Aalborg University
Department of Mathematical Sciences

The Chromatic Number and Brooks' Theorem

Author: Helle Blicher

Master of Science Thesis

June 5th, 2015

Pages:	37
Printings:	4

Department of Mathematical Sciences
Fredrik Bajers Vej 7G
Phone 99 40 99 40
Fax 98 15 81 29
www.math.aau.dk

Title:
The Chromatic Number and Brooks' Theorem

Project period:
February 1st 2015 -
June 5th 2015

Written by:

Helle Blicher

Supervisor:
Leif Kjaer Joergensen

Number of copies: 4

Number of pages: 37

Submitted: June 5th, 2015

Synopsis:

This Master of Science Thesis reviews the concepts of vertex coloring and encompasses a thorough study of the chromatic number for a graph G. The main emphasis is on Brooks' theorem in various editions. Brooks proved, in 1941, the inequality between a graph's maximum degree and its chromatic number. The basic traits for graphs are accounted for, as well as fundamental subtopics and theorems essential for understanding and completing the proofs of interest.
An examination of a more recent version of Brooks' theorem by Chartrand and Zhang reveals that although terminology and phrasing is very diverse, the disparity in mathematical technique is minimal as both utilizes paths, greedy coloring, and permutation.
In 1976 Catlin proved an extension to the theorem of Brooks and recently a unified version of these was published by Vaidy Sivaraman. He utilizes independence in graphs, matchings, and Hall's condition in order to complete the proof. As the original unified proof seemed insufficient in the sense that the author left out some intermediate results and explanations, an elaborate version is presented here.

Dansk resumé

Dette speciale er skrevet ved Institut for Matematiske Fag, Aalborg Universitet og behandler begreber indenfor punktfarvning af grafer. Hvis man ønsker at punktfarve en graf, står man over for følgende optimeringsproblem: Hvad er det mindste antal farver, der er nødvendige for at kunne tildele en farve til hvert punkt, således at to nabopunkter ikke har samme farve?
Dette mindste tal er kendt som det kromatiske tal og der gives en grundig gennemgang af egenskaber ved dette kromatiske tal $\chi(G)$ for en graf G.
Kromatisk grafteori kan dateres tilbage til 1850'erne, hvor Francis Guthrie opdagede det velkendte firfarveproblem: Kan ethvert landkort farves med højst fire farver, således at lande, der deler grænse, ikke har samme farve?

Hovedvægten lægges på Brooks' sætning i forskellige udgaver. Brooks beviste i 1941 uligheden mellem maksimumsgraden $\Delta(G)$ og det kromatiske tal $\chi(G)$ for en graf G. Der redegøres naturligvis for det basale indenfor grafteori og for fundamentale underemner, såsom grådig farvning og kritiske grafer, og for sætninger, der er essentielle for at forstå og gennemføre de beviser, der har interesse i dette speciale.

En gennemgang af en nyere version af Brooks' sætning, udarbejdet af Chartrand og Zhang, viser, at på trods af stor diversitet i terminologi og formulering, er forskellen i den matematiske teknik minimal, da begge anvender veje, grådig farvning og permutation.

I 1976 beviste Paul A. Catlin en sætning, der betragtes som en udvidelse af Brooks' sætning. Catlins sætning siger, at hvis en graf G opfylder betingelserne i Brooks' sætning, så indeholder G en monokromatisk, maksimal, uafhængig delmængde. For nyligt blev en forenet version af disse bevist og publiceret af Vaidy Sivaraman. Han anvender uafhængighed i grafer, matching og Halls betingelse for at gennemføre beviset. Hans bevis er dog utilskrækkeligt, forstået på den måde, at visse mellemregninger og forklaringer er udeladt. Derfor gives der i dette speciale en noget mere uddybende og detaljeret version af beviset.

Preface

This Master of Science Thesis is written by Helle Blicher in spring 2015 during the last semester of a Master's degree programme in Mathematics at the Department of Mathematical Sciences at Aalborg University.

The primary literature is "Chromatic Graph Theory" by G. Chartrand and P. Zhang, although Reinhard Diestel's "Graph Theory" has been consulted. Full bibliography is provided on the last page.

The reader is expected to have a certain knowledge of Graph Theory beforehand and to possess the mathematical qualifications corresponding to completion of a bachelor education in Mathematical Sciences as a minimum.

The author wishes to thank her supervisor Leif Kjaer Joergensen for his help and supervision, and the staff at Department of Mathematical Sciences at Aalborg University for their support.

Reading instructions

The first chapter will introduce the reader to the basics of graph theory and the terminology used throughout.
The chromatic number is introduced in the second chapter along with greedy coloring and color-critical graphs.
The third chapter is devoted to Brooks' theorem while the fourth and final chapter recapitulates the contents of the thesis.

If nothing else is mentioned then all the graphs in question are simple and finite.

Sections, figures, mathematical definitions, etc., are numbered according to the chapter, i.e. the first figure in Chapter 2 has number 2.1 etc. References to an equation is on the form (x.y).

Contents

Introduction

Within the mathematical field of Graph Theory the most well-known and studied area is inarguably coloring. Whether it be vertex coloring or edge coloring, new and improved studies and proofs are being published regularly. In this thesis the subject of vertex coloring is fundamental. When coloring the vertices of a graph G, one is faced with an optimization problem; what is the minimum number of colors necessary in order to assign a color to each vertex such that no to adjacent vertices have the color? This number is known as the chromatic number of G, denoted as $\chi(G)$.

Chromatic graph theory dates back to the 1850's where Francis Guthrie discovered the well-known Four Color Problem: Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?
Although it is desirable to obtain a $\chi(G)$-coloring, it may not be as easy to do as it might seem. Hence, the method of greedy coloring comes in handy as it, in addition to coloring without using an excessive number of colors, provides an upper bound for the chromatic number.

The purpose of this thesis is to examine a specific and fundamental result in the theory of graph coloring, namely Brooks' Theorem of 1941. He proved the, intuitively known, inequality between a graph's maximum degree and its chromatic number. Obviously, the terminology and the way mathematical texts were phrased at the time, was quite different from that of modern times. Due to this diversity within the mathematical methods and language, various proofs will be scrutinized, thus covering a greater area of graph theory.

In 1976 Catlin proved an extension to the theorem of Brooks and recently, a unified version of these was published by Vaidy Sivaraman. An elaborate version of Sivaraman's unified proof will be presented in this thesis.

Chapter 1

Terminology and Essential Graph Theory

1.1 Terminology

A **graph** G consists of a finite, nonempty set V of **vertices** and a set E of **edges**. If u and v are vertices in G, then the joining edge between them is denoted $e = uv$. Hence, an element of the set E is a 2-element subset of V. G is often written as $G = (V, E)$, V as $V(G)$, and E as $E(G)$. The number of vertices in G determines the **order** of G denoted as $|G|$ or n, while the number of edges determines the **size** of G denoted as $||G||$ or m. Two vertices, u and v, connected by an edge uv are **adjacent** and are referred to as **neighbors**. Similarly, if two distinct edges share a vertex they are also adjacent.

Another characteristic for a graph G is the **degree** which is based in the degrees of the vertices. The degree of a vertex v is the number of vertices adjacent to v; i.e. v's neighbors. Equivalently, the degree of v is the number of edges incident with v. A vertex v is incident with an edge e if and only if $e = uv$ or $e = vu$. The vertex with the largest degree determines the **maximum degree** of G denoted as $\Delta(G)$ while the **minimum degree** of G, determined by the vertex with the smallest degree, is denoted by $\delta(G)$. Thus for a graph G of order n the following applies:

$$0 \leq \delta(G) \leq deg\ v \leq \Delta(G) \leq n - 1.$$

A very important member of the graph family is the **complete graph**. A graph G is complete if and only if every two distinct vertices in G are adjacent and a complete graph of order n will be denoted as K_n. The size m of K_n is easily calculated by

$$\binom{n}{2} = \frac{n\,(n-1)}{2}.$$

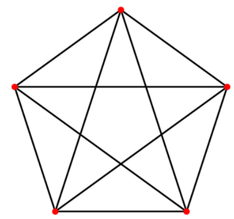

Figure 1.1: *A complete graph of order 5.*

A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is a **subgraph** of G and if $V(H) = V(G)$ then H is a **spanning subgraph** of G. H can also be a **proper subgraph** of G if either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$; i.e. $H \neq G$. Furthermore, H is an **induced subgraph** if S is a nonempty subset of $V(G)$ such that the subgraph $H = G[S]$ of G induced by S has S as vertex set and two vertices are adjacent i $G[S]$ if and only if the same two vertices are adjacent in G.

