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Singular Integral Operators on Matrix-weighted L^p Spaces

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Abstract: We consider the problem of extending weighted inequalities for a singular integral operator T to the vector-valued operator $\vec{T}$ defined componentwise by $(\vec{T}f)_i = Tf_i$, for functions $f = (f_1, \ldots, f_d)$ from $\mathbb{R}^n$ into $\mathbb{C}^d$. We introduce the notion of a matrix weight and the associated weighted norm space $L^p(W)$. The classic Muckenhoupt A_p condition is extended to matrix weights and several alternative characterizations of the A_p class is given. The main result is that $\vec{T}$ is bounded from $L^p(W)$ into $L^p(W)$ whenever W is an A_p matrix weight. We also show that, with one additional assumption on the kernel of T, the converse holds; if $\vec{T}$ is bounded on $L^p(W)$, then necessarily W is an A_p weight. As basic tools in our analysis, we introduce the notions of maximal functions and interpolation, and show several fundamental results concerning these. Also, standard results from the technique of "truncating integrals" are covered here.

Preface

This thesis is the result of my Mat-6 Project at The Department of Mathematical Sciences at Aalborg University. Some of the results here are of fundamental type in the theory of singular integrals, while others are of more recent appearing. In particular, the main theorems are based on the paper "Matrix A_p Weights Via Maximal Functions", by Michael Goldberg [1].

The reader is assumed to be familiar with basic measure and integral theory, basic functional analysis and also some theory of distributions. The last page of the report contains a list over references, which is referred to by a number in brackets, [*reference number*]. Also, in the back of the report, there is an index of notation.

Anders G. Aaen
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Chapter 1

Introduction

In the present thesis we consider singular integral operators of convolution type. Formally, these are operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy, \tag{1.1}$$

defined for suitable complex-valued functions f on $\mathbb{R}^n$. The *kernel* K is singular near the origin in the sense that the integral in (1.1) need not to converge absolutely. The subject of interest concerning such operators is their boundedness properties as linear operators between (weighted) L^p spaces. A thorough study of this has had great impact in the theory of partial differential equations. In what follows, we consider a classical example of a singular integral operator.

1.1 The Riesz Transform(s)

In $\mathbb{R}^n$ there exists n Riesz transforms similarly defined. For $1 \leq j \leq n$, we define the function $K = K^{(j)}$ on $\mathbb{R}^n \backslash \{0\}$ by

$$K(x) = c_n \frac{x_j}{|x|^{n+1}}, \quad x = (x_1, \dots, x_n),$$

where $|\cdot|$ denotes The Euclidean norm on $\mathbb{R}^n$ and

$$c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

is a normalization constant. The j'th *Riesz transform* $R = R^{(j)}$ is then given by

$$Rf(x) = p.v. \int K(x-y)f(y)\,dy := \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} K(x-y)f(y)\,dy,$$

whenever this limit exists. In order to obtain an explicit domain of definition for R, it is convenient to introduce a tempered distribution (also denoted by K) by

$$\langle K, f\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} K(x)f(x)\,dx,$$

defined for Schwartz functions $f \in \mathcal{S}$. To see that this indeed is a well-defined tempered distribution, we first notice that

$$\int_{|x|\geq\varepsilon} K(x)f(x)\,dx = \int_{\varepsilon\leq|x|\leq 1} K(x)(f(x)-f(0))\,dx + \int_{|x|>1} K(x)f(x)\,dx,$$

since K has integral zero over the set $\varepsilon \leq |x| \leq 1$. By The Mean Value Theorem, we see that

$$|f(x)-f(0)| \leq C_f|x|, \quad \text{where} \quad C_f := \sum_{i=1}^{n}\sup_{x\in\mathbb{R}^n}|\partial_i f(x)|.$$

It follows that $|K(x)(f(x)-f(0))| \leq c_n C_f |x|^{-n+1}$ and, by The Dominated Convergence Theorem, we conclude that

$$\int_{\varepsilon\leq|x|\leq 1} K(x)(f(x)-f(0))\,dx \to \int_{|x|\leq 1} K(x)(f(x)-f(0))\,dx \quad \text{as } \varepsilon\to 0.$$

For the integral over the set $|x| > 1$, we simply notice that

$$\int_{|x|>1}|K(x)f(x)|\,dx \leq c_n\int_{|x|>1}\frac{|xf(x)|}{|x|^{n+1}}\,dx \leq nv_nc_nC'_f,$$

where $C'_f := \sup_{x\in\mathbb{R}^n}|xf(x)|$ and v_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Thus K is well-defined and

$$|\langle K, f\rangle| \leq nv_nc_n(C_f + C'_f) \to 0, \quad \text{whenever } f\to 0 \text{ in } \mathcal{S},$$

showing that K is a tempered distribution. We can now express Rf in terms of the distribution K as

$$Rf(x) = (K * f)(x) := \langle K, f(x-\cdot)\rangle, \quad \text{for } f\in\mathcal{S}.$$

That is, R is the operator given by convolution with the tempered distribution K.

Employing the Fourier transform $\mathcal{F}$ and the co-Fourier transform $\bar{\mathcal{F}}$, it is possible to give an alternative characterization of the Riesz transform, namely

$$Rf(x) = \bar{\mathcal{F}}\big(-i\frac{\xi_j}{|\xi|}\mathcal{F}f(\xi)\big)(x), \quad \text{for } f\in S.$$

This is shown in e.g. [2]. By The Plancherel Theorem, we then conclude that the Riesz transform is bounded on $L^2(m)$, i.e.

$$\int|Rf|^2\,dm \leq C\int|f|^2\,dm, \quad \text{for all } f\in\mathcal{S},$$

where m denotes the Lebesgue measure.

The Riesz transforms illustrates perfectly the kind of singular integral operators we will be considering. In general, we say that T is a *singular integral operator (of convolution type) associated with a regular kernel* K if $Tf = K * f$, for some tempered distribution K, such that

(i) away from the origin, K agrees with an ordinary function, also denoted by K. This means that there exists a measurable function $K : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ such that

$$\langle K, f \rangle = \int K f \, dm,$$

whenever f is a compactly supported C^∞ function that vanishes near the origin.

(ii) The function K is C^1 and there exists constants $B, C > 0$ such that

$$|K(x)| \leq B|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leq C|x|^{-n-1}, \quad \text{for all } x \neq 0.$$

Having introduced the notion of singular integral operators, we now briefly clarify the aim of this thesis.

1.2 Weighted Inequalities

A *weight* is a positive measurable function on $\mathbb{R}^n$. Associated to each weight w and each exponent $1 \leq p < \infty$, we define the *weighted* space $L^p(w)$ as the set of Borel functions $f : \mathbb{R}^n \to \mathbb{C}$, for which

$$\|f\|_{L^p(w)} := \left(\int |f|^p w \, dm \right)^{1/p} < \infty.$$

Now, assume that T is a singular integral operator associated with a regular kernel, and assume that

$$\int |Tf|^p \, dm \leq C \int |f|^p \, dm, \quad \text{for all } f \in \mathcal{S},$$

for some $1 < p < \infty$ and some constant $C > 0$. It is of interest to characterize the set of all weights w such that T is bounded from $L^p(w)$ into itself, i.e.

$$\|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}, \quad \text{for all } f \in L^p(w). \tag{1.2}$$

This problem was solved in the 1970's by Hunt-Muckenhoupt-Weeden [5]. For $1 < p < \infty$, we let $p' = p/(p-1)$, and we let $|E|$ denote the Lebesgue measure of any measurable set E.

Theorem 1.1 (Hunt-Muckenhoupt-Weeden)**.** *T is bounded on $L^p(w)$ if w satisfies the A_p condition, i.e. if there exists a constant $A_p > 0$ such that*

$$\left(\frac{1}{|B|}\int_B w\,dm\right)^{1/p}\cdot\left(\frac{1}{|B|}\int_B w^{-p'/p}\,dm\right)^{1/p'} \leq A_p,$$

for all Euclidean balls $B \subset \mathbb{R}^n$.

Furthermore, it was shown that, with one additional assumption on the kernel (stated in Chapter 6), the converse is also true: if T is bounded from $L^p(w)$ into itself, then necessarily w satisfies the A_p condition. The main result of this thesis is a generalization of Theorem 1.1 (and its converse), showed recently in [1]. Given the operator T, we define a new operator $\vec{T}$ by

$$\vec{T}f = (Tf_1, \ldots, Tf_d),$$

for vector functions $f = (f_1, \ldots, f_d)$ from $\mathbb{R}^n$ into $\mathbb{C}^d$. The above notion of a weight generalizes to a matrix-valued function from $\mathbb{R}^n$ into the set of positive definite $d \times d$ matrices. Each matrix weight W induces a weighted space $L^p(W)$ of vector functions. We state a matrix analogue of the A_p condition and show that Theorem 1.1 generalizes perfectly to the vector-valued operator $\vec{T}$.

Chapter 2

Interpolation and Maximal Functions

In this chapter we introduce the notions of interpolation and maximal functions. These are fundamental tools in our analysis of singular integral operators. The main results are Theorem 2.14, The Marcinkiewicz Interpolation Theorem, and Theorem 2.17, The Maximal Theorem.

We let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{C}^d \times \mathbb{C}^d$, i.e.

$$\langle (u_1, \ldots, u_d), (v_1, \ldots, v_d) \rangle = u_1 \bar{v}_1 + \ldots + u_d \bar{v}_d,$$

and $|\cdot|$ denotes the Euclidean norm $|v| = \sqrt{\langle v, v \rangle}$. For any $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x, r)$ denote the Euclidean ball in $\mathbb{R}^n$ with center x and radius r, i.e.

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

The Lebesgue measure on the Borel algebra in $\mathbb{R}^n$ is denoted by m and we will consistently write $|E| = m(E)$, for measurable sets $E \subseteq \mathbb{R}^n$. We also use dx as a shorthand for $dm(x)$.

2.1 L^p and Weak L^p

Let μ denote a nonnegative σ-finite measure on a sigma-algebra over a nonempty set X. We are primary interested in the case where μ is the Lebesgue measure. However, we will keep the general setup in this and the following section, since it is just as easy. Denote by $\mathcal{M}(\mu)$ the set of all μ-measurable functions from X into $\mathbb{C}^d$.

Definition 2.1. For $1 \leq p < \infty$ we define *the space* $L^p(\mu)$ as the set of all $f \in \mathcal{M}(\mu)$ with

$$\|f\|_p := \|f\|_{L^p(\mu)} := \left(\int |f|^p d\mu \right)^{1/p} < \infty.$$

Functions in $L^p(\mu)$ are called *equal in* $L^p(\mu)$ if they are equal μ-a.e. in the usual sense. Notice that any measurable function f defined a.e. on X can be extended to a function $\tilde{f} \in \mathcal{M}(\mu)$, and we may define $\|f\|_p = \|\tilde{f}\|_p$ independent of the actual extension.

In the scalar case $d = 1$, it is a well-known fact that $(L^p(\mu), \|\cdot\|_p)$ is a Banach space. This is easily extended to the general case, since

$$|f_i| \leq |f| \leq \sqrt{d} \max_i |f_i| \leq \sqrt{d} \sum_i |f_i|$$

and, as a consequence,

$$\|f_i\|_p \leq \|f\|_p \leq \sqrt{d} \sum_i \|f_i\|_p,$$

for any function $f = (f_1, \dots, f_d) \in \mathcal{M}(\mu)$.

If $f = (f_1, \dots, f_d) \in L^1(\mu)$, then we define *the* μ*-integral of* f by

$$\int f \, d\mu = \Big(\int f_1 \, d\mu, \dots, \int f_d \, d\mu \Big).$$

We still have the property that

$$|\int f \, d\mu| \leq \int |f| \, d\mu.$$

To see this, fix an arbitrary $f \in L^1(\mu)$ with $\int f \, d\mu \neq 0$, and let

$$u = |\int f \, d\mu|^{-1} \int f \, d\mu.$$

Then $|u| = 1$ and

$$|\int f \, d\mu| = |\langle u, \int f \, d\mu \rangle| \leq \int |\langle u, f(x) \rangle| \, d\mu(x) \leq \int |f| \, d\mu.$$

Definition 2.2. Given any function $f \in \mathcal{M}(\mu)$, we define the *distribution function* d_f on $[0, \infty)$ by

$$d_f(\alpha) = \mu\left(\{x \in X : |f(x)| > \alpha\}\right).$$

Notice that

$$d_f(\alpha) = \int_X \chi_G(x, \alpha) d\mu(x),$$

where $G := \{(x, \alpha) \in X \times (0, \infty) : |f(x)| > \alpha\}$ is a $\mu \otimes m$-measurable subset of $X \times (0, \infty)$. By Tonelli's Theorem, this implies in particular that d_f is measurable on $(0, \infty)$.

Proposition 2.3. *The distribution function* d_f *is decreasing and right-continuous.*

Proof. It is clear that d_f is decreasing. To see that it is right-continuous, we first assume that $\alpha_n \searrow \alpha$. For each $n \in \mathbb{N}$ we let

$$E_n = \{x \in X : |f(x)| > \alpha_n\}.$$

Then $\chi_{E_n} \nearrow \chi_E$, where $E := \{x \in X : |f(x)| > \alpha\}$ and, by The Monotone Convergence Theorem, $d_f(\alpha_n) = \mu(E_n) \nearrow \mu(E) = d_f(\alpha)$. Now assume more generally that $\alpha_n \to \alpha^+$ and fix an arbitrary $\varepsilon > 0$. Since $(\alpha + 1/n) \searrow \alpha$, the preceding implies that $d_f(\alpha + 1/n) \nearrow d_f(\alpha)$, and therefore we can find $m \in \mathbb{N}$ such that

$$d_f(\alpha) - d_f(\alpha + 1/m) < \varepsilon.$$

Then choose $N \in \mathbb{N}$ such that $\alpha_n \leq \alpha + 1/m$ whenever $n \geq N$. Since d_f is decreasing, this implies that

$$d_f(\alpha) - d_f(\alpha_n) \leq d_f(\alpha) - d_f(\alpha + 1/m) < \varepsilon,$$

whenever $n \geq N$. □

The following properties are easily verified using the definition of the distribution function.

Proposition 2.4. *For any $f, g \in \mathcal{M}(\mu)$ and $\alpha, \beta \geq 0$ we have*

(i) $d_f \leq d_g$ *whenever* $|f| \leq |g|$ *μ-a.e.,*

(ii) $d_{cf}(\alpha) = d_f(\alpha/|c|)$, *for any* $c \in \mathbb{C}\backslash\{0\}$, *and*

(iii) $d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta)$.

Proposition 2.5. *For any $f \in \mathcal{M}(\mu)$ and $1 \leq p < \infty$, we have*

$$\|f\|_p^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha. \tag{2.1}$$

Proof. Let $G = \{(x, \alpha) \in X \times (0, \infty) : |f(x)| > \alpha\}$. Then

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X \chi_G(x, \alpha) d\mu(x) d\alpha \\ &= p \int_X \int_0^{|f(x)|} \alpha^{p-1} d\alpha d\mu(x) \\ &= \int_X |f(x)|^p d\mu(x), \end{aligned}$$

where we have employed Tonelli's Theorem to interchange the order of integration. □

Definition 2.6. The *space* $L^\infty(\mu)$ is the set of all $f \in \mathcal{M}(\mu)$ with

$$\|f\|_\infty := \|f\|_{L^\infty(\mu)} := \inf\{\alpha > 0 : d_f(\alpha) = 0\} < \infty.$$

Notice that $d_f(\|f\|_\infty) = 0$. We call functions in $L^\infty(\mu)$ *equal* if they are equal a.e. in the usual sense. From Proposition 2.4 it readily follows that $L^\infty(\mu)$ is a normed space. Furthermore, since

$$d_{f_i}(\alpha) \le d_f(\alpha) \le \sum_i d_{f_i}(\alpha/\sqrt{d}),$$

and consequently,

$$\|f_i\|_\infty \le \|f\|_\infty \le \sqrt{d} \sum_i \|f_i\|_\infty,$$

for any $f = (f_1, \dots, f_d) \in \mathcal{M}(\mu)$, we easily extend the well-known fact that $(L^\infty(\mu), \|\cdot\|_\infty)$ is a Banach space, when $d = 1$, to the general case.

We now define a space somewhat larger that L^p.

Definition 2.7. For $1 \le p < \infty$ we define *the space* $L^{p,\infty}(\mu)$ as the set of all $f \in \mathcal{M}(\mu)$ with

$$\|f\|_{p,\infty} := \|f\|_{L^{p,\infty}(\mu)} := \inf\{C > 0 : d_f(\alpha) \le C^p/\alpha^p \text{ for all } \alpha > 0\} < \infty.$$

For convenience we let $L^{\infty,\infty}(\mu) = L^\infty(\mu)$.

The space $L^{p,\infty}(\mu)$ is called *weak* $L^p(\mu)$. Notice that, for $p < \infty$,

$$d_f(\alpha) \le \frac{\|f\|_{p,\infty}^p}{\alpha^p}, \quad \text{for all } \alpha > 0,$$

and

$$\|f\|_{p,\infty} = \sup\{\alpha d_f(\alpha)^{1/p} : \alpha > 0\}.$$

As for $L^p(\mu)$, we consider functions *equal in* $L^{p,\infty}(\mu)$ if they are equal a.e. in the usual sense.

Proposition 2.8. *For each* $1 \le p \le \infty$, $L^{p,\infty}(\mu)$ *is a quasi-normed space.*

Proof. If $\|f\|_{p,\infty} = 0$ then $d_f(0) = d_f(\|f\|_{p,\infty}) = 0$ and hence $f = 0$ μ-a.e. Combined with Proposition 2.4 (ii), this shows in particular that $\|cf\|_{p,\infty} = |c|\|f\|_{p,\infty}$, for any $c \in \mathbb{C}$. To verify the quasi-triangle inequality, we apply Proposition 2.4 (iii) to obtain

$$\alpha d_{f+g}(\alpha)^{1/p} \le 2\big(\|f\|_{p,\infty}^p + \|g\|_{p,\infty}^p\big)^{1/p} \le 2\big(\|f\|_{p,\infty} + \|g\|_{p,\infty}\big),$$

for all $\alpha > 0$, and hence

$$\|f + g\|_{p,\infty} \le 2(\|f\|_{p,\infty} + \|g\|_{p,\infty}),$$

as desired. □

Proposition 2.9. *For any* $f \in \mathcal{M}(\mu)$ *and* $1 \le p \le \infty$, *we have* $\|f\|_{p,\infty} \le \|f\|_p$ *and, as a consequence,* $L^p(\mu) \subseteq L^{p,\infty}(\mu)$.

Proof. Since

$$\alpha^p \chi_{\{x\in X:|f(x)|>\alpha\}}(x) \le |f(x)|^p,$$

for all $x \in X$ and $\alpha > 0$, it follows that $\alpha^p d_f(\alpha) \le \|f\|_p^p$, for all $\alpha > 0$, and hence $\|f\|_{p,\infty} \le \|f\|_p$. □

When $p < \infty$, the space $L^p(\mu)$ is in general a proper subset of $L^{p,\infty}(\mu)$ as illustrated by the following example.

Example 2.10. Define f m-a.e. on $\mathbb{R}^n$ by $f(x) = |x|^{-n/p}$. Using polar coordinates we see that

$$\|f\|_p^p = \int_X |x|^{-n} d\mu(x) = \int_{S^{n-1}} \int_0^\infty \frac{r^{n-1}}{|r\xi|^n} dr d\omega(\xi) = nv_n \int_0^\infty \frac{1}{r} dr = \infty,$$

where ω denotes the surface measure on the unit sphere S^{n-1} and v_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. However, since

$$d_f(\alpha) = |\{x \in \mathbb{R}^n : |x|^{-n/p} > \alpha\}| = |B(0, \alpha^{-p/n})| = v_n/\alpha^p,$$

it follows that $\|f\|_{p,\infty} = v_n^{1/p} < \infty$.

2.2 Interpolation

The notion of interpolation provides us with a useful tool regarding L^p norms, for p ranging over some interval: it turns out that a great deal of information can be extracted just by considering the endpoints of the interval. As an easy application of Proposition 2.5, we have the following result.

Proposition 2.11. *Let $1 \le p < q \le \infty$ and let $f \in L^{p,\infty}(\mu) \cap L^{q,\infty}(\mu)$. Then $f \in L^r(\mu)$, for any $p < r < q$.*

Proof. If $q < \infty$, then

$$\begin{aligned}\|f\|_r^r &= r\int_0^1 \alpha^{r-1} d_f(\alpha)\, d\alpha + r\int_1^\infty \alpha^{r-1} d_f(\alpha)\, d\alpha \\ &\le r\|f\|_{p,\infty}^p \int_0^1 \alpha^{r-1-p}\, d\alpha + r\|f\|_{q,\infty}^q \int_1^\infty \alpha^{r-1-q}\, d\alpha,\end{aligned}$$

and the integrals are both finite, since $r - p > 0$ and $r - q < 0$. If $q = \infty$, then $d_f(\alpha) = 0$, for all $\alpha \ge \|f\|_\infty$, and hence

$$\|f\|_r^r \le r\|f\|_{p,\infty}^p \int_0^{\|f\|_\infty} \alpha^{r-1-p}\, d\alpha < \infty,$$

as desired. □

Shortly we will present The Marcinkiewicz Interpolation Theorem, but first we introduce some notation. In addition to the measure μ, we let ν denote a nonnegative σ-finite measure on a sigma-algebra over a set Y.

