IMPLEMENTATION OF A TWO-SUFACE PLASTICITY MODEL FOR CYCLIC LOADING ON SAND INTO THE FINITE ELEMENT METHOD



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Synopsis:

The objective of this paper is to implement the *Critical state two-surface plasticity model for sands* presented by Manzari and Prachathananukit [2000] into the Finite Element Method.

Firstly, the fundamental concepts of plasticity theory are presented along with the theory of three different stress update schemes namely, Modified Forward Euler, Forward Euler and Radial Return Method. Initially the von Mises criterion is applied and perfect plasticity and hardening plasticity are compared in the case of a patch test. In addition, a strip footing is tested to verify the bearing capacity and computational time of the different stress update schemes.

At last, the paper aims to verify the implementation of the model in drained and undrained conditions under monotonic and cyclic loading with different confining pressures.

This paper depicts the 4th semester thesis of M.Sc. in Civil and Structural Engineering at Aalborg University. For this final semester, an innovative and advanced topic has been chosen, which is the implementation of a multiple surface constitutive model for sand and cyclic loading within the Finite Element Method (FEM). One representative from Aalborg University guided us along the semester, our main supervisor, Associate Professor Johan Clausen.

We would like to thank Johan Clausen for his guidance and contribution to our work.

All files including calculations, plots, numerical models etc. are available on the Annex-CD.

Reading guide

Source citation

Source references are developed by the Harvard method and refer to the full source list at the back of the report. A passive source will be indicated as follows: [Surname, Year]. By an active source, for example a website, it will be referred in the text with the specific date or year of the entry. An example is Surname, [Year].

Figure, tables and equations

Figures, tables and equations in the report will be numbered under which chapter they belong and which number in the sequence of tables, figures and equations they are in chapter. As an example, "figure 2.2" can be found in chapter 2 and is the second figure within that same chapter. Tables are referred to in the same manner as figures. Equation numbers appear as "(3.1)" and are shifted to the right side of the document.

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INTRODUCTION

The most common constitutive models that engineers use today are not relevant for cyclic loads as they do not account stress history. It becomes an important topic when dealing with project demanding numerical analyses, e.g. the Offshore Wind Turbine industry and the DNV-GL new standards.

Indeed, offshore structures are subjected to time varying loads from wind and waves which means that their foundations will experience cyclic loads. Related observations to cyclic loading are accumulation of pore pressure, cyclic mobility and soil liquefaction and may lead to disastrous consequences.

For granular materials, only simple classical elasto-plastic models are supported by most commercial engineering programs, e.g. Mohr-Coulomb. Regarding pure strength calculations the Mohr-Coulomb criterion is often sufficient but insufficient for more complex phenomena as mentioned earlier.

A recently developed constitutive model named *Critical state two-surface plasticity model for sands* by Manzari and Prachathananukit [2000] is based on the framework of Critical State Soil Mechanics (CSSM) and takes into consideration the complex behaviour of granular materials. The soil model is capable of simulating the stress-strain behaviour of granular materials submitted to monotonic or cyclic loading and drained/undrained conditions.

To use a more advanced model like the critical state soil model in engineering calculations it requires a form of numerical implementation, i.e. the well-known Finite Element Method (FEM).

Soil behaviour is highly non-linear when deformed and by that no linear relation between stresses and strains exist. This means that an iterative solution is necessary to obtain the unknown stress increments from the constitutive relations in the non-linear analyse. Stress update or solution schemes are normally classified whether as explicit or implicit methods and differs in the approach to obtain the stress increment.

In an explicit integration scheme the constitutive matrix is initially evaluated by a known stress point and for every step the constitutive matrix is re-evaluated to check if the new stress point is elastic or plastic. On the other hand an implicit integration scheme defines a final stress point from a state of an unknown stress. By a direct solution procedure involving no iterations the final stress point accordingly fits the given yield criterion.

1.1 Presentation of thesis

Several research papers have presented new advanced models that simulate soil response under cyclic loading such as the butterfly shape, the accumulation of pore pressure and cyclic liquefaction. To be useful in engineering calculations and moreover commercial programs these complex models requires an efficient numerical implementation.

The aim of this thesis is to implement the *Critical state two-surface plasticity model for sands* by Manzari and Prachathananukit [2000] into Matlab with FEM. The explicit Forward Euler scheme is chosen as the method to solve the constitutive relations of the plasticity model.

Initially the von Mise elasto-plastic behaviour is investigated and extended to consider linear hardening. Three different stress update schemes, the Forward Euler, Modified Forward Euler and Radial Return methods are implemented for that particular model. This elasto-plastic model serves as a basis in the implementation of the more complex critical state model.

For each models used in this paper, two different tests are built in Matlab. Firstly a simple patch test is generated to ensure the models are implemented correctly. Secondly a more realistic example of a strip footing is tested to capture the soil behaviour under monotonic and cyclic loading.

1.2 Syntax

Throughout the report a number of vectors, matrices, tensors and variables are used.

A scalar is presented in normal text writing, e.g. σ_2 . A vector or tensor is written in bold $\boldsymbol{\sigma}$ and a matrix in bold capital letter, e.g. \boldsymbol{D} . By default, 2D plane strain is assume on the *xy* plane, hence, vectors are defined as 4×1 and matrices, 4×4 . Thereby the strain tensor, $\boldsymbol{\varepsilon}$, and stress tensor, $\boldsymbol{\sigma}$, are defined as

$$\boldsymbol{\varepsilon} = \{ \boldsymbol{\varepsilon}_x \ \boldsymbol{\varepsilon}_y \ \boldsymbol{\varepsilon}_z \ 2\boldsymbol{\varepsilon}_{xy} \ 2\boldsymbol{\varepsilon}_{xz} \ 2\boldsymbol{\varepsilon}_{yz} \}^{\mathsf{T}}$$
(1.1)

$$\boldsymbol{\sigma} = \{ \boldsymbol{\sigma}_x \ \boldsymbol{\sigma}_y \ \boldsymbol{\sigma}_z \ \boldsymbol{\sigma}_{xy} \ \boldsymbol{\sigma}_{xz} \ \boldsymbol{\sigma}_{yz} \}^{\mathsf{T}}$$
(1.2)

Symmetric properties makes it possible to express the stress and strain 2^{nd} order tensors as vectors and the constitutive relation, **D**, a 4th order tensor as a matrix. Further explanation of the constitutive relation is elaborated in Chapter 3.

1.2.1 Time-independency

From dynamics a dot denotes the first time-derivative, i.e. $\dot{\boldsymbol{\sigma}}_{ij} = d\boldsymbol{\sigma}_{ij}/dt$. In elasto-plastic constitutive theory, on the contrary of visco-plasticity, the dot is time independent but incremental dependant, e.g. load increments. The material response is assumed independent of the time during the loading process. The changes in the pseudo time reflects a change as an increment, i.e. $\dot{\boldsymbol{\sigma}}_{ij} = d\boldsymbol{\sigma}_{ij}$.

1.2.2 Sign convention

In continuum mechanics compression is negative but in geotechnics it is reversed. In this paper, the usual sign conventionn found in continuum mechanics is preferred. Therefore, if it is not

stipulated otherwise then the usual sign convention from mechanics is applied so that tension and compression are respectively positive and negative.

BASICS OF PLASTICITY

In this chapter fundamental concepts of the basic plasticity theory is implemented in the FEM. The ingredients needed in the plasticity theory are a yield criterion, a plastic potential function, a flow rule and a hardening law.

2.1 Stress invariants

Describing a given yield criterion in form of stress invariants makes it independent of which coordinate system is initially selected. Since an isotropic material has no directional properties, the principal stresses are given in terms of the three stress invariants I_1, I_2, I_3 . By adopting a given order of the principal stresses as $\sigma_1 \ge \sigma_2 \ge \sigma_3$ it is more convenient to express the yield criterion in form of another set of invariants, given by (2.1). [Ottosen and Ristinmaa, 2005]

$$f(I_1, J_2, \cos 3\theta) = 0$$
 (2.1)

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z \tag{2.2}$$

$$J_2 = \frac{1}{2} \mathbf{s}_{ij} \mathbf{s}_{ji} = \frac{1}{6} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$
(2.3)

$$J_{3} = \frac{1}{3} \mathbf{s}_{ij} \mathbf{s}_{jk} \mathbf{s}_{ki} = \det(\mathbf{s}_{ij}) = \begin{vmatrix} \sigma_{x} - \frac{1}{3} I_{1} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} - \frac{1}{3} I_{1} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{y} - \frac{1}{2} I_{1} \end{vmatrix}$$
(2.4)

$$\cos(3\theta) = \frac{3 \cdot \sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$$
(2.5)

 I_1 is the first invariant of the stress tensor σ_{ii} and J_2 , J_3 are the second and third stress invariants of the deviatoric stress tensor s_{ij} . θ is defined as the Lode angle.

By introducing a Haigh-Westergaard coordinate system the latter stress invariants implies a more physical interpretation and thereby a geometrical interpretation of the different stress invariants. [Ottosen and Ristinmaa, 2005] A diagram of this coordinate system is illustrated in figure 2.1 in principal stress orientation.



Figure 2.1. a) Haigh-Westergaard coordinate system. b) deviatoric plane perpendicular to hydrostatic axis.

By figure 2.1 it is shown that the magnitude of the hydrostatic load is equal to the stress invariant ξ .

$$\xi = \frac{1}{\sqrt{3}}I_1 = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)$$
(2.6)

Consider the plane that is perpendicular to the hydrostatic axis, here the magnitude of this distance is equal to the invariant ρ , which makes this stress invariant a measure of the deviatoric stresses. [Ottosen and Ristinmaa, 2005]

$$\rho = \sqrt{2J_2} = \sqrt{\frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]}$$
(2.7)

The invariant θ in (2.5) is determined by the principal stress values. When $\sigma_1 \ge \sigma_2 = \sigma_3$ the value for θ becomes 60°. Furthermore if $\sigma_1 = \sigma_2 \ge \sigma_3$, the value for θ becomes 0°. So the Lode angle θ , is an indication of the intermediate principal stress in relation to σ_1 and σ_3 . [Ottosen and Ristinmaa, 2005]

According to (2.1) the contributions of the invariants defined above are all essential to identify failure and the initial yield criterion.

2.2 The yield function

A yield function defines the limit of elasticity and the beginning of plastic deformation under any possible combination of stresses.[Yu, 2006] Considering the uniaxial stress-strain curve in figure 2.2 the yielding is defined by a point.



Figure 2.2. Elastic and plastic strains.

According to figure 2.2 if the stress is below σ_{Y0} , the material is assumed to behave linear elastic with a stiffness given by Young's Modulus *E*. If the material is loaded to σ_Y , yielding occurs and after unloading the plastic strain ε^p remains. The unloading and reloading are assumed to be elastic with the inclination of *E* until the point of σ_Y and contain the elastic uniaxial strain ε^e . [Ottosen and Ristinmaa, 2005]

As illustrated in figure 2.2 the total strain ε , contains the sum of elastic and in generalised stress space, this scalar becomes a tensor and is written as follow as in (2.8).

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{2.8}$$

The development of the plastic strain is controlled by a yield function f = 0. An example could be von Mises yield criterion, presented in (2.25). The yield function develops with the plastic strains and is expressed in terms of the stresses and some hardening parameters \mathcal{H} . [Ottosen and Ristinmaa, 2005] The yield function f is given as

$$f(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) = 0 \tag{2.9}$$

The hardening parameters are determined by some state variables κ , which characterize the internal conditions of the material.

$$\mathcal{H} = \mathcal{H}(\mathbf{\kappa}) \tag{2.10}$$

Per definition the state variables $\boldsymbol{\kappa}$ is zero before any plasticity is initiated and evolve along with the plastic loading history. So the choice of hardening parameters are equivalent to choosing hardening rules. [Ottosen and Ristinmaa, 2005]

2.3 Plastic potential

As known the elastic strain increment is related to stresses by the generalised Hooke's law. As for the plastic strain increment there is no direct way to determine this. Plastic deformations can occur as long as the stress point is located on the yield surface and moves along the yield surface, redistributing the stresses. [Krabbenhøft, 2002] Mathematically, the conditions for plastic loading can be written as

$$f(\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}) = f(\boldsymbol{\sigma}) + \nabla f \dot{\boldsymbol{\sigma}} = 0$$
(2.11)

Here $\dot{\sigma}$ is a stress increment and ∇f is the partial derivative of the yield function with respect to the stress vector $\boldsymbol{\sigma}$, which gives the normal to the yield surface, see figure 2.3. The following expression is obtained and is called the consistency condition and fulfils the yield criterion of f = 0. [Krabbenhøft, 2002]

$$\dot{f} = \nabla f \dot{\sigma} = 0 \tag{2.12}$$

In order to describe the plastic strains a potential field is introduced called the plastic potential and noted g, it depends on a given stress state and hardening parameter.

$$g = g(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) \tag{2.13}$$

A common choice for the plastic potential is to use the yield function. If this is the case, it is referred to as associated plasticity. If another function is chosen, it is referred to as non-associated plasticity. Thus the plastic strain increment is given by

$$\dot{\varepsilon}^{p} = \dot{\Lambda} \frac{\partial g}{\partial \sigma} \tag{2.14}$$

Assuming that g = f, this relation is known as the associated flow rule. The length of the incremental plastic strain is controlled by the plastic multiplier $\dot{\Lambda}$, which is a non-negative scalar. In figure 2.3 associated and non-associated plasticity is illustrated with given flow rule and plastic potential. [Krabbenhøft, 2002]



Figure 2.3. a) Associated plastcity. b) Non-associated plasticity.

2.4 Hardening laws

Materials have the capability to respond in different ways, see Figure 2.4. First picture from the left shows hardening plasticity. In this case the initial yield stress σ_{Y0} , increases with the increase

in plastic strain until an ultimate strength is reached. Second picture shows perfect plasticity, which means the material maintain the same initial yield strength. The third picture is known as softening. Firstly the material reach a peak strength, which for further loading makes the soil weakens until a residual strength is reached. [Ottosen and Ristinmaa, 2005]



Figure 2.4. Material behaviour under plastic loading.

