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Spectral Analysis of a δ-interaction in One Dimension Defined as a Quadratic Form

by

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Master Thesis

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Abstract:
In this thesis, the theory of sesquilinear forms on Hilbert spaces is addressed with an application to δ-interactions in one dimension. In the first part, sesquilinear forms is studied systematically starting with sesquilinear forms in general, and specializing to sectorial forms. Closed forms, and closability of forms is treated e.g. by demonstrating different criteria of closability. Representation of forms will be investigated, culminating with the so-called first representation theorem. As an application, The Friedrichs extension will be introduced.
In the second part, the δ-interaction Hamiltonian is under consideration. Its domain will be specified, the resolvent constructed in order to compute the discrete spectrum as singularities in the resolvent. Furthermore, some results on integration by parts in Sobolev spaces and a proof of existence of non-trivial testfunctions are included in the appendices.

Preface

The present thesis is the result of my final project work at Department of Mathematical Sciences at Aalborg University. The project has lasted from the beginning of september 2007 untill the end of february 2008.

It is my master thesis in mathematics, and I have chosen to work within the framework of mathematical analysis. In particular, mathematical physics.

Aalborg, February 29th, 2008.

Peter Jensen

Dansk resumé/Danish abstract

Den foreliggende specialerapport er resultatet af projektarbejdet på MAT6-semesteret på Institut for Matematiske Fag, Aalborg Universitet i efterårs-semesteret 2007 under temaet ”Anvendt matematisk analyse og geometri”.

Emnet for specialet er kvadratiske former og disses anvendelse til konstruktion af Hamilton operatorer for punkt-vekselvirkninger i én dimension, samt spektralteori for sådanne operatorer.

Rapporten er inddelt i to dele, hvoraf første del er en gennemgang af de væsentligste forudsætninger for behandling af problemet i anden del. Mere konkret er der tale om en grundig gennemregning af de relevante dele af kapitel VI i T. Katos ”Perturbation Theory for Linear Operators”, som drejer sig om teorien for sektorielle kvadratiske former på Hilbert rum, herunder hvorledes disse former kan repræsenteres ved hjælp af operatorer. Der er lagt vægt på omhyggelig bevisførelse, og selve teorien er essentielt set selv-indeholdt.

Det bemærkes, at der arbejdes i større generalitet end nødvendigt, idet den form der indføres i anden del er et specialtilfælde af sektoriel, nemlig symmetrisk og nedadtil begrænset. Således er første del af rapporten ikke så økonomisk som muligt.

I anden del defineres en kvadratisk form, som vises at opfylde visse betingelser der giver anledning til at repræsentere formen ved en operator kaldet Hamilton operatoren. Dennes domæne specificeres, og efterfølgende opstilles og løses et egenværdi-problem for denne operator.

Rapporten består af følgende kapitler:

Kapitel 2 - Sesquilinear forms on Hilbert spaces I dette kapitel indføres teorien for sesquilinearformer på Hilbert rum. Der begyndes med de mest basale definitioner og resultater. Først for sesquilinearformer i almindelighed, og dernæst specialiseres til en bestemt type af former - såkaldte sektorielle former. Denne type former har deres numeriske billeder indeholdt i bestemte sektorer i den komplekse plan, og denne egenskab gør det muligt at definere et nyt konvergens-begreb, som giver anledning til indførelse af lukkede former.

Lukkethed af former viser sig at være afgørende i forbindelse med repræsentation af former ved operatorer. Der findes former, der ikke er lukkede, men som dog har den egenskab at der eksisterer lukkede udvidelser af dem. Sådanne former kaldes aflukkelige. Egenskaber for aflukkellige former etableres, og forskellige kriterier for aflukkelighed bevises.

Kapitel 3 - Representation of forms Dette kapitel har til formål at bevise en repræsentations sætning for en bestemt type af former: tæt definerede, lukkede, sektorielle sesquilinearformer. Denne repræsentations sætning omtales som ”The first representation theorem”.

Dette gøres ved først at vise en tilsvarende sætning for begrænsede former, og anvende den tidligere gennemgåede teori. Desuden spiller Riesz-Frechets repræsentations sætning en vigtig rolle.

Kapitel 4 - The Friedrichs extension I dette kapitel indføres Friedrichs udvidelsen af en given sektoriel operator T. Denne kan opfattes som en bestemt type af aflukning af T. Den sædvanlige aflukning af operatorer kunne kaldes for "operator-aflukningen" af T, mens Friedrichs udvidelsen passende kunne kaldes for "form-aflukningen" af T.

Det viser sig, at sektorielle operatorer altid er form-aflukkelige, hvilket betyder at en bestemt form, som er defineret udfra T er aflukkelig. Denne form opfylder de betingelser, der indgår i første repræsentations sætning, som frembringer en ny operator. Denne operator udvider T, og besidder visse plausible egenskaber, som for eksempel selv-adjungerethed, hvis T er symmetrisk og halv-begrænset.

Kapitel 5 - Hamiltonian of the δ-interaction Hamilton operatoren bliver introduceret i dette kapitel. Dette foregår ved, at der defineres en sesquilinearform på $\mathcal{H}^1(\mathbb{R})$; denne form vises at opfylde bestemte betingelser som via første repræsentations sætning inducerer en selv-adjungeret operator, som er den ønskede Hamilton operator. Domænet for denne operator er a priori ukendt, og det præcise domæne bliver bestemt i dette kapitel.

Kapitel 6 - The eigenvalue problem Hamilton operatoren har præcis én egenværdi; dette vil blive bevist i dette kapitel. Metoden er at udlede et udtryk for resolventen for H ved hjælp af den frie resolvent, som er resolventen for den frie Hamilton operator (fysisk set svarende til potentialet $V = 0$ som for en fri partikel). Den frie resolvent udtrykkes som en integral operator med en bestemt integral-kerne kaldet den frie Green's funktion.

Resolventen for H konstateres at have netop én singularitet, og det vises at Riesz projektionen for H i denne singularitet har rang én, hvormed singulariteten tilhører det diskrete spektrum.

Kapitel A - Appendix A I dette appendiks vises nogle sætninger angående integration af funktioner tilhørende Sobolev rum. Blandt andet sætninger vedrørende partiel integration, som bliver anvendt flere steder i rapporten.

Kapitel B - Appendix B Eksistensen af ikke-trivielle testfunktioner bliver taget for givet i flere matematik bøger. Ikke desto mindre er det ikke lykkedes forfatteren af denne rapport at opdrive et bevis for deres eksistens i litteraturen. Dette appendiks kompenserer for denne mangel.

Desuden konstrueres der en type af glatte funktioner, som opfylder at de er identisk lig med en udenfor et symmetrisk interval centreret om origo, er identisk lig med nul på et mindre symmetrisk interval omkring origo, og som antager værdier i intervallet $[0, 1]$. Disse funktioner anvendes i vid udstrækning i kapitel 5 til bestemmelse af domænet for Hamilton operatoren.

Contents

Chapter 1

Introduction

In this thesis, the theory of sectorial forms on Hilbert spaces with an application to δ-interactions is addressed. The first part consists of a thorough treatment of forms on Hilbert spaces including their representation in terms of operators and the Friedrichs extension.

The second part treats the problem of giving meaning to the operator "$-\frac{d^2}{dx^2} + \lambda\delta$", where $\lambda < 0$ and δ is the δ-distribution that represents a one-point interaction, or δ-potential, in one dimension centered around the origin. In addition, an eigenvalue problem involving the above mentioned operator will be solved.

The thesis is organized in the following way:

Chapter 2 - Sesquilinear forms on Hilbert spaces: In this chapter the theory of sesquilinear forms on Hilbert spaces is addressed. First, various definitions are given, results about forms are stated and proved. In particular the theory of sectorial forms is treated. These are forms for which the numerical range is not required to be a subset of $\mathbb{R}$, but are semibounded in the sense that the numerical range is contained in some sector that is symmetric with respect to the real axis and contained in a half-plane. This property of being sectorial is particularly useful, since it allows for introducing a new notion of convergence, that gives rise to "closedness" of forms.

Closedness of a form is of major importance, when one wishes to represent the forms in terms of operators. In the meantime, it is not always the case that forms are closed - some of those admit closed extensions though. Forms that admit closed extensions are called "closable". Various criterions for closability of a form will be proven.

Chapter 3 - Representation of forms: This chapter is devoted to the statement and proof of the first representation theorem. Any operator in $\mathcal{B}(\mathcal{H})$ gives rise to a bounded form in an obvious manner, and conversely: any bounded form induces an operator in $\mathcal{B}(\mathcal{H})$. This bijection between bounded forms and operators does not generalize to unbounded forms without complications.

The answer is given by the first representation theorem that gives some sufficient conditions that makes this generalization possible. These conditions are that the sesquilinear form should be densely defined, closed and sectorial. The first representation theorem gives little information on the domain of the associated operator. A necessary and sufficient condition for having membership of the domain will be deduced though. This becomes relevant when the domain of the Hamiltonian is to be specified in chapter 5.

Chapter 4 - The Friedrichs extension: In this chapter the socalled Friedrichs extension is treated. It is a method that furnishes a self-adjoint operator as an extension of a given symmetric semibounded operator. In this thesis a more general version of this extension is demonstrated. The operators are not required to be symmetric and semibounded, but merely to be sectorial - a notion described in chapter 2. In some cases, a symmetric operator may be essentially self-adjoint, which means that it admits a closure, and that the closure is self-adjoint. In other cases, this closure may not be self-adjoint. The closure always exists though, since symmetry of an operator means that it is densely defined (implying the existence of the adjoint) and that the operator itself is contained in the adjoint, which is known always to be a closed operator.

This type of closure could be called the 'operator closure' to distinguish it from another type of closure. The Friedrichs extension is exactly this other type of closure. The procedure is as follows: given an operator that enjoys certain properties, a sesquilinear form is associated to the operator. This form shows up to be a closable form. Then one closes this form, to obtain a closed form; when this has been done, the conditions occuring in the first representation theorem are fulfilled, providing a new operator that represents this form. This operator is an extension of the original operator, and it is called the Friedrichs extension of the given operator. In the special case of a symmetric semibounded operator the Friedrichs extension is in fact self-adjoint. Since the Friedrichs extension is obtained via the closure of a certain form, this type of closure is called the 'form-closure' of the given operator.

Chapter 5 - Hamiltonian of the δ-interaction: The Hamiltonian that arises from a certain form is investigated in chapter. In particular, its domain will be specified. In order to do so, results from appendix A and appendix B will be used extensively. Integration by parts of functions belonging to certain Sobolev spaces will be used. These integrations are justified in appendix A.

Furthermore, some smooth functions that vanish on symmetric intervals around the origin, being identically equal to one outside a larger symmetric interval around the origin and taking values between zero and one will be used often. The purpose is to get around the difficulties the origin gives rise to. The existence of these is a consequence of the existence of non-trivial testfunctions. Appendix B supplies the reader with these functions.

Chapter 6 - The eigenvalue problem: Once the Hamiltonian has been defined precisely, an eigenvalue problem is formulated and solved. The strategy is to express the resolvent of the Hamiltonian H in terms of the free resolvent - the resolvent of the Laplacian on $\mathcal{H}^2(\mathbb{R})$. This expression contains exactly one singularity, which can be shown to be a discrete eigenvalue of multiplicity one by computing the Riesz-projection in that singularity. The corresponding normalized eigenfunction can be determined using the boundary conditions that are specified on the domain of H.

Appendix A - Prerequisites: In this appendix some techniques of integration will be stated. More precisely, The Fundamental Theorem of Integral Calculus that states that an absolutely continuous function on $[a, b]$ can be recovered by integrating its derivative will be generalized to include functions that belong to Sobolev spaces.

In addition, it will be proved that integration by parts remains to be true for functions that belong to Sobolev spaces on $[a, b]$. Once this result is established, it will be demonstrated that integration by parts can be carried out unrestrictedly of functions defined on the whole of $\mathbb{R}$ that, again, belong to Sobolev spaces; membership of the Sobolev spaces will, via one of the Sobolev embedding theorems, lead to the vanishing of the boundary terms that usually arise in connection with integration by parts.

The reason for these generalizations is that when dealing with distributions, then one is interested in moving the differentiation from one function to another in certain integrals - more specifically, to move the differentiation from one slot in an inner product to the other slot (on the cost of a change of sign). A concrete example of this is in the proof of symmetry of the free Hamiltonian.

Appendix B - Existence of non-trivial testfunctions: In this appendix it is proven that there is at least one test function on the real axis. Here, a test function is a function defined on the real axis with values in the real axis such that it is smooth (i.e. arbitrarily often differentiable) and has compact support; the space of such functions is typically denoted $C_0^\infty(\mathbb{R})$ or $C_c^\infty(\mathbb{R})$ in the literature. The proof is fully constructive in the sense that an explicit formula of a function is being given and succesively proved directly to be endeed a test function.

Furthermore, it will be shown how one can construct a test function with support of a given size called the radius of mollification. This will become relevant in specifying the domain of the operator that is associated to the form $\mathfrak{h}$ that was defined in chapter 5. In doing so, some smooth functions that are equal to one outside a symmetric interval around the origin, equal to zero on some strictly smaller symmetric interval around the origin and taking values between zero and one are needed - and here it is important to be able to control the size of the support (radius of mollification) of the test function. Finally, such functions will be constructed by convolution.

The reason for presenting a proof of the existence of test functions is the authors discontent with not being able to find a solid proof anywhere in the literature, together with the importance of the supply of test functions in a variety of situations, e.g. in the theory of distributions (first of all).

Notation

Throughout this thesis, the notation $(\cdot|\cdot)$ will mean the inner product on a given Hilbert space, unless otherwise is explicitly stated. Also, when nothing else has been stated, then $\|\cdot\|$ means the usual norm on $L^2(\mathbb{R})$.

The smooth functions that are constructed in appendix B are usually called φ, and for emphasis, $\varphi_{m,M}$ to denote where the functions are non-zero.

Part I

Preliminaries

Chapter 2

Sesquilinear forms on Hilbert spaces

In this chapter, the theory of sesquilinear forms on Hilbert spaces will be addressed. The presentation given here is based on [7] chapter VI, and is a detailed elaboration of the theory developed in that book. The purpose is to provide a part of the prerequisites for the problem to be solved later.

Definition 2.0.1 *Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. A map $\mathfrak{t} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is said to be a sesquilinear form on $\mathcal{H}$, if $\mathfrak{t}(\cdot, v)$ is linear for each $v \in \mathcal{D}$ and $\mathfrak{t}(u, \cdot)$ is conjugate linear for each $u \in \mathcal{D}$. The subspace $\mathcal{D}$ is called the domain of $\mathfrak{t}$, and will be denoted by $\mathcal{D}(\mathfrak{t})$.*

Remark: Strictly speaking, the domain of $\mathfrak{t}$ is $\mathcal{D} \times \mathcal{D}$, but it is custom to call it $\mathcal{D}$. This custom will be followed in this thesis.

Definition 2.0.2 *Let $\mathfrak{t}$ be a sesquilinear form on a Hilbert space $\mathcal{H}$. A sesquilinear form $\mathfrak{s}$ on $\mathcal{H}$ is said to be*

a restriction of $\mathfrak{t}$, if $\mathcal{D}(\mathfrak{s}) \subseteq \mathcal{D}(\mathfrak{t})$ and $\mathfrak{s}(u, v) = \mathfrak{t}(u, v)$ for all $u, v \in \mathcal{D}(\mathfrak{s})$.

an extension of $\mathfrak{t}$, if $\mathcal{D}(\mathfrak{t}) \subseteq \mathcal{D}(\mathfrak{s})$ and $\mathfrak{s}(u, v) = \mathfrak{t}(u, v)$ for all $u, v \in \mathcal{D}(\mathfrak{t})$.

Definition 2.0.3 *A quadratic form $\mathfrak{t}'$ on $\mathcal{H}$ is defined to be the restriction of a sesquilinear form $\mathfrak{t}$ on $\mathcal{H}$ to the diagonal in $\mathcal{D} \times \mathcal{D}$, i.e.*

$$\mathfrak{t}'(u) := \mathfrak{t}(u, u), \quad u \in \mathcal{D}.$$

The "$'$" on the quadratic form $\mathfrak{t}'$ will be suppressed, since the sesquilinear form can be recovered from the quadratic form:

Proposition 2.0.4 (Polarization identity) *Let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{H}$. For any $u, v \in \mathcal{D}(\mathfrak{t})$ we have*

$$\mathfrak{t}(u, v) = \frac{1}{4} \sum_{k=0}^{3} \mathfrak{t}(u + i^k v).$$

Proof: Expanding the terms $\mathfrak{t}(u + i^k v)$ using sesquilinearity and rearranging them gives the desired. ∎

Definition 2.0.5 *Let $\mathfrak{t}, \mathfrak{t}_1, \mathfrak{t}_2$ be sesquilinear forms on $\mathcal{H}$ and $\alpha \in \mathbb{C}$. Then we define $\mathfrak{t}_1 + \mathfrak{t}_2$ and $\alpha\mathfrak{t}$ as follows:*

$$\begin{cases} \mathcal{D}(\mathfrak{t}_1 + \mathfrak{t}_2) = \mathcal{D}(\mathfrak{t}_1) \cap \mathcal{D}(\mathfrak{t}_2) \\ (\mathfrak{t}_1 + \mathfrak{t}_2)(u,v) = \mathfrak{t}_1(u,v) + \mathfrak{t}_2(u,v) \end{cases} \quad \textit{and} \quad \begin{cases} \mathcal{D}(\alpha\mathfrak{t}) = \mathcal{D}(\mathfrak{t}) \\ (\alpha\mathfrak{t})(u,v) = \alpha\mathfrak{t}(u,v) \end{cases}.$$

Furthermore, the unit form $\mathbf{1}$ and the zero form $\mathbf{0}$ are defined as follows: $\mathcal{D}(\mathbf{1}) = \mathcal{H}$, $\mathbf{1}(u,v) = (u|v)$, and $\mathcal{D}(\mathbf{0}) = \mathcal{H}$, $\mathbf{0}(u,v) = 0$ respectively. Here $(u|v)$ denotes the inner product on $\mathcal{H}$.

Note that with the definition above, $0\mathfrak{t} \subseteq \mathbf{0}$, where strict inclusion occurs, if and only if $\mathcal{D}(\mathfrak{t})$ is a proper subspace of $\mathcal{H}$.

Definition 2.0.6 *A sesquilinear form $\mathfrak{t}$ on $\mathcal{H}$ is said to be symmetric, if*

$$\forall u, v \in \mathcal{D}(\mathfrak{t}) : \mathfrak{t}(u,v) = \overline{\mathfrak{t}(v,u)},$$

where the bar denotes complex conjugation.