Definition 2.12. Let D be a subspace of $\mathcal{M}(\mu)$. A map $T : D \to \mathcal{M}(\nu)$ is called *sublinear* if

$$|T(\alpha f + \beta g)| \leq |\alpha| \, |T(f)| + |\beta| \, |T(g)|,$$

for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in D$.

Notice that any linear operator is sublinear. Whenever T is a sublinear operator, we will frequently write Tf instead of $T(f)$.

Definition 2.13. Let D be a subspace of $\mathcal{M}(\mu)$ and fix exponents $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. A sublinear operator $T : D \to \mathcal{M}(\nu)$ is said to be of *type* $(L^p(\mu), L^q(\nu))$ if there exist a constant $A > 0$ such that

$$\|Tf\|_{L^q(\nu)} \leq A\|f\|_{L^p(\mu)} \quad \text{for all } f \in L^p(\mu).$$

T is said to be of *weak type* $(L^p(\mu), L^q(\nu))$ if there exist a constant $B > 0$ such that

$$\|Tf\|_{L^{q,\infty}(\nu)} \leq B\|f\|_{L^p(\mu)} \quad \text{for all } f \in L^p(\mu).$$

When no chance of confusion, we shall frequently refer to the two above types of operators simply as type (p, q) respectively weak type (p, q). As the name suggests, any operator of type (p, q) is also of weak type (p, q), since $\|Tf\|_{q,\infty} \leq \|Tf\|_q$. If T is of type (p, q) (respectively weak type (p, q)) then we shall also say that T is *bounded* from $L^p(\mu)$ into $L^q(\nu)$ (respectively weak $L^q(\nu)$).

In the following we let

$$L^p(\mu) + L^q(\mu) = \{f + g : f \in L^p(\mu),\, g \in L^q(\mu)\}.$$

Theorem 2.14 (The Marcienkiewicz Interpolation Theorem)**.** *Let* $1 \leq p < q \leq \infty$ *and assume that* $T : L^p(\mu) + L^q(\mu) \to \mathcal{M}(\nu)$ *is sublinear and simultaneously of weak type* (p, p) *and weak type* (q, q)*. Then* T *is of type* (r, r)*, for any* $p < r < q$*.*

Proof. Fix an arbitrary $f \in L^p(\mu)$. For each $\alpha > 0$ we let $E_\alpha = \{x \in X : |f(x)| > \delta\alpha\}$ and define

$$f_1^\alpha = f\chi_{E_\alpha} \quad \text{and} \quad f_2^\alpha = f\chi_{E_\alpha^c},$$

where $\delta > 0$ is chosen appropriately later. Notice that $f = f_1^\alpha + f_2^\alpha$. Since $p - r < 0$ we have $|f|^{p-r}\chi_{E_\alpha} \leq (\delta\alpha)^{p-r}$, and hence

$$\|f_1^\alpha\|_p^p = \int_X |f|^r |f|^{p-r} \chi_{E_\alpha} \, d\mu \leq (\delta\alpha)^{p-r}\|f\|_p^r,$$

showing that $f_1^\alpha \in L^p(\mu)$. For $q < \infty$, a similar argument shows that $f_2^\alpha \in L^q(\mu)$, and clearly $f_2^\alpha \in L^\infty(\mu)$. By sublinearity of T and Proposition 2.4 we obtain

$$d_{Tf}(\alpha) \leq d_{T(f_1^\alpha)}(\alpha/2) + d_{T(f_2^\alpha)}(\alpha/2),$$

which combined with Proposition 2.5 yields

$$\|Tf\|_r^r \leq r\int_0^\infty \alpha^{r-1} d_{T(f_1^\alpha)}(\alpha/2)d\alpha + r\int_0^\infty \alpha^{r-1} d_{T(f_2^\alpha)}(\alpha/2)d\alpha. \qquad (2.2)$$

Since T is of weak type (p,p),

$$\begin{aligned} d_{T(f_1^\alpha)}(\alpha/2) &\leq \frac{\|T(f_1^\alpha)\|_{p,\infty}^p}{(\alpha/2)^p} \\ &\leq (2A)^p\alpha^{-p}\|f_1^\alpha\|_p^p \\ &= (2A)^p\alpha^{-p}\int_X |f(x)|^p\chi_{E_\alpha}(x)d\mu(x), \end{aligned}$$

for some constant $A > 0$. By Tonelli's Theorem, the last expression in the above is a measurable function of α, and therefore the first term on the right in (2.2) is dominated by

$$r\int_0^\infty \alpha^{r-1}\left((2A)^p\alpha^{-p}\int_X |f(x)|^p\chi_{E_\alpha}(x)d\mu(x)\right)d\alpha,$$

which, again by Tonelli's Theorem, equals

$$r(2A)^p\int_X |f(x)|^p\int_0^{|f(x)|/\delta}\alpha^{r-1-p}d\alpha d\mu(x) = \frac{r(2A)^p}{r-p}\frac{1}{\delta^{r-p}}\|f\|_r^r.$$

For $q < \infty$, we can estimate the second term in (2.2) in a similar way to obtain the desired without any restrictions on δ. In the case $q = \infty$ we assume that

$$\|Tf\|_\infty \leq B\|f\|_\infty, \quad \text{for all } f \in L^q(\mu),$$

and put $\delta = (2B)^{-1}$. By noting that

$$\|T(f_2^\alpha)\|_\infty \leq B\|f_2^\alpha\|_\infty \leq B\delta\alpha = \alpha/2,$$

for all $\alpha > 0$, we see that

$$d_{T(f_2^\alpha)}(\alpha/2) \leq d_{T(f_2^\alpha)}(\|T(f_2^\alpha)\|_\infty) = 0,$$

for all $\alpha > 0$, and hence the second term on the right in (2.2) vanishes. □

2.3 The Maximal Function

The maximal function, first introduced by Hardy & Littlewood, is one of the cornerstones in our analysis of singular integral operators. In Chapter 4 we will also define the maximal function associated to a *weight*. We now leave the general setting from the previous sections and, unless otherwise stated, the measure under consideration is the Lebesgue measure.

Definition 2.15. Given a measurable function $f : \mathbb{R}^n \to \mathbb{C}^d$, we define *the maximal function* Mf on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f| \, dm,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. The map $f \mapsto Mf$ is called *the maximal operator*.

Notice that the maximal operator is sublinear. Also notice that the set $\{x \in \mathbb{R}^n : Mf(x) > \alpha\}$ is open, for each $\alpha \geq 0$. In particular, this shows that Mf is measurable. The fundamental property concerning the maximal operator is the fact that it is of weak type $(1,1)$ and of type (p,p), for any $1 < p \leq \infty$. To prove this we employ the following lemma.

Lemma 2.16 (The Vitali Covering Lemma)**.** *Any finite collection of balls* $\{B_j\}$ *in* $\mathbb{R}^n$ *has a subcollection* $\{B_{j_1}, \dots, B_{j_k}\}$ *of pairwise disjoint balls such that*

$$\Big|\bigcup_j B_j\Big| \leq 3^n \sum_{i=1}^k |B_{j_i}|. \tag{2.3}$$

Proof. Start by choosing B_{j_1} to be of maximal radius from $\{B_j\}$. If any of the remaining balls are disjoint from B_{j_1}, then we choose B_{j_2} to be of maximal radius among these. Next, if any of the remaining balls are disjoint from $B_{j_1} \cup B_{j_2}$, then we choose B_{j_3} to be of maximal radius among these. Proceeding in this way, we obtain a subcollection $\{B_{j_1}, \dots, B_{j_k}\}$ which, by construction, consists of pairwise disjoint balls. If a ball $B = B(x, \delta)$ is not selected, then B intersects some ball $B_{j_i} = B(y, r)$ with $r > \delta$, and hence $B \subseteq 3B_{j_i} := B(y, 3r)$. Thus we can cover $\cup_j B_j$ by the union of $3B_{j_1}, \dots, 3B_{j_k}$, leading to the estimate

$$\Big|\bigcup_j B_j\Big| \leq \Big|\bigcup_{i=1}^k 3B_{j_i}\Big| \leq \sum_{i=1}^k |3B_{j_i}| = 3^n \sum_{i=1}^k |B_{j_i}|,$$

as claimed. □

Theorem 2.17 (The Maximal Theorem)**.** *The maximal operator* $f \mapsto Mf$ *is of weak type* $(1,1)$ *and of type* (p,p)*, for each* $1 < p \leq \infty$*.*

Proof. Clearly M is of type (∞, ∞) and, by The Marcinkiewicz Interpolation Theorem, it then suffices to show that M is of weak type $(1,1)$. Fix an arbitrary $f \in L^1(m)$ and an $\alpha > 0$. Let $E_\alpha := \{x \in \mathbb{R}^n : Mf(x) > \alpha\}$. We must show that

$$d_{Mf}(\alpha) := |E_\alpha| \leq \frac{C}{\alpha}\|f\|_1, \tag{2.4}$$

for some constant $C > 0$ independent of f and α. By inner regularity of the Lebesgue measure, it suffices to verify (2.4) with E_α replaced by an arbitrary (nonempty) compact subset $K \subset E_\alpha$. For each $x \in K$ there exists a ball B_x containing x such that

$$\frac{1}{|B_x|}\int_{B_x} |f|\, dm > \alpha. \tag{2.5}$$

The collection $\{B_x\}_{x \in K}$ is an open covering of K and, by compactness of K, we can extract a finite subcover $\{B_j\}$. By The Vitali Covering Lemma, this subcover has a subcollection $\{B_{j_1}, \dots, B_{j_k}\}$ of pairwise disjoint balls satisfying (2.3). Thus

$$|K| \leq \big|\bigcup_j B_j\big| \leq 3^n \sum_{i=1}^k |B_{j_i}| \leq \frac{3^n}{\alpha}\sum_{i=1}^k \int_{B_{j_i}} |f|\, dm \leq \frac{3^n}{\alpha}\int |f|\, dm,$$

where the last inequality follows from the disjointness of the B_{j_i}'s. □

Corollary 2.18. *If $f \in L^p(m)$, for any $1 \leq p \leq \infty$, then Mf is finite a.e.*

Proof. Let $N = \{x \in \mathbb{R}^n : Mf(x) = \infty\}$. Since M is of weak type (p,p),

$$|N| \leq d_{Mf}(\alpha) \leq \frac{\|Mf\|_{p,\infty}^p}{\alpha^p} \leq \frac{C\|f\|_p^p}{\alpha^p},$$

for all $\alpha > 0$, and consequently $|N| = 0$. □

Remark 2.19. *The maximal function is not of type $(1,1)$. Indeed, any nonzero compactly supported function is not mapped into $L^1(m)$ by M. To see this, assume that supp$(f) \subset B(0,R)$. Then supp$(f) \subseteq B(x, |x|+R)$, for all $x \in \mathbb{R}^n$, and hence*

$$Mf(x) \geq \frac{1}{|B(x,|x|+R)|}\int_{B(x,|x|+R)} |f|\, dm = \frac{\|f\|_1}{v_n(|x|+R)^n},$$

for all $x \in \mathbb{R}^n$. Using polar coordinates we then see that

$$\begin{aligned} \|Mf\|_1 &\geq \frac{\|f\|_1}{v_n}\int_{S^{n-1}}\int_0^\infty \frac{r^{n-1}}{(|r\xi|+R)^n}\, dr\, d\omega(\xi) \\ &= n\|f\|_1\int_0^\infty \frac{r^{n-1}}{(r+R)^n}\, dr \geq n\|f\|_1\int_R^\infty \frac{r^{n-1}}{(r+R)^n}\, dr \\ &\geq \frac{n\|f\|_1}{2^n}\int_R^\infty \frac{1}{r}\, dr = \infty, \end{aligned}$$

as claimed.

Theorem 2.20 (The Lebesgue Differentiation Theorem)**.** *For any locally integrable function $f : \mathbb{R}^n \to \mathbb{C}^d$, we have*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy = f(x), \quad \textit{for almost all } x \in \mathbb{R}^n.$$

Proof. For each $r > 0$ we let

$$f_r(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\,dy.$$

Since

$$0 \leq \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy - f(x) \right| \leq f_r(x),$$

it suffices to show that

$$f^*(x) := \limsup_{r\to 0} f_r(x) = 0,$$

for almost all $x \in \mathbb{R}^n$. Notice that f_r is measurable. This follows readily from Tonelli's Theorem and the continuity of $|\cdot|$. Also notice that $f^* = 0$, whenever f is continuous. We may assume that $f \in L^1(m)$, since replacing f with $f\chi_{B(0,k)}$, for $k \in \mathbb{N}$, and noting that $(f\chi_{B(0,k)})^* = f^*$ on $B(0,k)$, allows us to conclude that $f^* = 0$ a.e. in $B(0,k)$ and consequently that $f^* = 0$ a.e. By right continuity of d_{f^*}, it suffices to show that $d_{f^*}(\alpha) = 0$, for an arbitrary $\alpha > 0$. Since $0 \leq f^* \leq Mf + |f|$, we have

$$\begin{aligned} d_{f^*}(\alpha) &\leq d_{Mf}(\alpha/2) + d_f(\alpha/2) \\ &\leq \frac{2\|Mf\|_{1,\infty}}{\alpha} + \frac{2\|f\|_{1,\infty}}{\alpha} \\ &\leq \frac{C\|f\|_1}{\alpha}, \end{aligned}$$

where the last inequality follows from Theorem 2.17. Now fix an arbitrary $\varepsilon > 0$ and choose $g \in C_c(\mathbb{R}^n; \mathbb{C}^d)$ with $\|f - g\|_1 < \varepsilon$. Since g is continuous and, since $(f+g)^* \leq f^* + g^*$, we have $(f-g)^* = f^*$. Thus

$$d_{f^*}(\alpha) = d_{(f-g)^*}(\alpha) \leq \frac{C}{\alpha}\|f - g\|_1 < \frac{C}{\alpha}\varepsilon,$$

and we are done. □

Corollary 2.21. *For any measurable function f, we have $|f| \leq Mf$ a.e.*

2.4 The Dyadic Maximal Function

In this section we introduce a variant of the maximal function defined in the last section. By a *cube Q in $\mathbb{R}^n$ of side length $l(Q) > 0$* we mean a set of the form $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$ where $l(Q) = b_1 - a_1 = \ldots = b_n - a_n$. Notice that $|Q| = l(Q)^n$. Also notice that the boundary of any cube has measure zero. For this reason we use the convention to call cubes disjoint, whenever their interiors are disjoint. For any nonempty set $S \subseteq \mathbb{R}^n$, we let $\operatorname{dia} S$ denote the *diameter* of S, i.e.

$$\operatorname{dia} S = \sup_{x,y \in S} |x - y|.$$

Cubes and balls in $\mathbb{R}^n$ are *equivalent* in the sense that there exist a constant $c > 1$ such that, for any cube Q, we can find balls B, B' with

$$B \subset Q \subset B' \quad \text{and} \quad |B'| \leq c|B|,$$

and visa-versa (with a possibly different constant $c' > 1$). The following special family of cubes are of particular interest to us.

Definition 2.22. A *dyadic cube in* $\mathbb{R}^n$ is a cube of the form

$$[m_1 2^{-k}, (m_1 + 1)2^{-k}] \times \ldots \times [m_n 2^{-k}, (m_n + 1)2^{-k}],$$

where $k, m_1, \ldots, m_n \in \mathbb{Z}$. A dyadic cube in $\mathbb{R}$ is called a *dyadic interval*.

It might be useful to consider a more geometric characterization of dyadic cubes: Let $\mathcal{D}_0$ denote the collection of cubes with vertices at $\mathbb{Z}^n$. Then let $\mathcal{D}_1$ denote the collection of cubes obtained by bisecting the sides in each cube in $\mathcal{D}_0$. We construct $\mathcal{D}_2$ by bisecting sides in $\mathcal{D}_1$ and so on. Starting again from $\mathcal{D}_0$, we let $\mathcal{D}_{-1}$ denote the collection of cubes obtained by gathering 2^n neighbor cubes from $\mathcal{D}_0$ into single cubes. In a similar way we construct $\mathcal{D}_{-2}$ from $\mathcal{D}_{-1}$ and so on. If $\mathcal{D}$ denotes the collection of all dyadic cubes, then

$$\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k.$$

Notice that the cubes in $\mathcal{D}_k$ have side length 2^{-k} and that a cube in $\mathcal{D}_k$ give rise to 2^n cubes in $\mathcal{D}_{k-1}$.

Let us note an important property of dyadic cubes: Any two dyadic cubes are either disjoint or one is contained in the other. To see this, we note that any two dyadic intervals of the same length are either disjoint or coincide. Given two arbitrary dyadic intervals I and J with, say, $l(I) \leq l(J)$ then J is composed of dyadic intervals of length $l(I)$, and hence $I \cap J = \emptyset$ or $I \subseteq J$. Since the sides in any cube are of equal length, the result is easily extended to general dyadic cubes.

Definition 2.23. Given a measurable function $f : \mathbb{R}^n \to \mathbb{C}^d$, we define *the dyadic maximal function* $\mathcal{M}f$ on $\mathbb{R}^n$ by

$$\mathcal{M}f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f| \, dm,$$

where the supremum is taken over all dyadic cubes Q containing x.

Remark 2.24. *Since cubes are closed sets it is not obvious that $\mathcal{M}f$ is measurable. In fact, we will not attempt to prove it. However, this difficulty may be avoided by restricting the domain of definition of $\mathcal{M}f$ to points not on the boundary of any dyadic cube. As seen by inspection of each particular case, this restriction has no effect in our applications of $\mathcal{M}f$, since the discarded set has measure zero.*

Since cubes and balls are equivalent, we see that $\mathcal{M}f \leq cMf$, for some constant $c > 1$. Thus we immediately conclude that The Maximal Theorem holds for the dyadic maximal operator and in particular that $\mathcal{M}f < \infty$ a.e., whenever $f \in L^p(m)$, for any $1 \leq p \leq \infty$. We also have the following variant of The Lebesgue Differentiation Theorem: For any locally integrable f,

$$\lim_{\operatorname{dia} Q \to 0} \frac{1}{|Q|} \int_Q f \, dm = f(x),$$

for almost all $x \in \mathbb{R}^n$. Here the limit is taken over any sequence of dyadic cubes containing x with diameters converging to zero.

Chapter 3

A_p Weights

This chapter is devoted to the notion of A_p weights. We generalize the classic Muckenhoupt A_p condition of scalar weights to matrix-valued weights. As in the scalar setting, the crucial property of matrix A_p weights is that they satisfy "Reverse Hölder Inequalities". In fact, these are obtained from the scalar case, and hence we start by a separate treatment of scalar weights.

For any $1 < p < \infty$ we let $p' = p/(p-1)$ denote the dual exponent of p.

3.1 Scalar A_p Weights

Definition 3.1. A *scalar weight* is a measurable function on $\mathbb{R}^n$ which is positive almost everywhere.

The property stated next is the classic *Muckenhoupt A_p condition.*

Definition 3.2. Let $1 < p < \infty$. A scalar weight w is called an A_p *weight* if there exists a (finite) constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w\,dm\right)^{1/p}\left(\frac{1}{|B|}\int_B w^{-p'/p}\,dm\right)^{1/p'} \le C, \quad \text{for all balls } B \subset \mathbb{R}^n.$$

The least of such constants is called the A_p *bound* of w and is denoted by $A_p(w)$. The set of all A_p weights is called the A_p *class* and is denoted by A_p.

Of course, any positive constant function is an A_p weight, for each $1 < p < \infty$. Let us consider a less trivial example.

Example 3.3. The function $w(x) = |x|^a$ is an A_p weight, for any $-n < a < (p-1)n$. To see this, we fix an arbitrary ball $B = B(x_0, R)$ and let

$$I_B = \left(\frac{1}{|B|}\int_B |x|^a\,dx\right)\left(\frac{1}{|B|}\int_B |x|^{-a\frac{p'}{p}}\,dx\right)^{p/p'}.$$

If $R < \operatorname{dist}(0,B) := \inf_{y\in B}|y|$, then

$$|x| \le |x-y| + |y| < 2R + |y| < 2\operatorname{dist}(0,B) + |y| \le 3|y|,$$

for all $x, y \in B$. In particular,

$$\frac{1}{3}|x| \le |x_0| \le 3|x|, \quad \text{for all } x \in B,$$

and consequently

$$I_B \le C|x_0|^a(|x_0|^{-a\frac{p'}{p}})^{p/p'} = C.$$

Now assume that $R \ge \operatorname{dist}(0,B)$. Since $|x_0| - R \le |y|$, for all $y \in B$, we have $|x_0| - R \le \operatorname{dist}(0,B)$ and hence

$$|x| \le R + |x_0| \le 2R + \operatorname{dist}(0,B) \le 3R,$$

for all $x \in B$. This shows that $B \subseteq B' := B(0,3R)$ and hence

$$\begin{aligned} I_B &\le \left(\frac{3^n}{|B'|}\int_{B'}|x|^a\,dx\right)\left(\frac{3^n}{|B'|}\int_{B'}|x|^{-a\frac{p'}{p}}\,dx\right)^{p/p'} \\ &= \frac{n}{R^n}\int_0^{3R} r^{a+n-1}\,dr\left(\frac{n}{R^n}\int_0^{3R} r^{-a\frac{p'}{p}+n-1}\,dr\right)^{p/p'}, \end{aligned} \tag{3.1}$$

where we have used polar coordinates. The assumption that $-n < a < (p-1)n$ is equivalent with $a + n - 1 > -1$ and $-ap'/p + n - 1 > -1$, and from this we easily see that the expression in (3.1) is bounded by a constant independent of R. This shows that $w \in A_p$.

