum_{g\in\mathcal{A}}\sum_{m,n'\in\mathbb{Z}}\sum_{l=0}^{q-1}\left|\langle f, g_{m,n'q+l}\rangle\right|^2 &= \frac{1}{p}\sum_{g\in\mathcal{A}}\sum_{l=0}^{q-1}\sum_{l'=0}^{q-1}\int_0^{1/q}\int_0^{1/p}\left|\left\langle G^f_{l'}(x,\omega), G^g_l(x,\omega)\right\rangle\right|^2 \mathrm{d}\omega\mathrm{d}x\\ &= \frac{1}{p}\sum_{g\in\mathcal{A}}\sum_{l=0}^{q-1}\int_{E'}\left|\langle P(x,\omega)v_0, G^g_l(x,\omega)\rangle\right|^2 \mathrm{d}\omega\mathrm{d}x\\ &\geq \frac{1}{p}(pb+\varepsilon)\times\int_{E'}\left\|P(x,\omega)v_0\right\|^2 \mathrm{d}\omega\mathrm{d}x\\ &= (b+\varepsilon/p)\int_0^{1/q}\int_0^{1/p}\left\|M(x,\omega)\right\|_F^2 \mathrm{d}\omega\mathrm{d}x\\ &= (b+\varepsilon/p)\left\|f\right\|^2,\end{aligned}$$

where the last equality follows by taking the Frobenius norm and using the unitary property of the Zibulski-Zeevi transform. This in turns means that b cannot be an upper bound for $\mathcal{G}(p/q,1,\mathcal{A})$, hence we arrive at a contradiction and thus Equation (5.8) holds.
To complete the proof for the frame sequence case, we have to show that if $\mathcal{G}(p/q,1,\mathcal{A})$ also satisfies the lower frame condition, then the system in Equation (5.2) is uniformly bounded from below by pa. However, a similar proof as above shows that if the lower condition of the system in Equation (5.2) is violated, then a cannot be a lower frame bound for $\mathcal{G}(p/q,1,\mathcal{A})$ hence arriving at a contradiction.
Now, for the result concerning frames, we notice by Theorem 5.4 that if the range function $J(x,\omega)=\mathbb{C}^p$ for a.e. (x,ω), then any $f\in L^2(\mathbb{R})$ is included in the SMI space V, defined in Equation (5.3), hence $\mathcal{G}(p/q,1,\mathcal{A})$ is complete. To prove that completeness of $\mathcal{G}(p/q,1,\mathcal{A})$ implies that $J(x,\omega)=\mathbb{C}^p$ for a.e. (x,ω) we assume, to the contrary that $\mathcal{G}(p/q,1,\mathcal{A})$ is not complete in $L^2(\mathbb{R})$. Then there exists an $f\in L^2(\mathbb{R}), f\neq 0$ such that

$$f\in(\overline{\mathrm{span}}\mathcal{G}(p/q,1,\mathcal{A}))^{\perp}.$$

By the unitary property of the Zak transform, this implies that $G^f_{l'}(x,\omega)\neq 0$ a.e. But then

$$\sum_{g\in\mathcal{A}}\sum_{l=0}^{q-1}\sum_{l'=0}^{q-1}\left|\left\langle G^f_{l'}(x,\omega), G^g_l(x,\omega)\right\rangle\right|^2 = 0, \quad \text{for a.e. } (x,\omega). \tag{5.10}$$

This in turn means that $J(x,\omega)\neq\mathbb{C}^p$ for a.e. (x,ω). Thus we conclude that $\mathcal{G}(p/q,1,\mathcal{A})$ is a frame with bounds a,b, if and only if $J(x,\omega)=\mathbb{C}^p$ for a.e. (x,ω), or equivalently, the system in Equation (5.2) is a frame with uniform bounds pa,pb. ∎

By incorporating a finite set of generators g, we obtain a version of Theorem 5.2 that is valid for Riesz bases. We note that a Riesz basis is just a special case of a frame, in that a Riesz basis is an exact frame, that is, if an arbitrary element is removed from the basis, the remaining sequence is not complete, and therefore not a frame.

Theorem 5.5: Let $\mathcal{A}=\{g_1,\dots,g_n\}\subset L^2(\mathbb{R})$. Then $\mathcal{G}(p/q,1,\mathcal{A})$ is a Riesz sequence (respectively Riesz basis) if and only if the system in Equation (5.2) is a Riesz sequence (respectively Riesz basis).

Proof: We note that the system $\{G^g_l(x,\omega) : 0\leq l\leq q-1, g\in\mathcal{A}\}$ is a Riesz sequence if and only if the corresponding nq vectors are linearly independent, which is equivalent to the $nq\times p$ matrix $\mathcal{G}^{\mathcal{A}}$ having rank nq a.e. This again is, due to [4, Lemma 1.1], equivalent with $\mathcal{G}(p/q,1,\mathcal{A})$ being a Riesz sequence. ∎

Up until now we have studied a generalization of the single-window Gabor system. By incorporating several window functions of different shape it is possible to achieve high resolution in time (position) and frequency simultaneously, and thereby obtaining a better representation of signals. By using matrix algebra we have eased the examination of stating frame properties of the multi-window Gabor system. However, in practice this approach can become relatively time-consuming and difficult to handle, since one needs to create large systems based on matrices, where reasonable delicate operations are needed in the study of their spectrum. This leads us to the following chapter, where we will examine another approach in constructing and analysing multi-window Gabor systems.

6. Quilted Gabor frames

This chapter is based on [7], [8] and [10]. In this chapter we will examine quilted Gabor frames. These types of frames possess the same properties as multi-window Gabor frames, discussed in Chapter 4 and Chapter 5. They are designed such that they can achieve different resolutions in specific areas of the time-frequency plane, thus the construction of quitled Gabor frames allows the system to adapt to a given signal's properties.