Normally soils tend to loose strength when plastic straining occurs. On the other side metals tend to show an increase in strength. Since the plastic strains has an effect on the yield criterion it is evident that the yield surface changes due to plastic loading. From figure 2.2 the perfect plastic yield surface is related to the initial yield stress, σ_{Y0} . [Ottosen and Ristinmaa, 2005] To highlight the yield criterion with perfect plasticity, as in (2.15) a capital *F* is used.

$$F(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) = f(\boldsymbol{\sigma}) = F(\boldsymbol{\sigma}) = 0$$
(2.15)

On the other hand the current yield surface is related to σ_Y which depends on the hardening parameter \mathcal{H} that characterize in which manner the yield surface changes size, shape, position and how the hardening evolves. [Ottosen and Ristinmaa, 2005]

If hardening is considered in a given model, different rules are known, namely isotropic, kinematic and mixed hardening.

For isotropic hardening, the position and shape of the yield surface remain fixed whereas the size of the yield surface changes with plastic deformation. This can be expressed as

$$f(\boldsymbol{\sigma}, \mathcal{H}_{iso}) = F(\boldsymbol{\sigma}) - \mathcal{H}_{iso} = 0$$
(2.16)

Moreover, kinematic hardening shifts the yield surface from one location in stress space to another. The size and shape of the yield surface remain fixed.

$$f(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}_{kin}) = F(\boldsymbol{\sigma} - \boldsymbol{\mathcal{H}}_{kin}) = 0$$
(2.17)

The two different hardening models can be used simultaneously, in which case it is referred to as mixed hardening. Mixed hardening alters the size and position of the yield surface and leaves the shape unchanged. This can be written as

$$f(\boldsymbol{\sigma}, \mathcal{H}_{iso}, \mathcal{H}_{kin}) = F(\boldsymbol{\sigma} - \mathcal{H}_{kin}) - \mathcal{H}_{iso} = 0$$
(2.18)

The three different hardening laws are illustrated in figure 2.5.



Mixed hardening

Figure 2.5. The three hardening laws with von Mises yield criterion.

2.5 Elasto-plastic constitutive relation

To formulate an elasto-plastic constitutive relation it requires the following: a yield function, a plastic potential function and a hardening rule. All of these have been touch upon in the previously sections.

The relation between stress increment and total strain increment is obtained in the same way as for elasticity (Hooke's law), such as in (2.19)

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p) = \boldsymbol{D}(\dot{\boldsymbol{\varepsilon}} - \dot{\Lambda} \frac{\partial g}{\partial \boldsymbol{\sigma}})$$
(2.19)

D is the elastic constitutive matrix. By first substituting the flow rule (2.14) into the stress

increment (2.19) and secondly replacing this expression into the consistency relation (2.12), the plastic multiplier $\dot{\Lambda}$ may be found, see (2.20).

$$\dot{\Lambda} = \frac{\boldsymbol{a}^{\mathsf{T}} \boldsymbol{D} \dot{\boldsymbol{\varepsilon}}}{A + \boldsymbol{a}^{\mathsf{T}} \boldsymbol{D} \boldsymbol{b}} \tag{2.20}$$

where

$$\boldsymbol{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{2.21}$$

$$\boldsymbol{b} = \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{2.22}$$

And A is the hardening parameter that is assumed constant in case of linear hardening. If (2.20) is substituted back into stress increment (2.19) the solution give the elasto-plastic constitutive relation

$$\boldsymbol{D}^{ep} = \boldsymbol{D} - \frac{\boldsymbol{D}\boldsymbol{b}\boldsymbol{a}^{\mathsf{T}}\boldsymbol{D}}{\boldsymbol{A} + \boldsymbol{a}^{\mathsf{T}}\boldsymbol{D}\boldsymbol{b}}$$
(2.23)

The difference between hardening plasticity and perfect plasticity when linear hardening is assumed is to have a postitive hardening parameter *A*. The hardening modulus *A* is defined as

$$A = -\frac{\partial f}{\partial \mathcal{H}} \frac{\dot{\mathcal{H}}}{\dot{\Lambda}}$$
(2.24)

where \mathcal{H} is the hardening parameter depending on the yield function and plastic multiplier.

2.6 Example of von Mises plasticity

As an example of implementing plasticity, the von Mises elasto-plastic model is described in the following. Supportive plots and results from the implementation of the von Mises plasticity model are described in appendix A.

2.6.1 Yield surface

Since von Mises criterion is a circle in the deviatoric plane, see figure 2.6, the yield strength is the same considering tension or compression due to a constant radius. Moreover it is a cylinder along the hydrostatic axis in generalised stress space, so that it is independent of the mean stress. Thereby I_1 and $\cos 3\theta$ can be neglected in the von Mises criterion (2.25) and only depends on the second deviatoric stress invariant, J_2 . [Ottosen and Ristinmaa, 2005]

$$F = \sqrt{3J_2} - \sigma_{Y0} \tag{2.25}$$

The radius of the yield surface depends on the initial yield stress Y_0 , which is defined as $\sqrt{2/3}\sigma_{Y0}$.

2.6.2 Hardening laws

All three types of hardening have been implemented with the Von Mises yield criterion and are illustrated in appendix A in deviatoric and meridian planes with the following hardening laws. According to figure 2.5 similar yield surface response due to the different hardening laws are obtained.



Figure 2.6. Von Mises criterion in deviatoric plane.

Isotropic hardening

As illustrated in figure 2.6 the isotropic hardening develops with radius $\sqrt{2/3}(\sigma_{Y0} + \mathcal{H}(\varepsilon^p))$. Then the isotropic hardening law for von Mises is shown in 2.26. It is noted that $J_2(\varepsilon^p)$ refers to the second deviatoric invariant of the ε^p tensor. If not stipulated by parenthesis, e.g. J_2 , it refers to the stress invariant stated in (2.4).

$$f(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) = \sqrt{3J_2} - \boldsymbol{\sigma}_{Y0} - A J_2(\boldsymbol{\varepsilon}^p) = 0$$
(2.26)

According to figure 2.4 the hardening parameter A may be pictured as the slope of the plastic deformation with respect to stresses in a uniaxial plot. The radius of the current yield surface $\sqrt{\frac{2}{3}}\sigma_Y$ is controlled by a scalar A. The hardening modulus is set to 10×10^3 Mpa which is approximately 1/20 of Young's Modulus.

Kinematic hardening

The kinematic hardening parameter is controlled by the stress $\boldsymbol{\alpha}$ known as the back-stress or shiftstress. Similar to (2.17) the von Mises kinematic hardening law can be described by the relative deviatoric stress $\boldsymbol{\eta} = \boldsymbol{s}_{ij} - \boldsymbol{\alpha}_{ij}$.

$$f(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) = \sqrt{\frac{3}{2} (\boldsymbol{s}_{ij} - \boldsymbol{\alpha}_{ij}) (\boldsymbol{s}_{ij} - \boldsymbol{\alpha}_{ij})} - \boldsymbol{\sigma}_{Y0} = 0$$
(2.27)

where the deviatoric stress tensor s_{ij} is defined in (2.5) and the back-stress is given by

$$\dot{\boldsymbol{\alpha}} = \dot{\Lambda} \sqrt{\frac{2}{3}} A \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} = \frac{\sqrt{3J_2(\boldsymbol{\eta})} - \sigma_Y}{A + 3G} \sqrt{\frac{2}{3}} A \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}$$
(2.28)

Mixed hardening

As the hardening name refer to it is a mix of isotropic and kinematic hardening. According to (2.18) the von Mises mixed hardening can be expressed as

$$f(\boldsymbol{\sigma}, \boldsymbol{\mathcal{H}}) = F(\boldsymbol{\sigma}_{ij} - \boldsymbol{\alpha}_{ij}) - A J_2(\boldsymbol{\varepsilon}^p) = 0$$
(2.29)

As the elasto-plastic constitutive relations in (2.23) defines a non-linear relation between stress and strain increment. In order to use it in the FEM, an iterative procedure must be applied. Three different stress update schemes, two explicit and one implicit has been used. In the next chapter a review of the numerical methods are described.

NUMERICAL METHODS

This chapter summarises techniques when dealing with non-linearity in the finite element method where solutions to these particular systems are investigated and implemented with an arbitrary elasto-plastic model for illustration purposes.

Here, the material non-linearity is accounted, thus focus is made upon the update of stresses during analyses, two explicit methods have been used and they are described in the following sections. These two are similar but one stands out as an improved version for higher results accuracy. Another type of stress update method is called implicit, see section 3.5, one is used for the von Mises criterion and it is taken as an example to illustrate implicit methods outcomes.

3.1 Non-linear analyses

Two kind of equations need to be solved, Ottosen and Ristinmaa [2005] refers to it as global and local equations. The latter refers to stress updates in each gauss point of the model and is described later in this chapter, whereas global equations involve the system equilibrium.

Here non-linearity is due to the material constitutive behaviour, therefore the stiffness matrix changes along the analysis and is called the tangential stiffness matrix K_t .

$$\boldsymbol{K}_{t} = \int_{V} \boldsymbol{B}^{\mathsf{T}} \boldsymbol{D}^{*} \boldsymbol{B} \, \mathrm{d} V \tag{3.1}$$

where D^* is either the elastic or elasto-plastic constitutive matrix, respectively D or D^{ep} from (2.23), **B** is the strain interpolation matrix and V is the body volume.

Ottosen and Ristinmaa [2005] formulates a solution based on residual forces, see (3.2), so that equilibrium is reached when these are zero. Residual forces r are the difference between external forces p, e.g. body forces such as weight, and internal forces q, i.e. due to stresses.

$$\boldsymbol{r}(\boldsymbol{\sigma}) = \boldsymbol{q}(\boldsymbol{\sigma}) - \boldsymbol{p} \tag{3.2}$$

In order to reach an acceptable level of residual forces, i.e. a small value compared to the final forces in the system, an iterative loop must be added preventing static equilibrium errors, this iteration scheme is based on the Newton-Raphson method. The calculations are described in table 3.1.

The figure 3.1(*a*) illustrates the force in the system F with respect to the nodal displacement U. Hence, the slope of the curve is the tangential stiffness K_t . Thus, this figure represents a standard Forward Euler method. The iteration scheme illustrated by figure 3.1(*b*) for the global equilibrium is the one based on the Newton-Raphson method. It is noted that different procedure may be used regarding the stiffness matrix K_t .[Ottosen and Ristinmaa, 2005] One may calculate the stiffness matrix before starting the global iterations and keep it constant. This is illustrated by the slope (1) and (2) in figure 3.1. Another way is to update the stiffness matrix, leading to less iterations as shown by the slopes (1) and (3). However the stiffness matrix must be built in each equilibrium loop increasing the computing cost for a single iteration.



Figure 3.1. Drift resulting from the Forward Euler method and residual correction from Ottosen and Ristinmaa [2005].

Table 3.1. General algorithm with Equilibrium iterations and stress update.

\circlearrowright <u>for</u> $k = 1, 2,$	Displacement increments
$\boldsymbol{U}_k = \boldsymbol{U}_{k-1} + \Delta \boldsymbol{U}_k$	Nodal displacements added from boundary conditions
\boldsymbol{p}_k	Initialisation of body force (self-weight)
$\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_{k-1}$	Updated stress
$\circlearrowright \text{ for } j = 1, 2,$	Global equilibrium loop
$m{r}=m{p}_k-m{q}(m{\sigma}_k)$	Residual forces ($\mathbf{r} = 0$ for $j = 1$)
$\boldsymbol{K}_t(\boldsymbol{D}^*)$	Form global stiffness matrix, equation (3.1)
$oldsymbol{\delta} oldsymbol{U} = (oldsymbol{K}_t)^{-1} oldsymbol{r}$	Solving FEM equations
$\Delta oldsymbol{U}_{j+1} = \Delta oldsymbol{U}_j + \delta oldsymbol{U}$	Update new nodal displacements
$\Delta \boldsymbol{\varepsilon} = \boldsymbol{B} \Delta \boldsymbol{U}_{j+1}$	Calculate strains via strain interpolation matrix B
$\boldsymbol{\sigma}_k(\boldsymbol{\sigma}_{k-1},\Delta\boldsymbol{\varepsilon}), \boldsymbol{D}^*=\boldsymbol{D}^*(\boldsymbol{\sigma}_k)$	Stress update scheme and calculate D^* (Table 3.2)
$\operatorname{\underline{if}} \ r\ < ETOL \ \boldsymbol{p}_k\ $	
break	Stop the global iteration loop
Θ	
(a)	end of global iterations
$oldsymbol{U}_k = oldsymbol{U}_{k-1} + \Delta oldsymbol{U}_{j+1}$	Store final nodal displacements
$oldsymbol{arepsilon}_k = oldsymbol{arepsilon}_{k-1} + \Delta oldsymbol{arepsilon}$	Store final strains
0	end of displacement increments

3.2 FEM formulation and application

3.2.1 Plane strain problem

When two-dimensional calculations are performed, a common assumption is the plane strain problem. Soil is assumed to have zero strains out-of-plane which is very often appropriate, i.e. when the out-of-plane dimension is long compared to the current working plane. In this paper, 2D plane strain analyses are conducted.

3.2.2 Linear Strain Triangles (LST)

The LST elements are used in this project to mesh the model. These elements each possess six nodes, namely three corner nodes and three midside nodes, hence the shape functions are quadratic [Cook et al., 2001]. Each node has two degrees of freedom along the in-plane axes and each element is composed of three Gauss points. Figure 3.2.



Figure 3.2. LST elements.