Proposition 2.0.7 *A sesquilinear form $\mathfrak{t}$ on $\mathcal{H}$ is symmetric if and only if $\mathfrak{t}(u) \in \mathbb{R}$ for all $u \in \mathcal{D}(\mathfrak{t})$.*

Proof: $\Rightarrow$: If $\mathfrak{t}$ is symmetric, then $\mathfrak{t}(u) = \mathfrak{t}(u,u) = \overline{\mathfrak{t}(u,u)}$, so $\mathfrak{t}(u) \in \mathbb{R}$.
$\Leftarrow$: Using the polarization identity, write $\overline{\mathfrak{t}(v,u)} = \overline{\frac{1}{4}\sum_{k=0}^{3} i^k \mathfrak{t}(v + i^k u)}$, which is equal to $\frac{1}{4}\sum_{k=0}^{3} \overline{i^k}\mathfrak{t}(v + i^k u)$, since $\mathfrak{t}(u) \in \mathbb{R}$. Expanding the terms, rearranging and collecting them gives that this equals $\frac{1}{4}\sum_{k=0}^{3} i^k \mathfrak{t}(u + i^k v)$, which again equals $\mathfrak{t}(u,v)$, by the polarization identity. ■

Definition 2.0.8 *Let $\mathfrak{t}$ be a sesquilinear form on $\mathcal{H}$; the adjoint form $\mathfrak{t}^*$ of $\mathfrak{t}$ is defined as*

$$\mathcal{D}(\mathfrak{t}^*) = \mathcal{D}(\mathfrak{t}), \qquad \mathfrak{t}^*(u,v) = \overline{\mathfrak{t}(v,u)}.$$

It is immediate, that $\mathfrak{t}$ is symmetric if and only if $\mathfrak{t} = \mathfrak{t}^*$, and trivial verifications show the identity $(\alpha_1\mathfrak{t}_1 + \alpha_2\mathfrak{t}_2)^* \equiv \overline{\alpha_1}\mathfrak{t}_1^* + \overline{\alpha_2}\mathfrak{t}_2^*$.

Definition 2.0.9 *For an arbitrary sesquilinear form $\mathfrak{t}$ on $\mathcal{H}$, define the forms $\mathfrak{h}$ and $\mathfrak{k}$ by:*

$$\begin{cases} \mathcal{D}(\mathfrak{h}) = \mathcal{D}(\mathfrak{t}) \\ \mathfrak{h} = \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*) \end{cases} \quad \textit{and} \quad \begin{cases} \mathcal{D}(\mathfrak{k}) = \mathcal{D}(\mathfrak{t}) \\ \mathfrak{k} = \frac{1}{2i}(\mathfrak{t} - \mathfrak{t}^*) \end{cases}.$$

For arbitrary $u, v \in \mathcal{D}(\mathfrak{t})$ we have that

$$\begin{aligned} \mathfrak{h}^*(u,v) &= (\frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*))^*(u,v) = (\frac{1}{2}t^* + \frac{1}{2}(t^*)^*)(u,v) \\ &= \frac{1}{2}\mathfrak{t}^*(u,v) + \frac{1}{2}(t^*)^*(u,v) = \frac{1}{2}\mathfrak{t}^*(u,v) + \frac{1}{2}\overline{t^*(v,u)} \\ &= \frac{1}{2}\mathfrak{t}^*(u,v) + \frac{1}{2}\overline{\overline{\mathfrak{t}(u,v)}} = \frac{1}{2}\mathfrak{t}^*(u,v) + \frac{1}{2}\mathfrak{t}(u,v) \\ &= \mathfrak{h}(u,v), \end{aligned}$$

so $\mathfrak{h}$ is symmetric; similarly it can be shown that $\mathfrak{k}$ is symmetric.

Notation 2.0.10 *Since* $\mathfrak{h} + i\mathfrak{k} = \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*) + i\frac{1}{2i}(t - t^*) = \mathfrak{t}$ *the forms* $\mathfrak{h}$ *and* $\mathfrak{k}$ *will be called the real and imaginary parts of* $\mathfrak{t}$ *respectively. They will be denoted by* $\mathfrak{h} := \operatorname{Re}\mathfrak{t}$ *and* $\mathfrak{k} := \operatorname{Im}\mathfrak{t}$.

Definition 2.0.11 *A sesquilinear form* $\mathfrak{t}$ *on* $\mathcal{H}$ *is called*

- *bounded, if*
$$\exists M \geq 0 \ \forall u, v \in \mathcal{D}(\mathfrak{t}) : |\mathfrak{t}(u, v)| \leq M\|u\|\|v\|. \tag{2.1}$$
- *unbounded, if it is not bounded.*

Definition 2.0.12 *Let* $\mathfrak{t}$ *be a sesquilinear form on* $\mathcal{H}$*. The quantity*

$$\|\mathfrak{t}\| = \sup\{|\mathfrak{t}(u, v)| \mid u, v \in \mathcal{D}(\mathfrak{t}), \ \|u\| = \|v\| = 1\},$$

which belongs to $[0, \infty]$ *is called the norm of* $\mathfrak{t}$.

Proposition 2.0.13 *A sesquilinear form* $\mathfrak{t}$ *on* $\mathcal{H}$ *is bounded if and only if* $\|\mathfrak{t}\| < \infty$.

Proof: $\Rightarrow$: Assume that $\mathfrak{t}$ is bounded; then $|\mathfrak{t}(u, v)| \leq M\|u\|\|v\|$ for all $u, v \in \mathcal{D}(\mathfrak{t})$ and for some $M \geq 0$, which is independent of u and v. In particular: $|\mathfrak{t}(u, v)| \leq M$, when $\|u\| = \|v\| = 1$. This implies that $\|\mathfrak{t}\| \leq M$, so $\|\mathfrak{t}\| < \infty$, since $M < \infty$.

$\Leftarrow$: Assume that $\|\mathfrak{t}\| < \infty$ and let $u, v \in \mathcal{D}(\mathfrak{t})$; if $u = 0$ or $v = 0$ (or both), then (2.1) is fulfilled, so assume that $u, v \in \mathcal{D}(\mathfrak{t}) \setminus \{0\}$; then $|\mathfrak{t}(\|u\|^{-1}u, \|v\|^{-1}v)| \leq \|\mathfrak{t}\| < \infty$. This implies that $|\mathfrak{t}(u, v)| \leq \|\mathfrak{t}\|\|u\|\|v\|$. Put e.g. $M := \|\mathfrak{t}\|$, and (2.1) is thus seen to be fulfilled. ■

An unbounded form may be semibounded in a sense defined in the following:

Definition 2.0.14 *A symmetric form* $\mathfrak{h}$ *is said to be bounded from below, if*

$$\exists \gamma \in \mathbb{R} \ \forall u \in \mathcal{D}(\mathfrak{h}) : \mathfrak{h}(u) \geq \gamma\|u\|^2.$$

Any number γ *obeying the formula above is called a lower bound of* $\mathfrak{h}$*. The number* $\sup\{\gamma | \gamma$ *is a lower bound of* $\mathfrak{h}\}$ *is called the lower bound of* $\mathfrak{h}$*, and is denoted by* $\gamma_{\mathfrak{h}}$*. That* $\mathfrak{h}$ *is bounded from below with a lower bound* γ *is denoted* $\mathfrak{h} \geq \gamma$*. If* $\mathfrak{h} \geq 0$*, then* $\mathfrak{h}$ *is called non-negative.*

Proposition 2.0.15 *For a symmetric non-negative form* $\mathfrak{t}$*, we have:*

$$\forall u, v \in \mathcal{D}(\mathfrak{t}) : |\mathfrak{t}(u, v)| \leq \mathfrak{t}(u)^{\frac{1}{2}}\mathfrak{t}(v)^{\frac{1}{2}}. \tag{2.2}$$

The proof of this proposition is inspired by the proof given in [13] for the Cauchy-Schwarz inequality for inner products. It is just adapted here for non-negative symmetric forms that need not be strictly positive, where strict positivity means non-negativity together with the condition that $\mathfrak{t}(u) = 0$ implies $u = 0$.

Proof: There is an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha \mathfrak{t}(v,u) = |\mathfrak{t}(v,u)| = |\mathfrak{t}(u,v)| \in \mathbb{R}$. Now, for any $r \in \mathbb{R}$ we have

$$\begin{aligned} & \mathfrak{t}(u - r\alpha v) = \mathfrak{t}(u - r\alpha v, u - r\alpha v) \\ = \; & \mathfrak{t}(u,u) + \mathfrak{t}(u, -r\alpha v) + \mathfrak{t}(-r\alpha v, u) + \mathfrak{t}(-r\alpha v, -r\alpha v) \\ = \; & \mathfrak{t}(u) + \overline{-r\alpha}\mathfrak{t}(u,v) + (-r\alpha)\mathfrak{t}(v,u) + |r\alpha|^2 \mathfrak{t}(v,v) \\ = \; & \mathfrak{t}(u) - r\overline{\alpha \mathfrak{t}(v,u)} - r\alpha \mathfrak{t}(v,u) + r^2|\alpha|^2 \mathfrak{t}(v) \\ = \; & \mathfrak{t}(u) - 2r|\mathfrak{t}(u,v)| + r^2 \mathfrak{t}(v) \geq 0. \end{aligned} \tag{2.3}$$

If $\mathfrak{t}(v) = 0$, then (2.3) is violated for $r > \frac{\mathfrak{t}(u)}{2|\mathfrak{t}(u,v)|}$, unless $|\mathfrak{t}(u,v)| = 0$. Therefore $|\mathfrak{t}(u,v)| = 0$, whence (2.2) is fulfilled.

If $\mathfrak{t}(v) > 0$, then putting $r := \frac{|\mathfrak{t}(u,v)|}{\mathfrak{t}(v)}$ one gets:

$$\mathfrak{t}(u) - 2\frac{|\mathfrak{t}(u,v)|^2}{\mathfrak{t}(v)} + \frac{|\mathfrak{t}(u,v)|^2}{\mathfrak{t}(v)} = \mathfrak{t}(u) - \frac{|\mathfrak{t}(u,v)|^2}{\mathfrak{t}(v)} \geq 0,$$

or equivalently: $|\mathfrak{t}(u,v)| \leq \mathfrak{t}(u)^{\frac{1}{2}} \mathfrak{t}(v)^{\frac{1}{2}}$. ■

Note that $\|\mathbf{1}\| = 1$ by Prop. 2.0.15, and $\|\mathbf{0}\| = 0$.

The following corollary corresponds to the triangle inequality for norms.

Corollary 2.0.16 *Let* $\mathfrak{t}$ *be a symmetric non-negative form. Then*

$$\forall u, v \in \mathcal{D}(\mathfrak{t}) : \mathfrak{t}(u+v)^{\frac{1}{2}} \leq \mathfrak{t}(u)^{\frac{1}{2}} + \mathfrak{t}(v)^{\frac{1}{2}}.$$

Proof:

$$\begin{aligned} \mathfrak{t}(u+v) &= \mathfrak{t}(u+v, u+v) = \mathfrak{t}(u) + \mathfrak{t}(u,v) + \mathfrak{t}(v,u) + \mathfrak{t}(v) \\ &= \mathfrak{t}(u) + \mathfrak{t}(u,v) + \overline{\mathfrak{t}(u,v)} + \mathfrak{t}(v) = \mathfrak{t}(u) + 2\,\mathrm{Re}[\mathfrak{t}(u,v)] + \mathfrak{t}(v) \\ &\leq \mathfrak{t}(u) + 2|\mathfrak{t}(u,v)| + \mathfrak{t}(v) \\ &\leq \mathfrak{t}(u) + 2\mathfrak{t}(u)^{\frac{1}{2}}\mathfrak{t}(v)^{\frac{1}{2}} + \mathfrak{t}(v), \quad \text{by Prop. 2.0.15} \\ &= \left(\mathfrak{t}(u)^{\frac{1}{2}} + \mathfrak{t}(v)^{\frac{1}{2}}\right)^2, \end{aligned}$$

which implies that $\mathfrak{t}(u+v)^{\frac{1}{2}} \leq \mathfrak{t}(u)^{\frac{1}{2}} + \mathfrak{t}(v)^{\frac{1}{2}}$. ■

Proposition 2.0.17 *Let* $\mathfrak{t}$ *be any sesquilinear form. Then*

$$\forall u, v \in \mathcal{D}(\mathfrak{t}) : \mathfrak{t}(u+v) + \mathfrak{t}(u-v) = 2[\mathfrak{t}(u) + \mathfrak{t}(v)].$$

Proof: Let $u, v \in \mathcal{D}(\mathfrak{t})$ be given; sesquilinearity of $\mathfrak{t}$ gives

$$\begin{aligned} & \mathfrak{t}(u+v) + \mathfrak{t}(u-v) = \mathfrak{t}(u+v, u+v) + \mathfrak{t}(u-v, u-v) \\ = \; & \mathfrak{t}(u) + \mathfrak{t}(u,v) + \mathfrak{t}(v,u) + \mathfrak{t}(v) + \mathfrak{t}(u) + \mathfrak{t}(u,-v) + \mathfrak{t}(-v,u) + \mathfrak{t}(-v,-v) \\ = \; & 2\mathfrak{t}(u) + 2\mathfrak{t}(v). \end{aligned}$$

■

Proposition 2.0.18 *Let $\mathfrak{t}$ be a symmetric non-negative form. Then*

$$\forall u, v \in \mathcal{D}(\mathfrak{t}) : \mathfrak{t}(u+v) \leq 2[\mathfrak{t}(u) + \mathfrak{t}(v)].$$

Proof: Since $\mathfrak{t} \geq 0$ one has that $\mathfrak{t}(u+v) \leq \mathfrak{t}(u+v) + \mathfrak{t}(u-v)$. Prop 2.0.17 now yields:

$$\mathfrak{t}(u+v) \leq \mathfrak{t}(u+v) + \mathfrak{t}(u-v) = 2[\mathfrak{t}(u) + \mathfrak{t}(v)].$$

■

Proposition 2.0.19 *Let $\mathfrak{h}, \mathfrak{k}$ be symmetric forms, where $\mathcal{D}(\mathfrak{h}) = \mathcal{D}(\mathfrak{k})$, and assume that $\mathfrak{k}$ is non-negative. Then the following is true:*

$$\left[\exists M \geq 0 \; \forall u \in \mathcal{D}(\mathfrak{t}) : |\mathfrak{h}(u)| \leq M\mathfrak{k}(u)\right] \Rightarrow \left[\forall u, v \in \mathcal{D}(\mathfrak{t}) : |\mathfrak{h}(u,v)| \leq M\mathfrak{k}(u)^{\frac{1}{2}}\mathfrak{k}(v)^{\frac{1}{2}}\right].$$

Proof: Since $\mathfrak{h}(u,v) \cdot c \in [0, \infty[$ for some $c \in \mathbb{C}$, $|c| = 1$, and $|\mathfrak{h}(u,v)| = |\mathfrak{h}(u,v) \cdot c|$ it can be assumed without loss of generality that $\mathfrak{h}(u,v) \in [0, \infty[$. The polarization identity

$$\mathfrak{h}(u,v) = \frac{1}{4}[\mathfrak{h}(u+v) + i\mathfrak{h}(u+iv) - \mathfrak{h}(u-v) - i\mathfrak{h}(u-iv)]$$

then reduces to $\mathfrak{h}(u,v) = \frac{1}{4}[\mathfrak{h}(u+v) - \mathfrak{h}(u-v)]$. Using the assumption $|\mathfrak{h}(u)| \leq M\mathfrak{k}(u)$ one gets:

$$\begin{aligned}
|\mathfrak{h}(u,v)| &= \frac{1}{4}|\mathfrak{h}(u+v) - \mathfrak{h}(u-v)| \\
&\leq \frac{1}{4}(\mathfrak{h}(u+v) + \mathfrak{h}(u-v)) \\
&= \frac{1}{4} \cdot 2(\mathfrak{h}(u) + \mathfrak{h}(v)), \qquad \text{by Prop. 2.0.17} \\
&\leq \frac{1}{2}(M\mathfrak{k}(u) + M\mathfrak{k}(v)), \qquad \text{by assumption} \\
&= \frac{1}{2}M(\mathfrak{k}(u) + \mathfrak{k}(v)).
\end{aligned}$$

Put $\alpha^2 := \mathfrak{k}(v)^{\frac{1}{2}}\mathfrak{k}(u)^{-\frac{1}{2}}$. Now,

$$\begin{aligned}
|\mathfrak{h}(u,v)| &= |\mathfrak{h}(\alpha u, \alpha^{-1}v)| \leq \frac{1}{2}M(\mathfrak{k}(\alpha u) + \mathfrak{k}(\alpha^{-1}v)) \\
&= \frac{1}{2}M\left(\alpha^2\mathfrak{k}(u) + \alpha^{-2}\mathfrak{k}(v)\right) \\
&= \frac{1}{2}M\sqrt{\frac{\mathfrak{k}(v)}{\mathfrak{k}(u)}}\mathfrak{k}(u) + \frac{1}{2}M\sqrt{\frac{\mathfrak{k}(u)}{\mathfrak{k}(v)}}\mathfrak{k}(v) \\
&= \frac{1}{2}M\mathfrak{k}(v)^{\frac{1}{2}}\mathfrak{k}(u)^{\frac{1}{2}} + \frac{1}{2}M\mathfrak{k}(u)^{\frac{1}{2}}\mathfrak{k}(v)^{\frac{1}{2}} \\
&= M\mathfrak{k}(u)^{\frac{1}{2}}\mathfrak{k}(v)^{\frac{1}{2}}.
\end{aligned}$$

■

Definition 2.0.20 *Let* $\mathfrak{t}$ *be sesquilinear form. The numerical range of* $\mathfrak{t}$ *is defined as*

$$\Theta(\mathfrak{t}) := \{\mathfrak{t}(u) \mid u \in \mathcal{D}(\mathfrak{t}),\ \|u\| = 1\}.$$

Definition 2.0.21 *A sesquilinear form* $\mathfrak{t}$ *is said to be sectorially bounded from the left, if*

$$\exists \gamma \in \mathbb{R}\ \exists \theta \in [0, \frac{\pi}{2}[\ : \Theta(\mathfrak{t}) \subseteq \{\zeta \in \mathbb{C} \mid |\mathrm{Arg}(\zeta - \gamma)| \leq \theta\},$$

where Arg *denotes the principal value.*

The number γ *is referred to as a vertex of* $\mathfrak{t}$*, and* θ *as a corresponding semi-angle of* $\mathfrak{t}$*.*

The numbers γ and θ are not unique, since any number smaller than γ is also a vertex, and any number in $[0, \frac{\pi}{2}[$ larger than θ is also a semi-angle.