Next we point out some simple properties of A_p. Let w, w_1, w_2 denote arbitrary A_p weights and let $\lambda > 0$.

(i) A_p is closed under multiplication by positive scalars. In fact, $A_p(\lambda w) = A_p(w)$, which follows directly from the definition.

(ii) A_p is closed under addition. Indeed, $w_1 + w_2$ is in A_p since $(w_1 + w_2)^{-p'/p}$ is dominated by both $w_1^{-p'/p}$ and $w_2^{-p'/p}$.

(iii) A_p is closed under translation. In fact, $A_p(w(\cdot + a)) = A_p(w)$, for any $a \in \mathbb{R}^n$, which is easily verified using the translation invariance of the Lebesgue measure.

(iv) A_p is closed under dilation by a positive scalar. Again $A_p(w(\lambda\cdot)) = A_p(w)$ as can be verified by change of variables.

We consider a few more properties of A_p weights. Given any scalar weight w and a measurable set $E \subseteq \mathbb{R}^n$, we use the notation $w(E) := \int_E w\,dm$. Notice that $E \mapsto w(E)$ is a Borel measure.

Definition 3.4. A scalar weight w is said to be in the *class* A_∞ if, for each $\alpha \in (0,1)$, there exists a $\beta \in (0,1)$ such that

$$|E| \geq \alpha|B| \quad \Longrightarrow \quad w(E) \geq \beta w(B),$$

for all balls $B \subset \mathbb{R}^n$ and for all measurable subsets $E \subseteq B$.

Lemma 3.5. *$A_p \subseteq A_\infty$ for each $1 < p < \infty$.*

Proof. If $w \in A_p$ then Proposition 3.23 (stated and proved in larger generality in Section 3.3) shows that there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B f\,dm\right)^p \leq \frac{C}{w(B)}\int_B f^p w\,dm, \tag{3.2}$$

for all balls $B \subset \mathbb{R}^n$ and for all nonnegative measurable functions f on $\mathbb{R}^n$. Fix an arbitrary ball $B \subset \mathbb{R}^n$ and a measurable subset $E \subseteq B$. If $|E| \geq \alpha|B|$, for some $\alpha \in (0,1)$, then the particular choice of $f = \chi_E$ in (3.2) yields

$$w(E) \geq \frac{w(B)}{C}\left(\frac{|E|}{|B|}\right)^p \geq \frac{\alpha^p}{C}w(B),$$

and hence $w \in A_\infty$ (we may assume that $C > 1$). □

Definition 3.6. A Borel measure μ is called a *doubling measure* if there exists a constant $c > 1$ such that

$$\mu(B(x,2\delta)) \leq c\mu(B(x,\delta)) \quad \text{for all } x \in \mathbb{R}^n \text{ and for all } \delta > 0.$$

Remark 3.7. *If $w \in A_p$, for some $1 < p < \infty$, then $E \mapsto w(E)$ is a doubling measure. Indeed, for an arbitrary $x \in \mathbb{R}^n$ and $\delta > 0$ we have $|B(x,\delta)| = 2^{-n}|B(x,2\delta)|$ and, since $w \in A_p \subseteq A_\infty$, there exists a constant $\beta \in (0,1)$ such that $w(B(x,\delta)) \geq \beta w(B(x,2\delta))$.*

3.1.1 The Reverse Hölder Inequality

The Reverse Hölder Inequality, Proposition 3.12, is the crucial property of scalar A_p weights. To prove it we need some preliminary results. The first lemma employs dyadic cubes and the dyadic maximal function, defined in Chapter 2. In the proof of this lemma and several other times throughout the report, we use the term *maximal dyadic cube* to mean a dyadic cube of maximal measure.

Lemma 3.8. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be any measurable function and let $\alpha \geq 0$. If the set*

$$\Omega_\alpha := \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \alpha\}$$

has finite (Lebesgue) measure, then either $\Omega_\alpha = \emptyset$ or Ω_α is the union of disjoint dyadic cubes $\{Q_j\}$ with

$$\alpha < \frac{1}{|Q_j|}\int_{Q_j} |f|\,dm \leq 2^n\alpha \quad \textit{for each } j \in \mathbb{N}. \tag{3.3}$$

Proof. For each $x \in \Omega_\alpha$ there exists a maximal dyadic cube Q containing x such that

$$\frac{1}{|Q|}\int_Q |f|\,dm > \alpha. \tag{3.4}$$

The maximality follows from the fact that $Q \subseteq \Omega_\alpha$ and hence $|Q| \leq |\Omega_\alpha| < \infty$, for any dyadic cube Q satisfying (3.4). Let $\{Q_j\}$ denote the collection of maximal dyadic cubes for points in Ω_α. Clearly these cubes are disjoint and their union equals Ω_α. Thus the first inequality in (3.3) holds by construction and, by letting Q'_j denote the next larger dyadic cube containing Q_j, we see that

$$\frac{1}{|Q_j|}\int_{Q_j} |f|\,dm \leq \frac{2^n}{|Q'_j|}\int_{Q'_j} |f|\,dm \leq 2^n\alpha,$$

as desired. □

There are two useful observations to be made about the decomposition guaranteed i Lemma 3.8.

Remark 3.9. *If Q_0 is a dyadic cube containing the support of f and $\alpha_0 = |Q_0|^{-1}\int_{Q_0}|f|\,dm$, then $\Omega_\alpha \subseteq Q_0$, for any $\alpha \geq \alpha_0$. To see this, assume that $x \notin Q_0$ and fix an arbitrary dyadic cube Q containing x. Since f is supported in Q_0 and, since either Q and Q_0 are disjoint or $Q_0 \subset Q$, we have*

$$\frac{1}{|Q|}\int_Q |f|\,dm \leq \frac{1}{|Q_0|}\int_{Q\cap Q_0} |f|\,dm \leq \alpha_0,$$

showing that $\mathcal{M}f(x) \leq \alpha_0 \leq \alpha$, and hence $x \notin \Omega_\alpha$.

Remark 3.10. *If $\alpha_1 \geq \alpha_2$ then $\Omega_{\alpha_1} \subseteq \Omega_{\alpha_2}$ and, by maximality, each cube in the decomposition of Ω_{α_1} is contained in some cube of the decomposition of Ω_{α_2}.*

Lemma 3.11. *Let $1 < p < \infty$ and let $w \in A_p$. Then there exists, for each $\gamma \in (0,1)$, some $\delta \in (0,1)$ such that*

$$|E| \leq \gamma|Q| \quad \Longrightarrow \quad w(E) \leq \delta w(Q),$$

for all cubes $Q \subset \mathbb{R}^n$ and for all measurable subsets $E \subseteq Q$.

Proof. Assume that $|E| \leq \gamma|Q|$. Since cubes and balls in $\mathbb{R}^n$ are equivalent, we can choose balls B, B' such that $B \subset Q \subset B'$ and $|B'| \leq c|B|$, for some constant $c > 1$ independent of Q. Since

$$|Q\backslash E| \geq (1-\gamma)|Q| \geq (1-\gamma)|B| \geq \frac{1-\gamma}{c}|B'|,$$

and since $w \in A_p \subseteq A_\infty$, there exists a $\beta \in (0,1)$ such that

$$w(Q\backslash E) \geq \beta w(B') \geq \beta w(Q),$$

or equivalently, $w(E) \leq \delta w(Q)$ with $\delta := 1 - \beta$. □

Proposition 3.12 (The Reverse Hölder Inequality)**.** *If $w \in A_p$, for some $1 < p < \infty$, then there exists constants $r > 1$ and $C > 0$ such that*

$$\left(\frac{1}{|B|}\int_B w^r\,dm\right)^{1/r} \le \frac{C}{|B|}\int_B w\,dm, \quad \textit{for all balls } B \subset \mathbb{R}^n. \tag{3.5}$$

Proof. It suffices to show (3.5) with cubes replacing balls. To see this, fix an arbitrary ball B and choose cubes Q, Q' with $Q \subset B \subset Q'$ and $|Q'| \le c|Q|$, for some constant $c > 1$. If (3.5) holds for cubes, then the doubling property of $E \mapsto w(E)$ implies that

$$\begin{aligned}\left(\frac{1}{|B|}\int_B w^r\,dm\right)^{1/r} &\le \left(\frac{c}{|Q'|}\int_{Q'} w^r\,dm\right)^{1/r}\\ &\le \frac{C}{|Q'|}\int_{Q'} w\,dm \le \frac{C'}{|B|}\int_B w\,dm.\end{aligned}$$

Since A_p is closed under dilation and translation, we may assume that $Q = Q_0$ is a dyadic cube with $|Q_0| = 1$. Furthermore, since A_p is closed under multiplication by positive scalars, we may also assume that $w(Q_0) = 1$. Thus it suffices to show that

$$\int_{Q_0} w^r\,dm \le C.$$

Let $f = w\chi_{Q_0}$ and define, for each $k \in \mathbb{N}_0$,

$$E_k = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2^{Nk}\},$$

where $N \in \mathbb{N}$ is to be chosen appropriately later. Notice that

$$\frac{1}{|Q_0|}\int_{Q_0} f\,dm = w(Q_0) = 1 \le 2^{Nk}$$

and hence $E_k \subseteq Q_0$, for all $k \in \mathbb{N}_0$, by Remark 3.9. Also notice that $E_k \subseteq E_{k-1}$, for all $k \in \mathbb{N}$. The crux of the proof is to show that $w(E_k) \le \delta^k$, for some $\delta \in (0,1)$ and, since this is trivial if $E_k = \emptyset$, we fix an arbitrary $k \in \mathbb{N}$ and assume that $E_k \neq \emptyset$. We then apply Lemma 3.8 to write E_k and E_{k-1} as disjoint unions of dyadic cubes,

$$E_k = \bigcup_j Q_j \quad \text{and} \quad E_{k-1} = \bigcup_i Q'_i.$$

We will show that

$$|E_k \cap Q| \le 2^{n-N}|Q|, \tag{3.6}$$

for each $Q \in \{Q'_i\}$. Notice that

$$|E_k \cap Q| = \Big|\bigcup_j (Q_j \cap Q)\Big| = \sum_j |Q_j \cap Q|.$$

By Remark 3.10, each Q_j is contained in some Q_i' and then, by disjointness, either $Q_j \cap Q = \emptyset$ or $Q_j \subseteq Q$. This observation combined with Lemma 3.8 yields

$$\begin{aligned}
|E_k \cap Q| &= \sum_{j:Q_j \subseteq Q} |Q_j| \le \sum_{j:Q_j \subseteq Q} 2^{-Nk} \int_{Q_j} f\,dm \\
&\le 2^{-Nk} \int_Q f\,dm \le 2^{-Nk} \cdot 2^n \cdot 2^{N(k-1)} |Q| \\
&= 2^{n-N} |Q|,
\end{aligned}$$

as desired. The third inequality above follows also from Lemma 3.8, with $\alpha = 2^{N(k-1)}$. Now choose any $\gamma \in (0,1)$ and let $\delta \in (0,1)$ be given as in Lemma 3.11. Choose N such that $2^{n-N} \le \gamma$. It follows that $w(E_k \cap Q) \le \delta w(Q)$ and consequently

$$\begin{aligned}
w(E_k) &= w(E_k \cap E_{k-1}) = w\left(\bigcup_i (E_k \cap Q_i')\right) \\
&= \sum_i w(E_k \cap Q_i') \le \delta \sum_i w(Q_i') \\
&= \delta w(E_{k-1}),
\end{aligned}$$

for each $k \in \mathbb{N}$. By induction we conclude that

$$w(E_k) \le \delta^k w(E_0) \le \delta^k w(Q_0) = \delta^k,$$

for each $k \in \mathbb{N}_0$ and, as a consequence of The Lebesgue Differentiation Theorem,

$$\begin{aligned}
\int_{Q_0} w^r\,dm &\le \int_{Q_0} (\mathcal{M}f)^{r-1} w\,dm \\
&= \int_{Q_0 \cap E_0} (\mathcal{M}f)^{r-1} w\,dm + \int_{Q_0 \cap E_0^c} (\mathcal{M}f)^{r-1} w\,dm.
\end{aligned}$$

The last integral on the right in the above is bounded by $w(Q_0) = 1$. To estimate the first integral, we notice that $f \in L^1(m)$ and so $\mathcal{M}f < \infty$ a.e. This implies that the characteristic function of $Q_0 \cap E_0 = E_0$ equals the characteristic function of $\bigcup_{k=0}^{\infty} E_k \backslash E_{k+1}$ a.e. and hence

$$\begin{aligned}
\int_{Q_0 \cap E_0} (\mathcal{M}f)^{r-1} w\,dm &\le \sum_{k=0}^{\infty} \int_{E_k \backslash E_{k+1}} (\mathcal{M}f)^{r-1} w\,dm \\
&\le \sum_{k=0}^{\infty} 2^{N(k+1)(r-1)} w(E_k) \le \sum_{k=0}^{\infty} 2^{N(k+1)(r-1)} \delta^k \\
&= 2^{N(r-1)} \sum_{k=0}^{\infty} \left(2^{N(r-1)} \delta\right)^k.
\end{aligned}$$

Since $\delta < 1$, the above series converges whenever $r > 1$ is sufficiently close to 1. □

Next we wish to generalize the notion of scalar weights to matrix-valued weights. In order to do so, we will need to make sense out of real powers of a matrix, and hence we make a short digression into this subject.

3.2 The Functional Calculus

Let $\mathbb{C}^{d\times d}$ denote the set of all complex $d \times d$ matrices. For any $A \in \mathbb{C}^{d\times d}$, we let

$$\|A\| = \sup_{|v|\leq 1} |Av| = \sup_{v\neq 0} \frac{|Av|}{|v|}$$

denote the operator norm of A. If A is self-adjoint, then $\|A\|$ equals the spectral radius of A, i.e.

$$\|A\| = \max_{\lambda\in\sigma(A)} |\lambda|,$$

where $\sigma(A)$ denotes the spectrum (the set of eigenvalues) of A. Recall that $A \in \mathbb{C}^{d\times d}$ is called positive definite (or just positive) if $\langle Av, v\rangle > 0$ for all $v \neq 0$. Notice that a positive matrix is invertible, and that its eigenvalues are all positive. Furthermore, as a consequence of The Polarization Identity, every positive (complex) matrix is also self-adjoint. Given a positive $A \in \mathbb{C}^{d\times d}$ and an exponent $r \in \mathbb{R}$, we wish to define a matrix $A^r \in \mathbb{C}^{d\times d}$ possessing some nice properties. There are several equivalent ways to do this. Here we employ the diagonalization approach.

We use the notation $D(\lambda_1, \ldots \lambda_d)$ for a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_d$. If $A = D(\lambda_1, \ldots, \lambda_d)$ is positive then we define

$$A^r = D(\lambda_1^r, \ldots, \lambda_d^r).$$

More generally, if A is any positive matrix in $\mathbb{C}^{d\times d}$, then A has d linearly independent eigenvectors $v_1, \ldots, v_d$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_d$, and hence $A = PDP^{-1}$, where the columns of P are the vectors $v_1, \ldots, v_d$ and $D = D(\lambda_1, \ldots, \lambda_d)$. We then define

$$A^r = PD^rP^{-1}.$$

The following properties of A^r are easily verified.

Proposition 3.13. *Assume that $A \in \mathbb{C}^{d\times d}$ is positive definite and let $r, s \in \mathbb{R}$. Then*

(i) $A^rA^s = A^{r+s}$

(ii) $A^0 = I := D(1, \ldots, 1)$

(iii) A^r *is invertible and* $(A^r)^{-1} = A^{-r}$

(iv) A^r *is positive definite*

(v) $\|A^r\| = \|A\|^r$ *for all* $r \geq 0$.

The above definition of A^r is simple, but it has the lack of one important property. We will need the fact that A^r may be expressed as a norm convergent power series. Let $A \in \mathbb{C}^{d\times d}$ be positive with spectrum contained in $(a, b) \subset (0, \infty)$. Let c denote the center point of (a, b). For any $r \in \mathbb{R}$, the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^r$ has Taylor series expansion,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - c)^k, \quad a_k := \frac{f^{(k)}(c)}{k!},$$

with radius of convergence $R = c$. It is then shown in [3] that

$$A^r = \sum_{k=0}^{\infty} a_k (A - cI)^k,$$

where the convergence is in operator norm.

3.3 Matrix A_p Weights

We begin by some notation. Any two nonnegative functions f and g are called *comparable* or *equivalent* if there exists a constant $c > 0$ such that $c^{-1}f(x) \leq g(x) \leq cf(x)$ for all x. We use the notation $f(x) \sim g(x)$ to indicate that f and g are comparable. The standard basis of $\mathbb{C}^d$ is denoted by $\{e_i\} = \{e_1, \ldots, e_d\}$.

Assume that W is a function from $\mathbb{R}^n$ into $\mathbb{C}^{d\times d}$. If f is a function from $\mathbb{R}^n$ into $\mathbb{C}^d$, then we let Wf denote the function $(Wf)(x) = W(x)f(x)$. For any $r \in \mathbb{R}$, we let W^r denote the function $W^r(x) = W(x)^r$. W is called measurable if the component functions of W, i.e. the functions

$$x \mapsto \langle W(x)e_j, e_i\rangle,$$

are measurable, for $1 \leq i, j \leq d$. We say that W is *locally integrable* if $\|W\|$ is.

Definition 3.14. A *matrix weight* is a measurable function $W : \mathbb{R}^n \to \mathbb{C}^{d\times d}$ such that $W(x)$ is positive definite for almost all $x \in \mathbb{R}^n$.

If W is a matrix weight then W^r is measurable, for each $r \in \mathbb{R}$. To see this, we employ the fact that $W(x)^r$ has a norm convergent power series expansion, i.e

$$W_N(x) := \sum_{k=0}^{N} a_k (W(x) - cI)^k \to W(x)^r$$

in operator norm as $N \to \infty$, for some $c, a_k \in \mathbb{R}$. Since W_N is measurable and, since

$$|\langle W(x)^r e_j, e_i\rangle - \langle W_N(x)e_j, e_i\rangle| \leq \|W(x)^r - W_N(x)\|,$$

for each $N \in \mathbb{N}$, it follows that each component function of W^r is the pointwise limit of measurable functions. Hence W^r is measurable.

Definition 3.15. Let W be a matrix weight and let $1 \leq p < \infty$. The *space* $L^p(W)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}^d$ with

$$\|f\|_{L^p(W)} := \|W^{1/p}f\|_{L^p(m)} = \left(\int |W^{1/p}f|^p \, dm\right)^{1/p} < \infty.$$

Notice that $L^p(W)$ is a normed space.

Definition 3.16. For any norm ρ on $\mathbb{C}^d$ we define its *dual norm* ρ^* on $\mathbb{C}^d$ by

$$\rho^*(v) = \sup_{u \neq 0} \frac{|\langle u, v\rangle|}{\rho(u)}.$$

Notice that

$$|\langle u, v\rangle| \leq \rho(u)\rho^*(v), \quad \text{for all } u, v \in \mathbb{C}^d. \tag{3.7}$$

Also notice that ρ^* is a norm and $(\rho^*)^* = \rho$. Indeed, (3.7) implies that $(\rho^*)^* \leq \rho$ and, as a consequence of The Hahn-Banach Theorem, there exists, for each $v \in \mathbb{C}^d$, a nonzero $u \in \mathbb{C}^d$, such that $\langle v, u\rangle = \rho^*(u)\rho(v)$. This shows that $(\rho^*)^* \geq \rho$. We are particular interested in norms of the following form.

Proposition 3.17. *Let $A \in \mathbb{C}^{d\times d}$ be positive definite and define ρ on $\mathbb{C}^d$ by $\rho(v) = |Av|$. Then $\rho^*(v) = |A^{-1}v|$.*

Proof. Since

$$\frac{|\langle u, v\rangle|}{\rho(u)} = \frac{|\langle A^{-1}Au, v\rangle|}{|Au|} = \frac{|\langle Au, A^{-1}v\rangle|}{|Au|} \leq |A^{-1}v|,$$

for all nonzero $u \in \mathbb{C}^d$, we have $\rho^*(v) \leq |A^{-1}v|$. However, for an arbitrary $v \neq 0$, the particular choice of

$$u = \frac{|Av|}{|A^{-1}v|}A^{-2}v$$

yields

$$\frac{|\langle u, v\rangle|}{\rho(u)} = |A^{-1}v|,$$

and hence $\rho^*(v) \geq |A^{-1}v|$. □

Let us fix a matrix weight W, a ball $B \subset \mathbb{R}^n$ and an exponent $1 < p < \infty$. We assume that W and $W^{-p'/p}$ are locally integrable. Throughout the report we use the notation

$$\begin{aligned} \rho_{p,B}(v) &:= |B|^{-1/p} \|\chi_B v\|_{L^p(W)} \\ \rho^*_{p',B}(v) &:= |B|^{-1/p'} \|\chi_B v\|_{L^{p'}(W^{-p'/p})}, \end{aligned} \tag{3.8}$$

for $v \in \mathbb{C}^d$. Notice that $\rho_{p,B}$ and $\rho^*_{p',B}$ are norms. Using (3.7), Proposition 3.17 and Hölders Inequality, we see that

$$\begin{aligned} |\langle u, v\rangle| &\leq \frac{1}{|B|} \int_B |W(x)^{1/p} u| \cdot \left(|W(x)^{1/p} v|\right)^* \, dx \\ &\leq \left(\frac{1}{|B|} \int_B |W(x)^{1/p} u|^p \, dx\right)^{1/p} \cdot \left(\frac{1}{|B|} \int_B |W(x)^{-1/p} v|^{p'} \, dx\right)^{1/p'} \\ &= \rho_{p,B}(u) \cdot \rho^*_{p',B}(v), \end{aligned}$$

and hence we always have $(\rho_{p,B})^* \leq \rho^*_{p',B}$. When the "opposite" statement is also true we call W an A_p *weight*.