3.2.3 Model setup

The procedure followed in order to check the computations validity is to focus at first on a single stress point with arbitrary strain increments. Secondly, when the results is coherent, a patch test is used. Then, finally a bigger test with a strip footing is set to investigate the model response of what could be a real geotechnical problem.

Patch test

A patch test is used in order to verify the algorithm stability in a small model. Besides, one single gauss points is followed through the computations to ensure the resulting stress is consistent. The patch test is made of 8 elements, with 50 degrees of freedom. The forced displacement is applied at the upper nodes of the mesh, i.e. nodes with y = 0 m such as in figure 3.3.

Regarding the boundary conditions, the upper nodes have a prescribed displacement along the *y*-axis, bottom nodes prevent displacement along the *y*-axis and the first bottom node on the left side of the patch restrict movement along the *x*-axis to avoid rigid body motion.



Figure 3.3. von Mises model: patch test setup.

Figure 3.4. Von Mises model: footing setup.

Footing test

In this test, a footing is submitted to a forced displacement, the prescribed nodes are the ones located below the footing on figure 3.4. The soil is initialised with body forces as a normal consolidated soil with Jaky's equation.

The model uses a principle of symmetry to reduce the computation cost. The final mesh is composed of 135 elements, each having 6 nodes and 3 gauss points, with 608 degrees of freedom. It is noted that the number of degree of freedom is low compared to the size of the model, a convergence analysis is conducted to control the results reliability, see figure B.2 in appendix B. The forced displacement is applied at the upper nodes of the mesh, where the footing is located. The mesh is also illustrated in appendix B.

Boundary conditions are illustrated in figure 3.4 where rolling supports are located along the boundaries of the model. The principle of symmetry is modelled by not allowing any displacement through it, hence rolling support are present.

It is noted that no interface (e.g. available in commercial software such as Plaxis) is made between the nodes representing the footing and the nodes representing the soil, therefore no slipping is allowed in the soil-structure interaction. Moreover the degrees of freedom under the footing are set to zero along the x-axis, simulating a rough footing.

The footing test enables a deeper analysis of the computations, with for instance stress or plastic strains distribution or to display the failure line.



Figure 3.5. Difference between stress update methods within one strain increment $\Delta \varepsilon$.

3.3 Stress update: Forward Euler method

3.3.1 Theory

The point here is to find a stress increment for which (3.3) is the solution, the constitutive matrix is not constant if elasto-plasticity is reached and leads to non-linearity. This section is related to the row highlighted in grey in table 3.1.

$$\Delta \boldsymbol{\sigma} = \int \boldsymbol{D} \Delta \boldsymbol{\varepsilon} \tag{3.3}$$

The Forward Euler method is based on the assumption that the behaviour of a future state will behave the same way as another known state close to the sought one. Figure 3.1 displays the method, where the tangent slope of a known point is taken as the curve on a specified range, by knowing the slope of each previous steps, a curve can be drawn, however a drift appears.

In terms of stress update, the method is better illustrated with Figure 3.5, showing the drift from the yield surface. Here the tangent slope is the constitutive matrix corresponding to the previous stress state. In order to perform accurately with the Forward Euler method, one may divide the incremental strain into smaller increments that can be called sub-step.

$$\delta \boldsymbol{\sigma} = \boldsymbol{D}^* \delta \boldsymbol{\varepsilon} \quad \text{with} \quad \delta \boldsymbol{\varepsilon} = \frac{\Delta \boldsymbol{\varepsilon}}{n_{ss}}$$
(3.4)

 $\Delta \boldsymbol{\varepsilon}$ | Strain increment, step

 $\delta \boldsymbol{\varepsilon}$ Strain increment, sub-step

 $\delta \sigma$ Stress increment, sub-step

 D^* Constitutive matrix at previous stress point ($D^* = D$ or D^{ep} depending on stress state)

n_{ss} Number of sub-steps

Considering a stress state near the boundary of the yield surface on the elastic side. The next incremental step is fully considered as an elastic behaviour, and the drift magnitude from the yield surface depends on this late increment size. It is obvious that a very small increment gives smaller error. By dividing the strain increment by a high number, the drift error is then reduced.

input	$(\boldsymbol{\sigma}_0,\Delta \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{D}^*, \boldsymbol{\sigma}_Y, \boldsymbol{lpha}_0, n_{ss})$	
steps	$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}_0 + \boldsymbol{D} \Delta \boldsymbol{\varepsilon}$ $\underbrace{\text{if } f \left(\boldsymbol{\sigma}^{\text{trial}}, \boldsymbol{\sigma}_y, \boldsymbol{\alpha}_0 \right) < -FTOL \ \underline{or} \ \mathcal{L} < -LTOL$ $\boldsymbol{D}^* = \boldsymbol{D}, \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}^{\text{trial}}$ $\underbrace{\text{else}}$	Compute trial stress Check if elastic or unloading Update + end of scheme else (elasto-plastic)
F	\circlearrowright for $i = 1, 2, n_{ss}$	
0	$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}_0 + \boldsymbol{D} \delta\boldsymbol{\varepsilon}$	
K	$\underline{\text{if }} f \left(\boldsymbol{\sigma}^{\text{trial}}, \boldsymbol{\sigma}_{Y}, \boldsymbol{\alpha}_{0} \right) < -FTOL$	Check if (elastic)
W	$D^* = D$	Elastic response
А	else	else (elasto-plastic)
R	$oldsymbol{D}^{*}=oldsymbol{D}^{ep}\left(oldsymbol{D},oldsymbol{\sigma}_{1},oldsymbol{lpha}_{1} ight)$	Elasto-plastic response
D E	Θ	
U	$\delta oldsymbol{\sigma} = oldsymbol{D}^* \delta oldsymbol{arepsilon}$	Substep increment
L	$\delta \pmb{lpha} =$	Substep increment
E	$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 + \delta \boldsymbol{\sigma}, \boldsymbol{\sigma}_Y(\boldsymbol{\sigma}_1)$	Update
R	$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_1 + \delta \boldsymbol{\alpha}$	Update
	۲	
	$\boldsymbol{\sigma}_0=\boldsymbol{\sigma}_1, \boldsymbol{lpha}_0=\boldsymbol{lpha}_1$	Initialise for next iteration
	Θ	
output	$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}^p, \boldsymbol{D}^*, \boldsymbol{\sigma}_y, \boldsymbol{lpha}_1)$	

<i>Table 3.2.</i>	Forward	Euler	stress	update	scheme.

The grey rows highlight the standard Forward Euler iteration procedure, however in order to save computation time when the strain increment generates a pure elastic stress point, an early "if" test is conducted. Thus, if the trial stress σ^{trial} is acceptable, a direct update is made.

3.3.2 Unloading

The scheme must react in a particular way if the material is unloaded. The unloading response of materials is elastic, the explicit Forward Euler does not stay on the yield surface and this gives inaccurate response to unloading since the stress increments will be considered elasto-plastic until reaching the upper yield surface. Hence, a loading parameter \mathcal{L} is introduced in (3.5) [Sloan et al., 2001].

$$\mathcal{L} = \cos \Theta = \frac{\boldsymbol{a}^{\mathsf{T}} \, \boldsymbol{\Delta} \boldsymbol{\sigma}}{\|\boldsymbol{a}\| \, \|\boldsymbol{\Delta} \boldsymbol{\sigma}\|} \tag{3.5}$$

The parameter \mathcal{L} gives the load increments direction, see figure 3.6. If its sign is negative, the angle between the surface normal and the elastic strain increment is bigger than 90 degrees. Thus, the incremental stress is directed towards the yield surface and unloading is detected even if f may be over zero due to a drift in the Forward Euler method, see table 3.4.1.



Figure 3.6. Illustration on the use of parameter \mathcal{L} .

3.3.3 Yield tolerance

In table 3.2 is represented the stress update scheme for an elasto-plastic material ruled by the von Mises criterion. The stress state is considered elastic if the value of the yield function is lower than -FTOL, i.e. the stress is inside the yield surface. Another line may be added to give an error if the stress state is superior to FTOL, i.e. the stress state is on the unacceptable side of the yield surface and too far from the surface tolerance, see section 3.4.3 with drift correctors and figure 3.7. If the number of sub-step is big, the computation can be heavy, therefore improvements were made.

3.4 Stress update: Modified Forward Euler

3.4.1 Efficiency improvement

Modifications can be made to improve the computational efficiency of the algorithms. Sloan et al. [2001] propose, in case the trial stress goes out of the yield surface, to calculate both the elastic and plastic ranges of the increment and to consider them separately in the iteration process. If the material is linear elastic, the contact stress at the yield surface can be calculated in one iteration and the hard computations are reserved for the plastic range. To do so, a Newton-Raphson method is described in Sloan et al. [2001] and Potts and Zdravkovic [1999], with both a linear and non-linear elastic constitutive behaviour.

If the constitutive matrix is constant, this Newton intersection method yields a result in the first iteration, a more complex function takes place for non-linear elastic behaviour in materials such as soils, and more iterations are needed. The iterative process stops when a value χ provide a yield function close to zero, a tolerance *PTOL* is established to do so, see table 3.4. This method is very efficient to avoid a long iterative process in the elastic range, which is interesting when many stress points of the model becomes plastic during the analysis, see appendix A and B.

A value of $\chi = 0$ means that the behaviour is entirely plastic while a value of $\chi = 1$ implies the behaviour is elastic over the strain increment.

input	$(\boldsymbol{\sigma}_0,\Delta \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{D}^*, \sigma_Y, \boldsymbol{lpha}_0, n_{ss})$	
steps	$oldsymbol{\sigma}^{ ext{trial}} = oldsymbol{\sigma}_0 + oldsymbol{D} \Delta oldsymbol{arepsilon}$	Compute trial stress
	$\underline{\operatorname{if}} f\left(\boldsymbol{\sigma}^{\operatorname{trial}}, \boldsymbol{\sigma}_{Y}, \boldsymbol{\alpha}_{0}\right) < -FTOL \underline{\operatorname{or}} \mathcal{L} < -LTOL$	Check if (elastic) or unloading
	$\boldsymbol{D}^{*}=\boldsymbol{D}, \boldsymbol{\sigma}_{1}=\boldsymbol{\sigma}^{ ext{trial}}$	Update + end of scheme
	else	else (elasto-plastic)
	$(\boldsymbol{\chi}) = NewtonIntersect(\boldsymbol{\sigma}_0, \boldsymbol{\alpha}_0, \Delta \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_y, \boldsymbol{D})$	
	$oldsymbol{\sigma}^{int} = oldsymbol{\sigma}_0 + oldsymbol{D} \left[\chi \; \Delta oldsymbol{arepsilon} ight]$	
	$\Delta oldsymbol{arepsilon}^p = [1-\chi] \Delta oldsymbol{arepsilon}$	strain increment out of surface
	$oldsymbol{\sigma}_0=oldsymbol{\sigma}^{int}, \deltaoldsymbol{arepsilon}=\Deltaoldsymbol{arepsilon}^p/n_{ss}$	Initialisation
	\circlearrowright for $i = 1, 2, n_{ss}$	
	$oldsymbol{D}^{*}=oldsymbol{D}^{ep}\left(oldsymbol{D},oldsymbol{\sigma}_{1},oldsymbol{lpha}_{1} ight)$	Elasto-plastic response
	$\delta oldsymbol{\sigma} = oldsymbol{D}^* \delta oldsymbol{arepsilon}$	Substep increment
	$\delta \alpha$	Substep increment, from (2.28)
	$oldsymbol{\sigma}_1 = oldsymbol{\sigma}_0 + \delta oldsymbol{\sigma}, \sigma_{\!Y}(oldsymbol{\sigma}_1)$	Update
	$oldsymbol{lpha}_1 = oldsymbol{lpha}_0 + \delta oldsymbol{lpha}$	Update
	$\boldsymbol{\sigma}_0=\boldsymbol{\sigma}_1, \boldsymbol{\alpha}_0=\boldsymbol{\alpha}_1$	Initialise for next iteration
	0	
	Θ	end of scheme
output	$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}^p, \boldsymbol{D}^*, \boldsymbol{\sigma}_Y, \boldsymbol{\alpha}_1)$	

Table 3.3. Stress update scheme with Newton intersection scheme for elasto-plastic models.

Table 3.4. Newton intersection scheme.

input	<i>NewtonIntersect</i> ($\boldsymbol{\sigma}_0, \boldsymbol{\alpha}_0, \Delta \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_Y, \boldsymbol{D}$)	
steps	$\chi_{0} = 0, \ \chi_{1} = 1$ $f_{0} = f(\boldsymbol{\sigma}_{0} + \boldsymbol{D} \ \chi_{0} \Delta \boldsymbol{\varepsilon}, \ \boldsymbol{\alpha}_{0}, \ \boldsymbol{\sigma}_{Y})$ $\circlearrowright \underline{\text{for}} \ i = 1, 2,$ $f_{i} = f(\boldsymbol{\sigma}_{0} + \boldsymbol{D} \ \chi_{i} \Delta \boldsymbol{\varepsilon}, \ \boldsymbol{\alpha}_{0}, \ \boldsymbol{\sigma}_{Y})$ $\chi_{i+1} = \chi_{i} - \frac{f_{i}}{f_{i}} [\chi_{i} - \chi_{i-1}]$	Initialise the bounds Maximum iteration number may be specified Yield function to iterate over Iterative process
	$\frac{ \mathbf{f} }{ \mathbf{f}_{i+1} } < PTOL$	Final value
	$\chi = \chi_{i+1}$ <u>break</u>	Iteration stopped
	⊖ ⊚	End of iterations
output	(X)	

3.4.2 Quality improvement

As stipulated above, the drift error is dependent on the sub-increment "length", an error control addition to the algorithm is detailed by Potts and Zdravkovic [1999] and Sloan et al. [2001] where a tolerance is set as the maximum stress difference between two future sub-increment iterations compared to the initial stress state. If this tolerance is small, the stress state may be assumed accurate enough.

input	$(SSTOL, n_{ss}, \boldsymbol{\sigma}_0, \Delta \boldsymbol{\varepsilon})$	
steps	$\circlearrowright \underline{\text{for } i = 1, 2,} \\ \delta \boldsymbol{\varepsilon} = \frac{\Delta \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}}, n_{ssold} = n_{ss}$	Maximum iteration number may be specified
	$\delta oldsymbol{\sigma}_1 = oldsymbol{\mathcal{D}}^{ep}(oldsymbol{\sigma}_0) \delta oldsymbol{arepsilon}$	First estimate
	$\delta \boldsymbol{\sigma}_2 = \boldsymbol{D}^{ep}(\boldsymbol{\sigma}_0 + \delta \boldsymbol{\sigma}_1) \ \delta \boldsymbol{\varepsilon}$	Second estimate
	$\delta oldsymbol{\sigma} = rac{1}{2} \left[\delta oldsymbol{\sigma}_1 + \delta oldsymbol{\sigma}_2 ight]$	
	$R = \frac{\ \frac{1}{2} [\delta \boldsymbol{\sigma}_1 - \delta \boldsymbol{\sigma}_2]\ }{\ \boldsymbol{\sigma}_0 + \delta \boldsymbol{\sigma}\ }$	Relative error
	$\underline{if} R > SSTOL$	$SSTOL = 10^{-3}$ to 10^{-8}
	$eta=0.8\sqrt{rac{SSTOL}{R}}$	
	$n_{ss}=n_{ssold}/eta$	New value of n_{ss} increases
	else	
	break	When $R < SSTOL$, function stops
	Θ	
	۲	
output	(n_{ss})	

It is possible to run these lines for each sub-increments since the relative stress difference may change where the constitutive matrix is not constant but the algorithm becomes heavier to compute.