Proposition 2.0.22 *Let* $\mathfrak{t}$ *be a possibly non-symmetric form. Then* $\mathfrak{t}$ *is sectorially bounded from the left with vertex* γ *and corresponding semi-angle* θ *if and only if*

$$\mathfrak{h} \geq \gamma \qquad \textit{and} \qquad |\mathfrak{k}(u)| \leq (\tan\theta)(\mathfrak{h} - \gamma)(u), \quad u \in \mathcal{D}(\mathfrak{t}),$$

where $\mathfrak{h} = \mathrm{Re}\,\mathfrak{t}$ *and* $\mathfrak{k} = \mathrm{Im}\,\mathfrak{t}$*.*

Proof: $\Rightarrow$: Assume $\Theta(\mathfrak{t}) := \{\mathfrak{t}(u) \mid u \in \mathcal{D}(\mathfrak{t}),\ \|u\| = 1\} \subseteq \{\zeta \in \mathbb{C} \mid |\mathrm{Arg}(\zeta - \gamma)| \leq \theta\}$. Then $|\mathrm{Arg}(\mathfrak{t}(u) - \gamma)| \leq \theta < \frac{\pi}{2}$ for all $u \in \mathcal{D}(\mathfrak{t})$ with $\|u\| = 1$; this implies that $\mathrm{Re}(\mathfrak{t}(u) - \gamma) \geq 0$, so $\mathrm{Re}(\mathfrak{t}(u)) \geq \gamma$ and also $\mathrm{Re}(\mathfrak{t}(u)) \geq \gamma\|u\|^2$, since $\|u\| = 1$. Since $\mathfrak{h}(u) := (\mathrm{Re}\,\mathfrak{t})(u) = \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*)(u) = \frac{1}{2}\mathfrak{t}(u) + \frac{1}{2}\mathfrak{t}^*(u) = \frac{1}{2}\mathfrak{t}(u,u) + \frac{1}{2}\mathfrak{t}^*(u,u) = \frac{1}{2}\mathfrak{t}(u,u) + \frac{1}{2}\overline{\mathfrak{t}(u,u)} = \mathrm{Re}(\mathfrak{t}(u))$ we get $(\mathrm{Re}\,\mathfrak{t})(u) \geq \gamma\|u\|^2$, or $\mathfrak{h}(u) \geq \gamma\|u\|^2$, or $\mathfrak{h} \geq \gamma$. We also have $\mathfrak{k}(u) := (\mathrm{Im}\,\mathfrak{t})(u) = \frac{1}{2i}(\mathfrak{t} - \mathfrak{t}^*)(u) = \frac{1}{2i}\mathfrak{t}(u) - \frac{1}{2i}\mathfrak{t}^*(u) = \frac{1}{2i}\mathfrak{t}(u) - \frac{1}{2i}\overline{\mathfrak{t}(u)} = \mathrm{Im}(\mathfrak{t}(u)) = \mathrm{Im}(\mathfrak{t}(u) - \gamma)$. For $u = 0$, the inequality is obvious. Now, given a $u \in \mathcal{D}(\mathfrak{t}) \setminus \{0\}$, put $u' := \|u\|^{-1}u$; then

$$\begin{aligned} |\mathfrak{k}(u')| &= |\mathrm{Im}(\mathfrak{t}(u') - \gamma)| = \mathrm{Re}(\mathfrak{t}(u') - \gamma)\tan|\mathrm{Arg}(\mathfrak{t}(u') - \gamma)| \\ &\leq (\tan\theta)(\mathfrak{h}(u') - \gamma) \\ &= (\tan\theta)(\mathfrak{h} - \gamma)(u'). \end{aligned}$$

This implies that

$$|\mathfrak{k}(u)| = \|u\|^2|\mathfrak{k}(u')| \leq \|u\|^2(\tan\theta)(\mathfrak{h} - \gamma)(u') = (\tan\theta)(\mathfrak{h} - \gamma)(u).$$

$\Leftarrow$: Assume that $\mathfrak{h} \geq \gamma$ and $|\mathfrak{k}(u)| \leq (\tan\theta)(\mathfrak{h} - \gamma)(u)$, for all $u \in \mathcal{D}(\mathfrak{t})$. Assume that $x \in \Theta(\mathfrak{t})$; then there is a $u \in \mathcal{D}(\mathfrak{t})$ with $\|u\| = 1$ such that $x = \mathfrak{t}(u)$. First, if $\mathfrak{t}(u) = \gamma$, then $|\mathrm{Arg}(\mathfrak{t}(u) - \gamma)| = 0 \leq \theta$. Now, if $\mathfrak{t}(u) \neq \gamma$, then

$$\begin{aligned} |\mathrm{Arg}(\mathfrak{t}(u) - \gamma)| &= \left|\mathrm{Arctan}\frac{\mathrm{Im}(\mathfrak{t}(u) - \gamma)}{\mathrm{Re}(\mathfrak{t}(u) - \gamma)}\right| = \left|\mathrm{Arctan}\frac{\mathrm{Im}(\mathfrak{t}(u) - \gamma)}{(\mathfrak{h} - \gamma)(u)}\right| \\ &= \left|\mathrm{Arctan}\frac{\mathrm{Im}(\mathfrak{t}(u))}{(\mathfrak{h} - \gamma)(u)}\right| \\ &\leq \left|\mathrm{Arctan}\frac{(\tan\theta)(\mathfrak{h} - \gamma)(u)}{(\mathfrak{h} - \gamma)(u)}\right| = |\theta| \\ &= \theta, \end{aligned}$$

so $x \in \{\zeta \in \mathbb{C} \mid |\mathrm{Arg}(\zeta - \gamma)| \leq \theta\}$. ■

Proposition 2.0.23 *Let $\mathfrak{t}$ be a sesquilinear form. Then $\mathfrak{t}$ is sectorial with vertex γ and corresponding semi-angle θ if and only if $\mathfrak{t}^*$ is sectorial with vertex γ and corresponding semi-angle θ.*

Proof: By definition $\mathfrak{t}$ is sectorial with vertex γ and corresponding semi-angle θ if and only if $\Theta(\mathfrak{t}) \subseteq \{\zeta \in \mathbb{C}|\ |\mathrm{Arg}(\zeta-\gamma)| \leq \theta\}$. Also, $\mathrm{Arg}(\mathfrak{t}(u)-\gamma) = -\mathrm{Arg}(\overline{\mathfrak{t}(u)-\gamma}) = -\mathrm{Arg}(\overline{\mathfrak{t}(u)}-\gamma)$, so $|\mathrm{Arg}(\mathfrak{t}(u)-\gamma)| \leq \theta$ if and only if $|-\mathrm{Arg}(\overline{\mathfrak{t}(u)-\gamma})| = |\mathrm{Arg}(\overline{\mathfrak{t}(u)}-\gamma)| \leq \theta$. Since $\Theta(\mathfrak{t}^*) = \{\mathfrak{t}^*(u) \in \mathbb{C}|\ u \in \mathcal{D}(\mathfrak{t}^*),\ \|u\| = 1\} = \{\overline{\mathfrak{t}(u)} \in \mathbb{C}|\ u \in \mathcal{D}(\mathfrak{t}),\ \|u\| = 1\}$, we have that $\Theta(\mathfrak{t}) \subseteq \{\zeta \in \mathbb{C}|\ |\mathrm{Arg}(\zeta-\gamma)| \leq \theta\}$ if and only if $\Theta(\mathfrak{t}^*) \subseteq \{\zeta \in \mathbb{C}|\ |\mathrm{Arg}(\zeta-\gamma)| \leq \theta\}$. ■

Proposition 2.0.24 *Let $\mathfrak{t}$ be a non-symmetric form, sectorially bounded from the left with vertex γ and semi-angle θ. Put $\mathfrak{h} := \mathrm{Re}\,\mathfrak{t}$ and $\mathfrak{k} := \mathrm{Im}\,\mathfrak{t}$; then for all $u, v \in \mathcal{D}(\mathfrak{t})$ it holds that*

(i) $|(\mathfrak{h}-\gamma)(u,v)| \leq (\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}$

(ii) $|\mathfrak{k}(u,v)| \leq (\tan\theta)(\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}$

(iii) $|(\mathfrak{t}-\gamma)(u,v)| \leq (1+\tan\theta)(\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}$.

Proof: Ad (*i*): Since $(\mathfrak{h}-\gamma)^* = (\mathfrak{h}+(-\gamma))^* = \mathfrak{h}^* + \overline{-\gamma} = \mathfrak{h}-\gamma$, $\mathfrak{h}-\gamma$ is symmetric; furthermore $\mathfrak{h}-\gamma \geq 0$ by assumption. Now (*i*) follows immediately from Prop. 2.0.15.

Ad (*ii*): $\mathfrak{t}$ is sectorial with vertex γ and corresponding semi-angle θ, so $|\mathfrak{k}(u)| \leq (\tan\theta)(\mathfrak{h}-\gamma)(u)$ for all $u \in \mathcal{D}(\mathfrak{t})$, by Prop. 2.0.22. Now Prop. 2.0.19 gives:

$$|\mathfrak{k}(u,v)| \leq (\tan\theta)(\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}, \quad \text{for all } u, v \in \mathcal{D}(\mathfrak{t}).$$

Ad (*iii*):

$$\begin{aligned}
& |(\mathfrak{t}-\gamma)(u,v)| \\
= \;& |(\mathfrak{h}+i\mathfrak{k}-\gamma)(u,v)| = |(\mathfrak{h}-\gamma+i\mathfrak{k})(u,v)| = |(\mathfrak{h}-\gamma)(u,v)+(i\mathfrak{k})(u,v)| \\
\leq \;& |(\mathfrak{h}-\gamma)(u,v)| + |(i\mathfrak{k})(u,v)| = |(\mathfrak{h}-\gamma)(u,v)| + |\mathfrak{k}(u,v)| \\
\leq \;& (\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}} + |\mathfrak{k}(u,v)|, \quad \text{by Prop. 2.0.15} \\
\leq \;& (\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}} + (\tan\theta)(\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}, \quad \text{by } (ii) \\
= \;& (1+\tan\theta)(\mathfrak{h}-\gamma)(u)^{\frac{1}{2}}(\mathfrak{h}-\gamma)(v)^{\frac{1}{2}}.
\end{aligned}$$

■

Proposition 2.0.25 *Let $\mathfrak{t}$ be a possibly non-symmetric sectorial form with vertex γ and corresponding semi-angle θ; then*

$$(\mathfrak{h}-\gamma)(u) \leq |(\mathfrak{t}-\gamma)(u)| \leq (\sec\theta)(\mathfrak{h}-\gamma)(u).$$

Proof: The first inequality is shown as follows:

$$\begin{aligned}
(\mathfrak{h}-\gamma)(u) &= (\mathrm{Re}\,\mathfrak{t}-\gamma)(u) = (\mathrm{Re}\,\mathfrak{t})(u) - \gamma\mathbf{1}(u) = \mathrm{Re}(\mathfrak{t}(u)) - \gamma\mathbf{1}(u) \\
&= \mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)) = \mathrm{Re}((\mathfrak{t}-\gamma)(u)) \\
&\leq |(\mathfrak{t}-\gamma)(u)|.
\end{aligned}$$

Next, the second inequality is shown. First, if $\mathfrak{h}(u) = \gamma\|u\|^2$, then

$$\begin{aligned} |(\mathfrak{t}-\gamma)(u)| &= |i\mathfrak{k}(u)| = |\mathfrak{k}(u)| \leq (\tan\theta)(\mathfrak{h}-\gamma)(u) = 0 \\ &\leq (\sec\theta)(\mathfrak{h}-\gamma)(u). \end{aligned}$$

Second, if $\mathfrak{h}(u) > \gamma\|u\|^2$, then

$$(\mathfrak{t}-\gamma)(u) = \mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)) + i\,\mathrm{Im}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)). \tag{2.4}$$

We have that

$$\begin{aligned} |\mathrm{Arg}((\mathfrak{t}-\gamma)(u))| &= \left|\mathrm{Arg}\big(\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)) + i\,\mathrm{Im}(\mathfrak{t}(u) - \gamma\mathbf{1}(u))\big)\right| \\ &= \left|\mathrm{Arg}\big(\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)) + i\,\mathrm{Im}(\mathfrak{t}(u))\big)\right| \\ &= \left|\mathrm{Arctan}\left(\frac{\mathrm{Im}(\mathfrak{t}(u))}{\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u))}\right)\right| \\ &= \mathrm{Arctan}\left(\frac{|\mathrm{Im}(\mathfrak{t}(u))|}{\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u))}\right). \end{aligned} \tag{2.5}$$

Since tan is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, the number (2.5) is less than or equal to θ if and only if

$$\frac{|\mathrm{Im}(\mathfrak{t}(u))|}{\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u))} \leq \tan\theta.$$

The above inequality is true if and only if

$$|\mathrm{Im}(\mathfrak{t}(u))| \leq (\tan\theta)\,\mathrm{Re}(\mathfrak{t}(u) - \gamma\mathbf{1}(u)) = (\tan\theta)(\mathfrak{h}-\gamma)(u),$$

which is true by Prop. 2.0.22. Therefore: $|\mathrm{Arg}((\mathfrak{t}-\gamma)(u))| \leq \theta$. Now, (2.4) implies that

$$\begin{aligned} |(\mathfrak{t}-\gamma)(u)| &= \sec(|\mathrm{Arg}((\mathfrak{t}-\gamma)(u))|) \cdot \mathrm{Re}((\mathfrak{t}-\gamma)(u)) \\ &\leq (\sec\theta)(\mathfrak{h}-\gamma)(u), \end{aligned}$$

where the inequality holds because $|\mathrm{Arg}((\mathfrak{t}-\gamma)(u))| \leq \theta$ and sec is increasing on $[0, \frac{\pi}{2}[$. ■

2.1 Closed forms

The notion of a 'closed form' is central in the theory of sesquilinear forms in that it is one of the conditions in the socalled first representation theorem (Th. 3.0.31) that makes it possible to represent forms in terms of operators. In that connection, a new notion of convergence is useful. This notion will play a key role in defining closedness of a form.

Definition 2.1.1 ($\mathfrak{t}$-convergence) *Let $\mathfrak{t}$ be a sectorial form on $\mathcal{H}$. A sequence $\{u_n\} \subseteq \mathcal{H}$ is said to $\mathfrak{t}$-converge to $u \in \mathcal{H}$, denoted by $u_n \xrightarrow{\mathfrak{t}} u$, if*

(i) $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$

(ii) $u_n \to u \in \mathcal{H}$

(iii) $\mathfrak{t}(u_n - u_m) \to 0$, *for* $m, n \to \infty$.

The sequence $\{u_n\}$ is said to be $\mathfrak{t}$-convergent, if $\{u_n\}$ $\mathfrak{t}$-converges to some $u \in \mathcal{H}$.

Proposition 2.1.2 *Let $\mathfrak{t}$ be a sectorial form on $\mathcal{H}$, and $\{u_n\} \subseteq \mathcal{H}$; then*

$$\{u_n\} \text{ is } \mathfrak{t}\text{-convergent if and only if } \{u_n\} \text{ is } (\mathfrak{t}+\alpha)\text{-convergent for all } \alpha \in \mathbb{C}.$$

Proof: From the definition of $\mathfrak{t}$-convergence it is seen that it is enough to show that $\mathfrak{t}(u_n - u_m) \to 0$ if and only if $(\mathfrak{t}+\alpha)(u_n - u_m) \to 0$.

$\Leftarrow$: Assume that $\{u_n\}$ is $(\mathfrak{t}+\alpha)$-convergent for all $\alpha \in \mathbb{C}$. Then $\{u_n\}$ is also $(\mathfrak{t}+0)$-convergent.

$\Rightarrow$: Assume that $\{u_n\}$ is $\mathfrak{t}$-convergent, and let $\alpha \in \mathbb{C}$ be given; then $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$, $u_n \to u \in \mathcal{H}$ and $\mathfrak{t}(u_n - u_m) \to 0$ for $m, n \to \infty$. Since $u_n \to u$, $\{u_n\}$ is $\|\cdot\|$−Cauchy. Now,

$$(\mathfrak{t}+\alpha)(u_n - u_m) = \mathfrak{t}(u_n - u_m) + \alpha\|u_n - u_m\| \to 0, \quad \text{for } m, n \to \infty.$$

Consequently: $\{u_n\}$ is $(\mathfrak{t}+\alpha)$-convergent. ■

Corollary 2.1.3 *With $\mathfrak{t}$ as above, $u_n \xrightarrow{\mathfrak{t}} u$ if and only if $u_n \xrightarrow{\mathfrak{t}+\alpha} u$, for all $\alpha \in \mathbb{C}$.*

Proof: $\Leftarrow$: If $u_n \xrightarrow{\mathfrak{t}+\alpha} u$ for all $\alpha \in \mathbb{C}$, then also $u_n \xrightarrow{\mathfrak{t}+0} u$, or simply $u_n \xrightarrow{\mathfrak{t}} u$.

$\Rightarrow$: Given $\alpha \in \mathbb{C}$; if $u_n \xrightarrow{\mathfrak{t}} u$, then $\|u_n - u\| \to 0$, and by Prop. 2.1.2 $\{u_n\}$ is also $\mathfrak{t}+\alpha$-convergent, whence $u_n \xrightarrow{\mathfrak{t}+\alpha} x$ for some $x \in \mathcal{H}$. In particular: $\|u_n - x\| \to 0$. In normed spaces limits of sequences are unique, so $x = u$. Hence $u_n \xrightarrow{\mathfrak{t}+\alpha} u$. ■

Proposition 2.1.4 *Let $\mathfrak{t}$ be a sectorial form on $\mathcal{H}$, and $\{u_n\} \subseteq \mathcal{H}$; then*

$$\{u_n\} \text{ is } \mathfrak{t}\text{-convergent if and only if } \{u_n\} \text{ is } \mathfrak{h} := \operatorname{Re}\mathfrak{t}\text{-convergent.}$$

Proof: By Prop. 2.1.2, $\{u_n\}$ is $\mathfrak{t}$-convergent if and only $\{u_n\}$ is $(\mathfrak{t}-\gamma)$-convergent; and by Prop. 2.0.25, $\{u_n\}$ is $(\mathfrak{t}-\gamma)$-convergent if and only if $\{u_n\}$ is $(\mathfrak{h}-\gamma)$-convergent. Finally, again by Prop. 2.1.2, $\{u_n\}$ is $(\mathfrak{h}-\gamma)$-convergent if and only if $\{u_n\}$ is $\mathfrak{h}$-convergent. ■

Corollary 2.1.5 *With $\mathfrak{t}$ as above, $u_n \xrightarrow{\mathfrak{t}} u$ if and only if $u_n \xrightarrow{\operatorname{Re}\mathfrak{t}} u$.*

Proof: Similar to the proof of Cor. 2.1.3. ■

Proposition 2.1.6 *Let $\mathfrak{t}$ be a sectorial form on $\mathcal{H}$, and $\{u_n\} \subseteq \mathcal{H}$; then*

$$\{u_n\} \text{ is } \mathfrak{t}\text{-convergent if and only if } \{u_n\} \text{ is } \mathfrak{t}^*\text{-convergent.}$$

Proof: Since $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(\mathfrak{t}^*)$ by definition of $\mathfrak{t}^*$, we have $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$ if and only if $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t}^*)$. Since $\mathfrak{t}^*(u_n - u_m) = \mathfrak{t}^*(u_n - u_m, u_n - u_m) = \overline{\mathfrak{t}(u_n - u_m, u_n - u_m)} = \overline{\mathfrak{t}(u_n - u_m)}$ we have $\mathfrak{t}^*(u_n - u_m) \to 0$, for $m, n \to \infty$ if and only if $\mathfrak{t}(u_n - u_m) \to 0$, for $m, n \to \infty$. Consequently: $\{u_n\}$ is $\mathfrak{t}$-convergent if and only if $\{u_n\}$ is $\mathfrak{t}^*$-convergent. ■

Corollary 2.1.7 *With $\mathfrak{t}$ as above, $u_n \xrightarrow{\mathfrak{t}} u$ if and only if $u_n \xrightarrow{\mathfrak{t}^*} u$.*

Proof: Similar to the proof of Cor. 2.1.3 ■

Proposition 2.1.8 *Let* $\mathfrak{t}_1, \mathfrak{t}_2$ *be sectorial forms on* $\mathcal{H}$*. Then*

$$u_n \xrightarrow{\mathfrak{t}_1} u \text{ and } u_n \xrightarrow{\mathfrak{t}_2} u \text{ implies } u_n \xrightarrow{\mathfrak{t}_1+\mathfrak{t}_2} u.$$

Proof: Assume that $u_n \xrightarrow{\mathfrak{t}_i} u$, $i = 1, 2$; then $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t}_i)$, $\|u_n - u\| \to 0$ and $\mathfrak{t}_i(u_n - u_m) \to 0$, for $m, n \to \infty$. And then $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t}_1) \cap \mathcal{D}(\mathfrak{t}_2)$, $\|u_n - u\| \to 0$ and $(\mathfrak{t}_1 + \mathfrak{t}_2)(u_n - u_m) \to 0$, for $m, n \to \infty$, which means that $u_n \xrightarrow{\mathfrak{t}_1+\mathfrak{t}_2} u$. ■

Proposition 2.1.9 *Let* $\mathfrak{t}$ *be a sectorial form. Then*

$$\left[u_n \xrightarrow{\mathfrak{t}} u \ \wedge \ v_n \xrightarrow{\mathfrak{t}} v\right] \Longleftrightarrow \left[\forall \alpha, \beta \in \mathbb{C} : \alpha u_n + \beta v_n \xrightarrow{\mathfrak{t}} \alpha u + \beta v\right].$$

Proof: $\Leftarrow$: Put $\alpha = 0$ and $\beta = 1$, then $v_n \xrightarrow{\mathfrak{t}} v$; similarly, if $\alpha = 1$ and $\beta = 0$, then $u_n \xrightarrow{\mathfrak{t}} u$.