We will no longer restrict us to the case where the product of time and frequency parameters ab is a rational number. However, for the construction of quilted Gabor frames we want to adapt a given window function g to a specific lattice Λ, hence in this chapter, we will use the notation $\mathcal{G}(g,\Lambda) := \{M_\eta T_\gamma g\}_{(\gamma,\eta)\in\Lambda}$ for the Gabor system with window function g. The lattice Λ is a discrete subgroup of $\mathbb{R}^{2d}$, i.e. $\Lambda \subset \mathbb{R}^{2d}$, of the form $\Lambda = A\mathbb{Z}^{2d}$, where A is an invertible $2d \times 2d$ matrix over $\mathbb{R}$. The special case

$$\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d,$$

where $a, b > 0$ are the lattice parameters, is called a separable or product lattice.

We will begin by defining two concepts, which will be used as a tool for assigning suitable Gabor systems to different regions in the time-frequency plane. Let $B_R(x) \subset \mathbb{R}^{2d}$ denote a ball of radius $R > 0$ and center $x \in \mathbb{R}^{2d}$.

Definition 6.1: Let the index set $\mathcal{I}$ be at most countable. A family $(B_{R_r}(x_r))_{r\in\mathcal{I}}$ of balls in $\mathbb{R}^{2d}$ is called an admissible covering of $\mathbb{R}^{2d}$ if

(i) $\bigcup_{r\in\mathcal{I}} B_{R_r} = \mathbb{R}^{2d}$

(ii) There exists $n_0 \in \mathbb{N}$ such that $|r^*| \leq n_0$ for all $r \in \mathcal{I}$, where

$$r^* := \{s : \ s \in \mathcal{I}, \ B_{R_r}(x_r) \cap B_{R_s}(x_s) \neq 0\}.$$

This is called the admissibility condition. △

By Definition 6.1 we note that the different balls B_{R_r} may overlap, however the admissible condition guarantees that the number of overlapping is bounded above, hence there is at most n_0 overlap for each $r \in \mathcal{I}$.

Definition 6.2: Let $(B_{R_r}(x_r))_{r\in\mathcal{I}}$ be a given admissible covering of $\mathbb{R}^{2d}$. A family of non-negative functions $\{\psi_r\}_{r\in\mathcal{I}}$ is called a bounded admissible partition of unity (BAPU) subordinate to $(B_{R_r}(x_r))_{r\in\mathcal{I}}$ in $L^2(\mathbb{R}^{2d})$, if the following conditions are satisfied:

(i) $\text{supp}(\psi_r) \subseteq \overline{(B_{R_r}(x_r))}$ for $r \in \mathcal{I}$.

(ii) $\sum_{r\in\mathcal{I}} \psi_r(x,\xi) \equiv 1$ for all $x, \xi \in \mathbb{R}^d$. △

We want to introduce a stronger assumption in relation to the admissible covering, such that the overlapping balls, which constitute the covering, are uniformly controlled. Hence for each $\rho > 0$, there exists $n_0 = n_0(\rho) \in \mathbb{N}$, called the height of the BAPU, such that $|r^*| \leq n_0$ for all $r \in \mathcal{I}$, where

$$r^* := \{s : \ s \in \mathcal{I}, B_{R_{r+\rho}}(x_r) \cap B_{R_s}(x_s + \rho) \neq 0\}.$$

This additional assumption is relevant for many results regarding BAPU's, but we note, however, that this will not be used as explicit in our further work.

Before discussing the concept of quilted frames, the following section will be dedicated to a brief analysis of a particular important function space, namely Feichtinger's algebra $S_0(\mathbb{R}^d)$. This function space have a variety of properties making it useful in the study of time-frequency analysis on $L^2(\mathbb{R}^d)$.

6.1 Characterization of $S_0(\mathbb{R}^d)$

The space S_0 is a Banach space. It carries properties, which makes it well suited in Gabor analysis in the sense that, by choosing window functions belonging to S_0, we ensure that the short-time Fourier transform (defined below) of a function $f \in L^2(\mathbb{R}^d)$ is well-behaved (compare with Definition 6.4).

Definition 6.3: Fix a function $g \in L^2(\mathbb{R}^d)$. The short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to the window function g is defined as a function of two real variables:

$$\mathcal{V}_g f(\gamma, \eta) = \langle f, M_\eta T_\gamma g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-\gamma)}\, \mathrm{e}^{-2\pi i t \cdot \eta}\, \mathrm{d}t, \quad \text{for } \gamma, \eta \in \mathbb{R}^d. \qquad \triangle$$

Now, considering the Gauss function defined by

$$g_{0,d}(x) := g_0(x) = \mathrm{e}^{-\pi \|x\|^2} = \mathrm{e}^{-\pi(x_1^2 + \cdots + x_d^2)},$$

for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define the space S_0 in terms of the STFT as follows.

Definition 6.4: Let g_0 be the Gauss function. The space $S_0(\mathbb{R}^d)$ is defined as

$$S_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \ \|f\|_{S_0} = \|\mathcal{V}_{g_0} f\|_{L^1(\mathbb{R}^{2d})} < \infty\}. \qquad \triangle$$

That is, $S_0(\mathbb{R}^d)$ consist of those functions with integrable STFT for sufficiently nice window functions. The choice of the Gaussian function g_0 in the definition of S_0, can be replaced with any non-zero function in S_0 by [8, Theorem 3.2.4]. Also different functions $g \in S_0(\mathbb{R}^d)\backslash\{0\}$ define equivalent norms on $S_0(\mathbb{R}^d)$. However, since the choice of window function influences the properties of the STFT, it it reasonable to choose the Gaussian function, since it offers the best time-frequency resolution, due to the Heisenberg uncertainty principle.