Definition 3.18. Let $1 < p < \infty$. A matrix weight W is called an A_p *weight* if W and $W^{-p'/p}$ are locally integrable and if there exists a constant $C > 0$ such that

$$\rho^*_{p',B} \leq C(\rho_{p,B})^* \quad \text{for all balls } B \subset \mathbb{R}^n.$$

The least of such constants is called the A_p *bound of* W and is denoted by $A_p(W)$. The *class* A_p is the set of all matrix A_p weights.

Remark 3.19. *For any scalar weight w we have*

$$(\rho_{p,B})^*(v) = |v| \left(\frac{1}{|B|} \int_B w \, dm\right)^{-1/p}$$

and

$$\rho^*_{p',B}(v) = |v| \left(\frac{1}{|B|} \int_B w^{-p'/p} \, dm\right)^{1/p'}.$$

This shows that Definition 3.18 is consistent with the definition of scalar A_p weights and, in particular, the assumption that w and $w^{-p'/p}$ are locally integrable is unnecessary in the scalar case.

It turns out that it is possible to give an alternative characterization of the class A_p in terms of matrices (Proposition 3.21). This is an easy consequence of the following lemma, which is a corollary to the famous Ellipsoid Theorem of Fritz John [4]. By an *ellipsoid* in $\mathbb{C}^d$, symmetric about the origin, we mean the image of the closed Euclidean unit ball in $\mathbb{C}^d$, by some invertible linear map $\Phi : \mathbb{C}^d \to \mathbb{C}^d$. Of course, $\Phi(v) = Av$, for some invertible $A \in \mathbb{C}^{d\times d}$. In fact, we may assume that A is positive definite. To see this we use polar decomposition and write $A = PU$, where P is positive definite and U is unitary. If $B \subset \mathbb{C}^d$ denotes the closed unit ball, then $UB = B$ and hence $AB = PUB = PB$.

Lemma 3.20. *For any norm ρ on $\mathbb{C}^d$ there exists a positive definite matrix $V \in \mathbb{C}^{d\times d}$ such that*

$$\rho(v) \leq |Vv| \leq \sqrt{d}\rho(v), \quad \text{for all } v \in \mathbb{C}^d.$$

Proof. Let $B_\rho = \{v \in \mathbb{C}^d : \rho(v) \leq 1\}$. Clearly B_ρ is convex and symmetric about the origin. Since all norms on $\mathbb{C}^d$ are equivalent to the Euclidean norm, B_ρ is compact and has a nonempty interior. By John's Theorem there exists an ellipsoid $E \subset \mathbb{R}^n$, also symmetric about the origin, such that

$$E \subseteq B_\rho \subseteq \sqrt{d}E.$$

By definition there exists a positive definite matrix $V \in \mathbb{C}^{d\times d}$ such that $VE = B := \{v \in \mathbb{C}^d : |v| \leq 1\}$. Now, fix an arbitrary nonzero $v \in \mathbb{C}^d$. For $\varepsilon > 0$ we define $v_\varepsilon = (1+\varepsilon)\rho(v)^{-1}v$. Since $\rho(v_\varepsilon) > 1$ and $E \subseteq B_\rho$, we must have $|Vv_\varepsilon| > 1$, which is equivalent to

$$\rho(v) < (1+\varepsilon)|Vv|.$$

Taking the limit as $\varepsilon \to 0$ yields $\rho(v) \leq |Vv|$. However, since $v/\rho(v) \in B_\rho \subseteq \sqrt{d}E$, we have $v/\rho(v) = \sqrt{d}V^{-1}u$, for some $u \in B$. Hence

$$\big|V\big(\frac{v}{\rho(v)}\big)\big| = \sqrt{d}|u| \leq \sqrt{d},$$

or equivalently, $|Vv| \leq \sqrt{d}\rho(v)$. □

Let $1 < p < \infty$ and let W be a matrix weight with W and $W^{-p'/p}$ locally integrable. For any ball $B \subset \mathbb{R}^n$ there exists, by Lemma 3.20, positive definite complex $d \times d$ matrices V_B and V_B' such that

$$|V_B v| \sim \rho_{p,B}(v) \quad \text{and} \quad |V_B' v| \sim \rho^*_{p',B}(v), \tag{3.9}$$

uniformly in B. Throughout the report we reserve the notation V_B and V_B' to mean *any two* matrices satisfying (3.9).

Proposition 3.21. *Let $1 < p < \infty$ and let W be a matrix weight with W and $W^{-p'/p}$ locally integrable. Then $W \in A_p$ if and only if there exists a constant $C > 0$ such that*

$$\|V_B V_B'\| \leq C, \quad \text{for all balls } B \subset \mathbb{R}^n. \tag{3.10}$$

Proof. Since $\rho_{p,B}(v) \sim |V_B v|$ uniformly in B we also have $(\rho_{p,B})^*(v) \sim (|V_B v|)^* = |V_B^{-1} v|$ uniformly in B. Notice that $\|V_B V_B'\| = \|V_B' V_B\|$, since both V_B and V_B' are self-adjoint. Now, if $W \in A_p$ then

$$|V_B' V_B v| \sim \rho^*_{p',B}(V_B v) \leq C(\rho_{p,B})^*(V_B v) \leq C|V_B^{-1} V_B v|,$$

and hence $\|V_B V_B'\| \leq C$. Conversely, if (3.10) holds then

$$\rho^*_{p',B}(v) \sim |V_B' v| \leq \|V_B' V_B\| \cdot |V_B^{-1} v| \leq C(\rho_{p,B})^*(v),$$

showing that $W \in A_p$. □

We will need yet another characterization of the class A_p and we employ the following lemma.

Lemma 3.22. *Let W be a matrix weight. For each $f \in L^p(W)$ we have*

$$\|f\|_{L^p(W)} = \sup \Big| \int \langle g, f \rangle \, dm \Big|, \tag{3.11}$$

where the supremum is taken over all g with $\|g\|_{L^{p'}(W^{-p'/p})} = 1$.

Proof. By Hölders Inequality we have

$$\Big| \int \langle g, f \rangle \, dm \Big| = \Big| \int \langle W^{-1/p} g, W^{1/p} f \rangle \, dm \Big| \leq \|f\|_{L^p(W)} \cdot \|g\|_{L^{p'}(W^{-p'/p})},$$

and hence the right side in (3.11) is dominated by $\|f\|_{L^p(W)}$. However, except in the trivial case where $f = 0$, the function

$$g := \|f\|^{1-p}_{L^p(W)} |W^{1/p} f|^{p-2} W^{2/p} f$$

satisfies $\|g\|_{L^{p'}(W^{-p'/p})} = 1$ and

$$\Big| \int \langle g, f \rangle \, dm \Big| = \|f\|_{L^p(W)},$$

showing the opposite inequality. □

Again let $1 < p < \infty$ and let W be a matrix weight with W and $W^{-p'/p}$ locally integrable. Given any ball $B \subset \mathbb{R}^n$ and a function $f = (f_1, \dots, f_d)$ from $\mathbb{R}^n$ into $\mathbb{C}^d$, we use the notation

$$f_B := \frac{1}{|B|} \int_B f \, dm = \frac{1}{|B|} \left(\int_B f_1 \, dm, \dots, \int_B f_d \, dm \right),$$

whenever the f_i's are locally integrable. Let us point out that f_i is in fact locally integrable when $f \in L^p(W)$. To see this, fix an arbitrary ball $B \subset \mathbb{R}^n$. Then Hölders Inequality implies that

$$\int_B |f_i| \, dm = \int |\langle f, \chi_B e_i \rangle| \, dm \leq \|f\|_{L^p(W)} \cdot \|\chi_B e_i\|_{L^{p'}(W^{-p'/p})} < \infty,$$

since $W^{-p'/p}$ is locally integrable. Hence the linear operator $f \mapsto \psi_B(f) := \chi_B f_B$ is well defined on $L^p(W)$.

Proposition 3.23. *With the above notation: $W \in A_p$ if and only if the operators ψ_B are uniformly bounded from $L^p(W)$ into $L^p(W)$, i.e. if there exists a constant $C > 0$ such that*

$$\|\psi_B\| := \sup_{\|f\|_{L^p(W)}=1} \|\psi_B(f)\|_{L^p(W)} \le C, \quad \textit{for all balls } B \subset \mathbb{R}^n. \tag{3.12}$$

In fact, $A_p(W)$ equals the supremum of $\|\psi_B\|$ over all balls $B \subset \mathbb{R}^n$.

Proof. Notice that

$$\begin{aligned} \|\psi_B(f)\|_{L^p(W)} &= |B|^{-1/p'} \rho_{p,B}\left(\int_B f\, dm \right) \\ &= |B|^{-1/p'} \sup_{u\neq 0} \frac{\left|\langle u, \int_B f\, dm\rangle\right|}{(\rho_{p,B})^*(u)} \\ &= |B|^{-1/p'} \sup_{u\neq 0} \frac{\left|\int \langle \chi_B u, f\rangle\, dm\right|}{(\rho_{p,B})^*(u)}, \end{aligned}$$

whenever $f \in L^p(W)$. Taking the supremum over all $f \in L^p(W)$ with $\|f\|_{L^p(W)} = 1$ and employing Lemma 3.22, we see that

$$\|\psi_B\| = \sup_{u\neq 0} |B|^{-1/p'} \frac{\|\chi_B u\|_{L^{p'}(W^{-p'/p})}}{(\rho_{p,B})^*(u)} = \sup_{u\neq 0} \frac{\rho^*_{p',B}(u)}{(\rho_{p,B})^*(u)}. \tag{3.13}$$

Taking the supremum over all balls $B \subset \mathbb{R}^n$ yields the desired. □

Corollary 3.24. *If W is a matrix A_p weight, then $w(x) := |W(x)^{1/p}v|^p$ is a scalar A_p weight, for each nonzero $v \in \mathbb{C}^d$. In fact, $A_p(w) \le A_p(W)$.*

Proof. Notice that $\|\phi\|_{L^p(w)} = \|\phi v\|_{L^p(W)}$, for any measurable scalar function ϕ on $\mathbb{R}^n$. Thus, for an arbitrary $\phi \in L^p(w)$ with $\|\phi\|_{L^p(w)} = 1$, we have

$$\|\chi_B \phi_B\|_{L^p(w)} = \|\chi_B \phi_B v\|_{L^p(W)} = \|\chi_B (\phi v)_B\|_{L^p(W)} \le A_p(W),$$

and hence $A_p(w) \le A_p(W)$, by Proposition 3.23. □

Corollary 3.25. *If W is a matrix A_p weight, then $\|W\|$ is a scalar A_p weight.*

Proof. If $A \in \mathbb{C}^{d\times d}$ is positive definite, then it is easily seen that

$$\|A\| \sim \operatorname{trace}(A) := \sum_{i=1}^{d} \langle Ae_i, e_i\rangle.$$

Thus we can estimate $\|W(x)\|$ pointwise as

$$\begin{aligned}
\|W(x)\| &= \|W(x)^{2/p}\|^{p/2} \sim \operatorname{trace}\big(W(x)^{2/p}\big)^{p/2} \\
&= \left(\sum_{i=1}^{d} |W(x)^{1/p} e_i|^2\right)^{p/2} \\
&\sim \sum_{i=1}^{d} |W(x)^{1/p} e_i|^p, \qquad (3.14)
\end{aligned}$$

where the last $\sim$ readily follows by noting that

$$\max_i c_i \le \sum_i c_i \le d \max_i c_i,$$

for any nonnegative numbers $c_1, \ldots, c_d$. By Corollary 3.24, each term in (3.14) is a scalar A_p weight and therefore the sum is as well. Thus $\|W\|$ is an A_p weight, since it is comparable to an A_p weight. □

Example 3.26. If $w_1, \ldots, w_d$ are scalar A_p weights, then the function $W : \mathbb{R}^n \to \mathbb{C}^{d\times d}$, given by

$$W(x) = D(w_1(x), \ldots, w_d(x)),$$

is a matrix A_p weight. To see this, first notice that

$$\langle W(x)v, v\rangle = \sum_i w_i(x)|v_i|^2 > 0,$$

for all nonzero $v = (v_1, \ldots, v_d) \in \mathbb{C}^d$ and for almost all $x \in \mathbb{R}^n$. Thus W is a weight. Furthermore, since

$$\|W(x)\| = \max_i w_i(x) \le \sum_i w_i(x)$$

and

$$\|W(x)^{-p'/p}\| = \max_i w_i(x)^{-p'/p} \le \sum_i w_i(x)^{-p'/p},$$

for almost all $x \in \mathbb{R}^n$, we see that W and $W^{-p'/p}$ are locally integrable. We now employ Proposition 3.23: Since $w_i \in A_p$, there exists constants $c_i > 0$ such that

$$\|\chi_B \phi_B\|_{L^p(w_i)} \le c_i \|\phi\|_{L^p(w_i)},$$

for all $\phi \in L^p(w_i)$ and for all balls $B \subset \mathbb{R}^n$. By noting that, for any function $f = (f_1, \ldots, f_d) : \mathbb{R}^n \to \mathbb{C}^d$,

$$|w_i(x)^{1/p} f_i(x)|^p \le |W(x)^{1/p} f(x)|^p \le d^{p/2} \max_i |w_i(x)^{1/p} f_i(x)|^p,$$

for almost all $x \in \mathbb{R}^n$, we see that

$$\begin{aligned}
\|\chi_B f_B\|^p_{L^p(W)} &\leq d^{p/2} \int \max_i |w_i(x)(f_i)_B(x)|^p \, dx \\
&\leq d^{p/2} \sum_i \int |w_i(x)^{1/p} \chi_B (f_i)_B(x)|^p \, dx \\
&\leq d^{p/2} \sum_i c_i^p \|f_i\|^p_{L^p(w_i)} \\
&\leq \left(d^{p/2} \sum_i c_i^p \right) \|f\|^p_{L^p(W)},
\end{aligned}$$

for any $f = (f_1, \dots, f_d) \in L^p(W)$. This shows that $W \in A_p$.

We close this chapter with some matrix analogies of The Reverse Hölder Inequality, Proposition 3.12.

Proposition 3.27 (The Reverse Hölder Inequalities)**.** *Let W be a matrix A_p weight. Then there exists a $\delta > 0$ and constants $C_q > 0$ such that*

$$\frac{1}{|B|} \int_B \|W(x)^{1/p} V'_B\|^q \, dx \leq C_q \quad \text{whenever } q < p + \delta \tag{3.15}$$

$$\frac{1}{|B|} \int_B \|V_B W(x)^{-1/p}\|^q \, dx \leq C_q \quad \text{whenever } q < p' + \delta \tag{3.16}$$

$$\frac{1}{|B|} \int_B \|W(x)^{1/p} V_B^{-1}\|^q \, dx \leq C_q \quad \text{whenever } q < p + \delta, \tag{3.17}$$

for all balls $B \subset \mathbb{R}^n$.

Proof. The proofs of these inequalities are similar, so we will show (3.15) only. By Corollary 3.24, the functions $x \mapsto |W(x)^{1/p} V'_B e_i|^p$ are scalar A_p weights and by The Reverse Hölder Inequality, Proposition 3.12, there exists constants $r_i > 1$ and $C_i > 0$ such that

$$\left(\frac{1}{|B|} \int_B |W(x)^{1/p} V'_B e_i|^{p r_i} \, dx \right)^{1/r_i} \leq \frac{C_i}{|B|} \int_B |W(x)^{1/p} V'_B e_i|^p \, dx,$$

or equivalently,

$$\left(\frac{1}{|B|} \int_B |W(x)^{1/p} V'_B e_i|^{q_i} \, dx \right)^{1/q_i} \leq C'_i \left(\frac{1}{|B|} \int_B |W(x)^{1/p} V'_B e_i|^p \, dx \right)^{1/p},$$

for all balls $B \subset \mathbb{R}^n$, where $q_i := pr_i$. Let $q_0 = \min_i q_i$ and let $C_q = \max_i C_i'$. Let $\delta = q_0 - p$ and fix an arbitrary $1 \leq q < p + \delta$. Then

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^q \, dx\right)^{1/q} &\leq \left(\frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^{q_i} \, dx\right)^{1/q_i} \\
&\leq C_i' \left(\frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^p \, dx\right)^{1/p} \\
&\leq C_q \left(\frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^p \, dx\right)^{1/p}.
\end{aligned}$$

Now, since

$$\|A\| \leq d \max_i |Ae_i|,$$

for any matrix $A \in \mathbb{C}^{d\times d}$, we can estimate

$$\begin{aligned}
\frac{1}{|B|}\int_B \|W(x)^{1/p} V_B'\|^q \, dx &\leq \frac{1}{|B|}\int_B \left(d \max_i |W(x)^{1/p} V_B' e_i|\right)^q dx \\
&\leq d^q \sum_i \frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^q \, dx \\
&\leq d^q C_q \sum_i \left(\frac{1}{|B|}\int_B |W(x)^{1/p} V_B' e_i|^p \, dm\right)^{q/p} \\
&\leq d^q C_q \sum_i |V_B V_B' e_i|^q \\
&\leq d^{q+1} C_q \|V_B V_B'\|^q \leq C_q',
\end{aligned}$$

as desired. $\square$

Chapter 4

Weighted Maximal Functions

Boundedness properties of maximal operators and singular integral operators are intimately connected. In this chapter we define maximal operators associated with a given matrix weight W and show that these operators are bounded from $L^q(m)$ into $L^q(m)$, whenever W is an A_p weight and q is sufficiently close to p.

Throughout this chapter we assume that $1 < p < \infty$ is fixed and W denotes and arbitrary A_p weight. Furthermore, we fix $\delta > 0$ such that The Reverse Hölder Inequalities (3.15) - (3.17) hold.

Definition 4.1. Given a measurable function $f : \mathbb{R}^n \to \mathbb{C}^d$, we define *the weighted maximal function* $M_W f$ on $\mathbb{R}^n$ by

$$M_W f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |W(x)^{1/p} W(y)^{-1/p} f(y)| \, dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

The objective is to show that M_W is of type (q, q), whenever $|p - q| < \delta$. Our first step is to establish this result for the auxiliary maximal operator M'_W defined by

$$M'_W f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |V_B W(y)^{-1/p} f(y)| \, dy.$$

However, by employing The Reverse Hölder Inequalities, this turns out to be more or less a repetition of the proof of The Maximal Theorem, Theorem 2.17.

Lemma 4.2. *M'_W is of type (q, q), whenever $q > p - \delta$.*

Proof. Fix an arbitrary $p - \delta < q < \infty$. Then $q' < p' + \delta$ and, by The Reverse Hölder Inequality 3.16,

$$\begin{aligned}\frac{1}{|B|}\int_B |V_B W(y)^{-1/p} f(y)|\,dy &\le \frac{1}{|B|}\int_B \|V_B W(y)^{-1/p}\|\cdot|f(y)|\,dy\\ &\le \left(\frac{1}{|B|}\int_B \|V_B W(y)^{-1/p}\|^{q'}\,dy\right)^{1/q'}\left(\frac{1}{|B|}\int_B |f(y)|^q\,dy\right)^{1/q}\\ &\le C\left(\frac{1}{|B|}\int_B |f(y)|^q\,dy\right)^{1/q},\end{aligned}$$

for all measurable f and for all balls $B \subset \mathbb{R}^n$. In particular, this shows that M'_W is of type (∞,∞), and hence it suffices to show that M'_W is of weak type (q,q). Fix an arbitrary $f \in L^q(m)$ and an $\alpha > 0$. We must show that $|K| \le C'\alpha^{-q}\|f\|_q^q$, for any compact subset

$$K \subset E_\alpha := \{x \in \mathbb{R}^n : M'_W f(x) > \alpha\}.$$

However, K can be covered by balls $\{B_x\}$ each satisfying

$$\alpha < \frac{1}{|B_x|}\int_{B_x} |V_{B_x} W(y)^{-1/p} f(y)|\,dy \le C\left(\frac{1}{|B_x|}\int_{B_x} |f|^q\,dm\right)^{1/q},$$

and thereby

$$|B_x| < \frac{C}{\alpha^q}\int_{B_x} |f|^q\,dm.$$

Thus we may extract a finite subcover from $\{B_x\}$, apply The Vitali Covering Lemma to obtain a disjoint subcollection of the subcover and estimate

$$|K| \le \frac{3^n C}{\alpha^q}\int |f|^q\,dm,$$

as desired. □

In analogue to the dyadic maximal operator we define the *weighted dyadic maximal operators* $\mathcal{M}_W$ and $\mathcal{M}'_W$ by taking supremum over dyadic cubes instead of balls. Since cubes and balls in $\mathbb{R}^n$ are equivalent and, since

$$E \mapsto \int_E |W(x)^{1/p} v|^p\,dx, \quad v \in \mathbb{C}^d\backslash\{0\},$$

is a doubling measure, by Corollary 3.24 and Remark 3.7, it follows that the A_p class can be characterized by cubes instead of balls. In particular, given any cube $Q \subset \mathbb{R}^n$ there exists positive definite complex $d \times d$ matrices V_Q and V'_Q such that

$$|V_Q v| \sim \rho_{p,Q}(v) \quad \text{and} \quad |V'_Q v| \sim \rho^*_{p',Q}(v), \tag{4.1}$$

independent of Q. Also, The Reverse Hölder Inequalities hold with cubes replacing balls.