$\Rightarrow$: By Prop. 2.1.2 and Prop. 2.1.4, $\{u_n\}$ is $\mathfrak{t}$-convergent if and only if $\{u_n\}$ is $(\operatorname{Re}\mathfrak{t} - \gamma)$-convergent. It can therefore without loss of generality be assumed, that $\mathfrak{t}$ is symmetric and non-negative.

Given $\alpha, \beta \in \mathbb{C}$. Assume that $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$.

Then $\{u_n\}, \{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$, which implies that $\{\alpha u_n + \beta v_n\} \subseteq \mathcal{D}(\mathfrak{t})$, since $\mathcal{D}(\mathfrak{t})$ is vector space.

Also, $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$, which implies that $\|(\alpha u_n + \beta v_n) - (\alpha u + \beta v)\| \to 0$, since $\|(\alpha u_n + \beta v_n) - (\alpha u + \beta v)\| = \|\alpha(u_n - u) + \beta(v_n - v)\| \le |\alpha|\|u_n - u\| + |\beta|\|v_n - v\|$.

Furthermore, $\mathfrak{t}(u_n - u_m) \to 0$ and $\mathfrak{t}(v_n - v_m) \to 0$, which implies that $\mathfrak{t}(\alpha(u_n - u_m)) \to 0$, since $\mathfrak{t}(\alpha(u_n - u_m)) = |\alpha|^2\mathfrak{t}(u_n - u_m) \to 0$ and $\mathfrak{t}(\beta(u_n - u_m)) \to 0$, since $\mathfrak{t}(\beta(u_n - u_m)) = |\beta|^2\mathfrak{t}(u_n - u_m) \to 0$.

Now, Prop. 2.0.18 gives that

$$\begin{aligned} \mathfrak{t}((\alpha u_n + \beta v_n) - (\alpha u_m + \beta v_m)) &= \mathfrak{t}(\alpha(u_n - u_m) + \beta(v_n - v_m)) \\ &\le 2\mathfrak{t}(\alpha(u_n - u_m)) + 2\mathfrak{t}(\beta(v_n - v_m)). \end{aligned}$$

Consequently, $\alpha u_n + \beta v_n \xrightarrow{\mathfrak{t}} \alpha u + \beta v$. ■

Definition 2.1.10 (Closed form) *A sectorial form* $\mathfrak{t}$ *is said to be closed, if*

$$u_n \xrightarrow{\mathfrak{t}} u \quad \text{implies} \quad u \in \mathcal{D}(\mathfrak{t}), \quad \mathfrak{t}(u_n - u) \to 0.$$

Proposition 2.1.11 *Let* $\mathfrak{t}$ *be a sectorial form. Then the following statements are equivalent:*

(i) $\mathfrak{t}$ *is closed*

(ii) $\mathfrak{t}^*$ *is closed*

(iii) $\operatorname{Re}\mathfrak{t}$ *is closed*

(iv) $\mathfrak{t} + \alpha$ *is closed for all* $\alpha \in \mathbb{C}$.

Proof: *(i)* ⇔ *(ii)* : Follows from $u_n \xrightarrow{\mathfrak{t}} u \Leftrightarrow u_n \xrightarrow{\mathfrak{t}^*} u$, which is true by Cor. 2.1.7.
(i) ⇔ *(iii)* : Follows from $u_n \xrightarrow{\mathfrak{t}} u \Leftrightarrow u_n \xrightarrow{\operatorname{Re}\mathfrak{t}} u$, which is true by Cor. 2.1.5.
(i) ⇔ *(iv)* : Follows from $u_n \xrightarrow{\mathfrak{t}} u \Leftrightarrow u_n \xrightarrow{\mathfrak{t}+\alpha} u$, which is true by Cor. 2.1.3. ■

Definition 2.1.12 *Let $\mathfrak{h}$ be a symmetric, non-negative form on $\mathcal{H}$. Define*

$$\begin{cases} (\cdot,\cdot)_{\mathfrak{h}} : \mathcal{D}(\mathfrak{h}) \times \mathcal{D}(\mathfrak{h}) \to \mathbb{C} \\ (u,v)_{\mathfrak{h}} = (\mathfrak{h} + \mathbf{1})(u,v) = \mathfrak{h}(u,v) + (u|v) \end{cases}.$$

It is trivial to verify, that $(\cdot,\cdot)_{\mathfrak{h}}$ is an inner product on $\mathcal{D}(\mathfrak{h})$. Consequently $\mathcal{H}_{\mathfrak{h}} := (\mathcal{D}(\mathfrak{h}), (\cdot,\cdot)_{\mathfrak{h}})$ is an inner product space. The inner product $(\cdot,\cdot)_{\mathfrak{h}}$ induces a norm on $\mathcal{D}(\mathfrak{h})$ given by

$$\|u\|_{\mathfrak{h}} = (u,u)_{\mathfrak{h}}^{\frac{1}{2}} = (\mathfrak{h}(u) + \|u\|^2)^{\frac{1}{2}}, \quad u \in \mathcal{D}(\mathfrak{h}).$$

For a sectorial form $\mathfrak{t}$, define $\mathcal{H}_{\mathfrak{t}} := \mathcal{H}_{\mathfrak{h}'}$, where $\mathfrak{h}' = \operatorname{Re}\mathfrak{t} - \gamma$, and γ is a vertex of $\mathfrak{t}$. Let γ_1 and γ_2 be two different vertices of $\mathfrak{t}$. Then $(\mathcal{D}(\mathfrak{t}), (\cdot,\cdot)_{\operatorname{Re}\mathfrak{t}-\gamma_1})$ and $(\mathcal{D}(\mathfrak{t}), (\cdot,\cdot)_{\operatorname{Re}\mathfrak{t}-\gamma_2})$ are equal considered as vector spaces. But the inner products are different. The induced norms $\|\cdot\|_{\operatorname{Re}\mathfrak{t}-\gamma_1}$ and $\|\cdot\|_{\operatorname{Re}\mathfrak{t}-\gamma_2}$ are equivalent though. This will not be proven.

Notation 2.1.13 *Since the inner product on $\mathcal{H}$ is denoted by $(\cdot|\cdot)$, the inner product $(\cdot,\cdot)_{\mathfrak{h}}$ will accordingly be denoted by $(\cdot|\cdot)_{\mathfrak{h}}$ in the future.*

Proposition 2.1.14 *Let $\mathfrak{t}$ be a sectorial form, and $\{u_n\} \subseteq \mathcal{H}$. Then*

$$\{u_n\} \textit{ is } \mathfrak{t}\textit{-convergent if and only if } \{u_n\} \textit{ is Cauchy in } \mathcal{H}_{\mathfrak{t}}.$$

Proof: ⇒: Assume that $\{u_n\}$ is $\mathfrak{t}$-convergent. Then $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$, $\exists u \in \mathcal{H} : \lim_{n\to\infty} \|u_n - u\| = 0$, and $\lim_{m,n\to\infty} \mathfrak{t}(u_n - u_m) = 0$. Then, since

$$\begin{aligned} & (\operatorname{Re}\mathfrak{t})(u_n - u_m) = \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*)(u_n - u_m) \\ = \; & \frac{1}{2}\mathfrak{t}(u_n - u_m) + \frac{1}{2}\mathfrak{t}^*(u_n - u_m) = \frac{1}{2}(\mathfrak{t}(u_n - u_m) + \overline{\mathfrak{t}(u_n - u_m)}) \\ = \; & \operatorname{Re}(\mathfrak{t}(u_n - u_m)), \end{aligned}$$

we also have that $\lim_{m,n\to\infty} (\operatorname{Re}\mathfrak{t})(u_n - u_m) = 0$. Also, since $\{u_n\}$ is $\mathfrak{t}$-convergent, $\{u_n\}$ is $\|\cdot\|$-convergent (by definition of $\mathfrak{t}$-convergence), in particular: $\lim_{m,n\to\infty} \|u_n - u_m\| = 0$. Now,

$$\|u_n - u_m\|_{\mathfrak{t}}^2 = \|u_n - u_m\|_{\operatorname{Re}\mathfrak{t}-\gamma}^2 = (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u_m) + \|u_n - u_m\|^2,$$

which implies that

$$\begin{aligned} \lim_{m,n\to\infty} \|u_n - u_m\|_{\mathfrak{t}}^2 &= \lim_{m,n\to\infty} \left\{ (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u_m) + \|u_n - u_m\|^2 \right\} \\ &= \lim_{m,n\to\infty} (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u_m) + \lim_{m,n\to\infty} \|u_n - u_m\|^2 \\ &= 0, \end{aligned}$$

so $\{u_n\}$ is Cauchy in $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_\mathfrak{t}) = \mathcal{H}_\mathfrak{t}$.

$\Leftarrow$: Assume that $\{u_n\}$ is Cauchy in $\mathcal{H}_\mathfrak{t}$. Now $\|u_n - u_m\|_\mathfrak{t}^2 = (\operatorname{Re}\mathfrak{t} - \gamma) + \|u_n - u_m\|^2$, and since $\|u_n - u_m\|_\mathfrak{t}^2 \to 0$, and both $(\operatorname{Re}\mathfrak{t} - \gamma)(\cdot) \geq 0$ and $\|\cdot\| \geq 0$, we have that $\|u_n - u_m\| \to 0$. And since $\mathcal{H}$ is complete, there is a $u \in \mathcal{H} : \|u_n - u\| \to 0$.

The only thing left is to show that $\mathfrak{t}(u_n - u_m) \to 0$. But since $\lim_{m,n\to\infty} \|u_n - u_m\|_\mathfrak{t}^2 = 0$, and

$$\begin{aligned} \|u_n - u_m\|_\mathfrak{t}^2 &= (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u_m) + \|u_n - u_m\|^2 \\ &= (\operatorname{Re}\mathfrak{t})(u_n - u_m) - \gamma\|u_n - u_m\|^2 + \|u_n - u_m\|^2, \end{aligned}$$

and $(\operatorname{Re}\mathfrak{t})(\cdot) \geq \gamma\|\cdot\|^2$, we have that $\lim_{m,n\to\infty}(\operatorname{Re}\mathfrak{t})(u_n - u_m) = 0$. From Prop. 2.0.24 we know that

$$|(\operatorname{Im}\mathfrak{t})(u_n - u_m)| \leq (\tan\theta)(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u_m),$$

so $\lim_{m,n\to\infty}(\operatorname{Im}\mathfrak{t})(u_n - u_m) = 0$, and then

$$\begin{aligned} \lim_{m,n\to\infty} \mathfrak{t}(u_n - u_m) &= \lim_{m,n\to\infty}(\operatorname{Re}\mathfrak{t} + i\operatorname{Im}\mathfrak{t})(u_n - u_m) \\ &= \lim_{m,n\to\infty}(\operatorname{Re}\mathfrak{t})(u_n - u_m) + i\lim_{m,n\to\infty}(\operatorname{Im}\mathfrak{t})(u_n - u_m) \\ &= 0. \end{aligned}$$

Consequently: $\{u_n\}$ is $\mathfrak{t}$-convergent. ■

Proposition 2.1.15 *Let $\mathfrak{t}$ be a sectorial form. If $\mathfrak{t}$ has a closed extension $\mathfrak{s}$, then*

$$\forall\{u_n\} \subseteq \mathcal{H}\ \forall u \in \mathcal{D}(\mathfrak{t}) : \left(u_n \xrightarrow{\mathfrak{t}} u \Longleftrightarrow \|u_n - u\|_\mathfrak{t} \to 0\right). \tag{2.6}$$

Proof: $\Rightarrow$: Assume that $\mathfrak{t}$ is sectorial, has a closed extension $\mathfrak{s}$, that $u \in \mathcal{D}(\mathfrak{t})$ and that $\{u_n\} \subseteq \mathcal{H}$ fulfills $u_n \xrightarrow{\mathfrak{t}} u$. We want to show that $\|u_n - u\|_\mathfrak{t} \to 0$. Now, since $\mathfrak{s} \supseteq \mathfrak{t}$, we have $\{u_n\} \subseteq \mathcal{D}(\mathfrak{s})$, $\|u_n - u\| \to 0$ and $\mathfrak{s}(u_n - u_m) \to 0$, for $m, n \to \infty$, so $u_n \xrightarrow{\mathfrak{s}} u$. And $\mathfrak{s}$ is closed, so $u \in \mathcal{D}(\mathfrak{s})$, $\mathfrak{s}(u_n - u) \to 0$, and $\mathfrak{t}(u_n - u) = \mathfrak{s}(u_n - u)$, so also $\mathfrak{t}(u_n - u) \to 0$; but then

$$\|u_n - u\|_\mathfrak{t}^2 = (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) + \|u_n - u\|^2 \to 0.$$

$\Leftarrow$: Assume that $\mathfrak{t}$ is sectorial, that $u \in \mathcal{D}(\mathfrak{t})$ and that $\{u_n\} \subseteq \mathcal{H}$ fulfills $\|u_n - u\|_\mathfrak{t} \to 0$. Then $\|u_n - u\|_\mathfrak{t}^2 = (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) + \|u_n - u\|^2 \to 0$. In particular: $u_n - u \in \mathcal{D}(\operatorname{Re}\mathfrak{t} - \gamma) = \mathcal{D}(\operatorname{Re}\mathfrak{t}) \cap \mathcal{D}(\gamma\mathbf{1}) = \mathcal{D}(\mathfrak{t})$, and since $u \in \mathcal{D}(\mathfrak{t})$ by assumption, then $(u_n - u) + u = u_n \in \mathcal{D}(\mathfrak{t})$. Therefore: $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$. Put $\mathfrak{h}' = \operatorname{Re}\mathfrak{t} - \gamma$. The triangle inequality (Cor. 2.0.16) implies that

$$\mathfrak{h}'(u_n - u_m)^{\frac{1}{2}} = \mathfrak{h}'(u_n - u + (u - u_m))^{\frac{1}{2}} \leq \mathfrak{h}'(u_n - u)^{\frac{1}{2}} + \mathfrak{h}'(u - u_m)^{\frac{1}{2}},$$

which implies

$$\mathfrak{h}'(u_n - u_m) \leq \mathfrak{h}'(u_n - u) + \mathfrak{h}'(u_n - u_m) + 2\mathfrak{h}'(u_n - u)^{\frac{1}{2}}\mathfrak{h}'(u - u_m)^{\frac{1}{2}}. \tag{2.7}$$

Now, by assumption

$$\begin{aligned} \lim_{n\to\infty} \|u_n - u\|_\mathfrak{t}^2 &= \lim_{n\to\infty}\left\{(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) + \|u_n - u\|^2\right\} \\ &= \lim_{n\to\infty}\left\{\mathfrak{h}'(u_n - u) + \|u_n - u\|^2\right\} \\ &= 0, \end{aligned}$$

which implies that $\lim_{n\to\infty} \mathfrak{h}'(u_n - u) = 0$. This together with (2.7) gives that $\lim_{m,n\to\infty} \mathfrak{h}'(u_n - u_m) = 0$. Prop. 2.0.24 now gives that

$$|(\operatorname{Im}\mathfrak{t})(u_n - u_m)| \leq (\tan\theta)\mathfrak{h}'(u_n - u_m).$$

Furthermore,

$$\begin{aligned}
\lim_{m,n\to\infty} (\operatorname{Re}\mathfrak{t})(u_n - u_m) &= \lim_{m,n\to\infty} \left\{\mathfrak{h}'(u_n - u_m) + \gamma\|u_n - u_m\|^2\right\} \\
&= \lim_{m,n\to\infty} \mathfrak{h}'(u_n - u_m) = 0,
\end{aligned}$$

so finally,

$$\begin{aligned}
\lim_{m,n\to\infty} \mathfrak{t}(u_n - u_m) &= \lim_{m,n\to\infty} (\operatorname{Re}\mathfrak{t} + i\operatorname{Im}\mathfrak{t})(u_n - u_m) \\
&= \lim_{m,n\to\infty} (\operatorname{Re}\mathfrak{t})(u_n - u_m) + i \lim_{m,n\to\infty} (\operatorname{Im}\mathfrak{t})(u_n - u_m) \\
&= 0.
\end{aligned}$$

■

Note that the assumption that $\mathfrak{t}$ has a closed extension was only needed in the first implication.

Proposition 2.1.16 *Let $\mathfrak{t}$ be a sectorial form, with vertex γ. Then $\mathfrak{t}$ is bounded on $\mathcal{H}_{\mathfrak{t}}$.*

Proof: Putting $\mathfrak{s} = \operatorname{Re}\mathfrak{t} - \gamma$, we get

$$\begin{aligned}
|\mathfrak{t}(u,v)| &\leq |\gamma(u|v)| + |(\mathfrak{t} - \gamma)(u,v)| \\
&\leq |\gamma||(u|v)| + (1 + \tan\theta)\mathfrak{s}(u)^{\frac{1}{2}}\mathfrak{s}(v)^{\frac{1}{2}}, \quad \text{by Prop. 2.0.24}(iii) \\
&\leq |\gamma|\|u\|\|v\| + (1 + \tan\theta)\|u\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}} \\
&= |\gamma|\|u\|\|v\| + (1 + \tan\theta)\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} \\
&\leq |\gamma|\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} + (1 + \tan\theta)\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} \\
&= (|\gamma| + 1 + \tan\theta)\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}}.
\end{aligned}$$

Consequently, $\mathfrak{t}$ is bounded on $\mathcal{H}_{\mathfrak{t}}$. ■

Proposition 2.1.17 *Let $\mathfrak{t}$ be a sectorial form, with vertex γ. Then*

$$\mathfrak{t} \textit{ is closed} \Longleftrightarrow \mathcal{H}_{\mathfrak{t}} \textit{ is complete.}$$

Proof: $\Rightarrow$: Let $\mathfrak{t}$ be a sectorial and closed form. Take a Cauchy sequence $\{u_n\} \subseteq \mathcal{H}_{\mathfrak{t}}$. We want to prove that $\|\cdot\|_{\mathfrak{t}} - \lim_{n\to\infty} u_n \in \mathcal{H}_{\mathfrak{t}}$. Since $\{u_n\} \subseteq \mathcal{H}_{\mathfrak{t}}$ is $\|\cdot\|_{\mathfrak{t}}$–Cauchy, we have that there is a $u \in \mathcal{H}$ such that $u_n \xrightarrow{\mathfrak{t}} u$, by Prop. 2.1.14. But $\mathfrak{t}$ was closed, so $u \in \mathcal{D}(\mathfrak{t})$, and $\mathfrak{t}(u_n - u) \to 0$. Now, $\|u_n - u\|_{\mathfrak{t}} := (\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) + \|u_n - u\|^2 \to 0$ if and only if both $(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) \to 0$ and $\|u_n - u\|^2 \to 0$. But $\|u_n - u\| \to 0$, since $u_n \xrightarrow{\mathfrak{t}} u$, and $(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) \to 0$, since $\mathfrak{t}(u_n - u) \to 0$. Therefore $\|u_n - u\|_{\mathfrak{t}} \to 0$, whence $\mathcal{H}_{\mathfrak{t}}$ is complete.