Without presenting details we will briefly mention the Wiener amalgam space, since $S_0(\mathbb{R}^d)$ corresponds to a space of this type. The main idea behind Wiener amalgam spaces is a separation of local and global properties of a function. The members of the space $W(L^p, \ell^q)$ are exactly the functions which belong locally to the L^p-space and have a global ℓ^q-behavior. Before stating the formal definition, we need to define the concept of a BUPU, which is in analogy to the definition of a BAPU (see Definition 6.2).

Definition 6.5: Given an admissible covering $(B_{R_r}(x_r))_{r\in\mathcal{I}}$ of $\mathbb{R}^{2d}$, we call a family of non-negative functions $\{\psi_r\}_{r\in\mathcal{I}}$ a bounded uniform partition of unity (BUPU) subordinate to $(B_{R_r}(x_r))_{r\in\mathcal{I}}$ in $L^2(\mathbb{R}^{2d})$, if the following conditions are satisfied:

(i) $\mathrm{supp}(\psi_r) \subseteq \overline{(B_{R_r}(x_r))}$ for $r \in \mathcal{I}$.

(ii) $\sum_{r\in\mathcal{I}} \psi_r(x,\xi) \equiv 1$ for all $x, \xi \in \mathbb{R}^d$.

(iii) $\psi_r = T_r\psi_0$ for $r \in \mathcal{I}$. △

Condition (iii) of Definition 6.5 says that the members of the BUPU are transformed into each other by translation. Thus we note that the difference between a BAPU and a BUPU is that the former is a more general method to decompose a function into a series of compactly supported pieces as opposed to the latter, which is more specific in the sense that the support of $(\psi_r)_{r\in\mathcal{I}}$ is of uniform size.

Definition 6.6: Let a BUPU $(\psi_r)_{r\in\mathcal{I}}$ for $\mathbb{R}^d$ be given. The Wiener amalgam space with local component L^p and global component ℓ^q is defined as

$$W(L^p, \ell^q)(\mathbb{R}^d) = \left\{ f \in L^p_{loc} : \|f\|_{W(L^p,\ell^q)} = \left(\sum_{r\in\mathcal{I}} \|f \cdot \psi_r\|_p^q \right)^{1/q} < \infty \right\}. \quad \triangle$$

We add that the spaces $W(L^\infty, \ell^q)(\mathbb{R}^d)$ and $W(L^p, \ell^\infty)(\mathbb{R}^d)$ are defined with the usual adjustment of the sup norm. The special type of Wiener amalgam spaces which coincides with the space $S_0(\mathbb{R}^d)$ is $W(\mathcal{F}L^1, \ell^1)$, where $\mathcal{F}L^1$ is the Fourier image of the space of integrable functions, i.e.

$$\mathcal{F}L^1 = \left\{ \hat{f} : f \in L^1, \left\|\hat{f}\right\|_{\mathcal{F}L^1} = \|f\|_{L^1} \right\}.$$

According to [8, Theorem 3.2.6] we have the following result.

Theorem 6.7: Let a BUPU $(\psi_r)_{r\in\mathcal{I}}$ for $\mathbb{R}^d$ be given. Then we have that $f \in S_0(\mathbb{R}^d)$ if and only if $\sum_{r\in\mathcal{I}} \|f \cdot \psi_r\|_{\mathcal{F}L^1} < \infty$. That is

$$S_0(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1),$$

with equivalence of the corresponding norms.

Since we will not need this particular characterization of the $S_0(\mathbb{R}^d)$-space in the following work, we will not give a proof for the theorem. We will, however, use the result of Proposition 6.8, inspired by [10, Proposition 12.1.4], which relates Feichtinger's algebra $S_0(\mathbb{R}^d)$ with the Wiener space $W(L^1)$ on $\mathbb{R}^d$, with norm given by

$$\|f\|_{W(L^1)} = \sum_{k\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in Q} |f(x+k)| = \sum_{k\in\mathbb{Z}^d} \|f \cdot T_k\chi_Q\|_\infty,$$

where $Q = [0,1]^d$ denotes the unit cube and χ_Q is the characteristic function of the set Q.

Proposition 6.8: If $f \in S_0(\mathbb{R}^d)$, then $f \in W(L^1)$.

Proof: We consider a function $f \in S_0(\mathbb{R}^d)$ and recall that we can choose any non-zero function $g \in S_0(\mathbb{R}^d)\backslash\{0\}$ as window function in the definition of $S_0(\mathbb{R}^d)$. We choose a compactly supported function $g \in C^\infty(\mathbb{R}^d)$ such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^d$ and $g(x) = 1$ for $x \in [-1,1]^d$. Then $\chi_Q(t) \le T_x g(t)$ for all $x \in Q$ and all $t \in \mathbb{R}^d$, and

$$\|f \cdot T_k\chi_Q\|_\infty \le \|f \cdot T_{k+x}\overline{g}\|_\infty.$$

Now, the Riemann-Lebesgue Lemma [see 10, Lemma 1.1.1] states that $\mathcal{F}$ maps $L^1(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$, a Banach space consisting of continuous functions that vanish at infinity. Since $C_0(\mathbb{R}^d)$ is a proper subspace of $L^\infty(\mathbb{R}^d)$ it follows that $\|h\|_\infty \le \left\|\hat{h}\right\|_1$. Hence the

L^∞-norm does not capture all information about the Fourier transform of an L^1-function. Using this we find the connection to the STFT of f as follows:

$$\begin{aligned}\|f\cdot T_{k+x}\overline{g}\|_\infty &\le \|\mathcal{F}(f\cdot T_{k+x}\overline{g})\|_1\\ &= \int_{\mathbb{R}^d} |\langle f, M_\eta T_{x+k} g\rangle| \mathrm{d}\eta\\ &= \int_{\mathbb{R}^d} |\mathcal{V}_g f(x+k,\eta)| \,\mathrm{d}\eta, \quad \text{for } x\in Q.\end{aligned}$$