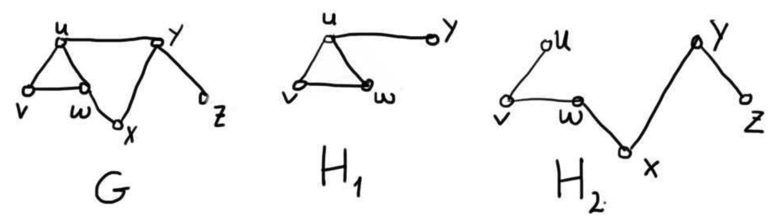

Figure 1.2: *A connected graph G, an induced subgraph H_1, and a spanning subgraph H_2.*

1.2 Connectivity

A graph G is connected if every two vertices of G are connected. This means that there must exist a $u - v$ walk between every two vertices u and v of G. A walk in which no edge is repeated is called a **trail** and if no vertices are repeated then the walk is considered a **path**. The minimum length of any path between to vertices u, v is called **distance** and is denoted $d(u, v)$. Such a path is also known as a $u - v$ **geodesic**.
If a graph G is not connected it contains **components** which are connected

independent subgraphs as seen on figure 1.3.

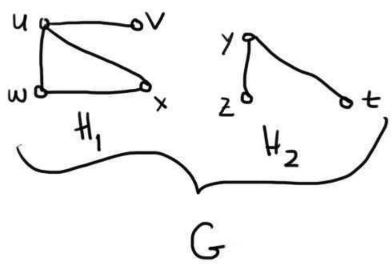

Figure 1.3: *An unconnected graph G consisting of two components, H_1 and H_2.*

Since Brooks' Theorem eliminates odd cycles and complete graphs one must understand the meaning of these. As explained earlier a complete graph is one where every two distinct vertices are adjacent. A **cycle** of a graph G is a path from one vertex v through distinct vertices and back to v. The lengt of a cycle is determined by the number of edges traveled and so an odd cycle would be one traveling along an odd number of edges.

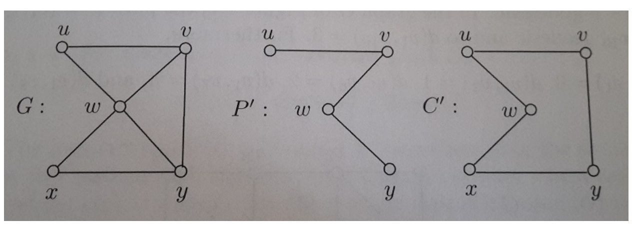

Figure 1.4: *A connected graph G, a path in G, and a cycle in G.*

If a graph contains no cycles it is referred to as a **tree**.

When determining the connectivity of a graph G one is actually examining the number of vertices or edges that must be removed for G to be disconnected or trivial ($n \leq 1$). A so called **vertex-cut** would then be a set S of vertices such that $G - S$ is disconnected and the minimum number of vertices for this to be possible is called the **vertex-connectivity** (or just connectivity) of G and is denoted by $\kappa(G)$.
Thus, for every graph G of order n,

$$0 \leq \kappa(G) \leq n - 1.$$

If $\kappa(G) = k \geq 1$, then G is considered k**-connected**. For any complete graph K_n the connectivity is defined as $n-1$, since the removal of less than $n-1$ vertices will result in a smaller complete graph, hence obtaining a trivial graph is necessary in order for K_n to be disconnected.

If a connected graph contains no cut-vertices (the removal of a single cut-vertex results in a disconnected graph) the graph is considered nonseparable. A connected nonseparable subgraph is called a **block** if it is not a part of a larger block.

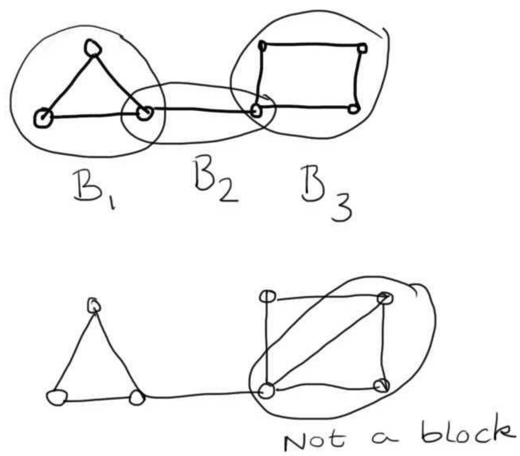

Figure 1.5: *Blocks*

A block that contains exactly one cut-vertex is called an **end-block** and a connected graph with cut-vertices must contain two or more end-blocks.

An edge-cut is a set X of edges such that $G - X$ is disconnected. The minimum number of edges for this to be possible is called the **edge-connectivity** of G and is denoted by $\lambda(G)$. Thus

$$0 \leq \lambda(G) \leq n - 1.$$

for every graph G of order n. A graph G is k-edge-connected if $\lambda(G) = k \geq 1$. The trivial complete graph K_1 does not contain an edge-cut. However, the edge-connectivity is defined as $\lambda(K_1) = 0$. Further, for every complete graph the following applies:

Theorem 1.1
Let n be a positive integer. Then, for every n,

$$\lambda(K_n) = n - 1.$$

Proof.
By definition $\lambda(K_1) = 0$, wherefore it is safe to assume that $n \geq 2$. If $n-1$ edges incident with any vertex of K_n is removed then a disconnected graph is obtained. Thus

$$\lambda(K_n) \leq n - 1.$$

Let X be a minimum edge-cut of K_n so that $|X| = \lambda(K_n)$. Now $G = K_n - X$ is a disconnected graph consisting of two components, G_1 and G_2. Suppose that G_1 has order k. Then G_2 must have order $n-k$ since the number of vertices is unaltered. Thus $|X| = k(n-k)$ since otherwise it would not have been a complete graph to begin with.
Now, since $k \geq 1$ and $n-k \geq 1$, it follows that $k-1 \geq 0$ and $n-k-1 \geq 0$ and so

$$\begin{aligned}
(k-1)(n-k-1) &\geq 0 \\
kn - k^2 - k - n + k + 1 &\geq 0 \\
k(n-k) - (n-1) &\geq 0 \\
k(n-k) &\geq (n-1)
\end{aligned}$$

This implies that

$$\lambda(K_n) = |X| = k(n-k) \geq n - 1.$$

Thus,

$$\lambda(K_n) = n - 1.$$

■

In the early 1930s American mathmatician Hassler Whitney (1907-1989) contributed to the studies of graph coloring and amongst his observations was the following theorem which establishes inequality between connectivity, edge-connectivity, and minimum degree of a graph G:

Theorem 1.2
For every graph G,

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof.
Let G be a graph of order n. If G is disconnected then obviously $\kappa(G) = \lambda(G) = 0$. If G is complete then $\kappa(G) = \lambda(G) = \delta(G) = n - 1$. Thus the theorem holds in these two cases.
Assume that G is a connected, non-complete graph. Since G is not complete, $\delta(G) \leq n - 2$. Now, let v be a vertex of G such that $deg(v) = \delta(G)$. If the edges incident with v are removed, a disconnected graph is produced. Hence $\lambda(G) \leq \delta(G) \leq n - 2$. It remains to show the first inequality; $\kappa(G) \leq \lambda(G)$. Let X be a minimum edge-cut of G such that $|X| = \lambda(G) \leq n - 2$. Now $G - X$ consists of two components, G_1 and G_2. Suppose that G_1 has order k. Then G_2 must have order $n - k$ and $k \geq 1$ and $n - k \geq 1$. Further, every edge in X joins a vertex of G_1 and a vertex of G_2. Two cases can be considered. Case 1: Every vertex of G_1 is adjacent to every vertex of G_2. Then $|X| = k(n - k)$. Since $k - 1 \geq 0$ and $n - k - 1 \geq 0$, it follows that, as shown in the previous proof;

$$k(n - k) \geq n - 1.$$

Thus $\lambda(G) = |X| = k(n-k) \geq n-1$ which contradicts $\lambda(G) \leq n-2$. Hence case 1 cannot occur.