$\Leftarrow$: Assume that $\mathfrak{t}$ is sectorial and that $\mathcal{H}_{\mathfrak{t}}$ is complete. Let there be given a sequence $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_n \xrightarrow{\mathfrak{t}} u$. We want to show that $u \in \mathcal{D}(\mathfrak{t})$ and $\mathfrak{t}(u_n - u) \to 0$. Now, since $u_n \xrightarrow{\mathfrak{t}} u$, we have that $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$, $\|u_n - u\| \to 0$ and $\mathfrak{t}(u_n - u_m) \to 0$, for $m, n \to \infty$; and by Prop. 2.1.14 $\{u_n\}$ is Cauchy in $\mathcal{H}_{\mathfrak{t}}$. The space $\mathcal{H}_{\mathfrak{t}}$ is complete, so $\exists x \in \mathcal{H}_{\mathfrak{t}} : \lim_{n\to\infty} \|u_n - x\|_{\mathfrak{t}} = 0$. But now, $\lim_{n\to\infty}\{(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - x) + \|u_n - x\|^2\} = 0$, which implies $\|u_n - x\| \to 0$ and $(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - x) \to 0$, and also $(\operatorname{Re}\mathfrak{t})(u_n - x) \to 0$. And then $\mathfrak{t}(u_n - x) \to 0$, since $|(\operatorname{Im}\mathfrak{t})(u_n - x)| \leq (\tan\theta)(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - x)$. Since both $\|u_n - x\| \to 0$, $\|u_n - u\| \to 0$ and that limits of sequences are unique in normed spaces, then $x = u$. And then $u \in \mathcal{D}(\mathfrak{t})$, and $\mathfrak{t}(u_n - u) \to 0$, whence $\mathfrak{t}$ is closed. ■

Proposition 2.1.18 *In a normed space, $u_n \to u$ implies that $\|u_n\| \to \|u\|$.*

Proof: Using the substitution $u' = u_n - u$, the triangle inequality reads

$$\|u' + u\| = \|u_n\| \leq \|u'\| + \|u\| = \|u_n - u\| + \|u\|.$$

This inequality implies $\|u_n\| - \|u\| \leq \|u_n - u\|$, and interchanging u and u_n, we also get that $\|u\| - \|u_n\| \leq \|u_n - u\|$. The last two inequalities give that

$$|\|u_n\| - \|u\|| \leq \|u_n - u\|.$$

Consequently, $u_n \to u$ implies $\|u_n\| \to \|u\|$. ■

Note that the proposition can also be stated as "In a normed space, the norm is sequentially continuous" (in fact uniformly continuous).

Proposition 2.1.19 *In an inner product space $\mathcal{H}$, if $\|\cdot\| - \lim_{n\to\infty} u_n$ and $\|\cdot\| - \lim_{n\to\infty} v_n$ exist, then $\lim_{n\to\infty}(u_n|v_n)$ also exists.*

Proof: Write:

$$\begin{aligned} (u_n|v_n) &= (u_n - u_m|v_n) + (u_m|v_n) \\ (u_m|v_m) &= (u_m|v_m - v_n) + (u_m|v_n). \end{aligned}$$

Now,

$$\begin{aligned} |(u_n|v_n) - (u_m|v_m)| &= |(u_n - u_m|v_n) - (u_m|v_m - v_n)| \\ &= |(u_n - u_m|v_n) + (u_m|v_n - v_m)| \\ &\leq |(u_n - u_m|v_n)| + |(u_m|v_n - v_m)| \\ &\leq \|u_n - u_m\|\|v_m\| + \|u_m\|\|v_n - v_m\| \end{aligned} \tag{2.8}$$

Since $\|\cdot\| - \lim_{n\to\infty} u_n$ and $\|\cdot\| - \lim_{n\to\infty} v_n$ exist by assumption, we have $u_n \to u$ and $v_n \to v$ for some $u, v \in \mathcal{H}$. By Prop. 2.1.18, $\|u_n\| \to \|u\|$ and $\|v_n\| \to \|v\|$. Also $\{u_n\}, \{v_n\}$ are $\|\cdot\|-$Cauchy. The expression in (2.8) now goes to zero, and thus $\{(u_n|v_n)\}$ is Cauchy, hence convergent by completeness of $\mathbb{C}$. ■

Proposition 2.1.20 *Let $\mathfrak{t}$ be a sectorial form. If $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$, then*

$$\lim_{n\to\infty} \mathfrak{t}(u_n, v_n) \text{ exists if and ony if } \left[\forall\alpha \in \mathbb{C} : \lim_{n\to\infty} (\mathfrak{t} + \alpha)(u_n, v_n) \text{ exists}\right].$$

Proof: $\Leftarrow$: Putting $\alpha = 0$ yields the desired.

$\Rightarrow$: The $\mathfrak{t}$-convergence of $\{u_n\}$ and $\{v_n\}$ imply that $\|\cdot\| - \lim_{n\to\infty} u_n$ and $\|\cdot\| - \lim_{n\to\infty} v_n$ exist. By Prop. 2.1.19, $\lim_{n\to\infty}(u_n|v_n)$ also exists. Given $\alpha \in \mathbb{C}$ we get

$$(\mathfrak{t}+\alpha)(u_n, v_n) = \mathfrak{t}(u_n, v_n) + \alpha(u_n|v_n),$$

and we see that $\lim_{n\to\infty} \mathfrak{t}(u_n, v_n)$ exists only if $\lim_{n\to\infty}(\mathfrak{t}+\alpha)(u_n, v_n)$ does. ■

Proposition 2.1.21 *Let $\mathfrak{t}$ be a sectorial form. If $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$, then $\lim_{n\to\infty} \mathfrak{t}(u_n, v_n)$ exists. If, in addition $\mathfrak{t}$ has a closed extension $\mathfrak{s}$ and $u, v \in \mathcal{D}(\mathfrak{t})$, then this limit is equal to $\mathfrak{t}(u, v)$.*

Proof: Let $\mathfrak{t}$ be a sectorial form with vertex γ, and assume that $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$. By Prop. 2.1.20, $\lim_{n\to\infty}(\mathfrak{t}-\gamma)(u_n, v_n)$ exists if and only if $\lim_{n\to\infty} \mathfrak{t}(u_n, v_n)$ exists. It can therefore without loss of generality be assumed that $\mathfrak{t} \geq 0$. Write

$$\begin{aligned}
\mathfrak{t}(u_n, v_n) &= \mathfrak{t}(u_n - u_m, v_n) + \mathfrak{t}(u_m, v_n) \\
\mathfrak{t}(u_m, v_m) &= \mathfrak{t}(u_m, v_m - v_n) + \mathfrak{t}(u_m, v_n).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathfrak{t}(u_n, v_n) - \mathfrak{t}(u_m, v_m) &= \mathfrak{t}(u_n - u_m, v_n) - \mathfrak{t}(u_m, v_m - v_n) \\
&= \mathfrak{t}(u_n - u_m, v_n) + \mathfrak{t}(u_m, v_n - v_m),
\end{aligned}$$

so putting $\mathfrak{h} = \operatorname{Re}\mathfrak{t}$ we get from Prop. 2.0.24(*iii*):

$$\begin{aligned}
|\mathfrak{t}(u_n, v_n) - \mathfrak{t}(u_m, v_m)| &\leq |\mathfrak{t}(u_n - u_m, v_n)| + |\mathfrak{t}(u_m, v_n - v_m)| \\
&\leq (1+\tan\theta)\left\{\mathfrak{h}(u_n - u_m)^{\frac{1}{2}}\mathfrak{h}(v_n)^{\frac{1}{2}} + \mathfrak{h}(u_m)^{\frac{1}{2}}\mathfrak{h}(v_n - v_m)^{\frac{1}{2}}\right\}
\end{aligned} \tag{2.9}$$

We know from Prop. 2.1.4 that $\mathfrak{t}$-convergence is equivalent to $\mathfrak{h}$-convergence, so therefore $\mathfrak{h}(u_n - u_m) \to 0$ and $\mathfrak{h}(v_n - v_m) \to 0$ as $m, n \to \infty$.

Furthermore,

$$\begin{aligned}
\mathfrak{h}(u_n - u_m) &= \mathfrak{h}(u_n) + \mathfrak{h}(u_n, -u_m) + \mathfrak{h}(-u_m, u_n) + \mathfrak{h}(u_m) \\
&= \mathfrak{h}(u_n) + \mathfrak{h}(u_n, -u_m) + \overline{\mathfrak{h}(u_n, -u_m)} + \mathfrak{h}(u_m), \qquad \text{since } \mathfrak{h} = \mathfrak{h}^* \\
&= \mathfrak{h}(u_n) + 2(\operatorname{Re}\mathfrak{h})(u_n, -u_m) + \mathfrak{h}(u_m) \qquad (2.10) \\
&\to 0,
\end{aligned}$$

where the three terms in (2.10) are all non-negative. Therefore $\mathfrak{h}(u_m) \to 0$, in particular: $\{\mathfrak{h}(u_m)\}$ is bounded. Similarly, $\{\mathfrak{h}(v_n)\}$ is bounded. From this we infer, that the right-hand side of (2.9) goes to zero. Consequently, $\{\mathfrak{t}(u_n, v_n)\}$ is Cauchy, hence convergent.

Suppose in addition, that $u, v \in \mathcal{D}(\mathfrak{t})$ and that $\mathfrak{t}$ has a closed extension $\mathfrak{s}$. Write

$$\begin{aligned}
\mathfrak{t}(u_n, v_n) &= \mathfrak{t}(u_n - u, v_n) + \mathfrak{t}(u, v_n) \\
\mathfrak{t}(u, v) &= \mathfrak{t}(u, v - v_n) + \mathfrak{t}(u, v_n).
\end{aligned}$$

Now,

$$\begin{aligned}\mathfrak{t}(u_n,v_n)-\mathfrak{t}(u,v) &= \mathfrak{t}(u_n-u,v_n)-\mathfrak{t}(u,v-v_n)\\ &= \mathfrak{t}(u_n-u,v_n)+\mathfrak{t}(u,v_n-v)\\ &= \mathfrak{s}(u_n-u,v_n)+\mathfrak{s}(u,v_n-v), \quad \text{since } \mathfrak{s}\supseteq\mathfrak{t},\end{aligned}$$

so with $\mathfrak{h}=\operatorname{Re}\mathfrak{s}$ one has

$$\begin{aligned}|\mathfrak{t}(u_n,v_n)-\mathfrak{t}(u,v)| &\le |\mathfrak{s}(u_n-u,v_n)|+|\mathfrak{s}(u,v_n-v)|\\ &\le (1+\tan\theta)\left\{\mathfrak{h}(u_n-u)^{\frac{1}{2}}\mathfrak{h}(v_n)^{\frac{1}{2}}+\mathfrak{h}(u)^{\frac{1}{2}}\mathfrak{h}(v_n-v)^{\frac{1}{2}}\right\}. \qquad (2.11)\end{aligned}$$

Now, since $\mathfrak{s}$ is closed, $\mathfrak{h}$ is also closed by Prop. 2.1.11; then $\mathfrak{h}(u_n-u)\to 0$, $\mathfrak{h}(v_n-v)\to 0$, $\{\mathfrak{h}(v_n)\}$ is bounded and $\mathfrak{h}(u)\ge 0$. Therefore the right-hand side of (2.11) goes to zero, so

$$\lim_{n\to\infty}\mathfrak{t}(u_n,v_n)=\mathfrak{t}(u,v).$$

■

2.2 Closable forms

A slightly more general notion than that of a 'closed form' is the notion of a 'closable form'. Closable forms permit closed extensions, which is important in the construction of the Friedrichs extension in chapter 4.

Definition 2.2.1 *Let $\mathfrak{t}$ be a sectorial form. Then $\mathfrak{t}$ is said to be closable, if there exists a closed extension of $\mathfrak{t}$.*

Definition 2.2.2 *For a closable form $\mathfrak{t}$, the closure $\tilde{\mathfrak{t}}$ of $\mathfrak{t}$ is defined as follows:*

$$\begin{cases}\mathcal{D}(\tilde{\mathfrak{t}})=\{u\in\mathcal{H}\mid \exists\{u_n\}\subseteq\mathcal{D}(\mathfrak{t}): u_n\xrightarrow{\mathfrak{t}}u\}\\ \tilde{\mathfrak{t}}(u,v)=\lim_{n\to\infty}\mathfrak{t}(u_n,v_n), \quad \textit{for some } u_n\xrightarrow{\mathfrak{t}}u,\ v_n\xrightarrow{\mathfrak{t}}v\end{cases}.$$

Proposition 2.2.3 *$\tilde{\mathfrak{t}}$ is a well-defined sesquilinear form.*

Proof: *$\mathcal{D}(\tilde{\mathfrak{t}})$ is a subspace of $\mathcal{H}$* : Given $u,v\in\mathcal{D}(\tilde{\mathfrak{t}})$ and $\alpha,\beta\in\mathbb{C}$. Since $u,v\in\mathcal{D}(\tilde{\mathfrak{t}})$, $u_n\xrightarrow{\mathfrak{t}}u$, $v_n\xrightarrow{\mathfrak{t}}v$ for some $\{u_n\},\{v_n\}\subseteq\mathcal{D}(\mathfrak{t})$. $\mathcal{D}(\mathfrak{t})$ is a subspace, so $\alpha u_n+\beta v_n\in\mathcal{D}(\mathfrak{t})$, and then by Prop. 2.1.9, $\alpha u_n+\beta v_n\xrightarrow{\mathfrak{t}}\alpha u+\beta v$. By definition of $\mathcal{D}(\tilde{\mathfrak{t}})$, $\alpha u+\beta v\in\mathcal{D}(\tilde{\mathfrak{t}})$.

$\tilde{\mathfrak{t}}$ is sesquilinear : Given $u',u'',v\in\mathcal{D}(\tilde{\mathfrak{t}})$ there are sequences $\{u'_n\},\{u''_n\},\{v_n\}\subseteq\mathcal{D}(\mathfrak{t})$ such that $u'_n\xrightarrow{\mathfrak{t}}u'$, $u''_n\xrightarrow{\mathfrak{t}}u''$ and $v_n\xrightarrow{\mathfrak{t}}v$; then by Prop. 2.1.21 $\lim_{n\to\infty}\mathfrak{t}(u'_n,v_n)$ and $\lim_{n\to\infty}\mathfrak{t}(u''_n,v_n)$ exist, and

$$\begin{aligned}&\tilde{\mathfrak{t}}(u',v)+\tilde{\mathfrak{t}}(u'',v)=\lim_{n\to\infty}\mathfrak{t}(u'_n,v_n)+\lim_{n\to\infty}\mathfrak{t}(u''_n,v_n)\\ &= \lim_{n\to\infty}\{\mathfrak{t}(u'_n,v_n)+\mathfrak{t}(u''_n,v_n)\}\\ &= \lim_{n\to\infty}\mathfrak{t}(u'_n+u''_n,v_n), \quad \text{by sesquilinearity of } \mathfrak{t}.\end{aligned}$$

By Prop. 2.1.9, $u_n' \xrightarrow{\mathfrak{t}} u'$ and $u_n'' \xrightarrow{\mathfrak{t}} u''$ implies $u_n' + u_n'' \xrightarrow{\mathfrak{t}} u' + u''$, so $u' + u'' \in \mathcal{D}(\tilde{\mathfrak{t}})$, and then by Prop. 2.1.21,

$$\tilde{\mathfrak{t}}(u' + u'', v) = \lim_{n\to\infty} \mathfrak{t}(u_n' + u_n'', v_n) = \tilde{\mathfrak{t}}(u', v) + \tilde{\mathfrak{t}}(u'', v).$$

This shows that $\tilde{\mathfrak{t}}$ is additive in the first variable. Homogeneity in the first variable and conjugate linarity in the second variable can be shown in a similar fashion.

$\tilde{\mathfrak{t}}$ *is well-defined*: It must be shown that $\lim_{n\to\infty} \mathfrak{t}(u_n, v_n)$ depends only on u and v and not on any choice of $\{u_n\}$ and $\{v_n\}$. So assume that $u_n \xrightarrow{\mathfrak{t}} u$, $u_n' \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$ $v_n' \xrightarrow{\mathfrak{t}} v$. By Prop. 2.1.21 the limits of $\{\mathfrak{t}(u_n, v_n)\}$ and $\{\mathfrak{t}(u_n', v_n')\}$ both exist. Next, we show that they coincide. Since $\mathfrak{t}$ is closable, there exists a closed sesquilinear form $\mathfrak{s}$ that extends $\mathfrak{t}$. Then $u_n \xrightarrow{\mathfrak{t}} u$ implies $u_n \xrightarrow{\mathfrak{s}} u$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \mathfrak{t}(u_n, v_n) &= \lim_{n\to\infty} \mathfrak{s}(u_n, v_n), \quad \text{since } \mathfrak{s} \supseteq \mathfrak{t} \\
&= \mathfrak{s}(u, v), \quad \text{by Prop. 2.1.21, since } \mathfrak{s} \text{ is closed} \\
&= \lim_{n\to\infty} \mathfrak{s}(u_n', v_n'), \quad \text{since } \mathfrak{s} \text{ is closed, } u_n' \xrightarrow{\mathfrak{s}} u, \text{ and } v_n' \xrightarrow{\mathfrak{s}} v \\
&= \lim_{n\to\infty} \mathfrak{t}(u_n', v_n').
\end{aligned}$$

■

Proposition 2.2.4 $\tilde{\mathfrak{t}}$ *extends* $\mathfrak{t}$.

Proof: Given $u, v \in \mathcal{D}(\mathfrak{t})$, then take $u_n = u$ and $v_n = v$ for all $n \in \mathbb{N}$. Then $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$, so $u, v \in \mathcal{D}(\tilde{\mathfrak{t}})$, and $\tilde{\mathfrak{t}}(u, v) = \lim_{n\to\infty} \mathfrak{t}(u_n, v_n) = \mathfrak{t}(u, v)$. ■

Proposition 2.2.5 *If* $\mathfrak{t}$ *is closable, then any closed extension of* $\mathfrak{t}$ *also extends* $\tilde{\mathfrak{t}}$.