Integrating over $x \in Q$ gives

$$\|f\cdot T_k\chi_Q\|_\infty \le \int_Q \left(\int_{\mathbb{R}^d} |\mathcal{V}_g f(x+k,\eta)|\,\mathrm{d}\eta\right)\,\mathrm{d}x,$$

and summing over k yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\|f\cdot T_k\chi_Q\|_\infty &\le \sum_{k\in\mathbb{Z}^d}\int_Q \left(\int_{\mathbb{R}^d} |\mathcal{V}_g f(x+k,\eta)|\,\mathrm{d}\eta\right)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{V}_g f(x,\eta)|\mathrm{d}x\mathrm{d}\eta.\end{aligned}$$

Thus we have shown that

$$\|f\|_{W(L^1)} \le \|\mathcal{V}_g f\|_{L^1(\mathbb{R}^{2d})} = \|f\|_{S_0}.$$

That is, the Feichtinger algebra S_0 is included in the Wiener space $W(L^1)$. ∎

Now that we are equipped with a suitable space for window functions, we will present an important result regarding the boundedness of the Gabor frame operator. This result will be applied in our following arguments. We assume that the STFT is sampled on a separable lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ of the time-frequency plane. The following proposition is inspired by [10, Proposition 6.2.2].

Proposition 6.9: Let Λ be a separable lattice in $\mathbb{R}^{2d}$ and consider the synthesis operator

$$T_g^* = T_{g,\Lambda}^* : (\langle f, M_\eta T_\gamma g\rangle)_{(\gamma,\eta)\in\Lambda} \mapsto \sum_{(\gamma,\eta)\in\Lambda} \langle f, M_\eta T_\gamma g\rangle M_\eta T_\gamma g.$$

Then, for $g \in S_0(\mathbb{R}^d)$, the operator T_g^* is bounded from $\ell^2(\mathbb{Z}^{2d})$ into $L^2(\mathbb{R}^d)$ and its operator norm, indicated as $\|\cdot\|_{\mathrm{op}}$, satisfies

$$\left\|T_g^*\right\|_{\mathrm{op}} \le \left(\frac{1}{a}+1\right)^{d/2}\left(\frac{1}{b}+1\right)^{d/2}\|g\|_{S_0}.$$

To prove this proposition, we will make use of the following lemma, inspired by [10, Lemma 6.1.2].

Lemma 6.10: If $g \in S_0(\mathbb{R}^d)$ and $\lambda > 0$, then

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\sum_{n\in\mathbb{Z}^d}|g(x-\lambda n)| \le \left(\frac{1}{\lambda}+1\right)^d\|g\|_{S_0}.$$

Proof: If we make a partition of an interval of length one, we know there can exist at most $\lambda^{-1}+1$ distinct points with minimum distance λ between each other. Hence we have at most $(\lambda^{-1}+1)^d$ points of the form $x+\lambda n, n\in\mathbb{Z}^d$, independent of $x\in\mathbb{R}^d$, in the cube $k+Q=\prod_{j=1}^d[k_j,k_j+1]$. Then, for almost all $x\in\mathbb{R}^d$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}|g(x-\lambda n)| &\leq \sum_{k\in\mathbb{Z}^d}\left(\frac{1}{\lambda}+1\right)^d \operatorname*{ess\,sup}_{\{n:x-\lambda n\in k+Q\}}|g(x-\lambda n)| \\ &\leq \left(\frac{1}{\lambda}+1\right)^d\sum_{k\in\mathbb{Z}^d}\|g\cdot T_k\chi_Q\|_\infty \\ &\leq \left(\frac{1}{\lambda}+1\right)^d\|g\|_{S_0},\end{aligned}$$

where the last inequality follows by Proposition 6.8. ∎

Proof: (Proof of Proposition 6.9) For notational convenience we set $(na,mb)_{n,m\in\mathbb{Z}^d}=(\gamma,\eta)\in\Lambda$ and $c_{m,n}=\langle f,M_{mb}T_{na}g\rangle$. To prove the proposition we assume that $c\in\ell^2(\mathbb{Z}^{2d})$ is a finitely supported sequence and then, by density, we can extend the result to any $c\in\ell^2(\mathbb{Z}^{2d})$. Consider the trigonometric polynomials

$$p_n(x)=\sum_{m\in\mathbb{Z}^d}c_{m,n}\,\mathrm{e}^{2\pi imb\cdot(x-na)}. \tag{6.1}$$

These $p_n(x)$ are periodic with period $\frac{1}{b}\mathbb{Z}^d$. Using Parseval's identity [see 5, Theorem 3.2.2] we calculate their L^2-norm over a period $Q_{1/b}=[0,1/b]^d$ and find

$$\|p_n\|^2_{L^2(Q_{1/b})}=\int_{Q_{1/b}}|p_n(x)|^2\mathrm{d}x=b^{-d}\sum_{m\in\mathbb{Z}^d}|c_{m,n}|^2. \tag{6.2}$$

Since it is assumed that $c\in\ell^2(\mathbb{Z}^{2d})$ has finite support, only finitely many p_n's are non-zero. Now let

$$f=\sum_{m,n\in\mathbb{Z}^d}c_{m,n}M_{mb}T_{na}g=\sum_{n\in\mathbb{Z}^d}p_n\cdot T_{na}g.$$