Remark 4.3. *Since $\mathcal{M}'_W f \leq c M'_W f$, we immediately conclude that $\mathcal{M}'_W$ is of type (q,q) whenever $q > p - \delta$.*

Our next step is to show that $\mathcal{M}_W$ is of type (q,q) whenever q is close to p. A substantial part of this task is established separately trough Lemma 4.4.

For each dyadic cube $Q \subset \mathbb{R}^n$ we define the function N_Q on $\mathbb{R}^n$ by letting

$$N_Q(x) = \sup_{x \in R \subseteq Q} \|W(x)^{1/p} V_R^{-1}\|,$$

for $x \in Q$ and $N_Q(x) = 0$ otherwise. Here the supremum is taken over all dyadic cubes $R \subseteq Q$ containing x.

Lemma 4.4. *For $q < p + \delta$ there exists a constant $C_q > 0$ such that*

$$\int_Q N_Q^q \, dm \leq C_q |Q|, \quad \textit{for all dyadic cubes } Q \subset \mathbb{R}^n.$$

Proof. Fix an arbitrary $q < p + \delta$. Denote by A a positive constant to be determined appropriately later. Let $\{R_j\}$ denote the collection of maximal dyadic cubes $R \subseteq Q$ satisfying $\|V_Q V_R^{-1}\| > A$, and let $D_1 = \cup_j R_j$. We take into account the possibility that $D_1 = \emptyset$. Notice that the cubes in $\{R_j\}$ are disjoint. For $x \in Q \backslash D_1$ we have

$$\|W(x)^{1/p} V_R^{-1}\| \leq \|W(x)^{1/p} V_Q^{-1}\| \cdot \|V_Q V_R^{-1}\| \leq A \|W(x)^{1/p} V_Q^{-1}\|,$$

for all dyadic $x \in R \subseteq Q$, and hence $N_Q(x) \leq A\|W(x)^{1/p} V_Q^{-1}\|$. By The Reverse Hölder Inequality (3.17), we then see that

$$\begin{aligned} \int_Q N_Q^q \, dm &\leq A^q \int_{Q \backslash D_1} \|W(x)^{1/p} V_Q^{-1}\|^q \, dx + \int_{D_1} N_Q^q \, dm \\ &\leq \frac{C}{2}|Q| + \int_{D_1} N_Q^q \, dm, \end{aligned}$$

for some $C > 0$ independent of Q. Of course, if $D_1 = \emptyset$ then we are done. Otherwise we continue to estimate

$$\int_{D_1} N_Q^q \, dm = \sum_j \int_{R_j} N_Q^q \, dm.$$

For each j we let

$$F_j = \{x \in R_j : N_Q(x) \neq N_{R_j}(x)\}.$$

Notice that $N_Q(x) \leq A\|W(x)^{1/p} V_Q^{-1}\|$ whenever $x \in F_j$. Indeed, if $N_{R_j}(x) < N_Q(x)$, then $N_Q(x)$ may be approximated by $\|W(x)^{1/p} V_R^{-1}\|$, for some dyadic cube R containing R_j as a proper subset and, by maximality of R_j,

$$\|W(x)^{1/p}V_R^{-1}\| \leq A\|W(x)^{1/p}V_Q^{-1}\|.$$

It follows that

$$\int_{R_j} N_Q^q\,dm \quad\leq\quad A^q\int_{F_j}\|W(x)^{1/p}V_Q^{-1}\|^q\,dx + \int_{R_j} N_{R_j}^q\,dm,$$

and consequently

$$\int_Q N_Q^q\,dm \leq \frac{C}{2}|Q| + \frac{C}{2}|Q| + \sum_j \int_{R_j} N_{R_j}^q\,dm \leq C|Q| + \int_{D_1} N_Q^q\,dm.$$

Let us show that $|D_1| \leq 1/2|Q|$ if A is sufficiently large. Since

$$\begin{aligned} |V_{R_j}^{-1}V_Q v| &= |V_{R_j}(V_Q v)|^* \leq (\rho_{p,R_j})^*(V_Q v) \\ &\leq \rho_{p',R_j}^*(V_Q v) \leq |V_{R_j}' V_Q v|, \end{aligned}$$

for all $v \in \mathbb{C}^d$, we have $\|V_Q V_{R_j}^{-1}\| = \|V_{R_j}^{-1}V_Q\| \leq \|V_{R_j}' V_Q\|$, and hence

$$\begin{aligned} |R_j|\cdot\|V_Q V_{R_j}^{-1}\|^{p'} &\leq \sup_{|v|\leq 1} |R_j|\cdot|V_{R_j}' V_Q v|^{p'} \\ &\leq d^{p'/2}\sup_{|v|\leq 1}\int_{R_j}|W(x)^{-1/p}V_Q v|^{p'}\,dx \\ &\leq d^{p'/2}\int_{R_j}\|W(x)^{-1/p}V_Q\|^{p'}\,dx. \end{aligned}$$

By disjointness of the R_j's we get

$$\begin{aligned} A^{p'}|D_1| &\leq \sum_j |R_j|\cdot\|V_Q V_{R_j}^{-1}\|^{p'} \leq d^{p'/2}\sum_j\int_{R_j}\|W(x)^{-1/p}V_Q\|^{p'}\,dx \\ &\leq d^{p'/2}\int_Q\|W(x)^{-1/p}V_Q\|^{p'}\,dx \leq C'|Q|, \end{aligned}$$

by The Reverse Hölder Inequality. Thus we may choose A independently of Q such that $|D_1| \leq 1/2|Q|$.

The crux of the proof is now over. Indeed, since Q is an arbitrary dyadic cube and C is independent Q, we may repeat the above argument to estimate

$$\int_{R_j} N_{R_j}^q\,dm \leq C|R_j| + \sum_i \int_{S_i^j} N_{S_i^j}^q\,dm \leq C|R_j| + \int_{\cup_i S_i^j} N_Q^q\,dm,$$

where $\{S_i^j\}$ denotes the collection of maximal dyadic cubes $S \subseteq R_j$ satisfying $\|V_{R_j}V_S^{-1}\| > A$. It follows that

$$\int_Q N_Q^q\,dm \leq C|Q| + \frac{1}{2}C|Q| + \int_{D_2} N_Q^q\,dm,$$

where $D_2 := \cup_j \cup_i S_i^j$ and $|D_2| \le 1/4|Q|$. This strategy may be employed into an induction argument to show that

$$\int_Q N_Q^q \, dm \le C\Big(\sum_{k=0}^{m} 2^{-k}\Big)|Q| + \int_{D_{m+1}} N_Q^q \, dm, \tag{4.2}$$

and $|D_{m+1}| \le 2^{m+1}|Q|$, for any $m \in \mathbb{N}_0$. Since $|D_m| \to 0$ as $m \to \infty$ and, since $Q\backslash D_1 \subseteq Q\backslash D_2 \subseteq \ldots$, The Monotone Convergence Theorem implies that $\chi_{Q\backslash D_m} \nearrow \chi_Q$ a.e., and consequently

$$\int_{D_m} N_Q^q \, dm = \int_Q N_Q^q \, dm - \int_{Q\backslash D_m} N_Q^q \, dm \to 0 \quad \text{as } m \to \infty.$$

Hence the proof is concluded by taking the limit as $m \to \infty$ in (4.2). □

Lemma 4.5. *The weighted dyadic maximal operator $\mathcal{M}_W$ is of type (q,q) whenever $|p-q| < \delta$.*

Proof. Fix an arbitrary $q \ge 1$ with $|p-q| < \delta$ and an arbitrary $f \in L^q(m)$. By virtue of Remark 4.3, it suffices to show that $\|\mathcal{M}_W f\|_q \le C\|\mathcal{M}'_W f\|_q$. For each $x \in \mathbb{R}^n$ we choose a dyadic cube Q_x containing x and satisfying

$$\begin{aligned} \mathcal{M}_W f(x) &\le 2\frac{1}{|Q_x|}\int_{Q_x} |W(x)^{1/p}W(y)^{-1/p}f(y)| \, dy \qquad (4.3) \\ &\le 2\|W(x)^{1/p}V_{Q_x}^{-1}\| \cdot \left(\frac{1}{|Q_x|}\int_{Q_x} |V_{Q_x}W(y)^{-1/p}f(y)| \, dy\right). \end{aligned}$$

For each $j \in \mathbb{Z}$ we let $\mathcal{F}_j$ denote the set of maximal dyadic cubes $Q \in \{Q_x\}$ for which

$$2^j < \frac{1}{|Q|}\int_Q |V_Q W(y)^{-1/p} f(y)| \, dy \le 2^{j+1}. \tag{4.4}$$

We may choose the cubes to be maximal, since any cube $Q = Q_x$ satisfying (4.4) must also satisfy

$$|Q| \le |\{x \in \mathbb{R}^n : \mathcal{M}'_W f(x) > 2^j\}| \le 2^{-jq}\|\mathcal{M}'_W f\|_{q,\infty}^q \le C_q 2^{-jq}\|f\|_q^q,$$

by Remark 4.3. Now, if $\mathcal{M}_W f(x) > 0$, then (4.3) implies that

$$2^j < \frac{1}{|Q_x|}\int_{Q_x} \|V_{Q_x}W(y)^{-1/p}f(y)\| \, dy \le 2^{j+1},$$

for some $j \in \mathbb{Z}$ and hence Q_x is contained in some cube $Q \in \mathcal{F}_j$. Combining this with (4.3) we see that

$$\mathcal{M}_W f(x) \le 2 \cdot 2^{j+1}\|W(x)^{1/p}V_{Q_x}^{-1}\| \le 2 \cdot 2^{j+1} N_Q(x),$$

and consequently

$$\mathcal{M}_W f(x)^q \le C 2^{jq} N_Q(x)^q \le C \sum_{j=-\infty}^{\infty} \sum_{Q' \in \mathcal{F}_j} 2^{jq} N_{Q'}(x)^q,$$

for all $x \in \mathbb{R}^n$. Employing Lemma 4.4 and the fact that the cubes in each $\mathcal{F}_j$ are disjoint, we get

$$\begin{aligned}
\|\mathcal{M}_W f\|_q^q &\leq C \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{Q' \in \mathcal{F}_j} \int N_{Q'}^q \, dm \\
&\leq C' \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{Q' \in \mathcal{F}_j} |Q'| = C' \sum_{j=-\infty}^{\infty} 2^{jq} \cdot \Big| \bigcup_{Q' \in \mathcal{F}_j} Q' \Big| \\
&\leq C' \sum_{j=-\infty}^{\infty} 2^{jq} \cdot |\{x \in \mathbb{R}^n : \mathcal{M}'_W f > 2^j\}| \\
&= C' \sum_{j=-\infty}^{\infty} 2^{jq} d_{\mathcal{M}'_W f}(2^j).
\end{aligned}$$

However, since the distribution function is decreasing,

$$\begin{aligned}
\int_{2^{j-1}}^{2^j} \alpha^{q-1} d_{\mathcal{M}'_W f}(\alpha)\, d\alpha &\geq (2^{j-1})^{q-1} \cdot d_{\mathcal{M}'_W f}(2^j) \int_{2^{j-1}}^{2^j} d\alpha \\
&= 2^{-q} \cdot 2^{jq} \cdot d_{\mathcal{M}'_W f}(2^j),
\end{aligned}$$

for each $j \in \mathbb{Z}$, and hence

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} 2^{jq} d_{\mathcal{M}'_W f}(2^j) &\leq 2^q \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} \alpha^{q-1} d_{\mathcal{M}'_W f}(\alpha)\, d\alpha \\
&= 2^q \int_0^{\infty} \alpha^{q-1} d_{\mathcal{M}'_W f}(\alpha)\, d\alpha \\
&\leq q^{-1} 2^q \|\mathcal{M}'_W f\|_q^q,
\end{aligned}$$

by Proposition 2.5. □

The final step is to obtain the L^q boundedness of M_W from its dyadic analogue.

Theorem 4.6. *The weighted maximal operator M_W is of type (q, q), whenever $|p - q| < \delta$.*

Proof. Fix an arbitrary $q \geq 1$ with $|p - q| < \delta$. Define $\tilde{M}_W$ as M_W, but with the supremum taken over all cubes in $\mathbb{R}^n$ containing x. Since cubes and balls in $\mathbb{R}^n$ are equivalent, it suffices to show that $\tilde{M}_W$ is of type (q, q). Furthermore, by The Monotone Convergence Theorem, it suffices to show that the operator $M_W^{2^k}$, given by

$$M_W^{2^k} f(x) = \sup_{l(Q) \leq 2^k} \frac{1}{|Q|} \int_Q |W(x)^{1/p} W(y)^{-1/p} f(y)|\, dy,$$

is of type (q,q), for each $k \in \mathbb{Z}$ with bound independent of k. Here the supremum is taken over all cubes in $\mathbb{R}^n$ containing x and with side length $l(Q) \leq 2^k$. For each $t \in \mathbb{R}^n$ we define $\mathcal{M}_{W,t}f$ by

$$\mathcal{M}_{W,t}f(x) = \sup_{x \in Q \in \mathcal{D}_t} \frac{1}{|Q|} \int_Q |W(x)^{1/p} W(y)^{-1/p} f(y)| \, dy,$$

where $\mathcal{D}_t$ denotes the set of all cubes $Q \subset \mathbb{R}^n$, for which $Q - t$ is a dyadic cube. The crucial property concerning dyadic cubes is the fact that they are nested, i.e. any two dyadic cubes are either disjoint or one contains the other. For each $t \in \mathbb{R}^n$, the cubes in D_t sustain this property. Hence we can imitate the proof of the L^q boundedness of $\mathcal{M}_W$ to show that $\mathcal{M}_{W,t}$ is also of type (q,q). In fact, the bound can be taken to be independent of t, as seen by separate inspection of each proof of this chapter.

For each $k \in \mathbb{Z}$ we let $Q_k = [-2^{k+2}, 2^{k+2}]^n$. We will show that there exists a constant $C > 0$ such that

$$M_W^{2^k} f(x) \leq C \int_{Q_k} \mathcal{M}_{W,t} f(x) \, \frac{dt}{|Q_k|}, \tag{4.5}$$

for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Once this is established, the result follows from Tonelli's Theorem and the uniform L^q boundedness of $\mathcal{M}_{W,t}$. To verify (4.5), we fix an arbitrary $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Choose a cube $Q \subset \mathbb{R}^n$ containing x with

$$\frac{1}{|Q|} \int_Q |W(x)^{1/p} W(y)^{-1/p} f(y)| \, dy > \frac{1}{2} M_W^{2^k} f(x)$$

and $2^{j-1} < l(Q) \leq 2^j$, for some integer $j \leq k$. Define

$$\Omega = \left\{ t \in Q_k : Q \subseteq Q_t, \text{ for some } Q_t \in D_t \text{ with } l(Q_t) = 2^{k+1} \right\}.$$

We claim that Ω is measurable and $|\Omega| \geq 2^{n(k+1)}$. First consider the case $n = 1$. By visualizing all intervals of length 2^{k+1} containing Q, it is geometrically evident that Ω is the union of two disjoint intervals of length

$$2^{k+1} - l(Q) \geq 2^{k+1} - 2^k = 2^k.$$

For a general $n \geq 1$, we project Ω onto the axes and conclude, by the preceding, that Ω is the union of 2^n disjoint cubes of side length at least 2^k. Thus $|\Omega| \geq 2^n \cdot (2^k)^n = 2^{n(k+1)}$. For each $t \in \Omega$, $|Q_t| \leq 4^n |Q|$, and hence

$$\begin{aligned} \mathcal{M}_{W,t} f(x) \quad &\geq \quad \frac{1}{|Q_t|} \int_{Q_t} |W(x)^{1/p} W(y)^{-1/p} f(y)| \, dy \qquad (4.6) \\ &\geq \quad \frac{4^{-n}}{|Q|} \int_Q |W(x)^{1/p} W(y)^{-1/p} f(y)| \, dy \geq (2 \cdot 4^n)^{-1} M_W^{2^k} f(x). \end{aligned}$$

Now it follows from (4.6) that

$$\frac{1}{|Q_k|} \int_{Q_k} \mathcal{M}_{W,t} f(x) \, dt \geq \frac{1}{|Q_k|} \int_\Omega \mathcal{M}_{W,t} f(x) \, dt \geq (4^n \cdot 2^{2n+1})^{-1} M_W^{2^k} f(x),$$

proving (4.5) and the theorem with it. □

Chapter 5

Truncation of Singular Integrals

Recall from Chapter 1 that the Riesz transform was defined in terms of "truncated integrals" over sets $|y| \geq \varepsilon$. This technique turns out to be useful also when dealing with general singular integral operators. The results in this chapter play a crucial role in the "weighted inequalities" in Chapter 6. We consider a singular integral operator T associated with a regular kernel K, as defined in Chapter 1. Furthermore, we assume that, for some $1 < p < \infty$, there exists a constant $A > 0$, such that

$$\|Tf\|_p \leq A\|f\|_p, \quad \text{for all } f \in \mathcal{S}.$$

It is a fundamental result, shown in e.g. [6], that T has a bounded linear extension of weak type $(1,1)$ and of type (q,q), for each $1 < q < \infty$. This extension is also denoted by T. The fact that K is a function away from the origin implies that

$$Tf(x) = \int K(x-y)f(y)\,dy, \quad \text{for all } x \notin \operatorname{supp}(f), \tag{5.1}$$

whenever $f \in C_c^\infty(\mathbb{R}^n;\mathbb{C})$. In fact, (5.1) holds a.e., for any compactly supported function $f \in L^p(m)$. To see this, first note that Hölder's Inequality and the estimate $|K(x)| \leq B|x|^{-n}$ implies that $y \mapsto K(x-y)f(y)$ is integrable whenever $x \notin \operatorname{supp}(f)$. Then choose a sequence $f_k \in C_c^\infty(\mathbb{R}^n;\mathbb{C})$ with $\operatorname{supp}(f_k) \subseteq \operatorname{supp}(f)$ such that $f_k \to f$ in L^p. By continuity of T, $Tf_k \to Tf$ in L^p, and hence

$$\int K(x-y)f_{k_j}(y)\,dy \to Tf(x) \quad \text{as } j \to \infty,$$

for almost all $x \notin \operatorname{supp}(f)$, for some subsequence f_{k_j}. However, we also have

$$\int K(x-y)f_{k_j}(y)\,dy \to \int K(x-y)f(y)\,dy \quad \text{as } j \to \infty,$$

provided that $x \notin \operatorname{supp}(f)$.

For each $\varepsilon > 0$ we define the measurable function K_ε on $\mathbb{R}^n \backslash \{0\}$ by letting $K_\varepsilon(x) = K(x)$ if $|x| \geq \varepsilon$ and $K_\varepsilon(x) = 0$ otherwise. By the assumption $|K(x)| \leq B|x|^{-n}$, the function $K_\varepsilon(x - \cdot)$ is bounded, for each $x \in \mathbb{R}^n$. Furthermore,

$$d_{K_\varepsilon(x-\cdot)}(\alpha) \leq |\{y \in \mathbb{R}^n : B|x-y|^{-n} > \alpha\}| = \frac{v_n B}{\alpha},$$

for each $\alpha > 0$, and hence $K(x - \cdot) \in L^{1,\infty}(m)$, for each $x \in \mathbb{R}^n$. By Proposition 2.11, we then conclude that $K_\varepsilon(x-\cdot)$ is in $L^r(m)$, for any $r > 1$ and, by Hölder's Inequality, we may therefore define *the truncated operator* T_ε on $L^p(m)$ by

$$T_\varepsilon f(x) = \int K_\varepsilon(x-y) f(y)\, dy, \quad \text{for } x \in \mathbb{R}^n.$$

We also define the sublinear operator T_* on $L^p(m)$ by

$$T_* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|, \quad \text{for } x \in \mathbb{R}^n.$$

It turns out that Tf in some sense is controlled pointwise by $T_* f$, and hence we will elaborate on various estimates concerning T_*. However, our first step is to show that the truncated operators T_ε are of type (p,p) with bound independent of ε. We need the following preliminary result.