3.4.3 Plastic correctors

For critical state models, a drift corrector can be added for each sub-increments because of the cumulative tolerated error at the yield surface. According to Sloan et al. [2001], even though the algorithm accuracy is increasing with the sub-increments number, a corrector gives more reliability in complex models such as the ones accounting for critical state soil mechanics. If the yield criterion is violated, correctors may be applied to the stress and hardening parameters. In (3.6), a Taylor series expansion of an arbitrary current stress point that crossed the yield surface is shown. The subscript index 0 is assigned to yet uncorrected values.

$$f = f_0 + \boldsymbol{a}_0^{\mathsf{T}} \delta \boldsymbol{\sigma}_{cor} + \frac{\partial f}{\partial \mathcal{H}} \delta \mathcal{H}_{cor} = f_0 + \left(\frac{\partial f}{\partial \boldsymbol{\sigma}^{\mathsf{T}}}\right)_0 \delta \boldsymbol{\sigma}_{cor} + \frac{\partial f}{\partial \mathcal{H}} \delta \mathcal{H}_{cor}$$
(3.6)

In order to perform an appropriate drift correction, the total strain increment must remain unchanged. The stress and hardening correctors are respectively calculated by (3.7) and (3.8)



Figure 3.7. Drift correction by plastic correctors.

following the theory of plasticity seen in Chapter 2.

$$\delta \boldsymbol{\sigma}_{cor} = -\dot{\Lambda} \boldsymbol{D} \boldsymbol{b}_0 \qquad \qquad \Rightarrow \quad \delta \boldsymbol{\sigma}_{cor} = \frac{-f_0 \boldsymbol{D} \boldsymbol{b}_0}{A_0 + \boldsymbol{a}_0^{\mathsf{T}} \boldsymbol{D} \boldsymbol{b}_0} \qquad (3.7)$$

$$\delta \mathcal{H}_{cor} = \dot{\Lambda} B_0 = -\dot{\Lambda} \frac{A_0}{df/d\mathcal{H}} \qquad \Rightarrow \quad \delta \mathcal{H}_{cor} = \frac{f_0 B_0}{A_0 + \boldsymbol{a}_0^{\mathsf{T}} \boldsymbol{D} \boldsymbol{b}_0} \tag{3.8}$$

And the final corrected stress and hardening parameter is calculated by (3.9) and (3.10).

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \delta \boldsymbol{\sigma}_{cor} \tag{3.9}$$

$$\mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H}_{cor} \tag{3.10}$$

These correctors can be used repeatedly until reaching an acceptable value for the yield criterion, e.g. the upper tolerance of the yield surface f = FTOL. Nevertheless, this method does not converge in some cases of non-associated plasticity by having a stress correction further than the initial one. Sloan et al. [2001] gives the example of the Mohr-Coulomb tip region. In that case, a correction normal to the yield surface is temporarily chosen. The change of stress is made according to (3.11) and the hardening parameter remains unchanged.

$$\delta \boldsymbol{\sigma}_{cor} = -\dot{\Lambda} \, \boldsymbol{a}_0 \quad \Rightarrow \quad \delta \boldsymbol{\sigma}_{cor} = \frac{-f_0 \, \boldsymbol{a}_0}{\boldsymbol{a}_0^{\mathsf{T}} \boldsymbol{a}_0} \tag{3.11}$$

3.5 Stress update: Implicit formulation

3.5.1 Principles

Implicit formulations are different from the explicit formulations. Indeed the trial stress may be corrected, if needed, to find another stress state on the yield surface which gives a realistic material behaviour. Return algorithms for many criteria exist, von Mises ([Krieg and Krieg, 1977]), Tresca and Mohr-Coulomb ([Clausen and Damkilde, 2006]), etc.. Its principle is illustrated in figure 3.5.

3.5.2 Radial return method

One of the characteristics of the von Mises criterion is that for the case of linear hardening, a closed-form formulae can be derived. [de Souza Neto et al., 2008] The linear hardening function (3.12) leads to the return mapping equation (3.14), where the plastic multiplier $\dot{\Lambda}$ can be analytically found.

$$\sigma_{Y}(\boldsymbol{\varepsilon}^{p}) = \sigma_{Y0} + A \, \boldsymbol{\varepsilon}_{\text{eff}}^{p} \tag{3.12}$$

where σ_{y0} is the initial yield stress and *A* is the hardening modulus, corresponding to the stress-strain slope in the elasto-plastic range, like the Young modulus *E* in the elastic range. The effective stress and plastic strain, respectively noted σ_{eff} and ε_{eff}^p , are defined by the von Mises criterion $\sqrt{3J_2}$ such as $\sigma_{eff} = \sqrt{3J_2(\sigma)}$.

$$\sigma_{\rm eff}^{\rm trial} - 3G\,\dot{\Lambda} - \left[\sigma_{\rm Y0} + \left(\varepsilon_{\rm eff}^{p(A)} + \dot{\Lambda}\right)A\right] = 0 \tag{3.13}$$

$$\sigma_{\text{eff}}^{\text{trial}} - 3G\,\dot{\Lambda} - \left[\sigma_Y^{(A)} + \dot{\Lambda}A\right] = 0 \tag{3.14}$$

leading to,

$$\dot{\Lambda} = \frac{\sigma_{\text{eff}}^{\text{trial}} - \sigma_Y^{(A)}}{3G + A} \tag{3.15}$$

The incrementation is done through the deviatoric stress:

$$\boldsymbol{s}^{(B)} = \left(1 - 3G\frac{\dot{\Lambda}}{\sigma_{\text{eff}}^{\text{trial}}}\right) \boldsymbol{s}^{\text{trial}}$$
(3.16)

This closed-form expression (3.16) does not produce any drift and no subincrements are needed. The patch and footing tests are available in appendix A and B.

3.6 Summary

Analyses of non-linear system demand an iterative solution process that is characterised by two sort of equations, i.e. global (the entire model) and local (one gauss point). It is seen in this chapter that the global equations can be solved by a Newton-Raphson method with residual forces presented in details by Ottosen and Ristinmaa [2005]. This method is used in this paper to check the system equilibrium during after each load increment.

The stress update or local equation accounts for the stress calculations along analyses. In plasticity, the constitutive matrix is not constant and therefore iterations are needed. This chapter presents a few methods to improve the well-known Forward Euler method. Explicit methods are easier to formulate but needs many iterations, nevertheless computers nowadays become faster every year, increasing the method's attractiveness. Besides, improvements regarding drifts from the yield surface or the control of subincrements in each gauss points were formulated and tested with the von Mises plasticity model. These improvements may be relevant in critical state models where many parameters are dependent on the stress state.

TWO-SURFACE CRITICAL STATE PLASTICITY MODEL

In this chapter the background theory and implementation of the two-surface critical state plasticity model is elaborated. In addition, the framework of the critical state soil mechanics theory is described to link the parameters used in the plasticity model.

4.1 Critical State Soil Mechanics

Critical State Soil Mechanics (CSSM) is a concept to understand how soil behaves when it is sheared. Studies have proved that the concepts of CSSM are suitable to describe the strength and volumetric behaviour of granular materials. [LeBlanc et al., 2008] Factors controlling the shear strength of granular soils are affected by

- Soil material Defined by the grain skeleton.
- *Initial state* Defined by initial void ratio and confining stress, which determines either the soil is loose or dense and contractive or dilative.
- *Loading type* Defined by the effective stress path and is depending on the condition of drained and undrained as well the type of loading, i.e. monotonic or cyclic loading.

For granular materials, Casagrande (1940) was first to introduce the concept of critical state. This state is reached when continuous shearing causes zero volumetric change and zero change in shear stress, see figure 4.1. [LeBlanc et al., 2008]



Figure 4.1. Behaviour of dense sand in triaxial compression test.

As illustrated in figure 4.1, the q- ε graph shows that only in the case of a dense sand, a peak strength is reach before the critical state. Secondly, the characteristic state is the point where the soil transits from contractive to dilative behaviour. In the graph ε_{v} - ε the dense sand contracts

before it start to dilate, which mathematically can be expressed as the horizontal tangent, defined as $\frac{\delta \varepsilon_v}{\delta \varepsilon_1} = 0$. From a triaxial point of view it means that the characteristic state is measured where $\delta \varepsilon_v = 0$ for the first time.

4.1.1 Critical State Line (CSL)

Several plots are useful to capture the soil state, those include the parameters (p',q,e). The stress components p' and q, displayed in (4.3), and the void ratio e, defined by the ratio between voids and the volume of solid, define the line called critical state line CSL. It forms an envelope representing the failure state and separates dilative and contractive soils, see figure 4.2 and 4.3. The failure envelope is a function of the stress state which is depending on the type of loading. The deviatoric stress q may be changed to τ , i.e. shear stress or p' changed to σ'_V , i.e. vertical effective stress. Secondly, the p'-e graph shows the volumetric changes of the soil due to effective stresses, see figure 4.2 and 4.3. The void ratio e is related to the relative density which characterises if the soil is loose or dense.



Figure 4.2. Undrained stress path of loose soil with the CSL.

Note that there is no change in void ratio in the above figure because it is undrained condition. This means that the volume is constant and it affect a build up in positive excess pore pressure. In drained conditions, the above initial stress state would be a straight line pointing downwards to
the CSL if a shear test were to be performed, see figure 4.3. In this case the void ratio, e changes due to consolidation and the volume change is contractive.



Figure 4.3. Drained stress path of dense soil with the CSL.

4.2 Model parameters

Significant soil behaviours from the CSSM theory as shear peak strength, dilative/contractive behaviour and the linear relation of p' and q are all parameters that are implemented in *The critical state two-surface plasticity model*. In the following CSSM is linked to the model parameters used in the soil model.

4.2.1 State parameter

The state parameter Ψ is the vertical distance from initial void ratio e_0 to critical state void ratio e_c , see figure 4.4. When the state of a sand is above the CSL, corresponding to a positive Ψ , the sand has a tendency to contract upon shearing. On the other hand if the state point is located below the CSL, corresponding to a negative Ψ , the sand tends to dilate. [Phan, 2015]



Effective mean normal stress (logarithmic scale)

Figure 4.4. Definition of the state parameter Ψ

The state parameter presented in figure 4.4 is determined as

$$\Psi = e_0 - e_c \tag{4.1}$$

The state parameter takes into account changes in mean effective stress p' and void ratio in a single variable. At the critical state of e_c the volume shears without further change in the effective stresses at the current p'. [Phan, 2015]

4.2.2 Critical state parameters

By combining the linear relation of the CSL in $e -\ln p'$ and p' - q space, as shown in figure 4.2 and figure 4.4, it is possible two assume two essential parameters given as

$$e_c = \Gamma - \lambda \ln\left(\frac{p'}{p_{atm}}\right) \tag{4.2}$$

$$M_c = q/p'$$
 with $q = \sqrt{J_2(\boldsymbol{\sigma})}$ and $p' = \frac{-I_1}{3}$ (4.3)

Since the CSL is independent of both the relative density I_D and the mean effective stress, p', in the *p*-*q* space the CSL inclination is determined by the critical stress ratio, M_c .

According to figure 4.4 the variation of the critical void ratio e_c and the effective mean stress p' are defined as in (4.2). Γ is the void ratio corresponding to a unit pressure of 1 kPa on the CSL and λ is the slope. [Phan, 2015]

4.2.3 Bounding line

The peak strength is associated with the maximum rate of dilation defined as $d\varepsilon_v/d\varepsilon_a$ with ε_v as the volumetric strain and, ε_a , is the axial strain. Correlation between the dilation void ratio *e* defines the magnitude of peak shear strength as shown in figure 4.5. [LeBlanc et al., 2008]



Figure 4.5. Variation of peak shear strength due to density of sand. [LeBlanc et al., 2008]

A dense sand exhibit a strong dilation because the packing density is high and thereby the void space between the grains are small. This means that the internal friction angle is higher for a dense sand compared with a loose sand and thus a larger shear strength is obtained for a dense sand. The peak shear strength can be illustrated as a threshold in the q-p' space and is referred as the bounding line, see figure 4.5. [LeBlanc et al., 2008]



Figure 4.6. Definition of bounding line and critical stress ratio line.