Case 2: There is a vertex $u \in G_1$ and a vertex $v \in G_2$ such that $uv \notin E(G)$. Let U be a set of vertices of G selected according to the following and let $e \in X$. If e is incident with $u \in G_1$ such that $e = uv'$, then v' is placed in the U. If e is not incident with u, $e = u'v'$ where $u' \in G_1$, then u' is placed in U. Hence, for every edge $e \in X$ one of its two incident vertices belongs in U but $u, v \notin U$. Thus $|U| \leq |X|$ and U is a vertex-cut. Ultimately, $\kappa(G) \leq |U| \leq |X| = \lambda(G)$.

■

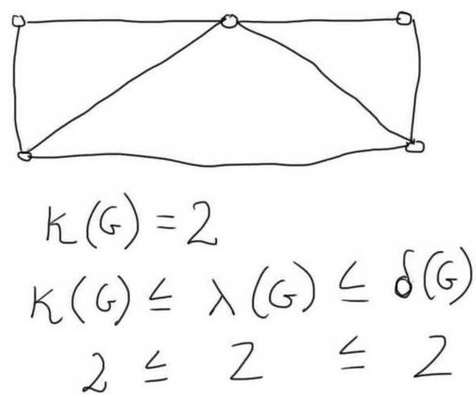

Figure 1.6: *An example of the inequality.*

Chapter 2

Vertex Coloring

2.1 The Chromatic Number

Basically, a **vertex coloring** of a graph G is the assignment of given colors to the vertices in such manner that no adjacent vertices are of same color. The colors used are commonly labeled with positive integers $1, 2, ..., k$. Hence, the following definition can be applied:

Definition 2.1 (Vertex Coloring)
A vertex coloring of a graph G is a function $c : V(G) \to \mathbb{N}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G.

A nonempty set of vertices with the same color can be referred to as a **color class** and each color class is an independent set of vertices of G.
If each color used to color a graph G is one of k given colors, it is referred to as a k-coloring of G.
If a coloring of G from a set of k colors exists, then G is said to be k-**colorable**. The minimum positive integer k with which G is k-colorable is called the **chromatic number** and is denoted by $\chi(G)$. No general formula for the chromatic number exists, hence determining the chromatic number for specific graphs of interest and examining the upper and lower bounds for the chromatic number must suffice. Obviously, for any graph G of order n, it must apply that

$$1 \leq \chi(G) \leq n.$$

Further, it is clear that any graph G containing one or more triangles must have $\chi(G) \geq 3$ since three vertices forming a triangle must be assigned different colors. Such triangles are also referred to as odd cycles; a key term, shortly mentioned in section 1.2, which will be further elaborated on later in this thesis. Furthermore, for any complete graph $G = K_n$ it must apply that $\chi(G) = n$.

The lower bound for the chromatic number of a graph can be determined by the chromatic numbers of its subgraphs, and the following applies:

Theorem 2.2
If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.

Proof.
Assume that $\chi(G) = k$. By definition there exists a k-coloring c of G. Since any two adjacent vertices of G are assigned distinct colors by c, the coloring c also assigns distinct colors to adjacent vertices of H. Hence, H is k-colorable and so $\chi(H) \leq k = \chi(G)$.

■

Further, as a direct consequence of the previous theorem:

Corollary 2.3
For every graph G, $\chi(G) \geq \omega(G)$.

This is obvious since $\omega(G)$, the clique number, is the order n of the largest complete subgraph H of G and, as already mentioned, $\chi(H) = n$ for a complete graph $H = K_n$.

A graph with chromatic number k is said to be **k-chromatic**, as well as **k-colorable**.

Should a graph G be made up of the union of k graphs, then the chromatic number of G will depend on the maximum chromatic number of any one of the k graphs:

Proposition 2.4
For graphs $G_1, G_2, ..., G_k$ and $G = G_1 \cup G_2 \cup ... \cup G_k$,
$\chi(G) = max\{\chi(G_i) : 1 \leq i \leq k\}$.

This proposition leads to the following:

Corollary 2.5
If G is a graph with components $G_1, G_2, ..., G_k$, then
$\chi(G) = max\{\chi(G_i) : 1 \leq i \leq k\}$.

The reader is reminded that a **component** of a graph is a connected, independent subgraph - see fig. 1.3.

Proposition 2.6
If G is a nontrivial (two or more vertices) connected graph with blocks $B_1, B_2, ..., B_k$, then

$$\chi(G) = max\{\chi(B_i) : 1 \leq i \leq k\}.$$

The reader is reminded that a **block** of a graph is a nonseparable, connected subgraph which is not a part of a larger block - see fig. 1.5.

While it is desirable to obtain a coloring of a graph G using exactly $\chi(G)$ colors, it is in fact difficult to do. However, there is a method that does not use an excessive number of colors, and although it may not result in a $\chi(G)$-coloring is will provide an upper bound for the chromatic number of a graph G.

2.1.1 Greedy Coloring

Let G be a graph of order n and let the vertices of G be listed in some specified order, say $v_1, v_2, ..., v_n$. A **greedy coloring** c assigns colors to the vertices successively, always assigning the smallest available color. The color 1 is in this case assigned to v_1. If v_2 is not adjacent to v_1 then it is assigned the color 1 too. Otherwise it is assigned the color 2. Following the list of vertices, they are now all assigned colors with respect to pre-assigned neighbors and the smallest available color. When done, G's vertices have been given colors from the set $\{1, 2, ..., k\}$ for some positive integer k. Thus $\chi(G) \leq k$ and k is said to be an upper bound for the chromatic number.
Intuitively, a greedy coloring will use at most $\Delta(G) + 1$ colors to color a graph G.
More formally, the greedy coloring is an algorithm and is defined as the following:

Definition 2.7
The Greedy Coloring Algorithm
Suppose that the vertices of a graph G are listed in the order $v_1, v_2, ..., v_n$.

1. *The vertex v_1 is assigned the color 1.*

2. *Once the vertices $v_1, v_2, ..., v_j$, where $1 \leq j < n$, have been assigned colors, the vertex v_{j+1} is assigned the smallest color that is not assigned to any neighbor of v_{j+1} belonging to the set $\{v_1, v_2, ..., v_j\}$.*

2.2 Color-Critical Graphs

If a graph G has chromatic number $\chi(G) = k \geq 2$ and thus is k-chromatic with $k \geq 2$, it means that the graph can be colored using exactly k colors and not $(k-1)$ colors. A graph G is considered **color-critical** if the chromatic number of any proper subgraph H of G is lower than the chromatic number of G itself.

Definition 2.8
*If for a graph G, $\chi(H) < \chi(G) = k$ for any subgraph H, then G is called **critically k-chromatic** or k-**critical**.*

Suppose that G is a k-chromatic graph, $k \geq 2$, and H is a k-chromatic subgraph of minimum size with no isolated vertices.
Then for every proper subgraph F of H, the chromatic number of F must be lower than the chromatic number of H, i.e. $\chi(F) < \chi(H)$. Hence H is color-critical, more specifically a k-critical subgraph of G. From this it can be derived that every k-chromatic graph, $k \geq 2$, contains a k-critical subgraph.
By corollary 2.5 it follows that every k-critical graph, $k \geq 2$, must be connected. If not, there would be a subgraph with an equal chromatic number which contradicts the aforementioned trait of a k-chritical graph.
The complete graph K_2 is the only 2-critical graph and obviously it is 1-connected and 1-edge-connected and so by proposition 2.6 it follows that every k-critical graph, $k \geq 3$, must be 2-connected. If not, there would be a subgraph with an equal chromatic number which again contradicts the aforementioned trait of a k-chritical graph. And finally, by theorem 1.2 it can be concluded that every k-critical graph, $k \geq 3$, must be 2-edge-connected.

The odd cycles, which are all 2-edge-connected, are 3-critical graphs. Any odd cycle G can be colored using colors c_1 and c_2 alternately, and only the last vertex before completing the cycle must be colored c_3, thus $\chi(G) = 3$. Further, every proper subgraph H of an odd cycle has $\chi(H) = 2$ since it is a tree, which is commonly known to be 2-colorable. Thus $\chi(H) < \chi(G)$ for any proper subgraph H of any odd cycle G, satisfying the conditions for a

k-critical graph. The following theorem and proposition will provide further information.

Theorem 2.9
A nontrivial graph G is a bipartite graph if and only if G contains no odd cycles.

A graph G is bipartite if it is possible to divide the set of vertices into two partitions U and W such that every edge of G joins a vertex of U and a vertex of W.