Proof: Assume that $\mathfrak{s}$ is a closed extension of $\mathfrak{t}$. Given $u \in \mathcal{D}(\tilde{\mathfrak{t}})$; then there is a sequence $\{u_n\} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_n \xrightarrow{\mathfrak{t}} u$, but then $u_n \xrightarrow{\mathfrak{s}} u$, since $\mathfrak{s}$ extends $\mathfrak{t}$. Since $\mathfrak{s}$ is closed, we have that $u \in \mathcal{D}(\mathfrak{s})$. Therefore, $\mathcal{D}(\tilde{\mathfrak{t}}) \subseteq \mathcal{D}(\mathfrak{s})$. Now, let $u, v \in \mathcal{D}(\tilde{\mathfrak{t}})$; then there are sequences $\{u_n\}, \{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_n \xrightarrow{\mathfrak{t}} u$ and $v_n \xrightarrow{\mathfrak{t}} v$. Then,

$$\begin{aligned}
\tilde{\mathfrak{t}}(u, v) &= \lim_{n\to\infty} \mathfrak{t}(u_n, v_n) = \lim_{n\to\infty} \mathfrak{s}(u_n, v_n), \quad \text{since } \mathfrak{s} \supseteq \mathfrak{t} \\
&= \mathfrak{s}(u, v), \quad \text{by Prop. 2.1.21, since } \mathfrak{s} \text{ is closed.}
\end{aligned}$$

Consequently, $\tilde{\mathfrak{t}} \subseteq \mathfrak{s}$. ■

Proposition 2.2.6 *Let* $\mathfrak{t}$ *be a closable sectorial form. Then* $\Theta(\mathfrak{t})$ *is a dense subset of* $\Theta(\tilde{\mathfrak{t}})$*, i.e.* $\Theta(\mathfrak{t}) \subseteq \Theta(\tilde{\mathfrak{t}})$ *and* $\overline{\Theta(\mathfrak{t})} \supseteq \Theta(\tilde{\mathfrak{t}})$.

Proof: If $x \in \Theta(\mathfrak{t})$, then $x = \mathfrak{t}(u)$ for some $u \in \mathcal{D}(\mathfrak{t})$, $\|u\| = 1$; but then $x = \tilde{\mathfrak{t}}(u)$ by Prop. 2.2.4, where $u \in \mathcal{D}(\mathfrak{t})$, $\|u\| = 1$, so $x \in \Theta(\tilde{\mathfrak{t}})$, whence $\Theta(\mathfrak{t}) \subseteq \Theta(\tilde{\mathfrak{t}})$.

Assume that $x \in \Theta(\tilde{\mathfrak{t}})$; then $x = \tilde{\mathfrak{t}}(u)$ for some $u \in \mathcal{D}(\tilde{\mathfrak{t}})$, $\|u\| = 1$. Since $u \in \mathcal{D}(\tilde{\mathfrak{t}})$, there is a sequence $\{u_n'\} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_n' \xrightarrow{\mathfrak{t}} u$. It can be assumed without loss of generality that

$\{u_n'\} \subseteq \mathcal{D}(\mathfrak{t}) \setminus \{0\}$. Put $u_n = \|u_n'\|^{-1}u_n'$. It will now be proven that $u_n \xrightarrow{\mathfrak{t}} u$. Let γ be a vertex of $\mathfrak{t}$; by Cor. 2.1.3, $u_n \xrightarrow{\mathfrak{t}} u$ if and only if $u_n \xrightarrow{\mathfrak{t}-\gamma} u$, so it can be assumed without loss of generality that $\mathfrak{t} \geq 0$. We have that

$$\begin{aligned} \|u_n - u\| &\leq \|u_n - u_n'\| + \|u_n' - u\| = \|\|u_n'\|^{-1}u_n' - u_n'\| + \|u_n' - u\| \\ &= \|(\|u_n'\|^{-1} - 1)u_n'\| + \|u_n' - u\| = |\|u_n'\|^{-1} - 1|\|u_n'\| + \|u_n' - u\| \\ &= |1 - \|u_n'\|| + \|u_n' - u\|. \end{aligned} \tag{2.12}$$

Since $u_n' \xrightarrow{\mathfrak{t}} u$, then in particular $u_n' \to u$, which implies by Prop. 2.1.18 that $\|u_n'\| \to \|u\| = 1$. Therefore, the expression (2.12) goes to zero, so $\|u_n - u\| \to 0$.

Since $\|u_n'\| \to 1$, we have that $\exists N(\frac{1}{2}) \in \mathbb{N}\ \forall n \geq N(\frac{1}{2}) : \frac{1}{2} < \|u_n'\| < \frac{3}{2}$, and this implies that $2 > \|u_n'\|^{-1} > \frac{2}{3}$, whenever $n \geq N(\frac{1}{2})$. Put $M = \max\{2, \|u_1'\|^{-1}, \ldots, \|u_{N(\frac{1}{2})-1}'\|^{-1}\}$. This shows that $\{\|u_n'\|^{-1}\}_{n\geq 1}$ is bounded with $\|u_n'\|^{-1} \leq M$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} & \mathfrak{t}(\|u_n'\|^{-1}u_n' - \|u_m'\|^{-1}u_m') \\ = {} & \mathfrak{t}(\|u_n'\|^{-1}u_n' - \|u_m'\|^{-1}u_m', \|u_n'\|^{-1}u_n' - \|u_m'\|^{-1}u_m') \\ = {} & \mathfrak{t}(\|u_n'\|^{-1}u_n') + \mathfrak{t}(\|u_n'\|^{-1}u_n', -\|u_m'\|^{-1}u_m') + \mathfrak{t}(-\|u_m'\|^{-1}u_m', \|u_n'\|^{-1}u_n') + \mathfrak{t}(-\|u_m'\|^{-1}u_m') \\ = {} & \|u_n'\|^{-2}\mathfrak{t}(u_n') + \|u_n'\|^{-1}\|u_m'\|^{-1}\mathfrak{t}(u_n, -u_m') + \|u_m'\|^{-1}\|u_n'\|^{-1}\mathfrak{t}(-u_m', u_n') + \|u_m'\|^{-2}\mathfrak{t}(u_m') \\ \leq {} & M^2\left(\mathfrak{t}(u_n') + \mathfrak{t}(u_n', -u_m') + \mathfrak{t}(-u_m', u_n') + \mathfrak{t}(u_m')\right) \\ = {} & M^2\mathfrak{t}(u_n' - u_m'), \end{aligned}$$

and since $\mathfrak{t}(\|u_n'\|^{-1}u_n' - \|u_m'\|^{-1}u_m') = \mathfrak{t}(u_n - u_m)$, and that $\lim_{m,n\to\infty}\mathfrak{t}(u_n' - u_m') = 0$, we also have $\lim_{m,n\to\infty}\mathfrak{t}(u_n - u_m) = 0$. Consequently: $u_n \xrightarrow{\mathfrak{t}} u$. By Prop. 2.2.5, $\tilde{\mathfrak{t}}$ has a closed extension, since $\mathfrak{t}$ is closable. Prop. 2.1.21 now gives that $x = \tilde{\mathfrak{t}}(u) = \lim_{n\to\infty}\tilde{\mathfrak{t}}(u_n) = \lim_{n\to\infty}\mathfrak{t}(u_n)$, where $\{\mathfrak{t}(u_n)\} \subseteq \Theta(\mathfrak{t})$; therefore $x \in \overline{\Theta(\mathfrak{t})}$. ■

Corollary 2.2.7 *Let* $\mathfrak{t}$ *be a closable sectorial form. Then* $\tilde{\mathfrak{t}}$ *is also a sectorial form, and a vertex* γ*, and corresponding semi-angle* θ *for* $\tilde{\mathfrak{t}}$ *can be chosen equal to the corresponding values for* $\mathfrak{t}$.

In particular: if $\mathfrak{h}$ *is a closable symmetric form bounded from below, then* $\mathfrak{h}$ *and* $\tilde{\mathfrak{h}}$ *have the same lower bound:* $\gamma_{\mathfrak{h}} = \gamma_{\tilde{\mathfrak{h}}}$.

Proof: Assume that $\mathfrak{t}$ is closable and sectorial with vertex γ and corresponding semi-angle θ. We have by Prop. 2.2.6 that

$$\overline{\Theta(\mathfrak{t})} \supseteq \Theta(\tilde{\mathfrak{t}}) \supseteq \Theta(\mathfrak{t}).$$

And since sector $:= \{z \in \mathbb{C} \mid |\mathrm{Arg}(z - \gamma)| \leq \theta\}$ is closed, and sector $\supseteq \Theta(\mathfrak{t})$, then sector $\supseteq \overline{\Theta(\mathfrak{t})} \supseteq \Theta(\tilde{\mathfrak{t}})$, whence $\tilde{\mathfrak{t}}$ is sectorial with vertex γ and corresponding semi-angle θ. ■

Proposition 2.2.8 $\tilde{\mathfrak{t}}$ *is a closed extension of* $\mathfrak{t}$.

Proof: $\tilde{\mathfrak{t}}$ *extends* $\mathfrak{t}$: True by Prop. 2.2.4.

$\tilde{\mathfrak{t}}$ *is closed* : By Prop. 2.1.11, $\tilde{\mathfrak{t}}$ is closed if and only if $\mathrm{Re}\,\tilde{\mathfrak{t}} - \gamma$ is closed, where γ is a vertex of $\tilde{\mathfrak{t}}$, so it can be assumed without loss of generality that $\tilde{\mathfrak{t}}$ is symmetric and non-negative.

Suppose now that $u_n \xrightarrow{\tilde{\mathfrak{t}}} u \in \mathcal{H}$. Now, $\{u_n\} \subseteq \mathcal{D}(\tilde{\mathfrak{t}})$, $\|u_n - u\| \to 0$ and $\tilde{\mathfrak{t}}(u_n - u_m) \to 0$, for $m, n \to \infty$. We know that for each $u_n \in \mathcal{D}(\tilde{\mathfrak{t}})$ there is a sequence $\{u_n^m\}_{m\geq 1} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_n^m \xrightarrow{\mathfrak{t}} u_n$, for $m \to \infty$. But $\tilde{\mathfrak{t}}$ extends $\mathfrak{t}$ by Prop. 2.2.4, so we also have $u_n^m \xrightarrow{\tilde{\mathfrak{t}}} u_n$. Since $\mathfrak{t}$ has a closed extension $\mathfrak{s}$ by assumption, we get from Prop. 2.2.5 that $\tilde{\mathfrak{t}} \subseteq \mathfrak{s}$; also: $\tilde{\mathfrak{t}}$ is sectorial by Cor. 2.2.7, so now Prop. 2.1.15 applies, and we see that $u_n^m \xrightarrow{\tilde{\mathfrak{t}}} u_n$, for $m \to \infty$ is equivalent to $\|u_n^m - u_n\|_{\tilde{\mathfrak{t}}} \to 0$, for $m \to \infty$, or explicitly:

$$\lim_{m\to\infty} \|u_n^m - u_n\| = 0 \quad \text{and} \quad \lim_{m\to\infty} \tilde{\mathfrak{t}}(u_n^m - u_n) = 0. \tag{2.13}$$

Let $\epsilon > 0$ be given. We now have from (2.13) that

$$\forall u_n \in \mathcal{D}(\tilde{\mathfrak{t}})\ \exists v_n \in \{u_n^m\}_{m\geq 1} \subseteq \mathcal{D}(\mathfrak{t}) : \|v_n - u_n\| < \frac{\epsilon}{2} \quad \wedge \quad \tilde{\mathfrak{t}}(v_n - u_n) < \frac{\epsilon^2}{9}. \tag{2.14}$$

And similarly for $\{u_m\}$:

$$\forall u_m \in \mathcal{D}(\tilde{\mathfrak{t}})\ \exists v_m \in \{u_m^k\}_{k\geq 1} \subseteq \mathcal{D}(\mathfrak{t}) : \|v_m - u_m\| < \frac{\epsilon}{2} \quad \wedge \quad \tilde{\mathfrak{t}}(v_m - u_m) < \frac{\epsilon^2}{9}. \tag{2.15}$$

Furthermore $\|u_n - u\| \to 0$, so

$$\exists N_1(\epsilon) \in \mathbb{N}\ \forall n \geq N_1(\epsilon) : \|u_n - u\| < \frac{\epsilon}{2}.$$

From the assumption $u_n \xrightarrow{\tilde{\mathfrak{t}}} u$ for $n \to \infty$, we have $\tilde{\mathfrak{t}}(u_n - u_m) \to 0$, for $m, n \to \infty$, so

$$\exists N_2(\epsilon) \in \mathbb{N}\ \forall m, n \geq N_2(\epsilon) : \tilde{\mathfrak{t}}(u_n - u_m) < \frac{\epsilon^2}{9}. \tag{2.16}$$

Put $N(\epsilon) := \max\{N_1(\epsilon), N_2(\epsilon)\}$. We claim that $v_n \xrightarrow{\mathfrak{t}} u$.
First, $\{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$, since $\{v_n\} \subseteq \{u_n^m\} \subseteq \mathcal{D}(\mathfrak{t})$.
Second, when $n \geq N(\epsilon)$ we have

$$\|v_n - u\| \leq \|v_n - u_n\| + \|u_n - u\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $\lim_{n\to\infty} \|v_n - u\| = 0$.
Third, when $m, n \geq N(\epsilon)$, the triangle inequality and the estimates (2.14), (2.15) and (2.16) give

$$\begin{aligned}
\mathfrak{t}(v_n - v_m)^{\frac{1}{2}} &= \tilde{\mathfrak{t}}(v_n - v_m)^{\frac{1}{2}} \\
&\leq \tilde{\mathfrak{t}}(v_n - u_n)^{\frac{1}{2}} + \tilde{\mathfrak{t}}(u_n - u_m)^{\frac{1}{2}} + \tilde{\mathfrak{t}}(u_m - v_m)^{\frac{1}{2}} \\
&< \left(\frac{\epsilon^2}{9}\right)^{\frac{1}{2}} + \left(\frac{\epsilon^2}{9}\right)^{\frac{1}{2}} + \left(\frac{\epsilon^2}{9}\right)^{\frac{1}{2}} \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon,
\end{aligned}$$

so $\lim_{m,n\to\infty} \mathfrak{t}(v_n - v_m) = 0$. As for now, it is shown that $\{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$, that $\|v_n - u\| \to 0$ and that $\lim_{m,n\to\infty} \mathfrak{t}(v_n - v_m) = 0$. Ergo: $v_n \xrightarrow{\mathfrak{t}} u$ as claimed, where $\{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$. Consequently: $u \in \mathcal{D}(\tilde{\mathfrak{t}})$.

Next, it will be proven that $\tilde{\mathfrak{t}}(u_n - u) \to 0$. Since $u \in \mathcal{D}(\tilde{\mathfrak{t}})$, there is a sequence $\{u_m\} \subseteq \mathcal{D}(\mathfrak{t})$ such that $u_m \xrightarrow{\mathfrak{t}} u$ for $m \to \infty$. Prop. 2.1.9, and the fact that $u_n \xrightarrow{\mathfrak{t}} u_n$ for $m \to \infty$ now gives that $u_n - u_m \xrightarrow{\mathfrak{t}} u_n - u$ for $m \to \infty$, which implies that $u_n - u_m \xrightarrow{\tilde{\mathfrak{t}}} u_n - u$ for $m \to \infty$. Now, for any $n \in \mathbb{N}$ we get that

$$\begin{aligned}
\tilde{\mathfrak{t}}(u_n - u) &:= \tilde{\mathfrak{t}}(u_n - u, u_n - u) \\
&= \lim_{m\to\infty} \tilde{\mathfrak{t}}(u_n - u_m, u_n - u_m), \quad \text{by Prop. 2.1.21} \\
&= \lim_{m\to\infty} \tilde{\mathfrak{t}}(u_n - u_m).
\end{aligned}$$

Since $u_n \xrightarrow{\tilde{\mathfrak{t}}} u$, we have $\lim_{m,n\to\infty} \tilde{\mathfrak{t}}(u_n - u_m) = 0$; and $\lim_{m\to\infty} \tilde{\mathfrak{t}}(u_n - u_m)$ exists for all $n \in \mathbb{N}$. Th. 8.39 in [2] now gives that $\lim_{n\to\infty} \tilde{\mathfrak{t}}(u_n - u)$ exists, and

$$\lim_{n\to\infty} \tilde{\mathfrak{t}}(u_n - u) = \lim_{n\to\infty} \lim_{m\to\infty} \tilde{\mathfrak{t}}(u_n - u_m) = \lim_{m,n\to\infty} \tilde{\mathfrak{t}}(u_n - u_m) = 0.$$

We have now proven that $u_n \xrightarrow{\tilde{\mathfrak{t}}} u \Rightarrow u \in \mathcal{D}(\tilde{\mathfrak{t}}),\ \tilde{\mathfrak{t}}(u_n - u) \to 0$, and this means by definition that $\tilde{\mathfrak{t}}$ is closed. ■

Note that Prop. 2.2.5 and Prop. 2.2.8 can be expressed by saying, that $\tilde{\mathfrak{t}}$ is the smallest closed extension of $\mathfrak{t}$.

Proposition 2.2.9 *Let $\mathfrak{t}$ be a sesquilinear form: Then $\mathfrak{t}$ is closed if and only if $\mathfrak{t}$ is closable and $\mathfrak{t} = \tilde{\mathfrak{t}}$.*

Proof: $\Leftarrow$: $\tilde{\mathfrak{t}}$ is closed by Prop. 2.2.8, so $\mathfrak{t}$ is closed since $\mathfrak{t} = \tilde{\mathfrak{t}}$ by assumption.

$\Rightarrow$: If $\mathfrak{t}$ is closed, then $\mathfrak{t}$ is a closed extension of $\mathfrak{t}$, so $\mathfrak{t}$ is closable. Since $\tilde{\mathfrak{t}}$ is a closed extension of $\mathfrak{t}$, we have $\mathfrak{t} \subseteq \tilde{\mathfrak{t}}$. By Prop. 2.2.5, $\mathfrak{t} \supseteq \tilde{\mathfrak{t}}$, since $\mathfrak{t}$ is a closed extension of $\mathfrak{t}$. Consequently: $\mathfrak{t} = \tilde{\mathfrak{t}}$. ■

Proposition 2.2.10 *Let $\mathfrak{t}$ be a sectorial sesquilinear form on $\mathcal{H}$. Then the following five statements are equivalent:*

(i) $\mathfrak{t}$ *is closable.*

(ii) $\forall \{u_n\} \subseteq \mathcal{H}\ \forall u \in \mathcal{D}(\mathfrak{t}) : \Big(u_n \xrightarrow{\mathfrak{t}} u \Longleftrightarrow \|u_n - u\|_{\mathfrak{t}} \to 0\Big)$

(iii) $\forall \{u_n\} \subseteq \mathcal{H}\ \forall u \in \mathcal{D}(\mathfrak{t}) : \Big(u_n \xrightarrow{\mathfrak{t}} u \Longrightarrow \mathfrak{t}(u_n - u) \to 0\Big)$

(iv) $\forall \{u_n\} \subseteq \mathcal{H} : \Big(u_n \xrightarrow{\mathfrak{t}} 0 \Longrightarrow \mathfrak{t}(u_n) \to 0\Big)$.

(v) The completion of $\mathcal{H}_{\mathfrak{t}}$, denoted by $\mathcal{H}_{\mathfrak{t}}^*$, is embeddable in $\mathcal{H}$, i.e. there is a linear map $\bar{i} : \mathcal{H}_{\mathfrak{t}}^* \to \mathcal{H}$ that is injective and continuous.

Proof: ***(i)*** $\Rightarrow$ ***(ii)*** **:** This is the content of Prop. 2.1.15.