Lemma 5.1. *Let $\mathcal{B}$ denote the collection of balls in $\mathbb{R}^n$ with fixed radius $r > 0$. Then $\mathcal{B}$ has a countable maximal disjoint subcollection, i.e. there exists a collection $\{B(x_k, r)\}_{k\in\mathbb{N}}$ of pairwise disjoint balls such that, for any $x \in \mathbb{R}^n$, $B(x,r) \cap B(x_k, r) \neq \emptyset$, for some $k \in \mathbb{N}$.*

Proof. This is a direct application Zorn's Lemma. Denote by X the set of all disjoint collections of balls of radius r. We order X partially by letting $\mathcal{F} \leq \mathcal{F}'$ whenever $\mathcal{F}, \mathcal{F}' \in X$ and $\mathcal{F} \subseteq \mathcal{F}'$. Each chain C in X is clearly bounded above by $\bigcup_{\mathcal{F}\in C} \mathcal{F} \in X$, and hence X has a maximal element $\mathcal{F}_0$. To see that $\mathcal{F}_0$ is countable, we define $\varphi : \mathbb{Z}^n \to \mathcal{F}_0$ by letting $\varphi(t) = B(x,r)$ if $t \in B(x,r)$ and $\varphi(t) = 0$ otherwise. By disjointness of the balls in $\mathcal{F}_0$, φ is well-defined and, by density of the rationals, it takes $\mathbb{Z}^n$ onto $\mathcal{F}_0$, showing that $\mathcal{F}_0$ is countable. □

Lemma 5.2. *T_ε is of type (p,p) with bound independent of ε.*

Proof. Fix an arbitrary $\varepsilon > 0$ and an $f \in L^p(m)$. By employing a density argument, we may assume that f is compactly supported. It suffices to show that $\Delta_\varepsilon := T - T_\varepsilon$ is of type (p,p) with bound independent of ε. Fix an arbitrary $\bar{x} \in \mathbb{R}^n$. For each $\delta > 0$ we let $\chi_\delta := \chi_{B(\bar{x},\delta)}$. We will show that

$$\|\chi_{a\varepsilon} \Delta_\varepsilon f\|_p \leq C \|\chi_{b\varepsilon} f\|_p, \tag{5.2}$$

for some constants $a, b, C > 0$ independent of $\varepsilon, \bar{x}$ and f. First notice that $\Delta_\varepsilon f(x) = 0$ for almost all x with $\operatorname{supp}(f) \subseteq B(x,\varepsilon)^c$. Also notice that $B(x,\varepsilon) \subseteq B(\bar{x}, b\varepsilon)$ whenever $x \in B(\bar{x}, a\varepsilon)$ and $b \geq 1 + a$. Since

$$f = f\chi_{B(x,\varepsilon)} + f\chi_{B(x,\varepsilon)^c},$$

the linearity of Δ_ε and the fact that $T_\varepsilon(f\chi_{B(x,\varepsilon)})(x) = 0$, shows that

$$\chi_{a\varepsilon}\Delta_\varepsilon f = \chi_{a\varepsilon}\Delta_\varepsilon\chi_{b\varepsilon} f \quad \text{a.e.}$$

Furthermore, since

$$\chi_{a\varepsilon}\Delta_\varepsilon\chi_{b\varepsilon} f = \chi_{a\varepsilon}\Delta_\varepsilon\chi_{d\varepsilon} f + \chi_{a\varepsilon}\Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f,$$

where $d > 0$ will be chosen appropriately later, it suffices to show (5.2) with $\Delta_\varepsilon f$ replaced by $\Delta_\varepsilon\chi_{d\varepsilon} f$ respectively $\Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f$. Regarding the first estimate, we assume that $a + d \leq 1$ and $d < b$. Then $B(\bar{x}, d\varepsilon) \subseteq B(x,\varepsilon)$ whenever $x \in B(\bar{x}, a\varepsilon)$ and, since $T_\varepsilon\chi_{d\varepsilon} f(x) = 0$ when $B(\bar{x}, d\varepsilon) \subseteq B(x,\varepsilon)$, this implies that

$$\chi_{a\varepsilon}\Delta_\varepsilon\chi_{d\varepsilon} f(x) = \chi_{a\varepsilon} T\chi_{d\varepsilon} f.$$

Since $\|Tf\|_p \leq A\|f\|$, for all $f \in L^p$, we get

$$\|\chi_{a\varepsilon}\Delta_\varepsilon\chi_{d\varepsilon} f\|_p = \|\chi_{a\varepsilon} T\chi_{d\varepsilon} f\|_p \leq A\|\chi_{d\varepsilon} f\|_p \leq A\|\chi_{b\varepsilon} f\|_p,$$

as desired.

For the second estimate, we first notice that

$$\Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f = \Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f\chi_{B(x,\varepsilon)} = Tf\chi_E,$$

where $E := B(x,\varepsilon) \cap B(\bar{x}, b\varepsilon) \backslash B(\bar{x}, d\varepsilon)$. If we assume that $a < d$, then $B(\bar{x}, a\varepsilon) \cap \operatorname{supp}(f\chi_E) = \emptyset$, and hence

$$\Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f(x) = \int_E K(x-y)f(y)\,dy,$$

for almost all $x \in B(\bar{x}, a\varepsilon)$. If $x \in B(\bar{x}, a\varepsilon)$ and $y \in E$, then

$$d\varepsilon \leq |\bar{x} - y| \leq a\varepsilon + |x - y|,$$

and therefore

$$|K(x-y)| \leq \frac{B}{|x-y|^n} \leq \frac{B'}{\varepsilon^n}.$$

We have now shown that

$$\left| \int_E K(x-y)f(y)\,dy \right| \leq \frac{B'}{\varepsilon^n}\|\chi_{b\varepsilon} f\|_1 \leq B''\varepsilon^{-n/p}\|\chi_{b\varepsilon} f\|_p,$$

for almost all $x \in B(\bar{x}, a\varepsilon)$, and consequently

$$\|\chi_{a\varepsilon}\Delta_\varepsilon(\chi_{b\varepsilon} - \chi_{d\varepsilon})f\|_p^p \leq \frac{B''}{\varepsilon^n}\|\chi_{b\varepsilon} f\|_p^p \cdot |B(\bar{x}, a\varepsilon)| = C\|\chi_{b\varepsilon} f\|_p^p.$$

Thus (5.2) holds provided that a, b, d are chosen appropriately. This can be done by letting $a < d < 1/2$ and $b \geq 1 + a$.

We may cover $\mathbb{R}^n$ with a countable collection of balls $\{B(\bar{x}^k, a\varepsilon)\}$. Indeed, Lemma 5.1 guaranties the existence of a maximal disjoint collection of balls $\{B(\bar{x}^k, a\varepsilon/2)\}$ such that, for each $y \in \mathbb{R}^n$, we have $B(y, a\varepsilon/2) \cap B(\bar{x}^k, a\varepsilon/2) \neq \emptyset$, for some $k \in \mathbb{N}$, and hence $y \in B(y, a\varepsilon/2) \subseteq B(\bar{x}^k, a\varepsilon)$. Also, there exists a number N such that no point $y \in \mathbb{R}^n$ belongs to more than N of the balls $B(\bar{x}^k, b\varepsilon)$. To see this, assume that

$$y \in \bigcap_{j=1}^{N} B(\bar{x}^{k_j}, b\varepsilon).$$

Then $B(\bar{x}^{k_j}, a\varepsilon/2) \subseteq B(y, r\varepsilon)$, for some $r > 0$ independent of y, and hence

$$\begin{aligned} N \cdot |B(y, r\varepsilon)| &= \sum_{j=1}^{N} |B(y, r\varepsilon)| = (2r/a)^n \sum_{j=1}^{N} |B(\bar{x}^{k_j}, a\varepsilon/2)| \\ &= (2r/a)^n \big|\big(\bigcup_{j=1}^{N} B(\bar{x}^{k_j}, a\varepsilon/2)\big| \leq (2r/a)^n |B(y, r\varepsilon)|, \end{aligned}$$

showing that $N \leq (2r/a)^n$. Then, finally,

$$\begin{aligned} \int |\Delta_\varepsilon f|^p \, dm &\leq \sum_{k=1}^{\infty} \int_{B(\bar{x}^k, a\varepsilon)} |\Delta_\varepsilon f|^p \, dm \leq C \sum_{k=1}^{\infty} \int_{B(\bar{x}^k, b\varepsilon)} |f|^p \, dm \\ &\leq CN \int |f|^p \, dm, \end{aligned}$$

and we are done. □

In what follows we let $\mathcal{B}(L^p, L^p)$ denote the set of all bounded linear operators from $L^p(m)$ into $L^p(m)$.

Definition 5.3. A sequence of operators $T_j \in \mathcal{B}(L^p, L^p)$ is said to *converge weakly in* L^p to $T_0 \in \mathcal{B}(L^p, L^p)$ if

$$\int (T_j f) h \, dm \to \int (T_0 f) h \, dm \quad \text{as } j \to \infty,$$

for all $f \in L^p$ and for all $h \in L^{p'}$.

As a consequence of the uniform L^p boundedness of T_ε and The Banach-Alaoglu Theorem, we have the following result. For the details see [2].

Lemma 5.4. *There exists a sequence $\varepsilon_j \to 0$ such that T_{ε_j} converges weakly in L^p to an operator $T_0 \in \mathcal{B}(L^p, L^p)$.*

The crucial role of the operator T_0 is stated in the following Lemma.

Lemma 5.5. *There exists a function* $b \in L^\infty(m)$ *such that*

$$Tf = T_0 f + bf \quad a.e., \quad \text{for all } f \in L^p(m).$$

Proof. For each $\varepsilon > 0$ we let $\Delta_\varepsilon = T - T_\varepsilon$. By Lemma 5.4, Δ_{ε_j} converges weakly in L^p to $\Delta := T - T_0$, for some sequence $\varepsilon_j \to 0$. For each $k \in \mathbb{N}$ we let $B_k = B(0,k)$. We define b a.e. on B_k by $b = \Delta(\chi_{B_k})$. Of course, if $x \in B_k$, then $x \in B_{k'}$, for each $k' \geq k$, so in order for b to be well-defined a.e., we must show that

$$\Delta(\chi_{B_k}) = \chi_{B_k}\Delta(\chi_{B_{k'}}) \quad \text{a.e.} \quad \text{whenever } k' \geq k.$$

Assume that g is a bounded and compactly supported function on $\mathbb{R}^n$. First notice that

$$\Delta_\varepsilon(g\chi_Q) = \chi_Q \Delta_\varepsilon g \quad \text{a.e.}, \tag{5.3}$$

for any cube $Q \subset \mathbb{R}^n$ and for ε sufficiently small. Indeed, $x \in Q^c$ implies that $\operatorname{supp}(g\chi_Q) \subseteq B(x,\varepsilon)^c$, for small ε, and hence $\Delta_\varepsilon(g\chi_Q)(x) = 0$. Similarly, $x \in Q^o$ (the interior of Q) implies that $\operatorname{supp}(g\chi_{Q^c}) \subseteq B(x,\varepsilon)^c$, for small ε, and so

$$\Delta_\varepsilon(g\chi_Q)(x) = \Delta_\varepsilon g(x) - \Delta_\varepsilon(g\chi_{Q^c})(x) = \Delta_\varepsilon g(x).$$

Next we show that (5.3) holds with Δ_ε replaced by Δ. For any $h \in L^{p'}$ we have

$$\begin{aligned}\int \Delta(g\chi_Q)h\,dm &= \lim_{j\to\infty}\int \Delta_{\varepsilon_j}(g\chi_Q)h\,dm = \lim_{j\to\infty}\int (\Delta_{\varepsilon_j} g)(\chi_Q h)\,dm \\ &= \int \Delta g(\chi_Q h)\,dm = \int (\chi_Q \Delta g)h\,dm\end{aligned}$$

and, as a consequence of The Hahn-Banach Theorem, we conclude that $\Delta(g\chi_Q) = \chi_Q\Delta g$ a.e. By linearity of Δ, this also holds with χ_Q replaced by any linear combination of characteristic functions of cubes. If $O \subset \mathbb{R}^n$ is nonempty and open with finite measure, then $O = \cup_{j=1}^\infty Q_j$, where $\{Q_j\}$ denotes the collection of (disjoint) maximal dyadic cubes contained in O. Hence $f_m := \sum_{j=1}^m \chi_{Q_j} \to \chi_O$ pointwise and, since both $|g|^p$ and $|\Delta g|^p$ are integrable, The Dominated Convergence Theorem yields

$$gf_m \to g\chi_O \quad \text{and} \quad f_m\Delta g \to \chi_O\Delta g \quad \text{in } L^p.$$

By continuity of Δ, we conclude that $\Delta(g\chi_O) = \chi_O\Delta g$ a.e. More generally, if $E \subset \mathbb{R}^n$ is measurable and of finite measure, then there exists a sequence O_m of open sets of finite measure containing E, such that $\chi_{O_m} \to \chi_E$ in L^p. Hence $\chi_{O_{m_k}} \to \chi_E$ a.e., for some subsequence O_{m_k} and, by The Dominated Convergence Theorem and the continuity of Δ, we conclude that

$$\Delta(g\chi_E) = \chi_E\Delta g \quad \text{a.e.}$$

By the particular choice of $g = \chi_{B_{k'}}$ and $E = B_k$, where $k' \geq k$, we obtain

$$\Delta(\chi_{B_k}) = \Delta(\chi_{B_{k'}}\chi_{B_k}) = \chi_{B_k}\Delta(\chi_{B_{k'}}) \quad \text{a.e.,}$$

and thus b is well-defined.

Let us show that $b \in L^\infty$. Clearly b is measurable. Notice that if $E \subset \mathbb{R}^n$ is of finite measure, then $E \subseteq B_k$, for some $k \in \mathbb{N}$, and hence

$$\Delta(\chi_E) = \Delta(\chi_{B_k}\chi_E) = \chi_E\Delta(\chi_{B_k}) = \chi_E b \quad \text{a.e.} \tag{5.4}$$

For each $C > 0$ we let $N_C = \{x \in \mathbb{R}^n : |b(x)| > C\}$. If $|N_C| > 0$ then there exists a compact subset $K \subseteq N_C$ with $|K| > 0$ and, by the L^p boundedness of Δ, we get

$$C^p|K| = \int C^p \chi_K \, dm \leq \int |b\chi_K|^p \, dm = \int |\Delta(\chi_K)|^p \, dm \leq A^p|K|,$$

showing that $C \leq A$, for some constant $A > 0$.

Finally we show that $\Delta f = bf$ a.e., for all $f \in L^p$. By (5.4) we know that this holds for $f = \chi_E$, where E is bounded and, by linearity of Δ, this holds if f is any linear combination of characteristic functions of bounded sets. Since any nonnegative $f \in L^p$ may be approximated in L^p norm by a sequence $\{f_m\}$ of simple nonnegative L^p functions, we conclude that

$$\Delta f = \lim_{m\to\infty} \Delta f_m = \lim_{m\to\infty} (bf_m) = bf,$$

where the last equality follows from the fact that b is bounded a.e. Applying linearity of Δ again, we see that $\Delta f = bf$, for all $f \in L^p$. □

We will now show that the operator T_* is of weak type $(1, 1)$. To this end, we employ three preliminary results, of which the first is a simple consequence of the gradient condition on K.

Lemma 5.6. *Let $x, \bar{x} \in \mathbb{R}^n$ with $x \neq \bar{x}$. If K satisfies $|\nabla K(x)| \leq C|x|^{-n-1}$, for all $x \neq 0$, then there exists a constant $C' > 0$ independent of x and $\bar{x}$ such that*

$$|K(x-y) - K(\bar{x}-y)| \leq \frac{C'|x-\bar{x}|}{|\bar{x}-y|^{n+1}},$$

for all $y \in \mathbb{R}^n$ with $2|x - \bar{x}| \leq |\bar{x} - y|$.

Proof. By splitting K into its real and imaginary parts, we may assume that K is real-valued. Let $\delta = |x - \bar{x}|$ and fix an arbitrary $y \in B(\bar{x}, 2\delta)^c$. Define $\varphi : [0, 1] \to \mathbb{R}^n$ by $\varphi(t) = \bar{x} + t(x - \bar{x}) - y$. Notice that the function $t \mapsto K(\varphi(t))$ is differentiable, since $\bar{x} + t(x - \bar{x}) \in B(\bar{x}, 2\delta)$, for all $t \in [0, 1]$. By The Mean Value Theorem there exists a point $\tau \in (0, 1)$ such that

$$K(x-y) - K(\bar{x}-y) = \langle \nabla K(\varphi(\tau)), (x - \bar{x}) \rangle,$$

and hence

$$|K(x-y)-K(\bar{x}-y)| \leq \frac{C|x-\bar{x}|}{|\varphi(\tau)|^{n+1}} \leq \frac{C'|x-\bar{x}|}{|\bar{x}-y|^{n+1}},$$

where the last inequality follows by noting that $|\bar{x}-y| \leq 2|\varphi(t)|$, for all $t \in [0,1]$. □

Lemma 5.7. *For each $0 < r \leq 1$ there exists a constant $A_r > 0$ such that*

$$T_* f \leq A_r\big[M(|Tf|^r)^{1/r} + Mf\big] \quad a.e.,$$

for all $f \in L^p(m)$.

Proof. Fix an arbitrary $\bar{x} \in \mathbb{R}^n$ and an $\varepsilon > 0$. Let $f_1 = f\chi_{B(\bar{x},\varepsilon)}$ and $f_2 = f\chi_{B(\bar{x},\varepsilon)^c}$. Notice that $T_\varepsilon f(\bar{x}) = Tf_2(\bar{x})$. We first show that

$$|Tf_2(x) - Tf_2(\bar{x})| \leq A'Mf(\bar{x}), \quad \text{whenever } x \in B(\bar{x}, \varepsilon/2),$$

for some $A' > 0$. To see this, fix an $x \in B(\bar{x},\varepsilon/2)$ with $x \neq \bar{x}$. Then

$$\begin{aligned}|Tf_2(x) - Tf_2(\bar{x})| &\leq \int_{B(\bar{x},\varepsilon)^c} |K(x-y)-K(\bar{x}-y)| \cdot |f(y)|\, dy \\ &= \sum_{k=0}^{\infty} \int_{2^k\varepsilon \leq |y-\bar{x}| < 2^{k+1}\varepsilon} |K(x-y)-K(\bar{x}-y)| \cdot |f(y)|\, dy.\end{aligned}$$

For $2^k\varepsilon \leq |y-\bar{x}| < 2^{k+1}\varepsilon$, Lemma 5.6 yields

$$|K(x-y)-K(\bar{x}-y)| \leq \frac{C'\frac{\varepsilon}{2}}{(2^k\varepsilon)^{n+1}} = \frac{2^{n-1}C'}{2^k(2^{k+1}\varepsilon)^n},$$

and hence

$$\begin{aligned}|Tf_2(x) - Tf_2(\bar{x})| &\leq C'' \sum_{k=0}^{\infty} \frac{2^{-k}}{|B(\bar{x},2^{k+1}\varepsilon)|} \int_{B(\bar{x},2^{k+1}\varepsilon)} |f(y)|\, dy \\ &\leq C''Mf(\bar{x}) \sum_{k=0}^{\infty} 2^{-k} = A'Mf(\bar{x}),\end{aligned}$$

as claimed. We can now estimate

$$\begin{aligned}|T_\varepsilon f(\bar{x})| &\leq |Tf_2(\bar{x}) - Tf_2(x)| + |Tf_2(x)| \\ &\leq |Tf(x)| + |Tf_1(x)| + A'Mf(\bar{x}),\end{aligned} \tag{5.5}$$

whenever $x \in B(\bar{x},\varepsilon/2)$. For each $\alpha > 0$ we let $E_\alpha = \{x \in B(\bar{x},\varepsilon/2) : |Tf(x)| > \alpha\}$ and $F_\alpha = \{x \in B(\bar{x},\varepsilon/2) : |Tf_1(x)| > \alpha\}$. Notice that

$$|E_\alpha| \leq \alpha^{-r} \int_{B(\bar{x},\varepsilon/2)} |Tf|^r\, dm \leq \alpha^{-r}|B(\bar{x},\varepsilon/2)|M(|Tf|^r)(\bar{x})$$

and, since T is of weak type $(1,1)$,

$$|F_\alpha| \leq \frac{B}{\alpha}\int |f_1|\,dm = \frac{B}{\alpha}\int_{B(\bar{x},\varepsilon)} |f|\,dm \leq \frac{B}{\alpha}|B(\bar{x},\varepsilon)|\cdot Mf(\bar{x}),$$

for some $B>0$. Since $f \in L^p$, Mf is finite a.e. Furthermore, since $Tf \in L^p$, $|Tf|^r$ is in $L^{p/r}$ and hence $M(|Tf|^r)$ is also finite a.e. Thus, for almost all $\bar{x} \in \mathbb{R}^n$, the particular choice of

$$\alpha = \max\{4^{1/r}[M(|Tf|^r)(\bar{x})]^{1/r}, 4\cdot 2^n B\cdot Mf(\bar{x})\}$$

yields

$$|E_\alpha \cup F_\alpha| \leq |E_\alpha| + |F_\alpha| \leq \frac{1}{2}|B(\bar{x},\varepsilon/2)|.$$

Therefore there exists an $x \in B(\bar{x},\varepsilon/2)$ such that $|Tf(x)| \leq \alpha$ and $|Tf_1(x)| \leq \alpha$. Substituting in (5.5) yields

$$|T_\varepsilon(\bar{x})| \leq A\big[M(|Tf|^r)^{1/r} + Mf\big],$$

for some constant $A>0$. □

Lemma 5.8. *For each $1<q<\infty$ there exists a constant $C_q>0$ such that*

$$\|Mf\|_{q,\infty} \leq C_q\|f\|_{q,\infty}, \quad \textit{for all } f \in L^{q,\infty}(m).$$

Proof. Fix an arbitrary $f \in L^{q,\infty}(m)$. We first notice that

$$\int_E |f|\,dm \leq \frac{q}{q-1}|E|^{1-1/q}\|f\|_{q,\infty}, \tag{5.6}$$

for all measurable subsets $E \subseteq \mathbb{R}^n$. To see this, assume that $0<|E|<\infty$ and let $B=|E|^{-1/q}\|f\|_{q,\infty}$. Since

$$d_{f\chi_E}(\alpha) \leq \min\{|E|, \alpha^{-q}\|f\|_{q,\infty}^q\},$$

for all $\alpha>0$, Proposition 2.5 yields

$$\begin{aligned}\int_E |f|\,dm &= \int_0^B d_{f\chi_E}(\alpha)\,d\alpha + \int_B^\infty d_{f\chi_E}(\alpha)\,d\alpha\\ &\leq |E|B + \frac{\|f\|_{q,\infty}^q}{q-1}B^{1-q}\\ &= \frac{q}{q-1}|E|^{1-1/q}\|f\|_{q,\infty},\end{aligned}$$

as claimed. Now fix an arbitrary $\alpha>0$ and let $E=\{x\in\mathbb{R}^n : |f(x)|>\alpha/2\}$. By sublinearity of M, we have

$$Mf \leq M(f\chi_E) + M(f\chi_{E^c}) \leq M(f\chi_E) + \frac{\alpha}{2},$$

and hence

$$d_{Mf}(\alpha) \le d_{M(f\chi_E)}(\alpha/2) \le \frac{2\|M(f\chi_E)\|_{1,\infty}}{\alpha} \le \frac{C}{\alpha}\int_E |f|\,dm,$$

by The Maximal Theorem. However, by employing (5.6) and noting that

$$|E| = d_f(\alpha/2) \le \frac{2\|f\|_{q,\infty}^q}{\alpha^q},$$

we get

$$d_{Mf}(\alpha) \le \frac{C'}{\alpha}\left(\frac{2\|f\|_{q,\infty}^q}{\alpha^q}\right)^{1-1/q} \cdot \|f\|_{q,\infty} = C''\frac{\|f\|_{q,\infty}^q}{\alpha^q},$$

as desired. □

Proposition 5.9. *The operator T_* is of weak type $(1,1)$.*

Proof. Fix and arbitrary $f \in L^1(m)$ and choose $0 < r < 1$. By Lemma 5.7 and The Maximal Theorem, we have

$$\|T_*f\|_{1,\infty} \le C\|M(|Tf|^r)^{1/r}\|_{1,\infty} + C'\|f\|_1.$$

Notice that

$$\||g|^{1/r}\|_{1,\infty} = \|g\|_{1/r,\infty}^{1/r},$$

for any measurable function g on $\mathbb{R}^n$. By Lemma 5.8, we then have

$$\begin{aligned}\|M(|Tf|^r)^{1/r}\|_{1,\infty} &= \|M(|Tf|^r)\|_{1/r,\infty}^{1/r}\\ &\le C_2\||Tf|^r\|_{1/r,\infty}^{1/r}\\ &= C_2\|Tf\|_{1,\infty}\end{aligned}$$

and, since T is of weak type $(1,1)$, the result follows. □

Before we close this chapter, we show one more estimate concerning T_*.