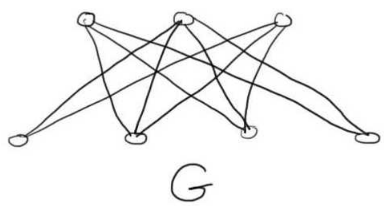

Figure 2.1: *A bipartite graph*

Proof. (of Thm. 2.9)
Suppose that G is bipartite with partite sets U and W and every edge of G joins a vertex of U and a vertex of W. Let $C = (v_1, v_2, ..., v_k, v_1)$ be a k-cycle of G. Assume that $v_1 \in U$. Thus $v_2 \in W$, $v_3 \in U$, and so on. For every odd integer i, $1 \leq i \leq k$, $v_i \in U$ and for every even integer j, $2 \leq j \leq k$, $v_j \in W$. Since $v_1 \in U$ it follows that v_k must be in W. Hence, k is even.
Now, let G be a nontrivial graph containing no odd cycles. Since it will suffice to show that every nontrivial component of G is bipartite one can, without loss of generality, assume that G is connected. Let u be a vertex of G and partition the remaining vertices based on the distance $d(u, x)$ such that

$$U = \{x \in V(G) | d(u, x) \ is \ even\}$$
$$W = \{x \in V(G) | d(u, x) \ is \ odd\},$$

and let $u \in U$. To show that G is bipartite with the above mentioned partite sets, it remains to show that no two vertices of U are adjacent and no two vertices of W are adjacent. Suppose that W contains two adjacent vertices, w_1 and w_2. Let P_1 be a $u - w_1$ geodesic and P_2 a $u - w_2$ geodesic. Let z

(possibly $z = u$) be the last vertex that P_1 and P_2 have in common. Then the length of the subpaths, $z - w_1$ and $z - w_2$, are both even or odd. This is illustrated in fig. 2.2.

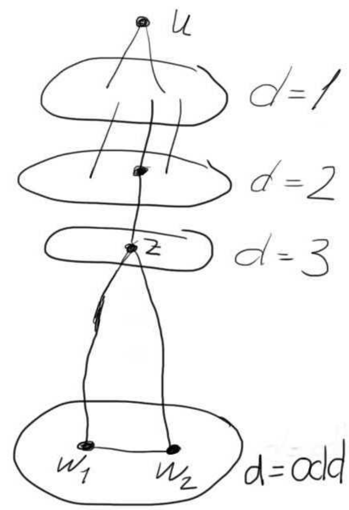

Figure 2.2: *Subpaths $z - w_1$ and $z - w_2$.*

Hence these two paths together with the edge w_1w_2 form an odd cycle which contradicts the presumption of G.
Similarly, it can be shown that no two vertices of U are adjacent.

■

Proposition 2.10
A nontrivial graph G is 2-colorable if and only if G is bipartite.

Therefore, if G contains no odd cycles, then $\chi(G) \leq 2$ and if a graph G has $\chi(G) = 3$ it must contain an odd cycle.
Furthermore, since no k-critical graph can contain another k-critical graph as a proper subgraph, the odd cycles are the only 3-critical graphs.

Theorem 2.11
Every k-critical graph with $k \geq 2$ is (k-1)-edge-connected.

Proof.
As per previous explanation, the theorem holds for $k = 2, 3$, thus it is safe to assume that $k \geq 4$ and it will be shown that the theorem holds by creating contradiction.
Let G be a k-critical graph, $k \geq 4$, that is NOT $(k-1)$-edge-connected. Then there exists a partition, V_1, V_2 of $V(G)$, such that the number of edges joining the vertices of V_1 and the vertices of V_2 is at most $k-2$.
G is k-critical, so the two induced subgraphs G_1 and G_2, containing the vertices of V_1 and V_2 repectively, are, per definition, $(k-1)$-colorable. Let G_1 and G_2 be given colorings from the same set of $k-1$ colors and assume that E' is the set of edges connecting the vertices in V_1 and the vertices in V_2. If every edge in E' joins vertices of different colors, G itself is $(k-1)$-colorable which, per definition, is impossible.
Hence some edges in E' must join vertices of the same color which contradicts the definition of coloring. It remains to show that by permutation of the colors assigned to the vertices of V_1, a proper coloring of G can be obtained. A coloring in which every edge of E' joins vertices of different colors, which again shows that G is $(k-1)$-colorable, creating a contradiction.
Let $U_1, U_2, ..., U_t$ denote the color classes of G_1 for which there is some vertex $u_i \in U_i$ $(1 \leq i \leq k-2)$ that is adjacent to a vertex of G_2. Let k_i denote the number of edges joining the vertices of U_i and the vertices of G_2. Now, for each k_i it must apply that

$$k_i \geq 1 \quad \text{and} \quad \sum_{i=1}^{t} k_i \leq k-2.$$

If the neighbors of every vertex $u_1 \in U_1$ are assigned a different color than u_1, an alteration of the color of the vertices in U_1 is unnecessary. However, if some vertex u_1 has a neighbor of same color, then a permutation of the $k-1$ colors used to color G_1 is conducted so that no vertex in U_1 is adjacent to a vertex of G_2 having the same color. This permutation is possible since there are at most k_1 (corresponding to the number of edges between U_1 and G_2) colors to avoid when recoloring the vertices of U_1 and there are $k-1-k_1 \geq 1$ colors available. Now U_2 is scrutinized. If no vertex u_2 has a neighbor of same color in G_2, then no permutation is performed. However, if some vertex u_2 is assigned the same color as one of its neighbors in G_2 then another permutation is implemented which leaves the vertices in U_1 unaltered. This is possible since there are at most $k_2 + 1$ (corresponding to the number of edges between U_2 and G_2 and the one color assigned to U_1)

colors to avoid but the number of colors available for U_2 is at least

$$(k-1)-(k_2+1) \geq (k-1)-(k_2+k_1) \geq 1.$$

This process of permuting the colors is continued until a $(k-1)$-coloring of G is obtained. This is, as aforementioned, impossible, thus a k-critical graph G, $k \geq 4$ must be $(k-1)$-edge-connected.

■

Chapter 3

Brooks' Theorem

3.1 The Original

In 1941 R. Leonard Brooks published a proof of what then became known as **Brooks' Theorem**. It states that there is a relationship between the chromatic number and the maximum degree of a graph. The theorem was at that time phrased as follows:

Theorem 3.1 (Brooks, 1941)
Let N be a network (or linear graph) such that at each node not more than n lines meet (where $n > 2$), and no line has both ends at the same node. Suppose also that no connected component of N is an n-simplex. Then it is possible to colour the nodes of N with n colours so that no two nodes of the same colour are joined.

It is obvious that N is equivalent to a simple graph G with maximum degree $\Delta(G) = k \geq 3$. Furthermore, a connected component of N is equivalent to a subgraph H of G and an *n-simplex* is a complete graph of order $k + 1$. Now the claim is that it is possible to conduct a vertex-coloring of G with no more than k colors. Brooks explains in his research note that without loss of generality one may assume that G is connected since otherwise the theorem can be proved for each connected component. Following his note, in which he allows multiple edges and applies induction, while at the same time using more modern terminology, the introductory remarks and corollary will be accounted for and the proof of the theorem will be given.

It is obvious that by greedy coloring G can be given a $(k+1)$-coloring with colors $c_0, c_1, ..., c_k$ such that each vertex is assigned a different color from those of its neighbors. The following three operations can be applied in order to minimize the occurence of the color c_0.
(1): A vertex v with no more than $k - 1$ neighbors can be recolored not-c_0. The term "recoloring" includes the case where no color is altered. Particu-

larly, a vertex with two neighbors of same color may be recolored not-c_0.

(2): Adjacent vertices v and u can be recolored without altering any other vertices so that v is not-c_0. Ignoring the edge uv, it is obvious that v can be recolored by (1) and then u can be recolored (possibly c_0).

(3): Let $v, v', v'', ..., u$ be a path in G. Beginning with v the vertices in this path can be recolored successively, without altering other vertices, so that at most u has the color c_0.

Corollary 3.2
If G is finite, choose u arbitrarily in G. Since there is a path joining u with every vertex $v \in G$, G can be recolored with at most u having the color c_0.

The proof of the theorem is by Brooks divided into two parts; one for finite graphs and one for infinite graphs. However, only the first part regarding finite graphs will be examined here.

Proof.
Let G be a finite, connected graph with $\Delta(G) = k$.

Case 1: If any vertex v has less than k neighbors then by (1), v can be colored not-c_0 and thus G is k-colored.