Since the bounding line is related to the dilative behaviour of the soil it is convenient to implement the state parameter Ψ . Secondly the critical stress ratio given in equation 4.3 is a convenient parameter to implement since it defines the stress ratio of q and p'. The bounding line

is defined as

$$M_b(\Psi) = M_c - k_b \langle \Psi \rangle \tag{4.4}$$

where k_b is a model parameter and together with M_c both parameters are assumed constant. The Macauley brackets $\langle \rangle$ sets all negative values to zero and provides $\langle x \rangle = x$ otherwise. [LeBlanc et al., 2008]

With the formulation of the bounding line given as in (4.4) it ensures that for loose sands the bounding line will coincide with the critical state line. Furthermore it ensures the curvature of the bounding line for dense sands due to the dependency of the state parameter, Ψ . It is due to the fact that the state parameter is depending on the critical state void ratio, e_c given in equation 4.2, which shows the dependency of the mean effective stress, p'. [LeBlanc et al., 2008]

4.2.4 Characteristic line

The characteristic line is defined from the characteristic state defined in figure 4.1. It describes the transition from compressive to dilative behaviour where the volumetric strain $\delta \varepsilon_v = 0$.

In undrained conditions the characteristic state cannot be measured but the pore water behaviour can. The state where maximum excess pore pressure is developed is called the Phase Transformation State. The phase transformation state plays a similar role for undrained tests as the characteristic state for drained tests.

As the critical stress ratio M_c and peak shear stress ratio M_b , the characteristic stress ratio M_d is modelled as a straight line in the p'-q space, thus the characteristic line is defined by: [LeBlanc et al., 2008]

$$M_d(\Psi) = M_c + k_d \Psi \tag{4.5}$$

The definition of the characteristic line is similar to the bounding line. The only difference is the model parameter k_c and an opposite sign convention.

4.2.5 Summary

The irrecoverable deformation can be linked approximately with the change in stress ratio M = q/p'. The change in stress ratio is related to the development of volumetric strains and this is defined as the volumetric hardening relationship. With this relationship it is proved that this stress ratio, such as peak stress, matches the one illustrated in figure 4.1. [Wood, 1994]

Besides, when volumetric strain develop, changes of the state parameter and volume affect the bounding stress ratio M_b . This means if the initial state parameter $\Psi < 0$ then the initial bounding stress ratio would be larger than the critical stress ratio $M_b > M_c$. It means the soil is heading for a peak stress ratio higher than the critical state value M_c . When the stress ratio exceeds M_d the volumetric expansion occur, i.e. $\delta \varepsilon_v < 0$ and the peak strength decrease until critical state. [Wood, 1994]

On the other hand if the initial state parameter $\Psi > 0$ then $M_b = M_c$ and $M_b < M_d$, the soil contracts until it reaches critical state. For small positive values of the state parameter which means at low pressures, a loose sand would act like a dense sand which leads to a peak in the stress-strain relationship similar to the one in figure 4.1 just with a smaller peak. [Wood, 1994]

In figure 4.7 the effects of different initial state parameters Ψ_0 is displayed where the volume change response $\delta \varepsilon_v$ and the variation of M_b are illustrated and is based on triaxial tests from Wood [1994].



Figure 4.7. Influence of initial state parameter Ψ_0 . a) Stress ratio M, b) Volumetric strains ε_v , c) Bounding stress ratio M_b .

4.3 Elasto-plastic framework

This following section deals with the model derivation and explains the steps followed in the algorithms. All tensors are here considered as column vector such as the strain and stress tensors (1.1) and (1.2) in chapter 2. The Euclidean norm of a tensor is represented by $||\mathbf{x}||$ and the inner and outer tensor product are written as $\mathbf{x}^{\mathsf{T}}\mathbf{y}$ and $\mathbf{x}\mathbf{y}^{\mathsf{T}}$ respectively.

4.3.1 Elastic formulation

The elastic behaviour is non-linear, with a dependency to the mean effective pressure within the soil. This behaviour is seen when test are performed in granular materials. As it is shown in (4.6), the constitutive matrix D can be derived using the bulk and shear moduli, respectively K and G. Hence, these two must be defined with respect to the mean effective stress.

$$\begin{cases} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{yz} \\ \sigma_{yz} \\ \end{cases} = \begin{bmatrix} K + \frac{4}{3}G & K - \frac{2}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K + \frac{4}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K + \frac{4}{3}G & 0 & 0 & 0 \\ sym. & G & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & & G & 0 \\ & & & & & & G \\ & & & & & & & G \\ \end{bmatrix} \begin{cases} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{yz} \\ \end{array} \}$$
(4.6)

Manzari and Prachathananukit [2000] give an hypo-elastic assumption in (4.7) to calculate the current moduli from one model parameter *b* (from 0.435 at very small strains to 0.765 at very large strains [LeBlanc et al., 2008]), the atmospheric pressure p_{atm} , and the reference bulk and shear moduli, noted as K_0 and G_0 . The bulk modulus is then dependent on the mean effective pressure p' as well as the shear modulus.

$$K = K_0 \left(\frac{p'}{p_{\text{atm}}}\right)^b \qquad \qquad G = G_0 \left(\frac{p'}{p_{\text{atm}}}\right)^b \qquad (4.7)$$

The moduli will increase with the mean effective pressure, which is consistent for sandy soils. [Manzari and Prachathananukit, 2000] These are integrated into D and form a so-called hypoelastic constitutive matrix. During the analyses the mean effective stress is calculated from the last known stress tensor.

4.3.2 Yield surface

The plastic range is defined by the yield function (4.8). Clearly, the pressure p' gives the strength to the soil, *m* is a pressure dependent yield stress related to the yield surface radius. The relative stress *r* takes into account the pressure, which is different from the kinematic hardening in the von Mises model, where the pressure has no impact.

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, m) = \|\boldsymbol{r}\| - \sqrt{\frac{2}{3}}mp' \quad \text{with} \qquad \boldsymbol{r} = \boldsymbol{s} - p'\boldsymbol{\alpha}$$
 (4.8)

- $\boldsymbol{\alpha} \mid \text{Back stress}$
- *m* Yield surface radius
- *r* Relative stress
- **s** Deviatoric stress

The stress tensor n is also defined according to (4.9) and represent the unit vector of r pointing out of the yield surface on the deviatoric plane.

$$\boldsymbol{n} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \tag{4.9}$$

It is noted that this formulation highlights deviatoric stress components, i.e. $\mathbf{r} = \mathbf{r}(\mathbf{s})$ and m and these are related to mean pressure. The deviatoric part may be compared to the J_2 invariant from the von Mises criterion. For illustrations of the different components in generalized stress space, see figure 4.8.

4.3.3 Flow rule

Following the yield function in (4.8), the derivative of yield and potential flow functions are given in (4.10).

$$\boldsymbol{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \boldsymbol{n} - \frac{1}{3}N\boldsymbol{I} \qquad \qquad \boldsymbol{b} = \frac{\partial g}{\partial \boldsymbol{\sigma}} = \boldsymbol{n} + \frac{1}{3}D\boldsymbol{I} \qquad (4.10)$$

where **I** is the identity tensor, D is the dilatancy parameter, and $N = \boldsymbol{\alpha}^{\mathsf{T}} \boldsymbol{n} + \frac{2}{3}m$

This gives the flow rule for the model in (4.11). If the flow rule were to be associative, D could be set to -N, however that is not relevant to represent soil plastic response.

$$\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\Lambda} \left(\boldsymbol{n} + \frac{1}{3} \boldsymbol{D} \boldsymbol{I} \right) \tag{4.11}$$



Figure 4.8. Yield surface in stress space.

4.3.4 Plastic modulus

The plastic modulus is calculated from the consistency relation in plasticity theory, see (2.24) chapter in 2.

$$A = -\left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\frac{\dot{\boldsymbol{\alpha}}}{\dot{\boldsymbol{\lambda}}} + \frac{\partial f}{\partial m}\frac{\dot{\boldsymbol{m}}}{\dot{\boldsymbol{\lambda}}}\right) = -\left(\frac{\partial f}{\partial \boldsymbol{\alpha}}\tilde{\boldsymbol{\alpha}} + \frac{\partial f}{\partial m}\tilde{\boldsymbol{m}}\right)$$
(4.12)

The plastic modulus for this model is defined in (4.13). Parameters \dot{m} and $\dot{\alpha}$ are defined later in the chapter.

$$A = p\left(\boldsymbol{n}^{\mathsf{T}}\tilde{\boldsymbol{\alpha}} + \sqrt{\frac{2}{3}}\tilde{\boldsymbol{m}}\right) \tag{4.13}$$

4.3.5 Critical state surfaces

This soil model outlines two surfaces that permit the particularity of critical states to be captured. The two surfaces are the dilatancy surface [Manzari and Prachathananukit, 2000], also called the characteristic surface [LeBlanc et al., 2008] or phase transformation surface in undrained conditions, and the bounding surface.

In order to represent surfaces that are dependent on the third deviatoric stress invariant, i.e. the Lode angle, Krenk formula (4.14) is used. $g(c, \theta)$ is a normalized function for which $g(1, \theta)$ is a circle such as in Von Mises. $g(0.7, \theta)$ is a triangular shape in which the Mohr-Coulomb yield surface may be circumscribed. The value of *c* is different for the bounding and dilatancy surfaces,

respectively c_b and c_d , and depends on the soil parameters as well as the state parameter Ψ , see figure 4.9. The values of M_i are calculated with (4.4) and (4.5).

$$g(c_i, \theta) = \frac{\cos(\gamma)}{\cos\left(\frac{1}{3}\arccos(\cos(3\gamma)\cos(3\theta))\right)}$$
(4.14)

$$\gamma = \frac{\pi}{3} + \arctan\left(\frac{1-2c_i}{\sqrt{3}}\right) \quad \text{and} \quad c_i(\Psi) = \frac{M_i^{ext}}{M_i} \quad \text{with} \quad i = b, c, d \quad (4.15)$$

Implementing the dilatancy surface is built from the critical state surface. To derive the soil behaviour, the stress state location with respect to the dilative surface must be known. On the other hand, the bounding surface defines the possible stress for the soil and approaching the bounding surface leads to reaching critical state, see section 4.2.3.

These two surfaces are not static, the state parameter Ψ , which can be think of a fourth dimension in generalized stress space, transforms their shape. Figure 4.9 outlines these changes by showing the surfaces dimensions in compression and extension in deviatoric plane with respect to the state parameter.



Figure 4.9. Surfaces in deviatoric plane.

The use of image points allow the stress state to be localised and hence the soil behaviour to be derived. LeBlanc et al. [2008] gives a formula that expresses the different back stress (or image vectors). They are calculated in the same way as the back stress in the yield function, noted α but they are directed toward the current stress deviatoric direction n. The intersection between an image vector and its corresponding surface is called an image point, see figure 4.10.

$$\boldsymbol{\alpha}_{i} = \sqrt{\frac{2}{3}} \left(M_{i}(\Psi) g(c_{i}, \boldsymbol{\theta}_{n}) - m \right) \boldsymbol{n} \qquad \text{with} \quad i = b, c, d$$
(4.16)

In addition, a link is made between these image vectors $\boldsymbol{\alpha}_i$ and the current location of the yield surface $\boldsymbol{\alpha}$ which leads to the image points. The distances $\boldsymbol{\beta}_i$, calculated in (4.17), play an important role in the soil response, e.g. dilation or kinematic hardening.

$$\boldsymbol{\beta}_i = \boldsymbol{\alpha}_i - \boldsymbol{\alpha} \tag{4.17}$$



Figure 4.10. Image points in deviatoric plane.

4.3.6 Hardening laws

Chapter 2 highlights kinematic and isotropic hardening laws and this particular model may use these two types. The rate of isotropic hardening, i.e. the yield stress, is a scalar which evolves with respect to one particular parameter being D. C_m is a positive model parameter, and e_0 is the initial void ratio.

$$\dot{m} = \tilde{m}\dot{\Lambda} = \left[C_m\left(1+e_0\right)D\right]\dot{\Lambda} \tag{4.18}$$

However, it is noted that isotropic hardening can be neglected for sandy soils [LeBlanc et al., 2008], as it almost preserves a constant radius *m*. In order to simplify this model, the isotropic hardening is neglected in this paper, i.e. $C_m = 0$.

The kinematic hardening may be compared to the one in chapter 2 for the Von Mises model. The tensor α is calculated through the deviatoric component β_b , which is illustrated in π -plane in figure 4.10.

$$\dot{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}}\dot{\boldsymbol{\Lambda}} = \left[C_{\alpha}\left(\frac{|\boldsymbol{\beta}_{b}^{\mathsf{T}}\boldsymbol{n}|}{b_{r} - |\boldsymbol{\beta}_{b}^{\mathsf{T}}\boldsymbol{n}|}\right)\boldsymbol{\beta}_{b}\right]\dot{\boldsymbol{\Lambda}}$$
(4.19)

 C_{α} is a positive model parameter and the inner tensor product norm $|\boldsymbol{\beta}_{b}^{\mathsf{T}} \boldsymbol{n}|$ gives the magnitude to the yield surface direction $\boldsymbol{\beta}_{b}$. It is noted that when the yield surface approaches the bounding surface, the tensor $\boldsymbol{\beta}_{b}$ tend to zero, hence the same for $\dot{\boldsymbol{\alpha}}$.