Then the L^2-norm of f is

$$\begin{aligned}\|f\|^2&=\int_{\mathbb{R}^d}\left|\sum_{n\in\mathbb{Z}^d}p_n(x)T_{na}g\right|^2\mathrm{d}x=\int_{\mathbb{R}^d}\sum_{j,n\in\mathbb{Z}^d}p_n(x)T_{na}g\,\overline{p_j(x)T_{ja}g}\,\mathrm{d}x\\&=\sum_{j,n\in\mathbb{Z}^d}\int_{\mathbb{R}^d}p_n(x)\overline{p_j(x)}g(x-na)\overline{g(x-ja)}\,\mathrm{d}x,\end{aligned} \tag{6.3}$$

where we have used that $p_n(x)T_{na}g$ and $\overline{p_j(x)T_{ja}g}$ are both $L^2(\mathbb{R}^d)$-functions thus, by the Cauchy-Schwarz inequality, the product of these two functions is in $L^1(\mathbb{R}^d)$. This justifies the use of Fubini's theorem [see 10, Appendix A.13] regarding the interchange of the sum and integral in the last equality of Equation (6.3). Now, recall that the functions $p_n(x)$ and $p_j(x)$ are $\frac{1}{b}\mathbb{Z}^d$-periodic; this follows immediately by Equation 6.1. Thus applying the periodization trick [see 10, Lemma 1.4.1] we find that

$$\|f\|^2=\sum_{j,n\in\mathbb{Z}^d}\int_{Q_{1/b}}p_n(x)\overline{p_j(x)}\sum_{r\in\mathbb{Z}^d}g(x-na-\frac{r}{b})\overline{g(x-ja-\tfrac{r}{b})}\mathrm{d}x.$$

Now define $\Gamma(x) = (\Gamma(x)_{j,n})_{j,n\in\mathbb{Z}^d}$ to be a matrix with elements given by

$$\Gamma_{j,n}(x) = \sum_{r\in\mathbb{Z}^d} \overline{g(x - ja - \tfrac{r}{b})} g(x - na - \frac{r}{b}), \quad \text{for } x \in \mathbb{R}^d.$$

The associated operators $\Gamma(x)$ acts on finite sequences $a = (a_n)_{n\in\mathbb{Z}^d}$ by

$$(\Gamma(x)a)_j = \sum_{n\in\mathbb{Z}^d} \Gamma_{j,n}(x)a_n.$$

We want to show that almost all $\Gamma(x)$ are bounded operators on $\ell^2(\mathbb{Z}^d)$ by applying Schur's test [see 10, Lemma 6.2.1]. First, by using Fubini's Theorem together with Lemma 6.10 we observe that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^d} |\Gamma_{j,n}(x)| &\leq \sum_{j\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d} \left|g(x - ja - \frac{r}{b})\right| \left|g(x - na - \frac{r}{b})\right| \\
&= \sum_{r\in\mathbb{Z}^d} \left(\sum_{j\in\mathbb{Z}^d} \left|g(x - ja - \frac{r}{b})\right|\right) \left|g(x - na - \frac{r}{b})\right| \\
&\leq \left(\frac{1}{a} + 1\right)^d \|g\|_{S_0} \cdot \sum_{r\in\mathbb{Z}^d} \left|g(x - na - \frac{r}{b})\right| \\
&\leq \left(\frac{1}{a} + 1\right)^d (b+1)^d \|g\|_{S_0}.
\end{aligned}$$

Since $\Gamma(x)$ is a self-adjoint matrix, the sum over k is estimated in the same manner. By Schur's test we find that $\Gamma(x)$ extends to a bounded operator on $\ell^p(\mathbb{Z}^d), 1 \leq p \leq \infty$ for a.e. x. The operator norm of $\Gamma(x)$ is bounded by

$$\|\Gamma(x)\|_{\text{op}} \leq \left(\frac{1}{a} + 1\right)^d (b+1)^d \|g\|_{S_0}^2.$$

Hence, for a.e. x, we have

$$0 \leq \sum_{j,n\in\mathbb{Z}^d} p_n(x)\overline{p_j(x)}\Gamma_{j,n}(x) \leq \left(\frac{1}{a} + 1\right)^d (b+1)^d \|g\|_{S_0}^2 \sum_{n\in\mathbb{Z}^d} |p_n(x)|^2.$$

By integrating over $Q_{1/b}$ and using Equation (6.2) we finally obtain

$$\begin{aligned}
\|f\|^2 &= \int_{Q_{1/b}} \sum_{j,n\in\mathbb{Z}^d} p_n(x)\overline{p_j(x)}\Gamma_{j,n}(x)\mathrm{d}x \\
&\leq \left(\frac{1}{a} + 1\right)^d (b+1)^d \|g\|_{S_0}^2 \sum_{n\in\mathbb{Z}^d} \int_{Q_{1/b}} |p_n(x)|^2 \mathrm{d}x \\
&= \left(\frac{1}{a} + 1\right)^d (b+1)^d b^{-d} \|g\|_{S_0}^2 \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |c_{m,n}|^2 \\
&= \left(\frac{1}{a} + 1\right)^d \left(\frac{1}{b} + 1\right)^d \|g\|_{S_0}^2 \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |c_{m,n}|^2.
\end{aligned}$$

This completes the proof. ∎

The result in Proposition 6.9 of the synthesis operator T_g^* can be transferred to an analogous result of the analysis operator $T_g : f \mapsto c_{m,n}$ since these two operators are adjoint to each other. Moreover, since the Gabor frame operator is just the composition of the synthesis- and analysis operator, i.e. $S_{g,g} = T_g^* \circ T_g$, we obtain the following.