Case 2: Suppose v, u, a, b are any four distinct vertices and that there is a path from v to u not including a or b. Since G is not complete, vertices v, u that are not adjacent can be found. Let G be $(k+1)$-colored such that only u has the color c_0. Hence v and all of its neighbors are not-c_0. Now, either a) v has less than k neighbors or b) there are two vertices, a, b, adjacent to v which have the same color.

Because there is a path from v to u not including a or b, G can be recolored by (3) without altering a, b so that now at most v is colored c_0. If a) occurs, v can be recolored by case 1. If b) occurs, v can be recolored by (1). Thus G is k-colored.

Case 3: Let v, u, a, b be four distinct vertices such that every path from v to u passes through a or b. Now, consider the subgraphs of G with the following specifications:

H_1: Contains the vertex v and all vertices joined to v by some path not passing through a, b as an intermediate point. Hence a can only be in H_1 if $a - v$ does not pass through b and vice versa. H_1 also contains all edges connecting the aforementioned vertices.
H_2: Contains a, b, u and all vertices of G not in H_1 and all edges connecting these vertices.
Thus H_1 and H_2 are non-empty subgraphs of G and $H_1 \cup H_2 = G$ with $\Delta(H_1) \leq k$ and $\Delta(H_2) \leq k$. They have at least one of the vertices a, b in common and at most both vertices and all edges ab. The possibility of multiple edges ab does not interfere with any coloring of G.
Let m_i denote $deg(a)$ in H_i and let m_0 denote the number of edges ab . Then

$$m_1 + m_2 \leq m_0 + k \tag{3.1}$$

Now, the following three subcases must be considered:

Case 3.1: Suppose H_1 and H_2 have only one vertex, say a, in common. Then in both subgraphs a has less than k neighbors. Thus by case 1 both H_1 and H_2 can be k-colored. If a should not be the same color in both subgraphs then a permutation of the colors in one of them can ensure that a is of same color, hence G is k-colored.

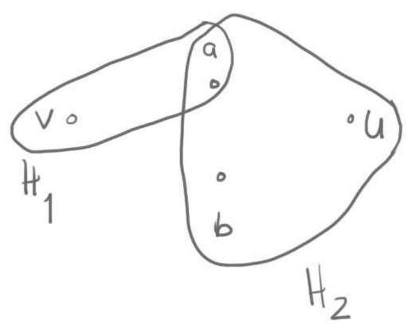

Figure 3.1: *Case 3.1.*

Case 3.2: One of the subgraphs, say H_1, is such that if the edge ab is added then it becomes a complete graph of order $k+1$. Thus a and b have $k-1$ neighbors in H_1. Now, H_1 and H_2 have both a and b in common and the edge ab does not exist.

Obviously, H_1 can be k-colored by assigning $k-1$ colors arbitrarily to the $k-1$ other vertices and the last remaining color to a and b. Clearly, because a and b have $k-1$ neighbors in H_1, there is only one edge connected to a in H_2 and also only one edge connected to b in H_2. Thus by case 1 H_2 can be k-colored with a and b the same color as in H_1. Now G is k-colored.

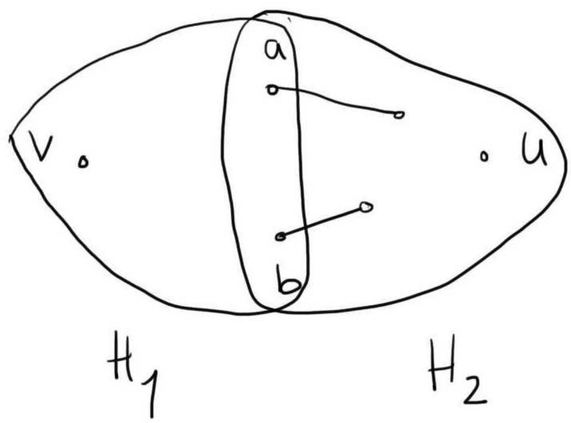

Figure 3.2: *Case 3.2.*

Case 3.3: Neither H_1 nor H_2 becomes a complete graph when adding the edge ab. However, suppose the edge ab is added. Then they both contain a, b and ab and by 3.1 they are of degree not greater than k.

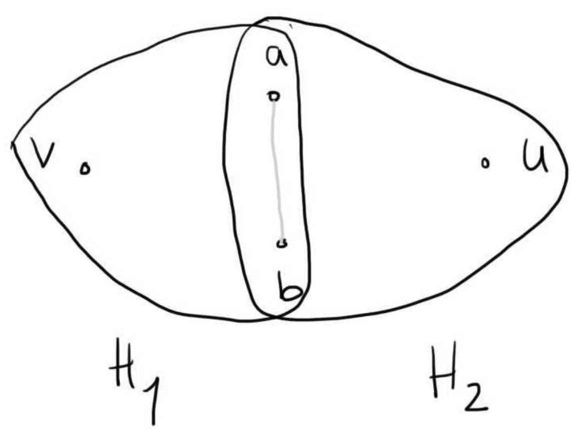

Figure 3.3: *Case 3.3.*

Now, if H_1 and H_2 are k-colorable so is G and by strong induction due to the fact that both H_1 and H_2 contain fewer vertices than G, both subgraphs must be k-colorable. Furthermore, since ab is in both subgraphs, the vertices a and b must have different colors in any k-coloring. By permutation of the colors in H_1 the colors of a and b can become the same as in H_2 and thus G is k-colorable.

■

3.2 A Contemporary Version

Over the years, the proof for this theorem has been given numerous times using a variety of techniques and terminology. The following is from Chartrand and Zhang's book "Chromatic Graph Theory".

Theorem 3.3 (Brooks' Theorem)
For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.

Proof. (Brooks by Chartrand & Zhang)
Let $\chi(G) = k \geq 2$ and let H be a k-critical subgraph of G as described in section 2.2. Obviously, $\Delta(H) \leq \Delta(G)$.
Suppose H is a complete graph K_k. Then $\Delta(H) = k - 1$. However, since G is connected, at least one vertex of H will have at least one more edge such that $\Delta(G) \geq k$ and so the following applies;

$$k = \chi(H) = \chi(G) \leq \Delta(G).$$

Suppose H is an odd cycle with $\chi(H) = 3$. Then $\Delta(H) = 2$ and, again, since G is connected, at least one vertex of H will have at least one more edge such that $\Delta(G) \geq 3$ and so the following applies;

$$3 = \chi(H) = \chi(G) \leq \Delta(G).$$

In both cases $\chi(G) \leq \Delta(G)$. Hence it is safe to assume that H is neither an odd cycle nor a complete graph. And since K_2 is the only 2-critical graph and odd cycles are the only 3-critical graphs it follows that $k \geq 4$. Furthermore, H is evidently 2-connected as clarified in section 2.2.

Now, suppose that H has order n. It has been established that $\chi(G) = k \geq 4$ which means that G can be colored using minimum 4 colors. Then, since H

is not complete and H is a k-critical subgraph of G, n must be greater than k such that $n \geq 5$. Otherwise H would be $(k-1)$-critical.
Since H is 2-connected, one of two cases can happen when removing two vertices from H. Either H is still connected, hence it is 3-connected or it is disconnected, thus H has connectivity 2.

Case 1: H is 3-connected.
H is not complete which means there are two vertices, $u, w \in H$, such that the shortest path between u and w has $d_H(u, w) = 2$. Now let (u, v, w) be a $u - w$ geodesic in H. Since H is 3-connected, $H - u - w$ is connected. Let $v = u_1$ and let $u_1, u_2, ..., u_{n-2}$ be the vertices of $H - u - w$. They must be listed in such manner that each vertex u_i, $2 \leq i \leq n-2$, is adjacent to some vertex preceding it. Furthermore, let $u_{n-1} = u$ and $u_n = w$. Consequently, for each set

$$U_j = \{u_1, u_2, ..., u_j\}, 1 \leq j \leq n,$$

the induced subgraph $H[U_j]$ is connected, see fig. 3.4.

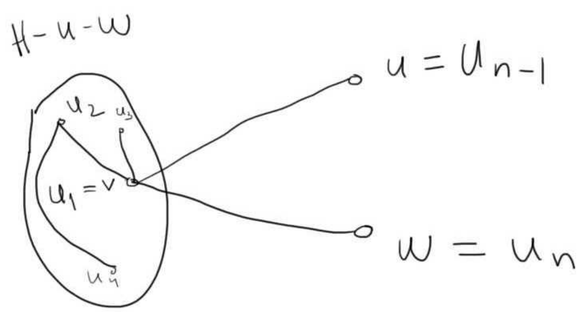

Figure 3.4: *An example of* U_j.