4.3.7 Volumetric behaviour

The dilatancy parameter *D* gives the volumetric reaction of the soil, A_0 and A_z are defined as positive model parameters, therefore dilation is only governed by the sign of $\boldsymbol{\beta}_d^{\mathsf{T}} \boldsymbol{n}$

$$D = (A_0 + A_z)(\boldsymbol{\beta}_d^{\mathsf{T}} \boldsymbol{n}) \qquad \text{with} \quad A_z = \boldsymbol{z}^{\mathsf{T}} \boldsymbol{n}$$
(4.20)

The model needs to account for cyclic loading response of sandy soils and therefore stress history must be accounted while the analysis is running. This is reached by implementing a "fabric-dilatancy tensor update" [Dafalias and Manzari, 2004] that considers the contractive tendency of sand at load reversal. In this paper, the name fabric tensor is used and noted as z.

$$\dot{\boldsymbol{z}} = \tilde{\boldsymbol{z}} \dot{\boldsymbol{\Lambda}} = \left[-C_z \left(A_z^{max} \boldsymbol{n} + \boldsymbol{z} \right) \langle -D \rangle \right] \dot{\boldsymbol{\Lambda}}$$
(4.21)

4.3.8 Recap of model parameters

Label	Multiaxial formulation	Constants	
Critical State Line (4.2)	$e_c = \Gamma - \lambda \ln \left(rac{p'}{p_{atm}} ight)$	Γ, λ, p_{atm}	
Bulk modulus (4.7)	$K = K_0 \left(\frac{p'}{p_{\rm atm}}\right)^b$	K_0, b, p_{atm}	
Shear modulus(4.7)	$G=G_0\left(rac{p'}{p_{ m atm}} ight)^b$	G_0, b, p_{atm}	
Yield surface (4.8)	$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, m) = \ \boldsymbol{r}\ - \sqrt{\frac{2}{3}}mp'$	т	
Back stress rate (4.19)	$\dot{\boldsymbol{\alpha}} = \left[C_{\boldsymbol{\alpha}} \left(\frac{ \boldsymbol{\beta}_{b}^{T} \boldsymbol{n} }{b_{r} - \boldsymbol{\beta}_{b}^{T} \boldsymbol{n} } \right) \boldsymbol{\beta}_{b} \right] \dot{\boldsymbol{\Lambda}}$	C_a	
Fabric tensor rate (4.21)	$\dot{\boldsymbol{z}} = \tilde{\boldsymbol{z}} \dot{\boldsymbol{\Lambda}} = \left[-C_z \left(A_z^{max} \boldsymbol{n} + \boldsymbol{z} \right) \langle -D \rangle \right] \dot{\boldsymbol{\Lambda}}$	C_{α}, A_{z}^{max}	
Plastic strain increment (4.11)	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\Lambda} \left(\boldsymbol{n} + \frac{1}{3} D \boldsymbol{I} \right)$		
Dilatancy parameter (4.20)	$D = (A_0 + A_z)(\boldsymbol{\beta}_d^{T} \boldsymbol{n})$	A_0, A_z	
Plastic modulus (4.13)	$A = p\left(oldsymbol{n}^{\intercal} ilde{oldsymbol{lpha}} + \sqrt{rac{2}{3}} ilde{m} ight)$		

Table 4.1. Recap of model formulations.

Table 4.2. Recap of model parameters.			
Parameter	Variable	Value	
Reference bulk modulus	K_0	31.4 MPa	
Reference shear modulus	G_0	31.4 MPa	
Elastic mean stress dependency	b	0.5	
Critical void ratio at $p = 1$ kPa	Г	0.9	
Critical stress ratio (dependent on friction angle φ)	M _{cr}	$\frac{6\sin(\varphi)}{3-\sin(\varphi)}$	
Initial yield stress	m_0	0.05	
Bounding surface slope parameter	$k_b \ k_b^{ext}$	4 2	
Dilatancy surface slope parameter	$k_d \ k_d^{ext}$	4.2 0.07	
Initial dilation multiplier	A_0	2.64	
Fabric tensor volumetric constant	A_z^{max}	100	
Loading slope $(e-ln(p') \text{ plot})$	λ	0.025	
Fabric tensor evolution constant	C_z	1000	
Isotropic hardening constant	C_m	0	
Kinematic hardening constant	C_{α}	1200	

Table 4.2. Recap of model parameters.

4.4 Response of the soil plasticity model

Calculations in drained and undrained conditions are performed. The assumptions used in undrained conditions are explained later in this chapter. In order to examine whether the model gives good response, several tests are conducted. Monotonic loading and cyclic loading are presented with the boundary values displayed in figure 4.11.

The initial void ratio is taken as $e_0 = 0.8$ which is consistent with previous tests realised with this model, e.g. Dafalias and Manzari [2004] or LeBlanc et al. [2008]. However this particular void ratio does not model contractive soil except at very high confining pressure, thus, another void ratio is chosen that models contractive soil. This is taken as $e_0 = 0.88$ and corresponds to a maximum void ratio for very fine sands according to the Swiss Standard [1999].

Regarding the numerical implementation, drift correctors are used to ensure the stress state does not leave far from the yield surface, and therefore giving more stability during the iterative process, see section 3.4.3. The Newton-Raphson intersection scheme table 3.4 is not used due to the small yield surface in this model and the absence of isotropic hardening that leads to a low number of elastic iterations. Furthermore the error control scheme was not used in this model since it is implemented with the Newton-Raphson intersection scheme.

4.4.1 Stress update overview.

input	$(\boldsymbol{\sigma}_0, \boldsymbol{\alpha}, \boldsymbol{\varepsilon}_{v}, \Delta \boldsymbol{\varepsilon}, \boldsymbol{z}, \boldsymbol{D})$	
steps	$\boldsymbol{\sigma}_{0} = -\boldsymbol{\sigma}_{0} ; \Delta \boldsymbol{\varepsilon} = -\Delta \boldsymbol{\varepsilon} ; \Delta \boldsymbol{\varepsilon}_{v} = \operatorname{tr}(\Delta \boldsymbol{\varepsilon}_{v})$ $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{0}$	Mech. \rightarrow Geotech. sign convention Initialisation
	$ \Psi = e - e_c \text{ with } e = e_0 - \varepsilon_v (1 + e_0) $ $ p = \text{tr}(\boldsymbol{\sigma})/3 $ $ \mathbf{D} - \mathbf{D}(K(p') \ G(p')) $	State parameter Initial pressure Hypoelastic constitutive matrix
	$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma} + \boldsymbol{D} \boldsymbol{\delta} \boldsymbol{\varepsilon}$ ($\tilde{\boldsymbol{\alpha}}, \tilde{m}, \tilde{\boldsymbol{z}}, D$) = TraceSurfaces($\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{z}, \Psi$)	assumed that $\tilde{m} = 0$
	$egin{array}{l} m{a} ; m{b} \ egin{array}{l} m{if} f m{(\sigma^{ ext{trial}}, m{lpha})} < -FTOL \ m{or} < -LTOL \ m{D}^* = m{D} \end{array}$	Check if elastic or unloading Elastic constitutive matrix
	$\begin{split} \hat{\Lambda} &= 0 \\ \underline{\text{else}} \\ (\boldsymbol{\sigma}, \boldsymbol{\alpha}, f) &= \boldsymbol{DriftCorrect}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}, D, FTOL, \Psi) \end{split}$	no plastic multiplier
	$A = A(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$ $\boldsymbol{D}^* = \boldsymbol{D}^{ep}(\boldsymbol{D}, \boldsymbol{a}, \boldsymbol{b}, A)$ $\dot{\Lambda} = \dot{\Lambda}(f, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{D}, A)$	Plastic modulus Elasto-plastic constitutive matrix Plastic multiplier
	$ \begin{split} & \Theta \\ \boldsymbol{\sigma} = \boldsymbol{\sigma} + \boldsymbol{D}^* \delta \varepsilon \\ & \varepsilon_v = \varepsilon_v + i \frac{\Delta \varepsilon_v}{\varepsilon_v} \end{split} $	Update stress Update volumetric strain
	$p' = \operatorname{tr}(\boldsymbol{\sigma}_1)/3$ $\boldsymbol{z} = \boldsymbol{z} + \tilde{\boldsymbol{z}}\dot{\boldsymbol{\lambda}}$ $\boldsymbol{\alpha} = \boldsymbol{\alpha} + \tilde{\boldsymbol{\alpha}}\dot{\boldsymbol{\lambda}}$	Update pressure Update fabric tensor Update current back stress
	$oldsymbol{\circ}$ $oldsymbol{\sigma}_0 = -oldsymbol{\sigma}_0$	Geotech. \rightarrow Mech. sign convention
output	$(\boldsymbol{\sigma}_0, \boldsymbol{lpha}, \boldsymbol{z}, \boldsymbol{D})$	

Table 4.3. Forward Euler stress update.

Table 4.4. Image points relative location with TraceSurfaces scheme.

input	$TraceSurfaces(\sigma, \alpha, m, z, \Psi)$	Locate image points
steps	$M_{b} = M_{c} + k_{b}^{c} \langle -\Psi \rangle \text{ and } M_{b} = M_{c} + k_{d}^{c} \Psi$ $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, p') \text{ with } p' = \text{tr}(\boldsymbol{\sigma})/3$ $g(c_{b}, \theta_{n}), g(c_{d}, \theta_{n})$ $\boldsymbol{\alpha}_{b}, \boldsymbol{\alpha}_{d}$ $\boldsymbol{\beta}_{c} = \boldsymbol{\beta}_{c}$	Max. surface radius in π -plane (4.8) - (4.9) Krenk formula from (4.14) Image vectors from (4.16) from (4.17)
output	$ \begin{aligned} \boldsymbol{\rho}_{b}, \boldsymbol{\rho}_{d} \\ D &= (A_{0} + A_{z}) \left(\boldsymbol{\beta}_{d}^{T} \boldsymbol{n} \right) \text{ with } A_{z} = \boldsymbol{z}^{T} \boldsymbol{n} \\ \tilde{\boldsymbol{\alpha}}(C_{\alpha}, \boldsymbol{\beta}_{b}, \boldsymbol{\alpha}_{b}, \boldsymbol{n}) \tilde{\boldsymbol{z}}(C_{z}, \boldsymbol{n}, \boldsymbol{z}, D) \tilde{\boldsymbol{m}}(C_{m} = 0, D) \\ (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{m}}, \tilde{\boldsymbol{z}}) \end{aligned} $	Get hardening rate parameter

4.4.2 Patch test: Boundary conditions

The difference with the von Mises model setup in figure 3.3 is that the model requires an initial pressure, and that is explicitly stated by the yield function (4.8) dependency on p'. In figure 4.11, F_0 is the force representing the confining pressure p_0 . It is noted that prescribed displacement prevents the nodal force to be in the finite element equation, and gives instead a reaction force for that particular degree of freedom.



Figure 4.11. Boundary conditions of Patch tests for soil model (Axial and shear load cases).

The shear analysis provides kinematic boundary conditions for every single degree of freedom of the model. Although not visible on the figure 4.11, zero displacement are set along the *y*-axis and in each node. The forced displacement represented in blue arrows are different whether monotonic or cyclic loads are generated.

Table 4.5. Recap of model setting			
	Forced displacement	Void ratio	
Monotonic	shear: $u_x \approx 0.4 \mathrm{m}$ axial: $u_y \approx -0.2 \mathrm{m}$	0.8 (Dilative soil)0.88 (Contractive soil)	
Cyclic drained (mean \pm 50%)	shear: $\bar{u}_x \approx 0.2 \text{ m}$ axial: $\bar{u}_y \approx -0.05 \text{ m}$	0.8 (Dilative soil)0.88 (Contractive soil)	
Cyclic undrained (mean \pm 75%)	axial: $\bar{u}_y \approx -0.005 \mathrm{m}$	0.8 (Dilative soil)0.88 (Contractive soil)	

4.4.3 Patch test: Drained conditions

This section relates the observed behaviour of the model with respect to the theory of critical state soil mechanics. For an larger overview of results and plots, the appendix C shows a more complete set of plots. The black squares define the starting point of each tests.



Figure 4.12. $\varepsilon_V - \varepsilon_y$ diagram - axial test - contractive soil - monotonic loading.



Figure 4.13. $\varepsilon_V - \varepsilon_y$ diagram - axial test - dilative soil - monotonic loading.

Figures 4.12 and 4.13 shows the volumetric response under monotonic loading of the soil. Here it is obvious that the dilative and contractive characteristics satisfy critical state theory. Moreover, these soils have the same void ratio and hence, the consequences of a different confining pressure may be noticed.

A link can be made with figures 4.14 and 4.15, where the distance to reach critical void ratio differs so that contractive soil at $p_0 = 2000$ kPa contracts more than the others since e_c is low.



Figure 4.14. e-*p*[′] diagram - axial test - contractive soil - monotonic loading.



Figure 4.15. e-*p*[′] diagram - axial test - dilative soil - monotonic loading.



Figure 4.16. Ψ - ε_y diagram - axial test - contractive soil - monotonic loading.



Figure 4.17. Ψ - ε_y diagram - axial test - dilative soil - monotonic loading.

On figures 4.16 and 4.17, different behaviour occur regarding the convergence to critical state. The dilative soil reaches the critical line but the contractive soil does not, although the state parameter Ψ converges to zero. Hence, more shearing is needed for contractive soil to reach critical state. A few observations during calculations showed that low confining pressure does increase the computation needs. For instance, the dilative soil at $p_0 = 250$ kPa and $p_0 = 500$ kPa on figure 4.17 shows slight instabilities. These were overtaken by having more subincrements in each load steps.

300

250

200

[ed x] 1500



 $\begin{array}{c} 1000 \\ 500 \\ 0 \\ 0 \\ 500 \\ 0 \\ 500 \\ 1000 \\ 1500 \\ p'[kPa] \end{array} \begin{array}{c} -P_0 = 250 \ kPa \\ -P_0 = 1000 \ kPa \\ +P_0 = 2000 \ kPa \\ +P_0 = 200 \ kPa \\ +P_0 = 200$

Figure 4.18. τ_{xy} -p' diagram - shear test - dilative soil - monotonic loading.



Figure 4.20. q- ε_y diagram - axial test - dilative soil - cyclic loading.

Figure 4.19. τ_{xy} -p' diagram - shear test - dilative soil - cyclic loading.



Figure 4.21. q- ε_y diagram - shear test - dilative soil - cyclic loading.