Corollary 6.11: Let Λ be a separable lattice in $\mathbb{R}^{2d}$ and $g \in S_0(\mathbb{R}^d)$. The analysis operator $T_g : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d})$ and the Gabor frame operator $S_{g,g} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ are bounded operators with operator norms satisfying

$$\|T_g\|_{\text{op}} \leq \left(\frac{1}{a}+1\right)^{d/2} \left(\frac{1}{b}+1\right)^{d/2} \|g\|_{S_0},$$

and

$$\|S_{g,g}\|_{\text{op}} \leq \left(\frac{1}{a}+1\right)^{d} \left(\frac{1}{b}+1\right)^{d} \|g\|_{S_0}^2 .$$

The result of Corollary 6.11 will be useful in the subsequent section. However, we begin with an introduction of quilted Gabor frames in the following section.

6.2 The generel concept of quilted Gabor frames

This section and the following is inspired by [7]. However, adjustment have been made by the author such that the following theory presented covers the case of $\mathbb{R}^d$ rather than $\mathbb{R}$.

For the construction of quilted Gabor frames we start from a collection of Gabor frames. These Gabor frames can vary in terms of different window functions or same window function with varying time- and frequency parameters a, b. We divide the time-frequency plane into different parts through a given covering $(\Omega_r)_{r\in\mathcal{I}}$ of the time-frequency plane $\mathbb{R}^{2d}$, and assign a frame, from a family of Gabor frames, to each member of the covering. The new system will then locally resemble the original frames, and the resulting global system will be named a quilted Gabor system. A formal definition is stated next.

Definition 6.12: Assume we are given a set of Gabor frames $\mathcal{G}(g^j, \Lambda^j)_{j\in\mathcal{J}}$ for $L^2(\mathbb{R}^d)$ and an admissible covering $\Omega_r \subseteq (B_{R_r}(x_r))_{r\in\mathcal{I}}$ of $\mathbb{R}^{2d}$. Define the local index sets $\chi^r = \Omega_r \cap \Lambda^{m(r)}$, where $m : \mathcal{I} \to \mathcal{J}$ is a map, which assigns a frame from the given Gabor frames to each member of the covering. Then we call the set

$$\bigcup_{r\in\mathcal{I}} \mathcal{G}(g^{m(r)}, \chi^r)$$

a quilted Gabor frame for $L^2(\mathbb{R}^d)$ if there exists constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{r\in\mathcal{I}} \sum_{(\gamma,\eta)\in\chi^r} \left|\left\langle f, M_\eta T_\gamma g^{m(r)}\right\rangle\right|^2 \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}^d). \qquad \triangle$$

In the following section we will consider the case where the time-frequency plane is partitioned into strips, that is we restrict us to the case where only the resolution in time is changed.

6.3 Reduced multi-window Gabor frames: Windows with compact support

We now show that quilted Gabor frames can be constructed. Assume we are given a set of Gabor frames $\mathcal{G}(g^j, \Lambda^j)$ for $L^2(\mathbb{R}^d)$, $j \in \mathcal{J}$. In practical situations the window functions g^j will always be bounded in either the time or frequency domain, hence we may assume that g^j are compactly supported and $\|g^j\|_{s_0} \leq C_g < \infty$ for all $j \in \mathcal{J}$. We divide the time-frequency plane into strips through a partition of unity, that is, we assume $f = \sum_{r\in\mathcal{I}} \psi_r f$ with $\psi_r \leq 1$ and that the ψ_r have compact support in a rectangle $Q \subseteq \mathbb{R}^d$, where $Q = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$. Now, by using a mapping $m : \mathcal{I} \to \mathcal{J}$, we assign one specific Gabor frame to each of these strips.

We can construct subfamilies of the given Gabor frames in the following way. Assume that $m(0) = 0$, then we aim at representing $\psi_0 f$ by using the given Gabor frame $\mathcal{G}(g^0, \Lambda^0)$:

$$\begin{aligned}\psi_0 f &= \psi_0 \left(\sum_{(\gamma,\eta)\in\Lambda^0} \left\langle f, M_\eta T_\gamma g^0 \right\rangle M_\eta T_\gamma g^0 \right) \\ &= \sum_{(\gamma,\eta)\in\Lambda^0} \left\langle f, M_\eta T_\gamma g^0 \right\rangle \psi_0 M_\eta T_\gamma g^0.\end{aligned}$$

Since g^j and ψ_r have compact support there exists n_0^u and n_0^l such that for $(\gamma, \eta) = (na_0, mb_0)_{n,m\in\mathbb{Z}^d}$ with $n \notin N_0$, where

$$N_{m(0)} = N_0 = [n_{0,1}^l, n_{0,1}^u] \times \cdots \times [n_{0,d}^l, n_{0,d}^u] \ \subseteq \mathbb{Z}^d,$$

we have $\psi_0 M_\eta T_\gamma g^0 \equiv 0$. Thus, without loosing information we may represent $\psi_0 f$ as

$$\psi_0 f = \sum_{(\gamma,\eta)\in\chi^0} \left\langle f, M_\eta T_\gamma g^0 \right\rangle \psi_0 M_\eta T_\gamma g^0,$$

where $\chi^0 = [n_{0,1}^l \cdot a_0, n_{0,1}^u \cdot a_0] \times \cdots \times [n_{0,d}^l \cdot a_0, n_{0,d}^u \cdot a_0] \times b_0\mathbb{Z}^d$ is the subset of Λ^0, which corresponds to the non-zero contributions. In a similar way we create analogous subsets $\chi^r \subset \Lambda^{m(r)}$ for all r and hence we obtain:

$$f = \sum_{r\in\mathcal{I}} \psi_r f = \sum_r \sum_{(\gamma,\eta)\in\chi^r} \left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle \psi_r M_\eta T_\gamma g^{m(r)}. \tag{6.4}$$

Based on this construction we state the following proposition.