A greedy coloring is now applied to H with respect to the *reverse* ordering

$$w = u_n, u = u_{n-1}, u_{n-2}, ..., u_2, u_1 = v \tag{3.2}$$

of the vertices. w and u are not adjacent and each is therefore assigned the color 1.
In accordance with The Greedy Coloring Algorithm (definition 2.7) each vertex u_i, $2 \leq i \leq n-2$, is assigned the smallest color available in the set $\{1, 2, ..., \Delta(H)\}$; i.e. the smallest color that was not used to color any

preceding neighbor of u_i in the sequence (3.2). Each vertex u_i has at least one neighbor following it in the sequence (3.2) which entails that u_i has at most $\Delta(H) - 1$ neighbors preceding it, hence a color is available for u_i. Furthermore, $u_1 = v$ is adjacent to both $w = u_n$ and $u = u_{n-1}$ which both are assigned the color 1, thus at most $\Delta(H) - 1$ colors are assigned to v's neighbors, leaving a color for v. Now it is evident that $\chi(H) \leq \Delta(H)$ and thus, in combination with the initial assumptions

$$\chi(G) = \chi(H) \leq \Delta(H) \leq \Delta(G). \tag{3.3}$$

Case 2: H has connectivity 2, $\kappa(H) = 2$.
Claim: H contains a vertex x with degree greater than 2 but less than $n-1$, that is $2 < deg_H(x) < n-1$.
Suppose that this is not the case. Then every vertex of H has degree 2 or $n-1$. Since $\chi(H) \geq 4$ which means that the minimum number of colors used to color H is 4, H cannot contain only vertices of degree 2, since that would contravene the definition of the chromatic number. And since H is not complete, it cannot contain only vertices of degree $n-1$. Therefore, if H contains vertices of both degree 2 and $n-1$ and no others, then either

$$H = K_{1,1,n-2} \quad \text{or} \quad H = K_1 + \binom{n-1}{2} K_2.$$

An illustration can be seen in fig. 3.5.
In both cases $\chi(H) = 3$ and H is no longer critical, which is in conflict with the initial assumptions. Thus, H contains a vertex x such that $2 < deg_H(x) < n-1$ as claimed.
Since $\kappa(H) = 2$, the removal of a vertex x will result in either $\kappa(H-x) = 2$ or $\kappa(H-x) = 1$.
If $\kappa(H-x) = 2$, then x is not a part of a minimum vertex-cut of H, and since x cannot be a neighbor to all other vertices because $2 < deg_H(x) < n-1$, it implies that there is a vertex $y \in H$ such that $d_H(x, y) = 2$, see fig. 3.6.
Continuing as in Case 1 with $u = x$ and $w = y$, a coloring of H using at most $\Delta(H)$ colors is obtained and so (3.3) applies, that is, $\chi(G) \leq \Delta(G)$.

Finally, assume that $\kappa(H-x) = 1$. Then, since $H-x$ is still connected and contains cut-vertices, it must contain end-blocks as explained in section 1.2. These end-blocks, B_1 and B_2, contains cut-vertices x_1 and x_2, respectively. Since H is 2-connected there must exist vertices $y_1 \in V(B_1) - \{x_1\}$ and $y_2 \in V(B_2) - \{x_2\}$ such that x is adjacent to both y_1 and y_2, se fig. 3.7.

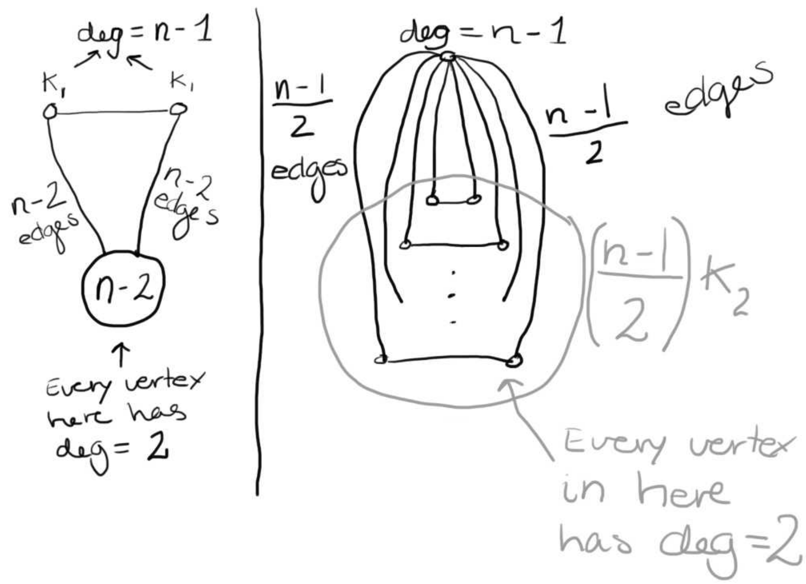

Figure 3.5: *An illustration of the two aforementioned possible graphs H.*

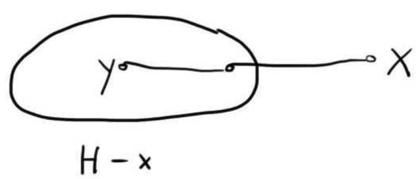

Figure 3.6: *An illustration of $d_H(x, y) = 2$.*

Obviously, $H - x - y_1 - y_1$ is connected. And since x has at least one more neighbor due to $2 < deg_H(x) < n - 1$, then $H - y_1 - y_2$ is also connected. Now it is possible to continue as in Case 1 with $u = y_1$ and $w = y_2$, thus a coloring of H using at most $\Delta(H)$ colors is obtained and so (3.3) applies, that is, $\chi(G) \leq \Delta(G)$.

■

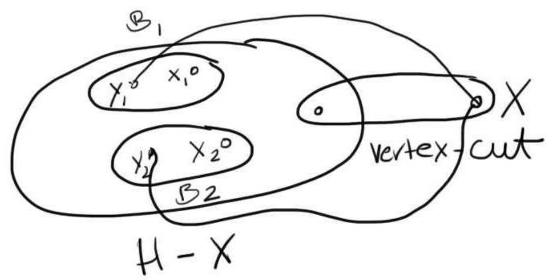

Figure 3.7: *An illustration of how x is adjacent to both y_1 and y_2.*

3.3 Catlin's Extension

In 1976 Paul A. Catlin proved a theorem that is considered an extension to Brooks' theorem. It was a part of his Ph.D. dissertation which he completed at Ohio State University. He showed that if a graph G, with $\Delta(G) = k$, does not contain $H = K_{k+1}$ as a subgraph and it has a Brooks-coloring, meaning it satisfies $\chi(G) \leq \Delta(G)$, then this graph contains a **monochromatic maximum independent set**. However, his proof is rather tedious and in 1978 John Mitchem of San Jose State University published a significantly shorter proof. In 2014 Vaidy Sivaraman, Department of Mathematical Sciences, Binghamton University, used these two proofs together with Brooks' original proof to complete a unified proof of Brooks' theorem and Catlin's theorem combined. This unified proof will be the focus of this section.

But first things first. What is a monochromatic maximum independent set? Independence in graphs applies to both edges and vertices. A set M of edges in G is independent if no two edges in M are adjacent, that is, no two edges share a vertex. Such a set is also referred to as a **matching**. If G contains no matching with more than $|M|$ edges, then M is a maximum matching and also a maximum independent set of edges.
A set U of vertices in G is independent if no two vertices in U are adjacent, that is, no two vertices share an edge. Such a set is also referred to as a **stable set**. If G contains no stable set with more than $|U|$ vertices, then U is a maximum stable set and also a maximum independent set of vertices. The number of vertices in such a maximum independent set is defined as the **independence number** and is denoted by $\alpha(G)$.

A monochromatic set is one where all the vertices or edges are of same color. Obviously the vertices or edges in a monochromatic set cannot be adjacent, hence it can only apply to an independent set.
In other words, a monochromatic maximum independent set of a graph G is a color class with size $\alpha(G)$.