(ii) $\Rightarrow$ ***(iii)*** **:** Given $\{u_n\} \subseteq \mathcal{H}$ and $u \in \mathcal{D}(\mathfrak{t})$ such that $u_n \xrightarrow{\mathfrak{t}} u$; then by assumption $\|u_n - u\|_{\mathfrak{t}} \to 0$. This implies that $(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) + \|u_n - u\|^2 \to 0$. And since both terms are ≥ 0, then in particular: $(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u) \to 0$. Since $u_n \xrightarrow{\mathfrak{t}} u$ by assumption, $\|u_n - u\| \to 0$, so $(\operatorname{Re}\mathfrak{t})(u_n - u) \to 0$ as well. Prop. 2.0.24(*ii*) gives that $|(\operatorname{Im}\mathfrak{t})(u_n - u)| \leq (\tan\theta)(\operatorname{Re}\mathfrak{t} - \gamma)(u_n - u)$, so $(\operatorname{Im}\mathfrak{t})(u_n - u) \to 0$. Consequently:

$$\begin{aligned}
\lim_{n\to\infty} \mathfrak{t}(u_n - u) &= \lim_{n\to\infty} \{(\operatorname{Re}\mathfrak{t})(u_n - u) + i(\operatorname{Im}\mathfrak{t})(u_n - u)\} \\
&= \lim_{n\to\infty} (\operatorname{Re}\mathfrak{t})(u_n - u) + i \lim_{n\to\infty} (\operatorname{Im}\mathfrak{t})(u_n - u) \\
&= 0.
\end{aligned}$$

(iii) $\Rightarrow$ ***(vi)*** **:** Tautology.

(vi) $\Rightarrow$ ***(v)*** **:** Consider the inclusion $i : (\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}}) \to (\mathcal{H}, \|\cdot\|)$, $i(u) = u$, $u \in \mathcal{D}(i) = \mathcal{H}_{\mathfrak{t}}$. Note that $\|u\|_{\mathfrak{t}} = [(\operatorname{Re}\mathfrak{t} - \gamma)(u) + \|i(u)\|^2]^{\frac{1}{2}} \geq \|i(u)\|$, so i is norm-decreasing. Furthermore, i is obviously injective and linear. But then $u_n \xrightarrow{\|\cdot\|_{\mathfrak{t}}} u$ implies $i(u_n) \xrightarrow{\|\cdot\|} i(u)$, so i is continuous. Now, let $\mathcal{C}$ denote the set of $\|\cdot\|_{\mathfrak{t}}-$Cauchy sequences in $(\mathcal{D}(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$, and write $\{u_n\} \sim \{v_n\}$, if $\lim_{n\to\infty} \|u_n - v_n\|_{\mathfrak{t}} = 0$, whenever $\{u_n\}, \{v_n\} \in \mathcal{C}$. It can easily be verified, that $\sim$ is an equivalence relation on $\mathcal{C}$. Consider $\mathcal{C}/\sim$. Equipped with the map

$$\begin{cases} \|\cdot\|_{\mathfrak{t}}^* : \mathcal{C}/\sim \to \mathbb{R} \\ \|u^*\|_{\mathfrak{t}}^* = \lim_{n\to\infty} \|u_n\|_{\mathfrak{t}}, \quad \{u_n\} \in u^* \in \mathcal{C}/\sim \end{cases},$$

which can be shown to be a norm on $\mathcal{C}/\sim$, the ordered pair $(\mathcal{C}/\sim, \|\cdot\|_{\mathfrak{t}}^*)$ becomes a Banach space (see e.g. [3] Th. 8.5). Put $\mathcal{H}_{\mathfrak{t}}^* := (\mathcal{C}/\sim, \|\cdot\|_{\mathfrak{t}}^*)$ and define

$$\begin{cases} \bar{i} : \mathcal{H}_{\mathfrak{t}}^* \to \mathcal{H} \\ \bar{i}(u^*) = \|\cdot\| - \lim_{n\to\infty} i(u_n), \quad \{u_n\} \in u^* \in \mathcal{H}_{\mathfrak{t}}^* \end{cases}.$$

First, $\bar{i}$ is well-defined, for given another $\{u_n'\} \in u^*$, then

$$\begin{aligned}
\|i(u_n') - \bar{i}(u^*)\| &\leq \|i(u_n') - i(u_n)\| + \|i(u_n) - \bar{i}(u^*)\| \\
&= \|i(u_n' - u_n)\| + \|i(u_n) - \bar{i}(u^*)\| \\
&\leq \|u_n' - u_n\|_{\mathfrak{t}} + \|i(u_n) - \bar{i}(u^*)\|,
\end{aligned}$$

which implies $\bar{i}(u^*) = \|\cdot\| - \lim_{n\to\infty} i(u_n')$, so $\bar{i}(u^*)$ does not depend on any representative of u^*. Second, $\bar{i}$ is linear: Given $\alpha, \beta \in \mathbb{C}$, $u^*, v^* \in \mathcal{H}_{\mathfrak{t}}^*$ there are $\|\cdot\|_{\mathfrak{t}}-$Cauchy sequences $\{u_n\}$ and $\{v_n\}$ such that $\{u_n\} \in u^*$ and $\{v_n\} \in v^*$. Now,

$$\begin{aligned}
\bar{i}(\alpha u^* + \beta v^*) &= \|\cdot\| - \lim_{n\to\infty} \big(i(\alpha u_n + \beta v_n)\big) = \|\cdot\| - \lim_{n\to\infty} \big(\alpha i(u_n) + \beta i(v_n)\big) \\
&= \alpha \Big(\|\cdot\| - \lim_{n\to\infty} i(u_n)\Big) + \beta \Big(\|\cdot\| - \lim_{n\to\infty} i(v_n)\Big) \\
&= \alpha \bar{i}(u^*) + \beta \bar{i}(v^*).
\end{aligned}$$

Third, $\bar{i}$ is continuous:

$$\begin{aligned}
\|\bar{i}(u^*)\| &= \Big\| \|\cdot\| - \lim_{n\to\infty} i(u_n) \Big\| = \lim_{n\to\infty} \|i(u_n)\|, \quad \text{by Prop. 2.1.18} \\
&\leq \lim_{n\to\infty} \|u_n\|_{\mathfrak{t}}, \quad \text{since } i \text{ was norm-decreasing} \\
&= \|u^*\|_{\mathfrak{t}}^*, \quad \text{by definition of } \|\cdot\|_{\mathfrak{t}}^*.
\end{aligned}$$

This shows that $\bar{i}$ is norm-decreasing, and by an argument similar to the one given for i, $\bar{i}$ is thus continuous.

Finally, it will be proven that $\bar{i}$ is injective, given that (iv) holds. Suppose that $[\forall\{u_n\} \subseteq \mathcal{H} : (u_n \xrightarrow{\mathfrak{t}} 0 \Rightarrow \mathfrak{t}(u_n) \to 0)]$ and let $\{u_n\} \in u^*$ be given such that $\bar{i}(u^*) = 0 \in \mathcal{H}$. Then $\{u_n\}$ is $\|\cdot\|_{\mathfrak{t}}$–Cauchy, and by Prop. 2.1.14, $\{u_n\}$ is $\mathfrak{t}$-convergent, so $\exists y \in \mathcal{H} : u_n \xrightarrow{\mathfrak{t}} y$. Now,

$$\lim_{n\to\infty} \|i(u_n)\| = \left\| \|\cdot\| - \lim_{n\to\infty} i(u_n) \right\| = \|\bar{i}(u^*)\| = 0.$$

Furthermore, $\|i(u_n) - y\| \to 0$, so $y = 0$. Therefore $u_n \xrightarrow{\mathfrak{t}} 0$. By assumption this implies that $\mathfrak{t}(u_n) \to 0$, so now

$$\|u^*\|_{\mathfrak{t}}^* = \lim_{n\to\infty} \|u_n\|_{\mathfrak{t}} = \lim_{n\to\infty} [(\mathrm{Re}\,\mathfrak{t} - \gamma)(u_n) + \|i(u_n)\|^2]^{\frac{1}{2}} = 0,$$

and since $\|\cdot\|_{\mathfrak{t}}^*$ is a norm, we have that $u^* = 0^*$. Therefore $\bar{i}$ is injective. It is now proven that $\bar{i}$ is a continuous injection from $\mathcal{H}_{\mathfrak{t}}^*$ into $\mathcal{H}$, which means that $\bar{i}$ is an embedding.

$(v) \Rightarrow (i)$: Assume that $\mathcal{H}_{\mathfrak{t}}^*$ is embeddable in $\mathcal{H}$. Extend $\mathfrak{t} : \mathcal{H}_{\mathfrak{t}} \to \mathbb{C}$, to

$$\begin{cases} \mathcal{D}(\mathfrak{t}^*) = \mathcal{C}/\sim \\ \mathfrak{t}^*(u^*, v^*) = \lim_{n\to\infty} \mathfrak{t}(u_n, v_n), \quad \{u_n\} \in u^* \in \mathcal{C}/\sim, \ \{v_n\} \in v^* \in \mathcal{C}/\sim \end{cases}.$$

Now define

$$\begin{cases} \mathcal{D}(\overline{\mathfrak{t}^*}) = \{\bar{i}(x^*) \in \mathcal{H} \mid x^* \in \mathcal{D}(\mathfrak{t}^*)\} \subseteq \mathcal{H} \\ \overline{\mathfrak{t}^*}(\bar{i}(u^*), \bar{i}(v^*)) = \lim_{n\to\infty} \mathfrak{t}(u_n, v_n) \end{cases}$$

and provide $\mathcal{D}(\overline{\mathfrak{t}^*})$ with the norm $\|\bar{i}(u^*)\|_{\overline{\mathfrak{t}^*}} = \|u^*\|_{\mathfrak{t}}^* = \lim_{n\to\infty} \|u_n\|_{\mathfrak{t}}$, $\{u_n\} \in u^*$. We claim that $\overline{\mathfrak{t}^*}$ extends $\mathfrak{t}$: Given $x \in \mathcal{D}(\mathfrak{t})$, then $i^*(x) = x^* \in \mathcal{D}(\mathfrak{t}^*)$ and $x = (\bar{i} \circ i^*)(x) = \bar{i}(x^*) \in \mathcal{D}(\overline{\mathfrak{t}^*})$. For $u, v \in \mathcal{D}(\mathfrak{t})$ we get

$$\mathfrak{t}(u, v) = \lim_{n\to\infty} \mathfrak{t}(u_n, v_n) = \overline{\mathfrak{t}^*}(\bar{i}(u^*), \bar{i}(v^*)) = \overline{\mathfrak{t}^*}(u, v),$$

so $\overline{\mathfrak{t}^*}$ extends $\mathfrak{t}$.

Next, $\overline{\mathfrak{t}^*}$ is closed: By Prop. 2.1.17 it is enough to show that $(\mathcal{D}(\overline{\mathfrak{t}^*}), \|\cdot\|_{\overline{\mathfrak{t}^*}})$ is complete. Given a Cauchy sequence in $(\mathcal{D}(\overline{\mathfrak{t}^*}), \|\cdot\|_{\overline{\mathfrak{t}^*}})$, $\{x_n\}$ say; then, because of the assumption, for each $x_m \in \{x_n\} \subseteq \mathcal{D}(\overline{\mathfrak{t}^*})$, there is a unique $x_m^* \in \mathcal{D}(\mathfrak{t}^*)$ such that $x_m = \bar{i}(x_m^*)$. Now,

$$\|x_n - x_m\|_{\overline{\mathfrak{t}^*}} = \|\bar{i}(x_n^*) - \bar{i}(x_m^*)\|_{\overline{\mathfrak{t}^*}} = \|x_n^* - x_m^*\|_{\mathfrak{t}}^* \to 0,$$

since $\{x_n\} \subseteq \mathcal{D}(\overline{\mathfrak{t}^*})$ is Cauchy. But $\mathcal{H}_{\mathfrak{t}}^*$ is complete, so there exists an $x^* \in \mathcal{H}_{\mathfrak{t}}^*$ such that $\|x_n^* - x^*\|_{\mathfrak{t}}^* \to 0$. Consider $\bar{i}(x^*) = \|\cdot\| - \lim_{n\to\infty} i(x_n) \in \bar{i}(\mathcal{D}(\mathfrak{t}^*)) = \mathcal{D}(\overline{\mathfrak{t}^*})$. Then

$$\begin{aligned} \lim_{n\to\infty} \|x_n - \bar{i}(x^*)\|_{\overline{\mathfrak{t}^*}} &= \lim_{n\to\infty} \|\bar{i}(x_n^*) - \bar{i}(x^*)\|_{\overline{\mathfrak{t}^*}} = \lim_{n\to\infty} \|\bar{i}(x_n^* - x^*)\|_{\overline{\mathfrak{t}^*}} \\ &= \lim_{n\to\infty} \|x_n^* - x^*\|_{\mathfrak{t}}^* = 0. \end{aligned}$$

Therefore $(\mathcal{D}(\overline{\mathfrak{t}^*}), \|\cdot\|_{\overline{\mathfrak{t}^*}})$ is complete. Consequently, $\overline{\mathfrak{t}^*}$ is closed and extends $\mathfrak{t}$, so $\mathfrak{t}$ is closable.

■

Proposition 2.2.11 $\mathfrak{t}$ *is closable if and only if* $\mathfrak{h} = \mathrm{Re}\,\mathfrak{t}$ *is closable.*

Proof: $\Rightarrow$: If $\mathfrak{t}$ is closable, then there is a form $\mathfrak{s}$ such that $\mathfrak{s} \supseteq \mathfrak{t}$, where $\mathfrak{s}$ is closed. Consider $\operatorname{Re}\mathfrak{s}$, which is closed by Prop. 2.1.11. Now,

$$\operatorname{Re}\mathfrak{s} = \frac{1}{2}(\mathfrak{s} + \mathfrak{s}^*) \supseteq \frac{1}{2}(\mathfrak{t} + \mathfrak{t}^*) = \operatorname{Re}\mathfrak{t},$$

so $\operatorname{Re}\mathfrak{t}$ is closable.

$\Leftarrow$: Assume that $\operatorname{Re}\mathfrak{t}$ is closable and that $u_n \xrightarrow{\mathfrak{t}} 0$. Then by Cor. 2.1.5, $u_n \xrightarrow{\operatorname{Re}\mathfrak{t}} 0$. Since $\operatorname{Re}\mathfrak{t}$ was assumed to be closable, $(\operatorname{Re}\mathfrak{t})(u_n) \to 0$ by Prop. 2.2.10, and by Prop. 2.0.24, $|(\operatorname{Im}\mathfrak{t})(u_n)| \leq (\tan\theta)(\operatorname{Re}\mathfrak{t} - \gamma)(u_n) = (\tan\theta)(\operatorname{Re}\mathfrak{t})(u_n) - \gamma(\tan\theta)\|u_n\|^2$, so $|(\operatorname{Im}\mathfrak{t})(u_n)| \to 0$, and now

$$|\mathfrak{t}(u_n)|^2 = |(\operatorname{Re}\mathfrak{t})(u_n)|^2 + |(\operatorname{Im}\mathfrak{t})(u_n)|^2 \to 0.$$

Therefore $\mathfrak{t}(u_n) \to 0$. By Prop. 2.2.10, $\mathfrak{t}$ is closable. ■

Proposition 2.2.12 *$\mathfrak{t}$ is closable if and only if $\mathfrak{t} + \alpha$ is closable for all $\alpha \in \mathbb{C}$.*

Proof: $\Leftarrow$: Tautology.

$\Rightarrow$: Assume that $\mathfrak{t}$ is closable, that $\alpha \in \mathbb{C}$ and that $u_n \xrightarrow{\mathfrak{t}+\alpha} 0$. By Cor. 2.1.3, $u_n \xrightarrow{\mathfrak{t}} 0$. Now, $(\mathfrak{t} + \alpha)(u_n) = \mathfrak{t}(u_n) + \alpha\|u_n\|^2$, and since $u_n \xrightarrow{\mathfrak{t}} 0$, then $\|u_n\| \to 0$; also $\mathfrak{t}$ is closable by assumption, so Prop. 2.2.10 gives that $\mathfrak{t}(u_n) \to 0$. Then $(\mathfrak{t} + \alpha)(u_n) \to 0$. By Prop. 2.2.10, $\mathfrak{t} + \alpha$ is closable. ■

Proposition 2.2.13 *A closable sectorial form $\mathfrak{t}$ with domain $\mathcal{H}$ is bounded.*

Proof: Since $\mathcal{H} = \mathcal{D}(\mathfrak{t}) \subseteq \mathcal{D}(\tilde{\mathfrak{t}})$, which implies that $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(\tilde{\mathfrak{t}})$, so $\mathfrak{t} = \tilde{\mathfrak{t}}$, the form $\mathfrak{t}$ is closed by Prop. 2.2.9. By Prop. 2.2.12 it can be assumed without loss of generality that a vertex $\gamma = 0$. Since $\mathcal{D}(\mathfrak{t}) = \mathcal{H}$, then $\|u\|_{\mathfrak{t}} \geq \|u\|$ for all $u \in \mathcal{H}$. Consider now $T : \mathcal{H}_{\mathfrak{t}} \to \mathcal{H}$ given by $Tu = u$, $u \in \mathcal{H}_{\mathfrak{t}}$. Then $\|Tu\| = \|u\| \leq \|u\|_{\mathfrak{t}}$, and assuming that $u \neq 0$ we get:

$$\|T\| := \sup_{u \in \mathcal{H}_{\mathfrak{t}} \setminus \{0\}} \frac{\|Tu\|}{\|u\|_{\mathfrak{t}}} \leq \sup_{u \in \mathcal{H}_{\mathfrak{t}} \setminus \{0\}} 1 = 1.$$

The operator T is therefore bounded. Its domain is $\mathcal{H}_{\mathfrak{t}}$, which is closed, so therefore T is closed. But then T^{-1} is also closed. Since $\mathcal{D}(T^{-1}) = \mathcal{H}$ the closed graph theorem implies that T^{-1} is bounded. Now, for some $M \geq 0$,

$$\|u\|_{\mathfrak{t}} = \|T^{-1}u\|_{\mathfrak{t}} \leq M\|u\|, \quad \text{for all } u \in \mathcal{H}.$$

Now, since $M^{-1}\|u\|_{\mathfrak{t}} \leq \|u\| \leq \|u\|_{\mathfrak{t}}$, we see that the norms $\|\cdot\|$ and $\|\cdot\|_{\mathfrak{t}}$ are equivalent. Now,

$$\mathfrak{h}(u) = (\operatorname{Re}\mathfrak{t})(u) \leq \|u\|_{\mathfrak{t}}^2 \leq M^2\|u\|^2.$$

Now, Prop. 2.0.24(ii) gives that

$$|\mathfrak{t}(u, v)| \leq (1 + \tan\theta)\mathfrak{h}(u)^{\frac{1}{2}}\mathfrak{h}(v)^{\frac{1}{2}} \leq (1 + \tan\theta)M^2\|u\|\|v\|,$$

so $\mathfrak{t}$ is bounded. ■

Definition 2.2.14 *Let $\mathfrak{t}$ be a closed sectorial form. A linear subspace $\mathcal{D}'$ of $\mathcal{D}(\mathfrak{t})$ is called a form-core of $\mathfrak{t}$ (or simply a core of $\mathcal{D}(\mathfrak{t})$) if the closure of the restriction of $\mathfrak{t}$ to $\mathcal{D}'$ is equal to $\mathfrak{t}$, i.e. if*

$$\widetilde{\mathfrak{t}_{|\mathcal{D}'}} = \mathfrak{t}.$$

Proposition 2.2.15 *Let $\mathfrak{t}$ be a closed sectorial form. For a subspace $\mathcal{D}' \subseteq \mathcal{D}(\mathfrak{t})$ we have:*

$$[\mathcal{D}' \text{ is a core of } \mathfrak{t}] \text{ if and only if } [\overline{\mathcal{D}'} \supseteq \mathcal{H}_{\mathfrak{t}}],$$

where the bar denotes closure in the topology induced by $\|\cdot\|_{\mathfrak{t}}$.