Proposition 6.13: Assume we are given a family of tight Gabor frames $\mathcal{G}(g^j, \Lambda^j), j \in \mathcal{J}$ for $L^2(\mathbb{R}^d)$. Let $\sup_{j\in\mathcal{J}} \|g^j\|_{S_0} = C_g < \infty$ and assume that each $C_{\Lambda^j}, j \in \mathcal{J}$ is uniformly bounded by a universal constant C_Λ, which is finite, i.e. $C_{\Lambda^j} = \left(\frac{1}{a_j} + 1\right)^d \left(\frac{1}{b_j} + 1\right)^d \leq C_\Lambda < \infty$ for all $j \in \mathcal{J}$. Let a partition of unity $(\psi_r)_{r\in\mathcal{I}}$ of compactly supported ψ_r with height n_0 be given and let a mapping $m : \mathcal{I} \to \mathcal{J}$ assign a frame $\mathcal{G}(g^{m(r)}, \Lambda^{m(r)})$ to each $r \in \mathcal{I}$. Choose the index sets

$$\chi^r = [n_{r,1}^l \cdot a_{m(r)}, n_{r,1}^u \cdot a_{m(r)}] \times \cdots \times [n_{r,d}^l \cdot a_{m(r)}, n_{r,d}^u \cdot a_{m(r)}] \times b_{m(r)}\mathbb{Z}^d$$

in such a way that for all $r \in \mathcal{I}$ and $(\gamma, \eta) = (na_{m(r)}, mb_{m(r)})_{n,m\in\mathbb{Z}^d}$ with $n \notin N_{m(r)}$, we have $\psi_r M_\eta T_\gamma g^{m(r)} \equiv 0$. Then, the union of the subfamilies $\bigcup_{r\in\mathcal{I}} (g^{m(r)}, \chi^r)$ is a frame for $L^2(\mathbb{R}^d)$ with lower frame bound $1/(n_0 C_\Lambda C_g^2)$.

Proof: In order to prove the proposition we first note that

$$\|\psi_r h\|^2 \leq \|h\|^2, \quad \text{for all } h \in L^2(\mathbb{R}^d). \tag{6.5}$$

Letting $h_r = \sum_{(\gamma,\eta)\in\chi^r} \left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle M_\eta T_\gamma g^{m(r)}$ and using Equations (6.4) and (6.5) we find that

$$\begin{aligned}
\|f\|^2 &\leq n_0 \sum_r \|\psi_r f\|^2 \\
&= n_0 \sum_r \left\| \sum_{(\gamma,\eta)\in\chi^r} \left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle \psi_r M_\eta T_\gamma g^{m(r)} \right\|^2 \\
&= n_0 \sum_r \|h_r \psi_r\|^2 \\
&\leq n_0 \sum_r \left\| \sum_{(\gamma,\eta)\in\chi^r} \left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle M_\eta T_\gamma g^{m(r)} \right\|^2 \\
&\leq n_0 \sum_r \left(\frac{1}{a_j} + 1\right)^d \left(\frac{1}{b_j} + 1\right)^d \left\|g^{m(r)}\right\|_{S_0}^2 \sum_{(\gamma,\eta)\in\chi^r} \left|\left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle\right|^2 \\
&\leq n_0 C_\Lambda C_g^2 \sum_r \sum_{(\gamma,\eta)\in\chi^r} \left|\left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle\right|^2,
\end{aligned}$$

where the second last inequality follows from Corollary 6.11, regarding the boundedness of the Gabor frame operator. This proves the existence of a lower frame bound. The upper frame bound can be estimated as a multiple of the maximum of all the upper frame bounds B^r:

$$\begin{aligned}
\sum_r \sum_{(\gamma,\eta)\in\chi^r} \left|\left\langle f, M_\eta T_\gamma g^{m(r)} \right\rangle\right|^2 &\leq \sum_r \sum_{(\gamma,\eta)\in\chi^r} \left|\left\langle \sum_{r\in r^*} \psi_r f, M_\eta T_\gamma g^{m(r)} \right\rangle\right|^2 \\
&\leq B \sum_r \left\| \sum_{r\in r^*} \psi_r f \right\|^2 \\
&\leq n_0 B \|f\|^2,
\end{aligned}$$

where the last inequality follows by the admissibility condition. This completes the proof.■

We notice by Proposition 6.13 that the estimate of the lower frame bound is influenced by the constants n_0, C_Λ and C_g. From an application point of view it is preferable to have frame bounds which are close to each other, and also to have a lower frame bound which is not too close to zero. By the estimate of the lower frame bound we see that it may be advantageous to control the sizes of n_0, C_Λ and C_g, for example by constructing the BAPU in such a way the number of overlaps n_0 is minimized.

Now that we have proved that the construction of systems, as the one depicted in Proposition 6.13, do indeed possess frame properties, we are able to reconstruct any signal $f \in L^2(\mathbb{R}^d)$ by means of a dual system with the same modulation and translation structure. However, complications might occur in calculating the dual frame, since there may be interference in the transition regions between the various overlapping of the Gabor systems. To address this problem numerical calculations have shown that one can reconstruct a given signal without having to calculate an exact dual frame. We refer the reader to [7] for the numerical results, which demonstrate some approaches to how one can reconstruct a signal without having to find an exact dual frame for the quilted Gabor frame.

7. Conclusion

The single-window Gabor system, as originated by D. Gabor [9], has had great impact in time-frequency analysis. However, the primary motivator for this thesis stems from the fact that the single-window Gabor system is not designed to provide precise representations of signals, which vary a lot in terms of high and low frequencies. This follows immediately from the fact that they only feature one window function g and hence one resolution, determined by the time- and frequency parameters a and b, over the whole time-frequency plane. The introduction of multi-window Gabor systems addresses this problem by incorporating several window functions of different shape in the system. We have studied two different approaches in constructing and analysing such systems.