In his published note, Sivaraman uses a slightly different notation than that of the previous writings in this thesis. However, modifications will be made in order to match the notation throughout the thesis. Further, he makes use of Hall's Condition, also known as a marriage condition, which states that for a bipartite graph G with partite sets U and W and for a subset $S \subseteq U$ where $N(S)$ denotes the set of all vertices in W having a neighbor in S, the number of vertices in $N(S)$ is greater than or equal to the number of vertices in S for all subsets $S \subseteq U$. More formally:

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq U.$$

Actually, in 1935 Philip Hall proved the following theorem:

Theorem 3.4 (Hall 1935)
A bipartite graph G with partite sets U and W contains a matching M of U if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$.

A matching M of U is when every vertex in U is an endpoint of some edge in the matching, hence the number of edges in M equals the number of vertices in U, that is, $|E(M)| = |V(U)|$.

Proof.
1: If G has a matching of U, then $|N(S)| \geq |S|$ for all $S \subseteq U$.
Assume M is a matching that contains all vertices of U. Let the set of all vertices of W matched by M to a given S be denoted $M(S)$. Thus, by definition, $|M(S)| = |S|$. Now, since all elements of $M(S)$ are adjacent to an element of S, then $M(S) \subseteq N(S)$. Now, $|N(S)| \geq |M(S)|$, hence $|N(S)| \geq |S|$.

2: If $|N(S)| \geq |S|$ for all $S \subseteq U$, then G has a matching of U.
In order to create a contradiction, assume that G is bipartite and has no matching of U. Let M be a maximum matching and $u \in U$ a vertex not covered by M. Consider all non-trivial paths in G that alternately uses edges not in M and edges in M and goes from u to another vertex in U, and let

U' denote the end-vertices of these paths. Further, let $W' \subseteq W$ denote the set of all penultimate vertices of the aforementioned paths. The last edges of these paths are contained in M, that is, $|U'| = |W'|$. Hence, by Hall's Condition, there must be an edge from a vertex $v \in S = U' \cup u$ to a vertex $w \in W - W'$.
Since $v \in U' \cup u$, there is an alternating path, say P, from u to v. Furthermore, either $P' = Pvw$ or $P' = Pw$ (if $w \in P$) is an alternating path from u to w. This is illustrated in fig. 3.8.

Figure 3.8: *The path P'.*

Now, let M' be the symmetric difference of M and $E(P')$,

$$M \oplus E(P') = (M \cup E(P')) - (M \cap E(P')).$$

The first edge of P' is not in M and if the last edge is not in M either, i.e. w is unmatched, then $M' > M$.
Suppose w was matched by $u'w \in M$. Then $P'wu'$ would be an alternating path placing u' in U' and w in W'. However, $w \notin W'$, thus w is unmatched and $M' > M$ which contradicts the initial assertion of M being a maximum matching.

∎

Theorem 3.5 (Sivaraman: Brooks and Catlin combined)
Let G be a simple finite graph with $\Delta(G) \geq 3$. If G does not contain $H = K_{k+1}$ as a subgraph, then G has a k-coloring in which one color class has size $\alpha(G)$. In particular, $\chi(G) \leq k$.

Proof.
The proof is accomplished by induction on $|V(G)|$. The induction step considers two cases; either $k \geq 4$ or G contains a subgraph $H = K_k$. Thus the base is when $k = 3$ and G contains no triangles, that is, no subgraph $H = K_3$.

Base case: If a maximum independent set I can be chosen such that $G - I$ is bipartite, then G can be 3-colored with I as one color class. For this purpose, choose an independent set I of size $\alpha(G)$ such that the number of odd cycles in $G - I$ is minimum.
Suppose that $G - I$ contains an odd cycle C. If it is possible to construct another independent set I' where $|I'| = |I|$ and $G - I'$ has fewer odd cycles than $G - I$, then it is a contradiction as to the choice of I and it proves that $G - I$ is bipartite.
Choose a $v \in V(C)$ and consider all paths P starting at v and alternating between vertices of I and non-isolated vertices of $G - I$, subject to $V(G - I) \cap V(P)$ being independent. Let P_0 be such a path of maximum length. Let I' be the symmetric difference of I and $V(P_0)$,

$$I \oplus V(P_0) = (I \cup V(P_0)) - (I \cap V(P_0))$$

such that I' contains the vertices of I that are not a part of P and the vertices of $G - I$ that are a part of P. This is illustrated in fig. 3.9.

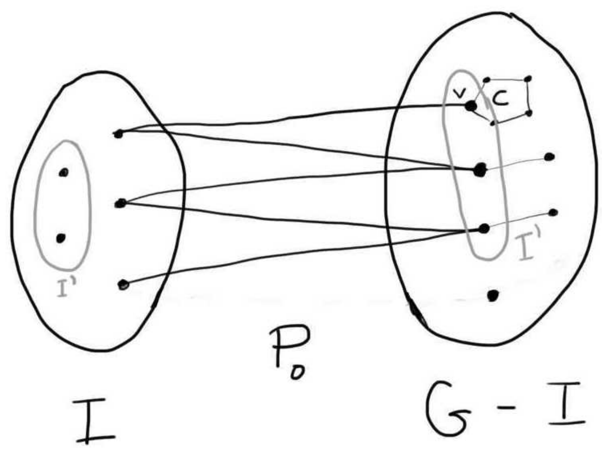

Figure 3.9: *An illustration of P_0 and I'.*

I' is independent because 1) v cannot have any neighbors in I' since it has two neighbors in C and one in $I - I'$. And 2) every vertex of $I' - I - v$ has two neighbors in $I \cap V(P_0)$, none of which are in I', and a neighbor in $G - I$, since the vertices of $P_0 \in G - I$ are non-isolated. And 3) the remaining vertices of I' were independent to begin with. I' cannot be greater than I, since I is maximum and since P_0 starts outside I, it must end in I, thus

$|I'| = |I| = \alpha(G)$.

No cycle in $G - I'$ contains a vertex of $I \cap V(P_0)$ because, by construction, each vertex, except the final one, of $I \cap V(P_0)$ is adjacent to two vertices of I'. The final vertex of P_0 has one neighbor in I' and if it has other neighbors, they must be isolated vertices of $G - I$, as shown in the following.
Let w be this final vertex. w can be adjacent to x and y in $G - I$ if x, y have neighbors, x', y' respectively, in $G - I$. x', y' are in P_0. If there is an $x - y$ path Q, then Q together with w would be a cycle in $G - I'$. However, since there can be no edge between x and y because that would create a triangle, then at least one of the neighbors, x' and y', must be in Q. Thus the cycle cannot be in $G - I'$. Therefore, it is safe to assume that Q is part of a longer path $x'xQyy'$. Thus x, y have degree 2 in $G - I$ and so their only neighbor in I is w. Hence, w can be replaced in I with x, y, thus creating an independent set exceeding I; a contradiction as to I being maximum.

This means that every odd cycle in $G - I'$ is also an odd cycle in $G - I$. But C is an odd cycle in $G - I$ that is NOT in $G - I'$. Thus the number of odd cycles in $G - I'$ is strictly less than in $G - I$, which contradicts the choice of I. Hence $G - I$ contains no odd cycle and is indeed bipartite and can be 2-colored (colors 1 and 2). I is a maximum independent set which can be given the color 3, thus G is 3-colored, $\chi(G) = 3$, and the color class 3 has size $\alpha(G)$.

Induction hypothesis: Assume that the theorem holds for any graph H, that satisfies the conditions and contains fewer vertices than G.

Induction step: Suppose $k \geq 4$ and G does not contain a copy of K_k. Let I be a maximum independent set.
Now, $G - I$ must have $\Delta(G - I) \leq k - 1$, so by induction hypothesis, $G - I$ has a $(k-1)$-coloring. And by applying the color k to I, the desired coloring is achieved. Should the maximum degree of $G - I$ be less than $k - 1$, then the induction is unnecessary; a $(k - 1)$-coloring can be obtained by using The Greedy Coloring Algorithm (see section 2.1.1).

Now, suppose G contains a subgraph $H = K_k$. Let $U = \{u_1, ..., u_k\}$ denote the vertices of H. For each u_i, let a_i be a neighbor outside U and let $A = \{a_i | 1 \leq i \leq k\}$. Suppose all $u_i's$ have a neighbor in A. The a_i does not have to be distinct, but since G does not contain $H = K_{k+1}$ they cannot all be equal. Let $G' = G - U$.

The goal is to color G' by induction hypothesis and then extend the coloring to G. Should the maximum degree of G' be less than $k - 1$, then (as above) it can be colored greedily beginning with a maximum independent set. Obviously, all the u_i's must be assigned distinct colors such that k colors are used. Thus the a_i's cannot all have the same color.
Suppose G' has a maximum independent set I that does not contain all of A. To ensure that the a_i's are not assigned the same color, create G'' from G' by adding some edge $a_i a_j$ where $a_i \notin I$, see fig. 3.10.