Figure 4.20 and figure 4.21 shows the contractive behaviour of sand under reversal loading which is numerically caused by the fabric tensor z. Besides, cyclic loads increases this effect which is of importance when undrained conditions and pore pressure are accounted.

4.4.4 Patch test: Undrained conditions

In undrained behaviour most of the load is carried by the water, i.e. increase in pore pressure. In a total undrained analysis the constitutive behaviour of the soil is expressed with respect to total stresses. This does not provide any information about the pore water pressure. [Potts and Zdravkovic, 1999]

To estimate the undrained behaviour an assumption is to neglect consolidation of the soil and thereby treat the soil as totally undrained without any seepage. By considering the principle of effective stress it is possible to separate the total stress increment into effective stress increments and pore pressure increments, see (4.22).

$$\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{\sigma}' + \Delta \boldsymbol{u} \tag{4.22}$$

Since the soil and water deform together and the relative movement is negligible, the strain increment of the two phases are the same. This means that the stress increments in each phase can be found by,

$$\Delta \boldsymbol{\sigma}' = \boldsymbol{D}' \Delta \boldsymbol{\varepsilon} \tag{4.23}$$

$$\Delta \boldsymbol{u} = \boldsymbol{D}_f \Delta \boldsymbol{\varepsilon} \tag{4.24}$$

and hence we may add the two contributions together,

$$\Delta \boldsymbol{\sigma} = \Delta \boldsymbol{\sigma}' + \Delta \boldsymbol{u} = \boldsymbol{D}' \Delta \boldsymbol{\varepsilon} + \boldsymbol{D}_f \delta \boldsymbol{\varepsilon} = (\boldsymbol{D}' + \boldsymbol{D}_f) \Delta \boldsymbol{\varepsilon} = \boldsymbol{D} \Delta \boldsymbol{\varepsilon}$$
(4.25)

where D is the total constitutive matrix D' is the elasto-plastic constitutive matrix for the soil skeleton and is given by (4.6). At last, D_f is the elastic constitutive matrix for the pore water such as

$$\boldsymbol{D}_{f} = K_{e} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
(4.26)

where K_e is an equivalent bulk modulus for the total soil response.

$$K_e = \frac{1}{\frac{1}{K_f} + \frac{(1-n)}{K_s}}$$
(4.27)

where K_s is the bulk modulus of the solid soil particles and K_f is the bulk modulus of the fluid. For saturated soils the magnitude of K_f and K_s is unimportant since it is much larger than the soil skeleton stiffness K. In that case K_e equals K_f and the fluid skeleton is defined as

$$K_f = \beta K \tag{4.28}$$

Where β is a value to obtain the bulk modulus relation $K_f \gg K$. Normally it is set to 100-1000. Too high value of β can cause numerical problems when Poisson ratio gets very close to 0.5. [Potts and Zdravkovic, 1999]

Plots

Different plots of the undrained behaviour are illustrated in the following.





Figure 4.22. q-*p*[′] diagram - contractive soil - cyclic loading.



From figure 4.22 and 4.23 both the dilative and contractive soil decreases in mean effective stress p'. Actually, the effective mean pressure p' for contractive soil with the lower confining pressures are negative which mean that the soil may reach cavitation.



Figure 4.24. q- ε_y diagram - contractive soil - monotonic loading.

Figure 4.25. q- ε_y diagram - dilative soil - monotonic loading.

According to figure 4.24 the contractive soil for monotonic loading shows both dilative and contractive behaviour. The dilative behaviour is clear within the peak stress that it reaches before going to contractive soil. In Appendix C figure C.2.4 this is illustrated by considering the stress paths and see that it crosses the CSL and returns afterwards.



Figure 4.26. q- ε_y diagram - contractive soil - cyclic loading.



Figure 4.28. q-e diagram - dilative soil - monotonic loading.



Figure 4.27. q- ε_y diagram - dilative soil - cyclic loading.



Figure 4.29. q-e diagram - dilative soil - monotonic loading.

Figure 4.28 and 4.29 shows the void ratio changes are insignificantly small which verify that for undrained conditions the change of volume strains are neglectable.



Figure 4.30. u-*n*_{inc} - contractive soil - monotonic loading.



Figure 4.32. u-n_{inc} diagram - contractive soil - cyclic loading.



Figure 4.31. u-n_{inc} diagram - dilative soil - monotonic loading.



Figure 4.33. u-n_{inc} diagram - dilative soil - cyclic loading.

According to figure 4.32 and 4.33 both dilative and contractive soil are increasing the pore pressure with respect to number of load increments n_{inc} . An observation regarding the pore pressure build-up is that the dilative soil displays an high amplitude compared to the contractive one. Furthermore, the pore pressure increases faster for the contractive soil with a low amplitude.

4.5 Summary

It is seen in this chapter that the elastic formulation in this model is non-linear. Therefore, an hypo-elastic constitutive matrix is set with a dependency on the mean effective pressure. Hence, for soil plasticity an iterative process is needed for both elastic and plastic response.

The Newton intersection scheme in chapter 3 may be derived to account for such non linearity and could be a further development in the numerical implementation of this model.

Regarding the plastic behaviour, the volumetric reaction of the soil is of a great importance so that image points are used to determine it. In this chapter, a patch test is used to implement the model in a small finite element mesh and results obtained supports the concepts of critical state soil mechanics theory.

In undrained conditions, the volumetric strain is zero but excess pore water pressure appears when submitted to loads. A method from Potts and Zdravkovic [1999] enables an undrained analysis where effective stresses and pore pressure may be investigated. Nevertheless, a better insight of this undrained behaviour could be approached by considering hydraulic conductivity of the soil, where local drained conditions could arise during cyclic loading or consolidation.

CONCLUSION

5.1 Soil plasticity

Plasticity theory responds with a framework seen in chapter 2 to the non-linear behaviour in materials. However granular materials are complicated to model as it is not a continuum but a material arranged with grains. The purpose of this paper is to implement an advanced soil plasticity model in order to model some particularity of cohesionless soil. The *Critical state two-surface plasticity model for sands* developed by Manzari and Prachathananukit [2000] is based on a multiple surface model.

An hypoelastic assumption is made regarding the elastic behaviour in the soil, dependent on the mean effective pressure. On the other hand, critical state soil mechanics is used to derive the different formulation in plasticity. A matter of importance is the dilation behaviour of the soil which has several consequences on the soil especially with undrained conditions and the critical state which defines failure. These are respectively characterized by a dilatancy and bounding surface in generalised space, thus covering any loading conditions. Moreover,

5.2 Results of the patch test

The patch test is used to verify the model implementation within a small mesh in plane strain. It is noted that the model could be made with an axi-symmetry assumption, enabling a triaxial test to be investigate. The transition from triaxial stress space, in which CSSM theory is built upon, is formulated to generalised stress space seems to provide fair results.

In drained conditions, the soil reaches critical state by increasing or decreasing its volume towards failure. Results obtained satisfy Casagrande formulation of critical state by reaching its critical void ratio, i.e. constant volumetric strain.

Undrained conditions is set up by assuming no water can escape the model, hence the solid and liquid phases have the same strain increment in the calculations. This assumption enables undrained tests to be performed and a satisfying accumulation of pore pressure under cyclic loading.

5.3 Further developments

The footing test, made with the von Mises model, may be implemented with the *Critical state two-surface plasticity model for sands*. Together with an better establishment of the fluid phase in the soil and an appropriate interpolation of each model parameter, this constitutive model has proven several important properties of soil could be modelled and thereby be useful for real geotechnical problems.

Finally, regarding the stress update, a simple Forward Euler method is implemented together with the model. The running time to perform the footing analysis could be greatly improved with

the Newton intersection and error control schemes both presented in chapter 3.

PATCH TEST WITH VON MISES MODEL

The von Mises theory is often called J_2 -plasticity because it is only described in terms of the second deviatoric stress invariant, J_2 . To ensure that the J_2 -plasticity is implemented correct, a patch test in MATLAB is done. The background of implementing the von Mises plasticity in the patch are based on Chapter 2 and Chapter 3.

The patch consists of a square divided into eight triangular LST-elements with a forced vertical displacement. The patch and its topology is illustrated in Figure A.1.



Figure A.1. Undeformed mesh with element numbers.

A.1 Assumptions

- Linear hardening is considered which makes the hardening modulus, H, a constant.
- · Associated plasticity
- Plane strain

A.2 Case 1 - Perfect Plasticity

In the case of perfect plasticity, no hardening parameters are considered and the yield criterion is given as Eq. A.1.

$$f = \sqrt{3J_2} - \sigma_{y0} \tag{A.1}$$

where J_2 in principal stresses is expressed as, see Eq. A.2

$$J_2 = \frac{1}{2} \boldsymbol{s}_{ij} \boldsymbol{s}_{ij} \tag{A.2}$$

where

$$\boldsymbol{s}_{ij} = \boldsymbol{\sigma} - \frac{1}{3} (\mathrm{tr} \boldsymbol{\sigma}) \boldsymbol{I}$$
(A.3)

Here, *s* is the deviatoric part of the stress vector $\boldsymbol{\sigma}$ and is used to obtain the second deviatoric stress invariant, J₂. The latter stress invariant and the derivative is used in the elasto-plastic constitutive relation, see (2.23) in chapter 2.

A.2.1 Forward Euler Scheme



Figure A.2. Stress path in deviatoric plane with Forward Euler Scheme.



Figure A.3. Stress path in meridian plane with Forward Euler.

Convergence analysis

With the simple Forward Euler scheme a convergence analysis of the drift between the magnitude of von Mises criterion, defined as $\sqrt{2/3}\sigma_{Y0}$ and the magnitude of the last stress point versus number of load increments. A percentage of the drift is obtained and is illustrated in figure A.4. Secondly the computational time due to number of load increments is shown in figure A.5.



Figure A.4. Percentage of drift from initial yield surface due to load increments.



Figure A.5. Computational time due to number of load increments.



A.2.2 Modified Forward Euler Scheme

Figure A.6. Stress path in deviatoric plane with Modified Euler Scheme.



Figure A.7. Stress path in Meridian plane with Modified Euler.

A.2.3 Radial Return Method



Figure A.8. Stress path in deviatoric plane with Radial Return Method.



Figure A.9. Stress path in Meridian plane with Radial Return Method.

As expected the results illustrated in the above figures shows that there is a slight drift in the Forward Euler scheme compared with the implicit and Modified Euler Scheme.

A.3 Case 2 - Hardening Plasticity

Three types of hardening plasticity is considered. Furthermore the only hardening parameter is the hardening modulus, A, which is stated to 10 MPa. The elasto-plastic constitutive relation (2.23) is stated in chapter 2. The following plots of the three hardening laws are only considered with the modified Euler scheme of 100 increments.

A.3.1 Isotropic hardening



Figure A.10. Isotropic hardening in deviatoric plane.

A.3.2 Kinematic hardening



Figure A.11. Kinematic hardening in deviatoric plane.

A.3.3 Mixed hardening



Figure A.12. Mixed hardening in deviatoric plane.

STRIP FOOTING

To test the J_2 hardening plasticity, in a more realistic example, a strip footing is carried out in plane strain. Two types of loading are considered, monotonic and cyclic loading.

B.1 The model

The strip footing is placed on top of a von Mises material. Due to the assumption of plane strain, some symmetry properties of the domain and footing size can be assumed. An illustration of the footing example with symmetry properties is shown in Figure B.1.



Figure B.1. Drawing of the footing model.

The strip footing is considered rough and rigid with a total width of 2 m and a half width of 1 m. The domain size i given by L and H, which has been set to 10 m and 12 m. The domain is meshed with 6-noded LST triangle with a gaussorder of 3. In total, the entire model consist of 135 LST elements, 304 nodes which gives 608 d.o.f and 405 Gauss points. This is considered as a very rough mesh and is illustrated in figure B.2



Figure B.2. Mesh used in footing model.

Furthermore at the footing a forced displacement of 0.1 m in the negative y-direction is applied.

B.2 Material parameters

The footing is originally made to work with a Mohr-Coulomb material. Since von Mises is not developed for soil material, some of the essential parameters as Young's Modulus, E, and cohesion, c have been modified. In table B.1 the assumed material parameters are listed.

Table B.1. Parameters used for the von Mises material.			
Ε	Young's Modulus	210	MPa
v	Poisson ratio	0.3	
G	Shear modulus	46.2	MPa
φ	Friction angle	20°	
Ψ	Dilation angle	20°	
С	Cohesion	80	kPa
k_0	Lateral earth pressure coefficient	0.658	
Ytot	Total saturated soil weight	20	kN/m ³
γ_w	Water weight	9.82	kN/m ³

Footing load

Normally for a footing case it is convenient to find the maximum displacement at a given load level. This is called a force-controlled solution. It is known that an one-to-one relation exists

B.3

between the force and displacement. In the case of the footing used for the von Mises it is a displacement-controlled solution. By gradually increase the displacement the reaction forces is obtained by considering the static FE-equation

With K as the global stiffness matrix and f_{res} as the global load vector defined as residual force (difference in external - internal forces) and u as the displacement vector.

Two load cases are evaluated with the Modified Euler Scheme and kinematic hardening.

B.3.1 Convergence analysis

A convergence analysis of the footing load is illustrated in figure B.3.



Figure B.3. Convergence analysis of footing test.

From the strip footing test, the coarseness and thereby number of d.o.f are controlled by the parameter k_{mesh} . In the figure above the k_{mesh} has been tested from 2 to 10 where 2 is a very coarse mesh. The black point equals the k_{mesh} factor of 3 which is used in the following calculations and has the mesh illustrated in figure B.2.

B.3.2 Monotonic loading

As illustrated in figure B.4 the reaction curve of the footing load with a forced displacement of 0.1 m and 100 load increments is shown.



Figure B.4. Footing load with no hardening.

No hardening is considered and the reaction curve have similar design as the stress-strain curve for perfect-plasticity.