Lemma 5.10. *If $f \in C_c^\infty(\mathbb{R}^n;\mathbb{C})$ then there exists a constant $C_f > 0$ such that*

$$T_*f(x) \le \frac{C_f}{(1+|x|)^n}, \quad \textit{for almost all } x \in \mathbb{R}^n. \tag{5.7}$$

Proof. Suppose that $\operatorname{supp}(f) \subseteq \overline{B(0,R)}$. Since the convolution of a tempered distribution with a compactly supported smooth function is smooth, we conclude that Tf is bounded on the compact set $\overline{B(0,2R)}$ and, by Lemma 5.7, it follows that T_*f is bounded a.e. on $\overline{B(0,2R)}$. Thus

$$T_*f(x) \le C \le \frac{C_R}{(1+|x|)^n},$$

for almost all x with $|x| \leq 2R$. For $|x| \geq 2R$ we employ the assumption $|K(x)| \leq B|x|^{-n}$ to estimate

$$|T_\varepsilon f(x)| \leq C \int_{|y|\leq R} |x-y|^{-n}\, dy \leq C'|x|^{-n},$$

where C' is independent of ε. The last inequality in the above follows by noting that $|y| \leq R$ implies that

$$2|x| \leq 2|x-y| + 2R \leq 2|x-y| + |x|,$$

and hence $|x| \leq 2|x-y|$. However, since $|x| \geq 2R$, we have $C'_R|x| \geq 1+|x|$, where $C'_R := 1 + 1/(2R)$. Thus we conclude that (5.7) holds for $|x| \geq 2R$ also. $\square$

Chapter 6

Weighted Inequalities

As in chapter 5, we consider a singular integral operator T associated with a regular kernel K and assume that T is of type $(L^p(m), L^p(m))$, for some $1 < p < \infty$. We then define the vector-valued operator $\vec{T}$ componentwise by $(\vec{T}f)_i = Tf_i$, for vector functions $f = (f_1, \ldots, f_d)$. We will show that $\vec{T}$ is bounded from $L^p(W)$ into itself, whenever W is an A_p matrix weight, and that the converse holds with one additional hypothesis on K.

For each $\varepsilon > 0$ we define the truncated operator $\vec{T}_\varepsilon$ componentwise by $(\vec{T}_\varepsilon f)_i = T_\varepsilon f_i$, and we define $\vec{T}_*$ by

$$\vec{T}_* f(x) = \sup_{\varepsilon>0} |\vec{T}_\varepsilon f(x)|,$$

for any vector function $f \in L^r(m)$ and $1 \le r < \infty$. By noting that

$$T_* f_i \le \vec{T}_* f \le \sum_i T_* f_i, \tag{6.1}$$

we see that $\vec{T}_*$ is of weak type $(1,1)$ and that the estimate

$$\vec{T}_* f(x) \le \frac{C_f}{(1+|x|)^n}, \quad \text{for almost all } x \in \mathbb{R}^n, \tag{6.2}$$

holds as in the scalar case (see Proposition 5.9 and Lemma 5.10). By Lemma 5.4, the truncated operators T_{ε_j} converges weakly in $L^p(m)$ to an operator $T_0 \in \mathcal{B}(L^p, L^p)$, for some sequence $\varepsilon_j \to 0$. By Lemma 5.5, there exists a function $b \in L^\infty(m)$ such that $Tf = T_0 f + bf$, for all $f \in L^p(m)$. We define $\vec{T}_0$ componentwise by $(\vec{T}_0 f)_i = T_0 f_i$, for any vector function $f \in L^p(m)$.

Given any matrix-valued function $W : \mathbb{R}^n \to \mathbb{C}^{d\times d}$, we let $W\vec{T}$ denote the operator $(W\vec{T})f(x) = W(x)\vec{T}f(x)$, and similarly for the operators $\vec{T}_\varepsilon$ and $\vec{T}_0$. We also let

$$(W\vec{T})_* f(x) = \sup_{\varepsilon>0} |W(x)\vec{T}_\varepsilon f(x)|.$$

Lemma 6.1. *If W is a matrix A_p weight then $|W^{1/p}\vec{T}_0 f| \leq (W^{1/p}\vec{T})_* f$ a.e., for all $f \in L^p(m)$.*

Proof. From the definition of $\vec{T}_0$, it follows that

$$\int \langle \vec{T}_{\varepsilon_j} f, g\rangle \, dm \to \int \langle \vec{T}_0 f, g\rangle \, dm \quad \text{as } j \to \infty,$$

for all $f \in L^p$ and $g \in L^{p'}$. Notice that $W^{p'/p}$ is locally integrable, by The Reverse Hölder Inequalities. Since $W^{1/p}$ is self-adjoint a.e., it follows that

$$\Delta_j := \Big| \int \langle W^{1/p}\vec{T}_0 f, g\rangle \, dm - \int \langle W^{1/p}\vec{T}_{\varepsilon_j} f, g\rangle \, dm\Big| \to 0 \quad \text{as } j \to \infty, \tag{6.3}$$

for all $f \in L^p$ and for all bounded, compactly supported functions g. By The Monotone Convergence Theorem, it suffices to show that the set

$$N_\alpha := \{x \in \mathbb{R}^n : |W^{1/p}\vec{T}_0 f(x)| > (W^{1/p}\vec{T})_* f(x) + \alpha\}$$

has measure zero, for each $\alpha > 0$. If $|N_\alpha| > 0$ then there exists a compact subset $K \subseteq N_\alpha$ with $|K| > 0$. Define the function g on $\mathbb{R}^n$ by

$$g = \big[\operatorname{sgn} \circ (W^{1/p}\vec{T}_0 f)\big]\chi_K,$$

where $\operatorname{sgn}(z) := z/|z|$, for $z \in \mathbb{C}^d \backslash \{0\}$, and $\operatorname{sgn}(0) := 0$. With this particular choice of g, we get

$$\begin{aligned}
\Delta_j &= \Big| \int_K |W^{1/p}\vec{T}_0 f| \, dm - \int \langle W^{1/p}\vec{T}_{\varepsilon_j} f, g\rangle \, dm\Big| \\
&\geq \int_K |W^{1/p}\vec{T}_0 f| \, dm - \Big| \int \langle W^{1/p}\vec{T}_{\varepsilon_j} f, g\rangle \, dm\Big| \\
&\geq \int_K |W^{1/p}\vec{T}_0 f| \, dm - \int_K |W^{1/p}\vec{T}_{\varepsilon_j} f| \, dm \\
&\geq \int_K \big(|W^{1/p}\vec{T}_0 f| - (W^{1/p}\vec{T})_* f\big) \, dm \geq \alpha |K|,
\end{aligned}$$

for all $j \in \mathbb{N}$, contradicting (6.3). □

6.1 The Relative Distributional Inequality

Let us fix a matrix A_p weight W and choose $\delta > 0$ such that The Reverse Hölder Inequalities (3.15) - (3.17) hold. The objective of this section is to show that, whenever $q < p + \delta$, there exists a constant $C_q > 0$ such that

$$\|(W^{1/p}\vec{T})_* f\|_q \leq C_q \|W^{1/p} f\|_q, \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^d). \tag{6.4}$$

We establish (6.4) via the so called *relative distributional inequality* (6.5).

Lemma 6.2. *Assume that F and G are nonnegative measurable functions on $\mathbb{R}^n$ and assume that $\|F\|_q < \infty$, for some $1 \le q < \infty$. If there exists constants $b, c > 0$ such that*

$$|\{x \in \mathbb{R}^n : F(x) > \alpha\,;\, G(x) \le c\alpha\}| \le \frac{1}{2} b^q |\{x \in \mathbb{R}^n : F(x) > b\alpha\}|, \tag{6.5}$$

for all $\alpha > 0$, then $\|F\|_q \le 2c^{-q}\|G\|_q$.

Proof. From (6.5) it follows that

$$\begin{aligned} d_F(\alpha) &= |\{x : F(x) > \alpha\,;\, G(x) \le c\alpha\}| + |\{x : F(x) > \alpha\,;\, G(x) > c\alpha\}| \\ &\le \frac{1}{2} b^q d_F(b\alpha) + d_G(c\alpha). \end{aligned}$$

Multiplying both sides in the above by $q\alpha^{q-1}$, integrating in α over $(0, \infty)$ and changing variables, we obtain

$$\|F\|_q^q \le \frac{1}{2}\|F\|_q^q + c^{-q}\|G\|_q^q.$$

Since $\|F\|_q < \infty$, the result follows. □

Now, fix an arbitrary $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C}^d)$ and $q < p + \delta$. We will show that the functions

$$F := (W^{1/p}\vec{T})_* f \quad \text{and} \quad G := \max\big(M'_W(W^{1/p}f), M_W(W^{1/p}f)\big) \tag{6.6}$$

satisfy (6.5), for some c independent of f. Then Lemma 6.2 and the L^q boundedness of the weighted maximal operators implies (6.4). Of course, we also have to check that $\|F\|_q < \infty$.

Lemma 6.3. *The function $F = (W^{1/p}\vec{T})_* f$ is in $L^q(m)$.*

Proof. From (6.2) we get

$$F(x) \le \frac{C\|W(x)\|^{1/p}}{(1 + |x|)^n}, \quad \text{for almost all } x \in \mathbb{R}^n.$$

Choose a nonzero scalar function $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$. Then

$$\frac{C'}{(1 + |x|)^n} \le M(\|W\|^{-1/p}\phi)(x), \quad \text{for all } x \in \mathbb{R}^n,$$

for some constant $C' > 0$ (see Remark 2.19). It follows that

$$F(x) \le C''\|W(x)\|^{1/p} M(\|W\|^{-1/p}\phi)(x), \quad \text{for almost all } x \in \mathbb{R}^n.$$

However, since W is a matrix A_p weight, $\|W\|$ is a scalar A_p weight and, by Theorem 4.6, the function

$$\|W\|^{1/p} M(\|W\|^{-1/p}\phi) = M_{\|W\|}(\phi)$$

is in $L^q(m)$. □

To prove the relative distributional inequality (6.5) we employ two preliminary results. Given a cube Q, we denote by $3Q$ the cube with the same center as Q and side length $3 \cdot l(Q)$.

Lemma 6.4. *Any measurable set $E \subset \mathbb{R}^n$, with $0 < |E| < \infty$, can be covered a.e. by a collection of disjoint cubes $\{Q_j\}$ satisfying*

(i) $\sum_j |Q_j| \le 2|E|$ *and*

(ii) $|3Q_j \cap E^c| \ge C_n |3Q_j|$,

for some dimensional constant $C_n > 0$.

Proof. For almost all $x \in E$ there exists, by The Lebesgue Differentiation Theorem, a dyadic cube $Q \subset \mathbb{R}^n$ containing x such that

$$|Q \cap E|/|Q| \ge 1/2. \tag{6.7}$$

Also, $|Q| \le 2|E| < \infty$, for any cube Q satisfying (6.7). Therefore we may define $\{Q_j\}$ as the set of maximal dyadic cubes satisfying (6.7). Thus $E \subseteq \cup_j Q_j$ up to sets of measure zero and, by disjointness of the Q_j's,

$$\sum_j |Q_j| \le 2 \sum_j |Q_j \cap E| = 2|E|.$$

To verify (ii) we fix an arbitrary cube $Q \in \{Q_j\}$ and let Q' denote the next larger dyadic cube containing Q. Then $Q' \subset 3Q$ and $|Q' \cap E| < 1/2|Q'|$, by maximality of Q. It follows that

$$|3Q \cap E^c| \ge |Q' \cap E^c| = |Q'| - |Q' \cap E| \ge 1/2|Q'| = 1/2 \cdot (2/3)^n |3Q|,$$

as desired. □

Lemma 6.5. *Let $f \in L^r(m)$, for some $1 \le r < \infty$. Assume that $Q \subset \mathbb{R}^n$ is a cube and fix arbitrary points $\bar{y} \in Q$ and $\bar{x} \in 3Q$. Let $B = B(\bar{x}, 4\,\mathrm{dia}\,Q)$ and let $f_2 = \chi_{B^c} f$. Then there exists a constant $A > 0$ independent of Q such that*

$$\vec{T}_* f_2(x) \le d \cdot \vec{T}_* f(\bar{x}) + AMf(\bar{y}), \quad \text{for all } x \in Q.$$

Proof. By (6.1) it suffices to consider the scalar case $d = 1$. Let $\delta = \mathrm{dia}\,Q$. Figure 6.1 illustrates the present construction. Notice that

$$|T_\varepsilon f_2(x) - T_\varepsilon f_2(\bar{x})| \le \sum_{i=1}^{4} \int_{S_i} |K_\varepsilon(x-y) - K_\varepsilon(\bar{x}-y)| \cdot |f(y)|\, dy,$$

where we have splitted B^c into four disjoint sets

$$\begin{aligned} S_1 &= B^c \cap B(x,\varepsilon) \cap B(\bar{x},\varepsilon) \\ S_2 &= B^c \cap B(x,\varepsilon)^c \cap B(\bar{x},\varepsilon)^c \\ S_3 &= B^c \cap B(x,\varepsilon) \cap B(\bar{x},\varepsilon)^c \\ S_4 &= B^c \cap B(x,\varepsilon)^c \cap B(\bar{x},\varepsilon). \end{aligned}$$

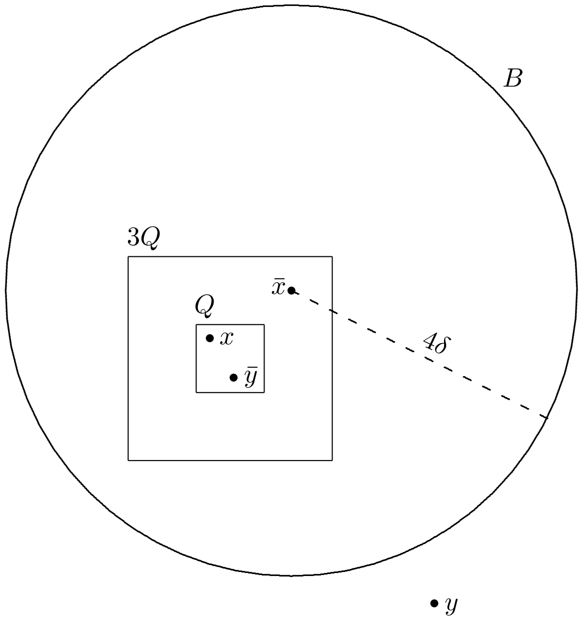

Figure 6.1: The construction from Lemma 6.5.

Clearly, the integral over S_1 vanishes. Let us consider the integral over S_2. Since $2|x-\bar{x}| \leq 4\delta$ and

$$|y-\bar{y}| \leq |y-\bar{x}| + 2\delta \leq 3/2|y-\bar{x}|,$$

for all $x \in Q$ and $y \in B^c$, Lemma 5.6 yields

$$|K_\varepsilon(x-y) - K_\varepsilon(\bar{x}-y)| \leq \frac{\delta A}{|y-\bar{y}|^{n+1}}, \quad \text{whenever } x \in Q \text{ and } y \in S_2. \tag{6.8}$$

Since $B^c \subseteq B(\bar{y},\delta)^c$, we get

$$\begin{aligned} \int_{S_1} &\leq \delta A \int_{B(\bar{y},\delta)^c} \frac{|f(y)|}{|y-\bar{y}|^{n+1}}\, dy \\ &= \delta A \sum_{k=0}^{\infty} \int_{2^k\delta \leq |y-\bar{y}| < 2^{k+1}\delta} \frac{|f(y)|}{|y-\bar{y}|^{n+1}}\, dy \\ &\leq 2^n A \Big(\sum_{k=0}^{\infty} 2^{-k}\Big) Mf(\bar{y}), \end{aligned} \tag{6.9}$$

whenever $x \in Q$. For the integral over S_3, we notice that $|y-\bar{y}| \leq 3/2|y-\bar{x}|$ and $|y-\bar{x}| \leq 2|x-y|$, for $x \in Q$ and $y \in S_3$. It follows that $S_3 \subseteq B(\bar{y}, 3\varepsilon)$

and hence

$$\int_{S_3} \leq \int_{B(\bar{y},3\varepsilon)} |K(\bar{x}-y)| \cdot |f(y)|\, dy \leq \frac{C}{\varepsilon^n} \int_{B(\bar{y},3\varepsilon)} |f(y)|\, dy \leq C' M f(\bar{y}),$$

where we have employed the bound $|K(x)| \leq B|x|^{-n}$ to obtain the second inequality. The integral over S_4 yields a similar estimate, and therefore

$$|T_\varepsilon f_2(x) - T_\varepsilon f_2(\bar{x})| \leq AMf(\bar{y}), \quad \text{whenever } x \in Q. \tag{6.10}$$

By noting that $T_\varepsilon f_2(\bar{x}) = T_{4\delta} f(\bar{x})$ when $\varepsilon < 4\delta$ and $T_\varepsilon f_2(\bar{x}) = T_\varepsilon f(\bar{x})$ when $\varepsilon \geq 4\delta$, we see that

$$|T_\varepsilon f_2(\bar{x})| \leq \sup_{\varepsilon \geq 4\delta} |T_\varepsilon f(\bar{x})| \leq T_* f(\bar{x}), \quad \text{for all } \varepsilon > 0,$$

and hence the estimate (6.10) implies that

$$T_* f_2(x) \leq T_* f(\bar{x}) + AMf(\bar{y}),$$

whenever $x \in Q$. □

Proposition 6.6. *With the notation in* (6.6)*: If* $1 \leq q < p + \delta$ *then there exists constants* $0 < b < 1$ *and* $c > 0$*, both independent of* f*, such that*

$$|\{x \in \mathbb{R}^n : F(x) > \alpha\, ;\, G(x) \leq c\alpha\}| \leq \frac{1}{2} b^q |\{x \in \mathbb{R}^n : F(x) > b\alpha\}|, \tag{6.11}$$

for all $\alpha > 0$.