In Chapter 4 we presented an approach developed by Meir Zibulski and Yehoshua Y. Zeevi. This approach analyses properties of the sequence $\{g_{r,m,n}\}$, from Equation 4.1, in the Zak transform domain. In this domain we are able to state frame properties of the sequence $\{g_{r,m,n}\}$ by considering the corresponding Gabor frame operator $S(x, \omega)$ as a matrix-valued function, under the additional assumption that the product of time- and frequency parameters, ab, is a rational number. We derived a connection between lower and upper frame bounds for the sequence in relation to the eigenvalues of $S(x, \omega)$. Moreover, we constructed an expression of a dual frame, which, in the context of matrix algebra induces the inverse of the matrix-valued function $S(x, \omega)$. My own contribution to this chapter is the addition of Lemma 4.5 and the corresponding proof. Moreover, in relation to showing that the eigenvalues of the matrix-valued function $S(x, \omega)$ coincides with the optimal frame bounds of the sequence $\{g_{r,m,n}\}$, I have added an argument which shows that the normalized eigenvector corresponding to the maximal eigenvalue of the matrix $S(x, \omega)$ is in fact measurable.

In Chapter 6 we introduced the concept of quilted Gabor frames. As opposed to the former approach, the requirement that the product of time- and frequency parameters ab is a rational number, is omitted. We considered a specific situation, where the time-frequency domain is partitioned into strips, such that a specific Gabor frame is assigned to each of these strips. This approach allows the use of appropriate window functions in particular areas of the time-frequency plane. We showed that this constructing do indeed yield frame properties. My own contribution to this chapter is an extension of the theory, such that it covers $\mathbb{R}^d$ for any $d \in \mathbb{N}$, whereas the original paper [7] switches between $\mathbb{R}^d$ and $\mathbb{R}$.

The aim of this thesis was to form theory of multi-window Gabor systems. A natural extension is to apply the established theory in practice. An obvious application of the theory is in relation to analysing audio signals, or more specific, musical signals. The analysis of musical signals may be in terms of transcription, where the problem is to transfer a piece of music to a musical score, which, symbolically, can be seen as a time- frequency representation of the musical signal. A future work could include examples which illustrate the advantages of using multi-window Gabor systems over single-window Gabor systems. Likewise, a comparison between the two approaches presented could be included, to see whether one of the approaches are more preferable than the other in some situations.

It should be noted that the theory presented in this thesis can be used in other areas than in the analysis of audio signals. In dimension $d = 1$ one may apply the presented theory on medical signal processing and in the case where $d = 2$ the function of interest, $f(x_1, x_2)$, may represent a color level at the pixel position $(x_1, x_2) \in \mathbb{R}^2$, and thus the theory may be used to analyse an image. Other important areas where the theory can be used include compression of digital signals and noise reduction in audio signals.

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Appendix A:

In this appendix we remind the reader of some properties of linear operators on $L^2(\mathbb{R})$ and basic linear algebra.

1 Properties of linear operators

We mainly focus on three classes of operators on $L^2(\mathbb{R})$. The three operators are defined as follows:

Definition A.1:

$$\textit{Translation by } a \in \mathbb{R}, \quad T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (T_a f)(x) = f(x-a). \tag{A.1}$$

$$\textit{Modulation by } b \in \mathbb{R}, \quad M_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (M_b f)(x) = \mathrm{e}^{2\pi i b x} f(x). \tag{A.2}$$

$$\textit{Dilation by } c \neq 0, \quad D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (D_c f)(x) = \frac{1}{\sqrt{|c|}} f\left(\frac{x}{c}\right). \tag{A.3}$$

△

Lemma A.2: The three operators T_a, M_b and D_c are unitary operators for all $a, b \in \mathbb{R}$ and $c \neq 0$.

Proof: We will only prove that the dilation operator D_c is unitary. Similar proofs hold for T_a and M_b. Since

$$\begin{aligned}
\langle D_c f, g\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{|c|}} f\left(\frac{x}{c}\right) \overline{g(x)} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{|c|}} f(t) \overline{g(ct)} |c| \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{|c|}}{|c|} f(t) \overline{g(ct)} |c| \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \sqrt{|c|} f(t) \overline{g(ct)} \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) \overline{\frac{1}{\sqrt{c^{-1}}} g\left(\tfrac{t}{c^{-1}}\right)} \mathrm{d}t \\
&= \langle f, D_{c^{-1}} g\rangle, \quad \forall f, g \in L^2(\mathbb{R}),
\end{aligned}$$

we see that $D_c^* = D_{c^{-1}}$. Furthermore, the dilation operator D_c is an invertible operator satisfying $D_c^{-1} = D_{c^{-1}}$, so we conclude that $D_c^{-1} = D_c^*$. ■

2 Trace of diagonalizable matrices

The next result concerns the properties of diagonalizable matrices, where we consider the trace of such matrices. We recall that the trace of an $n \times n$ square matrix A is defined to be the sum of the elements on the main diagonal.

Proposition A.3: Let A be an $n \times n$ square self-adjoint matrix. Then

$$\operatorname{tr}(A) = \sum_i \mu_i, \tag{A.4}$$

where $\mu_i, i = 0, \ldots n$ are the eigenvalues of A.

Proof: Since A is self-adjoint we know that it is diagonalizable, i.e. we may write

$$A = PDP^{-1},$$

where the entries of the diagonal $p \times p$ matrix D are the eigenvalues of A, and the column vectors of the $p \times p$ matrix P are the eigenvectors of A. Now

$$\mathrm{tr}(A) = \mathrm{tr}(PDP^{-1}) = \mathrm{tr}(DPP^{-1}) = \mathrm{tr}(D) = \sum_i \mu_i,$$

with μ_i the eigenvalues of A. ■