To verify this result, it is compared with the analytical solution by Terzaghi's bearing capacity formula

$$R/w = \frac{1}{2}\gamma w N_{\gamma} + q N_q + c N_c \tag{B.1}$$

where *R* is the bearing capacity, *w* is the footing width, *q* is the overburden pressure and N_{γ} , N_q and N_c are the bearing capacity factors determined by et. al [2007].

The first term in (B.1) is related to the width of the footing and the area of the rupture line. The second term is related to the depth of the footing and overburden pressure. The third term is related to the cohesion of the soil.

Since von Mises does not consider friction angle but only cohesion the limit state is given by the case of frictionless soil and the bearing capacity factors are given as

$$N_{\gamma} = 0$$

 $N_q = 1$
 $N_c = \pi + 2 \approx 5.14$

The factors are predetermined in et. al [2007]. The bearing capacity from the von Mises is 477.52 kPa and from Terzaghi it is 411.32 kPa. It is known that the analytical solution gives a conservative estimate of the bearing capacity. With this in mind the two bearing capacities mark the upper and lower bound of what to expect as the "real" bearing capacity of the von Mises material. It is known that the bearing capacity from the numerical analysis could even be closer to analytical solution by decrease the mesh size.

With hardening

In the other case hardening is considered and a constant value of the hardening parameter, A, is set to 10 kPa, see figure B.5.


Figure B.5. Footing load with a hardening modulus of 10 kPa.

Since the hardening modulus, A is considered strain hardening takes place and makes the reaction curve of the footing load evolve with an inclination of A in the elasto-plastic part. It is clear that the hardening modulus, A is lower than the Young's modulus, E due to a steeper inclination of the elastic part.

B.3.3 Cyclic loading

The cyclic loading is obtained by considering an harmonic function consisting of a sine wave given as

$$v = B \cdot \sin(\omega \cdot t - \psi) + v_{forced} \tag{B.2}$$

and is illustrated in figure B.6



Figure B.6. Illustration of the cyclic load applied by a sine wave.

With an amplitude, *B* of 0.05 m. ω as the angular frequency defined as $2\pi/T$ where the wave period, *T* is defined to be 5 seconds. ψ is the phase angle and is set to $\pi/2$. To never exceed a displacement larger than zero, *v* from equation B.2 is added with the forced displacement, *v*_{forced} of 0.05 m. In total it gives a maximum displacement of 0.1 m as in the monotonic case. In figure B.7 the Modified Euler Scheme with kinematic hardening.



Figure B.7. Cyclic loading with no hardening.

B.4 Computational time

The three schemes described in Chapter 3 have been tested with the case of monotonic loading and kinematic hardening. Without the material parameters listed in table B.1, some initial model parameters have been stated in table B.2.

<i>There</i> B.2. Initial stated model parameters.			
Model parameter	Abbreviation	Value	
Number of load increments Tolerance factor of the residual force Maximum number of global Eqm iterations Initial substep value	n _{ink} tolfac n _{maxglobal} N _{start}	$1000 \\ 10^{-5} \\ 30 \\ 50$	

Table B.2. Initial stated model parameters

The computational time of the three schemes are listed in table B.3 with the stated model parameters above.

Table B.3. Time of the three schemes		
Scheme	Time	
[-]	[h:min:sec]	
Forward Euler	08:12:36	
Modified Euler	00:10:52	
Implicit	00:02:39	

As expected the implicit formulation is faster than the two explicit methods since the implicit formulation is a closed form solution. In the stress update there is no iteration going on but the implicit formulation contain a direct solution to take the stress point back to the yield surface. The implicit method is also called the radial return method for von Mises since the yield surface is a circle in the deviatoric plane.

The computational time of the Forward Euler scheme and the Modified Euler Scheme have a significant time difference. The reason is explained in the following.

For the Forward Euler scheme it has to go through the entire stress update scheme for each strain increment, $\delta\varepsilon$ to see if the stress increment, $\delta\sigma$ is in the elastic or plastic range. This scheme does not differ from elastic or plastic range.

Since the von Mises is linear elastic the Modified Euler Scheme skip all the iterations in the elastic range and calculates directly the stress point at the yield surface by implementing the Newton-Raphson method. When plasticity is reach the Modified Euler Scheme uses same approach as the Forward Euler Scheme. The modification in the elastic range makes the scheme a lot faster.

B.4.1 Modified Euler scheme

Considering the Modified Euler Scheme the incorporated error control that calculates the number of sub increments have been investigated. Number of load increments and initial number of sub increments have been set. By considering different substep tolerance factors, Tol_{sub} an average substep has been calculated and is listed in table B.4 with the computational time.

Tol _{sub} [-]	Average substep [-]	Time [min:sec]
$n_{\rm ink} = 200$	$N_{start} = 10$	
10 ⁻³	10	09:42
10^{-5}	10.7	10:13
10^{-7}	25.6	64:15
$n_{\rm ink} = 500$	$N_{start} = 10$	
10^-3	10	08:08
10^{-5}	10	08:35
10^{-7}	11.6	16:32
$n_{\rm ink} = 1000$	$N_{start} = 5$	
10 ⁻³	5	08:36
10^{-5}	5	09:36
10^{-7}	5.9	19:22

Table B.4. Average values of the substep with computational time.

The initial number of subincrements is the minimum value of substep the Modified Euler Scheme can use for a given test. If the initial value of the subincrements is too low the global equilibrium iteration does not converge. Furthermore by increase the number of load increments the number of iterations needed to obtain equilibrium is decreased, which can be seen by the computational time.

Secondly, a correlation between load increments and tolerance factor are observed. Higher number of load increments decrease the minimum value of substeps even for very low tolerance factor.

B.5 Surface plots



Figure B.8. Effective stresses.



Figure B.9. Ruptureline with total deformation of 0.1 m.



Figure B.10. Accumulated effective plastic strains.

PLOTS OF THE SOIL PLASTICITY MODEL

C.1 Patch test: Drained conditions

C.1.1 Deviatoric stress q and Shear stress τ_{xy} - Axial strains



Figure C.1. q- ε_y diagram - axial test - contractive soil - monotonic loading.



Figure C.3. τ_{xy} - ε_y diagram - shear test - contractive soil - monotonic loading.



Figure C.5. q- ε_y diagram - axial test - contractive soil - monotonic loading.



Figure C.2. q- ε_y diagram - axial test - dilative soil - monotonic loading.



Figure C.4. τ_{xy} - ε_y diagram - shear test - dilative soil - monotonic loading.



Figure C.6. q- ε_y diagram - axial test - dilative soil - cyclic loading.



Figure C.7. τ_{xy} - ε_y diagram - shear test - contractive soil - cyclic loading.



Figure C.8. τ_{xy} - ε_y diagram - shear test - dilative soil - cyclic loading.

C.1.2 Deviatoric stress q - Mean effective stress p'



Figure C.9. q-p' diagram - axial test - contractive soil - monotonic loading.



Figure C.11. τ_{xy} -p' diagram - shear test - contractive soil - monotonic loading.



Figure C.13. q-p' diagram - axial test - contractive soil - cyclic loading.



Figure C.10. q-*p*['] diagram - axial test - dilative soil - monotonic loading.



Figure C.12. τ_{xy} -*p'* diagram - shear test - dilative soil - monotonic loading.



Figure C.14. q-*p*['] diagram - axial test - dilative soil - cyclic loading.



Figure C.15. τ_{xy} -*p'* diagram - shear test - contractive soil - cyclic loading.



Figure C.16. τ_{xy} -p' diagram - shear test - dilative soil - cyclic loading.

C.1.3 Void ratio e - Effective mean pressure p'



Figure C.17. e-p' diagram - axial test - contractive soil - monotonic loading.



Figure C.19. e-p' diagram - axial test - contractive soil - cyclic loading.



Figure C.21. e-p' diagram - shear test - contractive soil - cyclic loading.



Figure C.18. e-p' diagram - axial test - dilative soil - monotonic loading.



Figure C.20. e-p' diagram - axial test - dilative soil - cyclic loading.



Figure C.22. e-p' diagram - shear test - dilative soil - cyclic loading.

C.1.4 Deviatoric stress q - Void ratio e



Figure C.23. q-e diagram - axial test - contractive soil - monotonic loading.



Figure C.25. q-e diagram - axial test - contractive soil - cyclic loading.

Volumetric strain ε_V - Axial strains



Figure C.24. q-e diagram - axial test - contractive soil - dilative loading.



Figure C.26. q-e diagram - axial test - dilative soil - cyclic loading.



Figure C.27. $\varepsilon_V \cdot \varepsilon_y$ diagram - axial test - contractive soil - monotonic loading.



Figure C.28. $\varepsilon_V - \varepsilon_y$ diagram - axial test - dilative soil - monotonic loading.

C.1.5



Figure C.29. $\varepsilon_V - \varepsilon_x$ diagram - shear test - contractive soil - monotonic loading.



Figure C.31. $\varepsilon_V - \varepsilon_y$ diagram - axial test - contractive soil - cyclic loading.

C.1.6 State parameter Ψ - Axial strains



Figure C.30. $\varepsilon_V - \varepsilon_x$ diagram - shear test - dilative soil - monotonic loading.



Figure C.32. $\varepsilon_V - \varepsilon_y$ diagram - axial test - dilative soil - cyclic loading.



Figure C.33. Ψ - ε_y diagram - axial test - contractive soil - monotonic loading.



Figure C.34. Ψ - ε_y diagram - axial test - dilative soil - monotonic loading.



Figure C.35. Ψ - ε_y diagram - shear test - contractive soil - monotonic loading.



Figure C.37. Ψ - ε_y diagram - axial test - contractive soil - cyclic loading.



Figure C.39. Ψ - ε_y diagram - shear test - contractive soil - cyclic loading.



Figure C.36. Ψ - ε_y diagram - shear test - dilative soil - monotonic loading.



Figure C.38. Ψ - ε_y diagram - axial test - dilative soil - cyclic loading.



Figure C.40. Ψ - ε_y diagram - shear test - dilative soil - cyclic loading.

C.2 Patch test: Undrained Conditions

C.2.1 Deviatoric stress q - Axial strains



Figure C.41. q- ε_y - axial test - contractive soil - monotonic loading.



Figure C.42. q- ε_y - axial test - dilative soil - monotonic loading.



Figure C.43. $q - \varepsilon_y$ - axial test - contractive soil - cyclic loading.



Figure C.44. q- ε_y - axial test - dilative soil - cyclic loading.

C.2.2 Deviatoric stress q - Mean effective stress p'



Figure C.45. q-*p*′ diagram - axial test - contractive soil - monotonic loading.



Figure C.46. q-p' diagram - axial test - dilative soil - monotonic loading.



Figure C.47. q-p' diagram - axial test - contractive soil - cyclic loading.



Figure C.48. q-p' diagram - axial test - dilative soil - cyclic loading.





Figure C.49. e-p' diagram - axial test - contractive soil - monotonic loading.



C.2.4



Figure C.51. q-e diagram - axial test - contractive soil - monotonic loading.



Figure C.50. e-p' diagram - axial test - dilative soil - monotonic loading.



Figure C.52. q-e diagram - axial test - dilative soil - monotonic loading.

700

600



Figure C.53. q-e diagram - axial test - contractive soil - cyclic loading.



Figure C.54. q-e diagram - axial test - dilative soil - cyclic loading.



Figure C.55. $\varepsilon_V - \varepsilon_y$ - axial test - contractive soil - monotonic loading.



Figure C.56. $\varepsilon_V - \varepsilon_y$ - axial test - dilative soil - monotonic loading.



Figure C.57. $\varepsilon_V - \varepsilon_y$ - axial test - contractive soil - cyclic loading.



Figure C.58. $\varepsilon_V - \varepsilon_y$ - axial test - dilative soil - cyclic loading.

C.2.5 Volumetric strain ε_V - Axial strains

C.2.6 State parameter Ψ - Number of load increments n_{inc}



Figure C.59. Ψ-*n*_{inc} diagram - axial test - contractive soil - monotonic loading.



Figure C.61. Ψ-*n*_{inc} diagram - axial test - contractive soil - cyclic loading.



Figure C.60. Ψ-*n_{inc}* diagram - axial test - dilative soil - monotonic loading.



Figure C.62. Ψ-*n_{inc}* diagram - axial test - dilative soil - cyclic loading.

C.2.7 Vertical stresses σ_V - Number of load increments n_{inc}



Figure C.63. σ_V - n_{inc} diagram - axial test - contractive soil - monotonic loading - p_0 1000.



Figure C.64. σ_V - n_{inc} diagram - axial test - dilative soil - monotonic loading - p_0 1000.



Figure C.65. σ_V - n_{inc} diagram - axial test - contractive soil - cyclic loading - p_0 1000.

C.2.8 Pore pressure *u* - Axial strains



Figure C.67. u- ε_y - axial test - contractive soil - monotonic loading.

- p₀ = 250 kPa - p₀ = 500 kPa - p₀ = 1000 kPa

• p₀ = 2000 kPa

-0.5

120

100

80

40

200

0└ -4.5

-3.5

-3

u [kPa] 009



Figure C.66. σ_V - n_{inc} diagram - axial test - dilative soil - cyclic loading - p_0 1000.



Figure C.68. u- ε_y - axial test - dilative soil - monotonic loading.



Figure C.69. u- ε_y - axial test - contractive soil - cyclic loading.

-2.5 -2 ϵ_v [-] -1.5

-1

Figure C.70. u- ε_y - axial test - dilative soil - cyclic loading.

81

C.2.9 Pore pressure *u* - Number of load increments n_{inc}



Figure C.71. u-n_{inc} diagram - axial test - contractive soil - monotonic loading.



Figure C.73. u-n_{inc} diagram - axial test - contractive soil - cyclic loading.



Figure C.72. u-n_{inc} diagram - axial test - dilative soil - monotonic loading.



Figure C.74. u-*n*_{*inc*} diagram - axial test - dilative soil - cyclic loading.

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