Proof. Fix an arbitrary $\alpha > 0$. By Lemma 6.4 it suffices to show that there exists constants $0 < b < 1$ and $c > 0$ independent of α such that

$$|\{x \in Q : F(x) > \alpha\, ;\, G(x) \leq c\alpha\}| \leq \frac{1}{4} b^q |Q|, \tag{6.12}$$

for any cube $Q \subset \mathbb{R}^n$ with $|3Q \cap E^c| \geq C_n |3Q|$, where

$$E := \{x \in \mathbb{R}^n : F(x) > b\alpha\}.$$

Indeed, E may be covered a.e. by a collection of disjoint cubes $\{Q_j\}$ satisfying (i) and (ii) in Lemma 6.4 and, since

$$D := \big\{x \in \mathbb{R}^n : F(x) > \alpha\, ;\, G(x) \leq c\alpha\big\} \subseteq E,$$

we obtain

$$|D| = \big|D \cap \bigcup_j Q_j\big| = \sum_j |D \cap Q_j| \leq \frac{1}{2} b^q |E|.$$

Notice that $|E| < \infty$, since $F \in L^q$ and (6.12) is trivial if $|E| = 0$. Thus we assume that Q is a cube satisfying $|3Q \cap E^c| \geq C_n |3Q|$ and we will deduce suitable bounds on b and c. In what follows we let C denote a positive generic constant, which value may vary at different occurrences. Let O denote the ball with the same center as Q and radius $6\,\mathrm{dia}\,Q$. Notice that $3Q \subset O$. Now, if $\|V_O W(x)^{-1/p}\| > C$, for all $x \in 3Q \cap E^c$, then (ii) in Lemma 6.4 implies that

$$\begin{aligned}\int_O \|V_O W(x)^{-1/p}\|^{p'}\,dx &\geq \int_{3Q\cap E^c} \|V_O W(x)^{-1/p}\|^{p'}\,dx \\ &\geq C^{p'}|3Q\cap E^c| \geq C^{p'}C_n|3Q| \\ &= C^{p'}C'|O|,\end{aligned}$$

contradicting The Reverse Hölder Inequality (3.16) when C is sufficiently large. Thus there exists a point $\bar{x} \in 3Q$ such that

$$F(\bar{x}) \leq b\alpha \quad \text{and} \quad \|V_O W(\bar{x})^{-1/p}\| \leq C. \tag{6.13}$$

Let $g = W^{1/p}f$. We may assume that there exists a point $\bar{y} \in Q$ with $M'_W g(\bar{y}) \leq c\alpha$; otherwise $|\{x \in Q : M'_W g(x) \leq c\alpha\}| = 0$ and (6.12) is trivial. Let $B = B(\bar{x}, 4\,\mathrm{dia}\,Q)$. Notice that $Q \subset B \subset O$ and that these sets are of proportional measure. Let $f_1 = \chi_B f$ and $f_2 = \chi_{B^c} f$. Then $f = f_1 + f_2$ and, by sublinearity of $(W^{1/p}\vec{T})_*$, it suffices to show that

$$|\{x \in Q : F_i(x) > \alpha/2\}| \leq \frac{1}{8} b^q |Q|, \quad \text{for } i = 1, 2,$$

where $F_i := (W^{1/p}\vec{T})_* f_i$.

Estimate with F_1: For each $R > 0$ we let

$$S_R = \{x \in Q : (V_B\vec{T})_* f_1(x) > \alpha/(2R)\}$$

and

$$N_R = \{x \in Q : \|W(x)^{1/p}V_B^{-1}\| > R\}.$$

Since $\vec{T}_*$ is of weak type $(1,1)$ and, since $\vec{T}_*(V_B f_1) = (V_B\vec{T})_* f_1$, we see that

$$|S_R| \leq \frac{CR}{\alpha}\|V_B f_1\|_1.$$

Furthermore, since

$$\begin{aligned}\|V_B f_1\|_1 &= \int_B |V_B f(x)|\,dx = \int_B |V_B W(x)^{-1/p} g(x)|\,dx \\ &\leq |B| M'_W g(\bar{y}) \leq Cc\alpha|Q|,\end{aligned}$$

we must have $|S_R| \leq CcR|Q|$. By The Reverse Hölder Inequality (3.17),

$$R^p|N_R| \leq \int_{N_R} \|W(x)^{1/p}V_B^{-1}\|^p\,dx \leq \int_Q \|W(x)^{1/p}V_B^{-1}\|^p\,dx \leq C'|Q|,$$

and hence $|N_R| \le C'R^{-p}|Q|$. Now, since

$$F_1(x) \le \|W(x)^{1/p}V_B^{-1}\| \cdot (V_B\vec{T})_* f_1(x), \quad \text{for all } x \in \mathbb{R}^n,$$

we have

$$|\{x \in Q : F_1(x) > \alpha/2\}| \le |S_R \cup N_R| \le (CcR + C'R^{-p})|Q|.$$

Taking the infimum over all $R > 0$, we conclude that

$$|\{x \in Q : F_1(x) > \alpha/2\}| \le C_1 c^{p/(p+1)}|Q|.$$

Estimate with F_2: Since B and O are of proportional measure,

$$|V_B v| \le \sqrt{d}\rho_{p,B}(v) \le C\rho_{p,O}(v) \le C|V_O v|,$$

and hence $|V_B V_O^{-1} v| \le C|v|$, for all $v \in \mathbb{C}^d$. It follows that $\|V_B V_O^{-1}\| \le C$ and hence

$$\|V_B W(\bar{x})^{-1/p}\| \le C\|V_O W(\bar{x})^{-1/p}\|.$$

Combining this with (6.13), we see that

$$(V_B\vec{T})_* f(\bar{x}) \le \|V_B W(\bar{x})^{-1/p}\| \cdot F(\bar{x}) \le C_2 b\alpha.$$

Employing Lemma 6.5 we obtain

$$\begin{aligned} (V_B\vec{T})_* f_2(x) &\le d \cdot (V_B\vec{T})_* f(\bar{x}) + AM(V_B f)(\bar{y}) \\ &\le C_2' b\alpha + AM_W' g(\bar{y}) \\ &\le (C_2' b + Ac)\alpha, \end{aligned}$$

for all $x \in Q$. We now repeat the strategy employed in the estimate with F_1: Taking $q < r < p + \delta$, The Reverse Hölder Inequality yields $|N_R| \le C''R^{-r}|Q|$, and so

$$|\{x \in Q : F_2(x) > R(C_2' b + Ac)\alpha\}| \le |N_R| \le C''R^{-r}|Q|.$$

A particular choice of $R = (4bC_2')^{-1}$ yields

$$|\{x \in Q : F_2(x) > (1/4 + C_3 b^{-1} c)\alpha\}| \le C_4 b^r |Q|.$$

Thus the proof may be performed with $b = \min\{1/2, (8C_4)^{-1/(r-q)}\}$ and c so small that

$$C_1 c^{p/(p+1)} \le 1/8b^q \quad \text{and} \quad 1/4 + C_3 b^{-1} c \le 1/2.$$

This completes the proof of Proposition 6.6 and with it the estimate (6.4). □

6.2 The Main Theorems

Lemma 6.7. *If W is a locally integrable matrix weight, then $C_c^\infty(\mathbb{R}^n;\mathbb{C}^d)$ is dense i $L^q(W)$, for each $1 \le q < \infty$.*

Proof. We employ the well-known fact that $C_c^\infty(\mathbb{R}^n;\mathbb{C})$ is dense in (the scalar) $L^q(\mu)$, whenever μ is a Radon measure on the Borel algebra in $\mathbb{R}^n$, i.e. when $\mu(K) < \infty$, for any compact set $K \subset \mathbb{R}^n$. Notice that, if w is a nonnegative locally integrable function on $\mathbb{R}^n$, then the Borel measure $w\,dm$, given by $E \mapsto \int_E w\,dm$, is a Radon measure.

Choose an arbitrary $f = (f_1, \dots, f_d) \in L^q(W)$ and let $\{e_i\}$ denotes the standard basis of $\mathbb{C}^d$. Then

$$W^{1/q} = \begin{pmatrix} u_{11} & \dots & u_{1d} \\ \vdots & \ddots & \vdots \\ u_{d1} & \dots & u_{dd} \end{pmatrix},$$

where $u_{ij} := \langle W^{1/q} e_j, e_i\rangle$ are a.e. positive and q-locally integrable functions on $\mathbb{R}^n$. Since

$$|W^{1/q}(f-g)|^q \le d^{3q/2} \max_{i,j} |u_{ij}(f_j - g_j)|^q \le d^{3q/2} \sum_{i,j} |u_{ij}(f_j - g_j)|^q,$$

for any function $g = (g_1, \dots, g_d)$, it follows that

$$\|f-g\|^q_{L^q(W)} \le d^{3q/2} \sum_{i,j} \|f_j - g_j\|^q_{L^q(u_{ij}^q\,dm)},$$

for any measurable g and, by choosing an appropriately $g \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^d)$, we can make each term on the right in the above arbitrary small. ☐

We are now ready to show one of the main results of this thesis. Let $W^{1/p}\vec{T}W^{-1/p}$ denote the operator given by

$$(W^{1/p}\vec{T}W^{-1/p})g(x) = W(x)^{1/p}\vec{T}(W^{-1/p}g)(x).$$

Theorem 6.8. *Assume that W is an A_p matrix weight. Then there exists a $\delta > 0$ such that $W^{1/p}\vec{T}W^{-1/p}$ is a bounded linear operator from a dense subset of $L^q(m)$ into $L^q(m)$, whenever $|p-q| < \delta$. In particular, $\vec{T}$ has a unique linear extension that is bounded from $L^p(W)$ into $L^p(W)$.*

Proof. Let

$$D = \big\{ g \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^d) : (W^{-1/p}g) \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^d)\big\}.$$

By The Reverse Hölder Inequality (3.15), $W^{q/p}$ is locally integrable and hence C_c^∞ is dense in $L^q(W^{q/p})$. Notice that the map $f \mapsto \varphi(f) := W^{1/p}f$ is an invertible isometry from $L^q(W^{q/p})$ into L^q. Fix an arbitrary $h \in L^q$

and an $\varepsilon > 0$. Choose $f \in L^q(W^{q/p})$ such that $h = \varphi(f)$ and choose $g \in C_c^\infty$ with $\|f - g\|_{L^q(W^{q/p})} < \varepsilon$. Then $\varphi(g) \in D$ and

$$\|h - \varphi(g)\|_q = \|\varphi(f-g)\|_q = \|f-g\|_{L^q(W^{q/p})} < \varepsilon,$$

showing that D is dense in L^q.

Since $\vec{T}f = \vec{T}_0 f + bf$, for some a.e. bounded scalar function b, Lemma 6.1 yields

$$\begin{aligned} |(W^{1/p}\vec{T}W^{-1/p})g(x)| &= |(W^{1/p}\vec{T}_0 W^{-1/p})g(x) + b(x)g(x)| \\ &\le (W^{1/p}\vec{T})_*(W^{-1/p}g)(x) + C|g(x)|, \end{aligned}$$

for almost all $x \in \mathbb{R}^n$, whenever $(W^{-1/p}g) \in L^p$. By Minkowski's Inequality and (6.4) we have, in particular, that

$$\|(W^{1/p}\vec{T}W^{-1/p})g\|_q \le \|(W^{1/p}\vec{T})_*(W^{-1/p}g)\|_q + C\|g\|_q \le C_q\|g\|_q,$$

for all $g \in D$.

To verify the last assertion of the theorem, we simply notice that, by the above,

$$\int |W^{1/p}\vec{T}f|^p\,dm = \int |(W^{1/p}\vec{T}W^{-1/p})(W^{1/p}f)|^p\,dm \le C\int |W^{1/p}f|^p\,dm,$$

for all $f \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^d)$. Thus $\vec{T}$ admits a unique bounded linear extension from $L^p(W)$ into itself. □

In order to prove the "converse" of Theorem 6.8 we employ the following preliminary result.

Lemma 6.9. *Let $1 < p < \infty$ and fix an arbitrary ball $B \subset \mathbb{R}^n$. Assume that $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is measurable, supported in $B \times B$ and satisfies $|k(x,y)| \le |B|^{-1}$, for all $(x,y) \in B \times B$. Then the linear operator S, defined by*

$$Sf(x) = \int k(x,y)f(y)\,dy,$$

is bounded from $L^p(W)$ into $L^p(W)$ with operator norm $\|S\| \le C_d\|V_B V_B'\|$, for some constant $C_d > 0$ independent of the particular choice of S. In the special case where $k = |B|^{-1}\chi_{B\times B}$ we also have $\|S\| \ge C_d^{-1}\|V_B V_B'\|$.

Proof. Since any $f \in L^p(W)$ is locally integrable, it follows that S is well-defined on $L^p(W)$. Fix an arbitrary $f \in L^p(W)$ with $\|f\|_{L^p(W)} \le 1$. Then

$$\begin{aligned} |W(x)^{1/p}Sf(x)| &\le \frac{1}{|B|}\int_B |W(x)^{1/p}f(y)|\,dy \\ &\le \frac{1}{|B|}\left(\int_B \|W(x)^{1/p}W(y)^{-1/p}\|^{p'}\,dy\right)^{1/p'}, \end{aligned}$$

by Hölder's Inequality. Let $\{e_i\}$ denote the standard basis of $\mathbb{C}^d$ and let $C > 0$ denote a generic constant. Then

$$\begin{aligned}
\int_B \|W(x)^{1/p}W(y)^{-1/p}\|^{p'}\,dy &\le \int_B \big(d\max_i |W(y)^{-1/p}W(x)^{1/p}e_i|\big)^{p'}\,dy \\
&\le C\sum_i \int_B |W(y)^{-1/p}W(x)^{1/p}e_i|^{p'}\,dy \\
&\le C\sum_i |B|\cdot|V_B' W(x)^{1/p}e_i|^{p'} \\
&\le C|B|\cdot\|V_B'W(x)^{1/p}\|^{p'},
\end{aligned}$$

and consequently

$$|W(x)^{1/p}Sf(x)| \le C|B|^{-1/p}\|V_B'W(x)^{1/p}\|, \quad \text{for all } x\in\mathbb{R}^n.$$

Repeating the strategy employed above we obtain

$$\begin{aligned}
\|Sf\|^p_{L^p(W)} &\le C\frac{1}{|B|}\int_B \|V_B'W(x)^{1/p}\|^p\,dx \\
&\le C\sum_i \frac{1}{|B|}\int_B |W(x)^{1/p}V_B'e_i|^p \\
&\le C\sum_i |V_BV_B'e_i|^p \le C\|V_BV_B'\|^p,
\end{aligned}$$

as desired.

Now, for the particular choice of $k = |B|^{-1}\chi_{B\times B}$ we have

$$\|S\| = \sup_{u\neq 0}\frac{\rho^*_{p',B}(u)}{(\rho_{p,B})^*(u)},$$

by (3.13) in Section 3.3. However, since $(\rho_{p,B})^*(v) \sim |V_B^{-1}v|$ and $\rho^*_{p',B}(v) \sim |V_B'v|$, we get

$$\|V_BV_B'\| = \sup_{v\neq 0}\frac{|V_B'V_Bv|}{|v|} = \sup_{v\neq 0}\frac{|V_B'v|}{|V_B^{-1}v|} \le C\|S\|,$$

as claimed. □

Theorem 6.10. *Let W be a matrix weight and assume that the kernel K has the additional property that there exists a constant $a > 0$ and a unit vector $u \in \mathbb{R}^n$ such that*

$$|K(ru)| \ge a|r|^{-n}, \quad \textit{for all } r\in\mathbb{R}\backslash\{0\}. \tag{6.14}$$

If $\vec{T}$ is bounded from $L^p(W)$ into $L^p(W)$, then $W \in A_p$.

Proof. Choose $\varepsilon > 0$ such that $\varepsilon^2 + 2\varepsilon \le \frac{1}{2} C_d^{-2}$, where $C_d > 0$ is the constant appearing in Lemma 6.9. Choose $A > 0$ such that $|\nabla K(x)| \le A|x|^{-n-1}$, for all $x \ne 0$. Let $\lambda = \max\{4, 2^{n+2} A \cdot (a\varepsilon)^{-1}\}$. We claim that, for any $r \in \mathbb{R}\backslash\{0\}$,

$$|K(x) - K(r\lambda u)| \le \varepsilon |K(r\lambda u)|, \quad \text{whenever } x \in B(r\lambda u, 2|r|). \tag{6.15}$$

To see this, define $\varphi : [0,1] \to \mathbb{R}^n$ by $\varphi(t) = (x - r\lambda u)t + r\lambda u$. Notice that

$$\lambda|r| \le |\varphi(t)| + |(x - r\lambda u)t| \le |\varphi(t)| + 2|r| \le |\varphi(t)| + \frac{1}{2}\lambda|r|,$$

and hence $\lambda|r| \le 2|\varphi(t)|$, for all $t \in [0,1]$. Applying The Mean Value Theorem to the function $t \mapsto K(\varphi(t))$, we conclude that there exists a number $\tau \in (0,1)$ such that

$$|K(x) - K(r\lambda u)| \le \frac{A}{|\varphi(\tau)|^{n+1}} |x - r\lambda u| \le \frac{2^{n+1} A}{|\lambda r|^{n+1}} 2|r| \le \varepsilon |K(r\lambda u)|,$$

where the last inequality follows by the assumption (6.14).

Fix an arbitrary ball $B = B(\bar{x}, r)$ in $\mathbb{R}^n$ and let $B' = B(\bar{x} + r\lambda u, r)$. Notice that B and B' are disjoint. Define the linear operator S on $L^p(W)$ by

$$Sf(x) = \chi_B(x) \vec{T}(\chi_{B'} \vec{T}(\chi_B f))(x) = \int k(x,y) f(y)\, dy,$$

where

$$k(x,y) := \chi_{B\times B}(x,y) \int_{B'} K(x-z) K(z-y)\, dz.$$

If $\vec{T}$ is bounded on $L^p(W)$, then so is S and $\|S\| \le \|T\|^2$. Notice that

$$K(x-z)K(z-y) = K(r\lambda u)K(-r\lambda u) + K_1(x,y,z),$$

where

$$\begin{aligned} K_1(x,y,z) &:= \big[K(z-y) - K(r\lambda u)\big] \cdot \big[K(x-z) - K(-r\lambda u)\big] \\ &+ \big[K(x-z) - K(-r\lambda u)\big] \cdot K(r\lambda u) \\ &+ \big[K(z-y) - K(r\lambda u)\big] \cdot K(-r\lambda u). \end{aligned}$$

Thus $S = S_0 + S_1$, where

$$S_0 f(x) := |B| K(r\lambda u) K(-r\lambda u) \int \chi_{B\times B}(x,y) f(y)\, dy$$

and

$$S_1 f(x) := \int \left(\chi_{B\times B}(x,y) \int_{B'} K_1(x,y,z)\, dz \right) f(y)\, dy.$$

Since $x, y \in B$ and $z \in B'$ implies that

$$(z - y) \in B(r\lambda u, 2r) \quad \text{and} \quad (x - z) \in B(-r\lambda u, 2r),$$

we can apply the estimate (6.15) to obtain

$$|K_1(x,y,z)| \leq (\varepsilon^2 + 2\varepsilon)|K(r\lambda u)| \cdot |K(-r\lambda u)| \leq \frac{1}{2} C_d^{-2} C_0,$$

whenever $x, y \in B$ and $z \in B'$, where $C_0 := |K(r\lambda u)| \cdot |K(-r\lambda u)|$. By Lemma 6.9 there exists a constant $C_d > 0$ independent of B such that $\|S_0\| \geq C\|V_B V_B'\|$ and $\|S_1\| \leq \frac{1}{2} C\|V_B V_B'\|$, with

$$C := C_0 |B|^2 C_d^{-1} \geq \frac{a^2}{|r\lambda|^{2n}} \cdot (v_n r^n)^2 C_d^{-1} = \frac{a^2 v_n^2}{\lambda^{2n} C_d}.$$

It follows that

$$C\|V_B V_B'\| \leq \|S - S_1\| \leq \|S\| + \frac{1}{2} C\|V_B V_B'\|,$$

and hence

$$\|V_B V_B'\| \leq 2C^{-1}\|S\| \leq \frac{\lambda^{2n} C_d \|T\|^2}{a^2 v_n^2} < \infty,$$

independent of B. Thus we conclude, by Proposition 3.21, that $W \in A_p$. □

Remark 6.11. *The kernel of the Riesz transform,*

$$K^{(j)}(x) = c_n \frac{x_j}{|x|^{n+1}},$$

satisfies (6.14) *with* $u = e_j$ *and any* $0 < a \leq c_n$.

Appendix A

Index of Notation

$B(x,r)$	The Euclidean ball in $\mathbb{R}^n$ with center x and radius r
χ_S	The characteristic function of a set S
$C_c^\infty(\mathbb{R}^n;\mathbb{C}^d)$	The set of compactly supported C^∞ functions from $\mathbb{R}^n$ into $\mathbb{C}^d$
$\operatorname{dia} S$	The diameter of S
p'	The dual exponent, $p' = p/(p-1)$
$\langle \cdot, \cdot \rangle$	The Euclidean inner product on $\mathbb{C}^d \times \mathbb{C}^d$
$\lvert \cdot \rvert$	The Euclidean norm on $\mathbb{C}^d$ (or $\mathbb{R}^d$)
$l(Q)$	The side length of a cube $Q \subset \mathbb{R}^n$
m	The Lebesgue measure on the Borel algebra in $\mathbb{R}^n$
$\lvert E \rvert$	The Lebesgue measure of E, i.e. $\lvert E \rvert = m(E)$
v_n	The Lebesgue measure of the unit ball in $\mathbb{R}^n$
$\lVert A \rVert$	The operator norm of $A \in \mathbb{C}^{d\times d}$
$\partial_i f$	The partial derivative of f w.r.t. the ith variable
$\mathcal{S}$	The Schwartz space
$\{e_i\}$	The standard basis of $\mathbb{C}^d$
$\operatorname{supp}(f)$	The support of f
S^{n-1}	The unit sphere in $\mathbb{R}^n$.

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