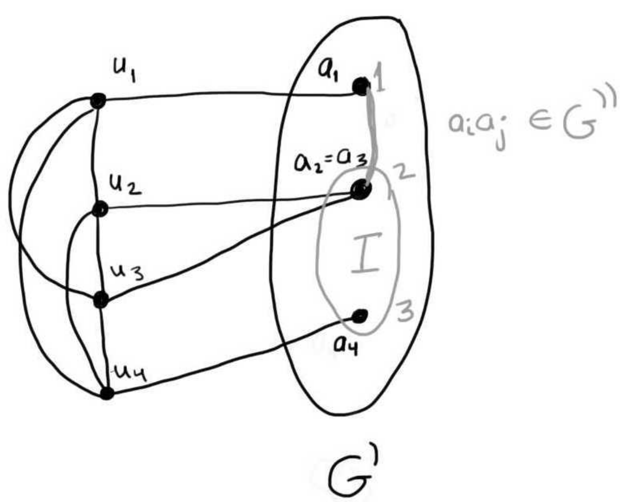

Figure 3.10: *Creating G''.*

Thus I is a maximum independent set of both G' and G'' which implies that $\alpha(G') = \alpha(G'')$. By the induction hypothesis there is a k-coloring of G'' with one color class of size $\alpha(G'')$. And if G'' has a k-coloring, then so does G'. This k-coloring can be extended to G as follows.
Each u_i has one neighbor in A, thus for every u_i there is $k - 1$ available colors. Construct a bipatite graph as in fig. 3.11 with U as one part and the k colors as the other. Let each vertex u_i be adjacent to colors not used on a_i.
If Hall's theorem (3.4) holds for the entire graph, such that the number of available colors are greater than or equal to the number of vertices in U, then there is a matching and the k-coloring can be extended to U. Hence G is k-colored.

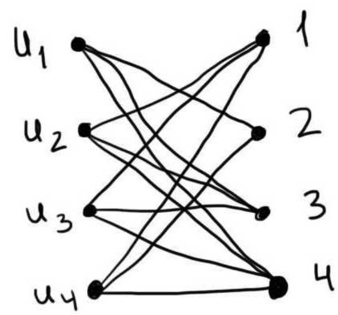

Figure 3.11: *The bipartite graph.*

The only way for Hall's theorem not to hold for the entire graph, is if all the a_i's are assigned the same color. This cannot occur due to the edge added when forming G''. Now, G is k-colored and the largest color class has size $\alpha(G') + 1 = \alpha(G)$, since every independent set in G contains at most one vertex of U.

Suppose instead that every maximum independent set of G' contains all of A. To ensure that the a_i's are not assigned the same color, form G'' from G' by adding an arbitrary edge a_ia_j, see fig.3.12. Thus $\alpha(G') = \alpha(G'') + 1$. Now, apply the induction hypothesis to G'' and extend the k-coloring by Hall's theorem as above.

Now the largest color class has size $\alpha(G') = \alpha(G)$. The equality holds because the maximum independent set of G' contains all of A but when forming G'', one vertex is removed but this one is replaced by one in U when extending the coloring to G.

Finally, assume some u_i has no neighbor a_i as in fig. 3.13.

G' can be k-colored by induction hypothesis. This k-coloring can be extended to G by greedily coloring the vertices in U beginning with a vertex adjacent to a maximum independent set in G' and ending with the vertex without a neighbor a_i. There are $k - 1$ colors available for the first vertex since it has only one neighbor in G' and none of the u_i's have been assigned a color. For the second vertex, at least $k - 2$ colors are available and so on, leaving exactly one color available for the last vertex in U.

■

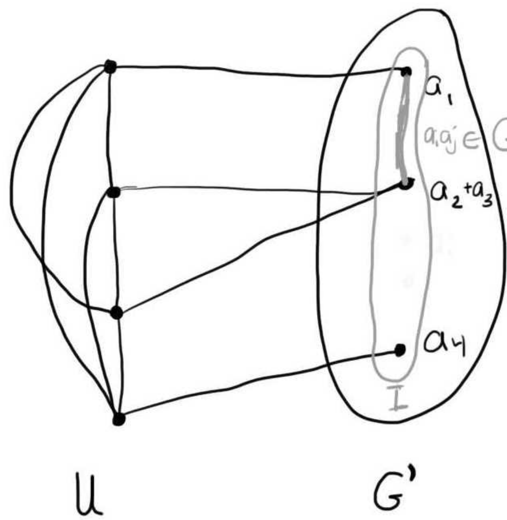

Figure 3.12: *G'' when G' contains all of A.*

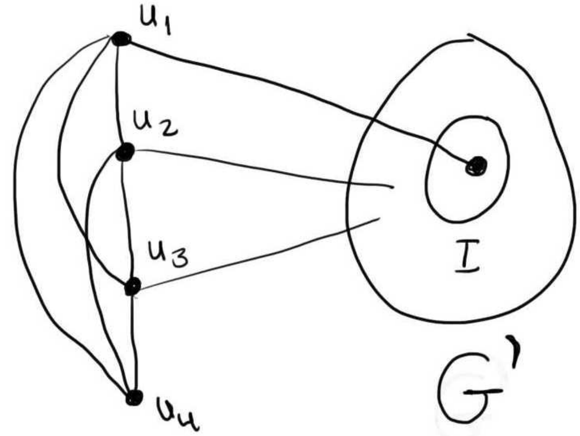

Figure 3.13: *Some u_i without a neighbor a_i.*

Chapter 4

Recapitulation

The first chapter provided the necessary knowledge for the rest of the thesis. The traits of paths and cycles in graphs was emphasized, as it was essential to understand these terms in order to complete the proofs of interest. Also Whitney's inequality between connectivity, edge-connectivity, and minimum degree of a graph G was accounted for.

Chapter 2 covered the fundamentals of graph coloring and offered a thorough study of the chromatic number $\chi(G) = k$ which is the minimum number of colors with which a graph G can be colored. It was shown that the lower bound for $\chi(G)$ can be determined by the chromatic numbers of G's subgraphs. Furthermore, the task of coloring a graph greedily was explained.
While writing this chapter it became apparent that the subject of color-critical graphs had to be covered, since these play an important role in the proofs of the contemporary and unified version of Brooks' theorem. As explained, a graph is considered color-critical if the chromatic number of any proper subgraph is lower than the chromatic number of the graph. Moreover, it was derived that every k-chromatic graph, $k \geq 2$, contains a k-critical subgraph.

Chapter 2 became the theoretical foundation on which chapter 3 was build. Starting of with an exhaustive review of Brooks' original theorem and proof, rewriting the latter using more modern terminology, the basis for examining a more recent version was formed. Although terminology and phrasing differs a lot from the original proof by Brooks, the newer version by Chartrand and Zhang is not so different since the utilization of for instance paths, greedy coloring, and permutation is evident in both proofs.
Looking into the theorem of Brooks' and various proofs, it is impossible not to stumble upon Catlin's extension of 1976, which is a clever strengthening of the original theorem by Brooks. It states that a graph G which satisfies the conditions of Brooks' theorem, contains a monochromatic maximum independent set. Catlin's original proof was in 1978 shortened significantly by John Mitchem. However, a combined version of Brooks' theorem and

Catlin's theorem had been proved and published by Sivaraman earlier this year, wherefore this unified proof became of interest.
Sivaraman utilizes matchings and independence in graphs along with Hall's condition in order to prove the theorem by induction. However, as the proof by Sivaraman seemed insufficient, the version given in this thesis ended out being much more detailed and elaborate.

Bibliography

R. L. Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(2):194–197, 1941.

Paul A. Catlin. Brooks' graph-coloring theorem and the independence number. *Journal of Combinatorial Theory, Series B*, 27(1):42–48, 1979.

G. Chartrand and P. Zhang. *Chromatic Graph Theory*. Chapman and Hall/CRC, 2009. ISBN 9781584888000.

Reinhard Diestel. *Graph Theory*. Springer, 2010. ISBN 9783642142789.

John Mitchem. A short proof of catlin's extension af brooks' theorem. *Discrete Mathematics*, 21:213–214, 1978.

Vaidy Sivaraman. A unified proof of brooks' theorem and catlin's theorem. *Discrete Mathematics*, 338(2):272–273, 2015.