Proof: Let $\mathfrak{t}$ be a closed sectorial form, and $\mathcal{D}'$ a subspace of $\mathcal{D}(\mathfrak{t})$.
$\Rightarrow$: Assume that $\widetilde{\mathfrak{t}_{|\mathcal{D}'}} = \mathfrak{t}$, and that $u \in \mathcal{H}_{\mathfrak{t}}$. In particular:

$$\mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}}) = \{u \in \mathcal{H} \mid \exists\{u_n\} \subseteq \mathcal{D}' : u_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} u\} = \mathcal{D}(\mathfrak{t}).$$

Now, $\exists\{u_n\} \subseteq \mathcal{D}' : u_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} u$, and then also $u_n \stackrel{\mathfrak{t}}{\to} u$. But $\mathfrak{t}$ was closed by assumption, so it is also closable. By Prop. 2.1.15 we have that $\|u_n - u\|_{\mathfrak{t}} \to 0$, so $u = \|\cdot\|_{\mathfrak{t}} - \lim_{n\to\infty} u_n$, or $u \in \overline{\mathcal{D}'}$. Therefore: $\overline{\mathcal{D}'} \supseteq \mathcal{H}_{\mathfrak{t}}$.
$\Leftarrow$: Assume that $\overline{\mathcal{D}'} \supseteq \mathcal{H}_{\mathfrak{t}}$. Given $u \in \mathcal{D}(\mathfrak{t})$, $\exists\{u_n\} \subseteq \mathcal{D}' : \|u_n - u\|_{\mathfrak{t}} \to 0$ (by assumption). Since $\mathfrak{t}$ is closed, $\mathfrak{t}$ is closable, and $u \in \mathcal{D}(\mathfrak{t})$, so Prop. 2.1.15 gives that $u_n \stackrel{\mathfrak{t}}{\to} u$. And since $\{u_n\} \subseteq \mathcal{D}'$ we get $u_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} u$. So now $u \in \mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}})$. Therefore $\mathcal{D}(\mathfrak{t}) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}})$. Now, given $u \in \mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}})$, $\exists\{u_n\} \subseteq \mathcal{D}' : u_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} u$. Then $u_n \stackrel{\mathfrak{t}}{\to} u$, and $\mathfrak{t}$ is closed so $u \in \mathcal{D}(\mathfrak{t})$. Therefore $\mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}}) \subseteq \mathcal{D}(\mathfrak{t})$. Consequently $\mathcal{D}(\widetilde{\mathfrak{t}_{|\mathcal{D}'}}) = \mathcal{D}(\mathfrak{t})$. Now,

$$\begin{aligned}
\widetilde{\mathfrak{t}_{|\mathcal{D}'}}(u,v) &= \lim_{n\to\infty} \mathfrak{t}_{|\mathcal{D}'}(u_n, v_n), \quad \text{for some } u_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} u,\ v_n \stackrel{\mathfrak{t}_{|\mathcal{D}'}}{\to} v \\
&= \lim_{n\to\infty} \mathfrak{t}(u_n, v_n), \quad \text{since } u_n \stackrel{\mathfrak{t}}{\to} u,\ v_n \stackrel{\mathfrak{t}}{\to} v \\
&= \mathfrak{t}(u,v).
\end{aligned}$$

Consequently: $\widetilde{\mathfrak{t}_{|\mathcal{D}'}} = \mathfrak{t}$. ■

The next proposition is included for completeness. No reference will be made to it in the future.

Proposition 2.2.16 *Let $\mathfrak{t}'$ and $\mathfrak{t}''$ be two sectorial forms such that $\mathfrak{t}' \subseteq \mathfrak{t}''$. Let $\mathcal{H}_{\mathfrak{t}'}$ and $\mathcal{H}_{\mathfrak{t}''}$ be the associated inner product spaces.*

(i) If $\mathfrak{t}''$ is closed, then:

$$\mathfrak{t}' \text{ is closed} \Longleftrightarrow \mathcal{H}_{\mathfrak{t}'} \text{ is a closed subspace of } \mathcal{H}_{\mathfrak{t}''}.$$

(ii) If $\mathfrak{t}''$ is closable, then:

$$\widetilde{\mathfrak{t}'} = \widetilde{\mathfrak{t}''} \Longleftrightarrow \mathcal{H}_{\mathfrak{t}'} \text{ is dense in } \mathcal{H}_{\mathfrak{t}''}.$$

Proof: Assume that $\mathfrak{t}'$ and $\mathfrak{t}''$ are sectorial with vertices $\gamma_{\mathfrak{t}'}$ and $\gamma_{\mathfrak{t}''}$ respectively, and that $\mathfrak{t}' \subseteq \mathfrak{t}''$. Put $\gamma := \gamma_{\mathfrak{t}''} \leq \gamma_{\mathfrak{t}'}$; then γ is a vertex of both $\mathfrak{t}'$ and $\mathfrak{t}''$.

Ad *(i)*: $\Leftarrow$: Assume that $\mathcal{H}_{\mathfrak{t}'}$ is a closed subspace of $\mathcal{H}_{\mathfrak{t}''}$, and that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{D}(\mathfrak{t}'), \|\cdot\|_{\mathfrak{t}'})$. Then $\{u_n\}$ is also a Cauchy sequence in $(\mathcal{D}(\mathfrak{t}''), \|\cdot\|_{\mathfrak{t}''})$, which is complete, so $u = \|\cdot\|_{\mathfrak{t}''} - \lim_{n\to\infty} u_n \in \mathcal{H}_{\mathfrak{t}''}$. Now, u is an accumulation point of $\mathcal{H}_{\mathfrak{t}'}$, which is closed by assumption, so $u \in \mathcal{H}_{\mathfrak{t}'}$. Furthermore: $\mathfrak{t}' \subseteq \mathfrak{t}''$ and $\{u_n\} \subseteq \mathcal{H}_{\mathfrak{t}'}$, so $u = \|\cdot\|_{\mathfrak{t}''} - \lim_{n\to\infty} u_n = \|\cdot\|_{\mathfrak{t}'} - \lim_{n\to\infty} u_n$. Therefore $\mathcal{H}_{\mathfrak{t}'}$ is complete, hence $\mathfrak{t}'$ is closed by Prop. 2.1.17.

$\Rightarrow$: Assume that $\mathfrak{t}'$ and $\mathfrak{t}''$ are closed. In particular $\mathcal{H}_{\mathfrak{t}'}$ is complete by Prop. 2.1.17. Given $u \in \mathcal{H}_{\mathfrak{t}''}$ such that u is an accumulation point of $\mathcal{H}_{\mathfrak{t}'}$ in the norm $\|\cdot\|_{\mathfrak{t}''}$. Then $\exists\{u_n\} \subseteq \mathcal{H}_{\mathfrak{t}'}$ such that $u = \|\cdot\|_{\mathfrak{t}''} - \lim_{n\to\infty} u_n$. We want to show that $u \in \mathcal{H}_{\mathfrak{t}'}$. Now,

$$\begin{aligned} \|u_n - u_m\|_{\mathfrak{t}''}^2 &= (\operatorname{Re}\mathfrak{t}'' - \gamma)(u_n - u_m) + \|u_n - u_m\|^2 \\ &= (\operatorname{Re}\mathfrak{t}' - \gamma)(u_n - u_m) + \|u_n - u_m\|^2 \\ &= \|u_n - u_m\|_{\mathfrak{t}'}^2. \end{aligned} \tag{2.17}$$

Since $u = \|\cdot\|_{\mathfrak{t}''} - \lim_{n\to\infty} u_n$, then $\|u_n - u_m\|_{\mathfrak{t}''} \to 0$, and then by (2.17), $\|u_n - u_m\|_{\mathfrak{t}'} \to 0$ as well. The sequence $\{u_n\}$ is thus seen to be $\|\cdot\|_{\mathfrak{t}'}$–Cauchy, and by completeness of $\mathcal{H}_{\mathfrak{t}'}$, we have that $\exists x \in \mathcal{H}_{\mathfrak{t}'} : x = \|\cdot\|_{\mathfrak{t}'} - \lim_{n\to\infty} u_n$. But $\mathfrak{t}' \subseteq \mathfrak{t}''$, so $\|\cdot\|_{\mathfrak{t}'} - \lim_{n\to\infty} u_n = \|\cdot\|_{\mathfrak{t}''} - \lim_{n\to\infty} u_n = u$, and then we have $x = u$. Consequently: $u \in \mathcal{H}_{\mathfrak{t}'}$, so $\mathcal{H}_{\mathfrak{t}'}$ is a closed subspace of $\mathcal{H}_{\mathfrak{t}''}$.

Ad *(ii)*: $\Rightarrow$: Assume that $\widetilde{\mathfrak{t}'} = \widetilde{\mathfrak{t}''}$. Given $u \in \mathcal{D}(\mathfrak{t}'')$, then also $u \in \mathcal{D}(\widetilde{\mathfrak{t}''}) = \mathcal{D}(\widetilde{\mathfrak{t}'})$, so $\exists\{u_n\} \subseteq \mathcal{D}(\mathfrak{t}') : u_n \stackrel{\mathfrak{t}'}{\to} u$, and then $u_n \stackrel{\mathfrak{t}''}{\to} u$. Since $\mathfrak{t}''$ is closable by assumption, and also $u \in \mathcal{D}(\mathfrak{t}'')$, Prop. 2.1.15 applies and we get that $\|u_n - u\|_{\mathfrak{t}''} \to 0$. Consequently: $\mathcal{H}_{\mathfrak{t}'}$ is dense in $\mathcal{H}_{\mathfrak{t}''}$.

$\Leftarrow$: Assume that $\mathcal{H}_{\mathfrak{t}'}$ is dense in $\mathcal{H}_{\mathfrak{t}''}$, and assume without loss of generality that $\gamma \neq 0$. Let $u \in \mathcal{D}(\widetilde{\mathfrak{t}''})$ and $\epsilon > 0$ be given. Then $\exists\{u_n\} \subseteq \mathcal{D}(\mathfrak{t}'') : u_n \stackrel{\mathfrak{t}''}{\to} u$; this means that

$$\begin{aligned} &\exists N(\epsilon) \in \mathbb{N}\ \forall m, n \geq N(\epsilon) : \\ &\|u_n - u\| < \min\left\{\frac{\epsilon}{2}, \sqrt{\frac{1}{|\gamma|}\frac{\epsilon}{144}\cos\theta}\right\} \quad \wedge \quad |\mathfrak{t}''(u_n - u_m)| < \frac{\epsilon}{36}\cos\theta. \end{aligned} \tag{2.18}$$

Now, $\mathcal{H}_{\mathfrak{t}'}$ is dense in $\mathcal{H}_{\mathfrak{t}''}$, so

$$\forall u_n \in \mathcal{D}(\mathfrak{t}'')\ \exists v_n \in \mathcal{D}(\mathfrak{t}') : \|v_n - u_n\|_{\mathfrak{t}''}^2 < \epsilon_1^2 + \epsilon_2^2 < \epsilon^2,$$

or explicitly:

$$(\operatorname{Re}\mathfrak{t}'' - \gamma)(v_n - u_n) < \epsilon_1^2 \quad \wedge \quad \|v_n - u_n\|^2 < \epsilon_2^2. \tag{2.19}$$

Put $\epsilon_1^2 = \min\{\frac{\epsilon^2}{4}, \frac{\epsilon}{18}\cos\theta\}$ and $\epsilon_2^2 = \min\{\frac{\epsilon^2}{4}, \frac{\epsilon}{18}\frac{1}{|\gamma|}\}$; note that with these choices of ϵ_1^2 and ϵ_2^2, we have $\epsilon_1^2 + \epsilon_2^2 \leq \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2$. Now,

$$(\operatorname{Re}\mathfrak{t}'' - \gamma)(v_n - u_n) < \frac{\epsilon}{18}\cos\theta. \tag{2.20}$$

Whenever $m, n \geq N(\epsilon)$ we have

$$\begin{aligned} \|u_n - u_m\|^2 &\leq \|u_n - u\|^2 + \|u - u_m\|^2 + 2\|u_n - u\|\|u - u_m\| < 4\frac{1}{|\gamma|}\frac{\epsilon}{144}\cos\theta \\ &= \frac{\epsilon}{36}\frac{1}{|\gamma|}\cos\theta. \end{aligned} \tag{2.21}$$

Furthermore Prop. 2.0.25 gives:

$$\begin{aligned}(\operatorname{Re}\mathfrak{t}''-\gamma)(u_n-u_m) &\le |(\mathfrak{t}''-\gamma)(u_n-u_m)| \le |\mathfrak{t}''(u_n-u_m)| + |\gamma|\|u_n-u_m\|^2\\ &\le \frac{\epsilon}{36} + |\gamma|\frac{\epsilon}{36}\frac{1}{|\gamma|}\cos\theta\\ &= \frac{\epsilon}{18}\cos\theta. \qquad (2.22)\end{aligned}$$

The triangle inequality for non-negative forms together with the estimates (2.20) and (2.22) now gives the following when $m,n \ge N(\epsilon)$:

$$\begin{aligned}&(\operatorname{Re}\mathfrak{t}''-\gamma)(v_n-v_m)^{\frac{1}{2}}\\ \le\ &(\operatorname{Re}\mathfrak{t}''-\gamma)(v_n-u_n)^{\frac{1}{2}} + (\operatorname{Re}\mathfrak{t}''-\gamma)(u_n-u_m)^{\frac{1}{2}} + (\operatorname{Re}\mathfrak{t}''-\gamma)(u_m-v_m)^{\frac{1}{2}}\\ <\ &\sqrt{\frac{\epsilon}{18}\cos\theta} + \sqrt{\frac{\epsilon}{18}\cos\theta} + \sqrt{\frac{\epsilon}{18}\cos\theta}\\ =\ &\sqrt{\frac{\epsilon}{2}\cos\theta}. \qquad (2.23)\end{aligned}$$

Since $\sqrt{\frac{1}{|\gamma|}\frac{\epsilon}{144}\cos\theta} < \frac{1}{6}\sqrt{\frac{\epsilon}{2|\gamma|}}$, one obtains by using (2.19) and (2.18) the following for $m,n \ge N(\epsilon)$:

$$\begin{aligned}\|v_n-v_m\| &\le \|v_n-u_n\| + \|u_n-u\| + \|u-u_m\| + \|u_m-v_m\|\\ &< \frac{1}{3}\sqrt{\frac{\epsilon}{2|\gamma|}} + \frac{1}{6}\sqrt{\frac{\epsilon}{2|\gamma|}} + \frac{1}{6}\sqrt{\frac{\epsilon}{2|\gamma|}} + \frac{1}{3}\sqrt{\frac{\epsilon}{2|\gamma|}}\\ &= \sqrt{\frac{\epsilon}{2|\gamma|}}. \qquad (2.24)\end{aligned}$$

Once again Prop. 2.0.25 and (2.23) yields

$$|(\mathfrak{t}''-\gamma)(v_n-v_m)| \le (\sec\theta)(\operatorname{Re}\mathfrak{t}''-\gamma)(v_n-v_m) < (\sec\theta)\cdot\frac{\epsilon}{2}\cos\theta = \frac{\epsilon}{2}. \qquad (2.25)$$

Now, for $m,n \ge N(\epsilon)$ we have by (2.24) and (2.25) that

$$\begin{aligned}|\mathfrak{t}'(v_n-v_m)| &= |\mathfrak{t}''(v_n-v_m)| \le |(\mathfrak{t}''-\gamma)(v_n-v_m)| + |\gamma|\|v_n-v_m\|^2\\ &\le \frac{\epsilon}{2} + |\gamma|\frac{\epsilon}{2|\gamma|}\\ &= \epsilon.\end{aligned}$$

Whenever $m,n \ge N(\epsilon)$ we also have by (2.18) and (2.19) that

$$\|v_n-u\| \le \|v_n-u_n\| + \|u_n-u\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It is now proven that $\exists\{v_n\} \subseteq \mathcal{D}(\mathfrak{t}') : v_n \xrightarrow{\mathfrak{t}'} u$, which implies by definition that $u \in \mathcal{D}(\widetilde{\mathfrak{t}'})$. Therefore $\mathcal{D}(\widetilde{\mathfrak{t}''}) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}'})$. The inclusion $\mathcal{D}(\widetilde{\mathfrak{t}'}) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}''})$ is obvious, so we have that $\mathcal{D}(\widetilde{\mathfrak{t}'}) = \mathcal{D}(\widetilde{\mathfrak{t}''})$. Now,

$$\begin{aligned}\widetilde{\mathfrak{t}'}(u,v) &= \lim_{n\to\infty}\mathfrak{t}'(u_n,v_n), \quad \text{for some } u_n \xrightarrow{\mathfrak{t}'} u,\ v_n \xrightarrow{\mathfrak{t}'} v\\ &= \lim_{n\to\infty}\mathfrak{t}''(u_n,v_n), \quad \text{since } u_n \xrightarrow{\mathfrak{t}''} u,\ v_n \xrightarrow{\mathfrak{t}''} v\\ &= \widetilde{\mathfrak{t}''}(u,v).\end{aligned}$$

Consequently: $\widetilde{\mathfrak{t}'} = \widetilde{\mathfrak{t}''}$. ■

Definition 2.2.17 *An operator T on $\mathcal{H}$ is said to be form-closable, if the form $\mathfrak{t}$ defined by*

$$\mathfrak{t}(u,v) = (Tu|v), \quad \mathcal{D}(\mathfrak{t}) = \mathcal{D}(T)$$

is closable.

Definition 2.2.18 *An operator T on $\mathcal{H}$ is said to be sectorial, if its numerical range*

$$\Theta(T) := \{(Tu|u) \in \mathbb{C} \mid u \in \mathcal{D}(T),\ \|u\| = 1\}.$$

is contained in the sector $\{\zeta \in \mathbb{C} \mid |\mathrm{Arg}(\zeta - \gamma)| \leq \theta\}$, for some $\gamma \in \mathbb{R}$ and some $\theta \in [0, \frac{\pi}{2}[$.

Proposition 2.2.19 *A sectorial operator T is form-closable.*

Proof: Assume without loss of generality that $\gamma_T = 0$. Define $\mathfrak{t}$ as in Def. 2.2.17, and assume that $u_n \xrightarrow{\mathfrak{t}} 0$. Prop. 2.0.24(*iii*) now gives

$$\begin{aligned} |\mathfrak{t}(u_n)| &\leq |\mathfrak{t}(u_n, u_n - u_m)| + |\mathfrak{t}(u_n, u_m)| \\ &\leq (1 + \tan\theta)(\mathrm{Re}\,\mathfrak{t})(u_n)^{\frac{1}{2}}(\mathrm{Re}\,\mathfrak{t})(u_n - u_m)^{\frac{1}{2}} + |(Tu_n|u_m)|. \end{aligned} \tag{2.26}$$

Since $u_n \xrightarrow{\mathfrak{t}} 0$, we have $\mathfrak{t}(u_n - u_m) \to 0$, for $m, n \to \infty$, in particular: $(\mathrm{Re}\,\mathfrak{t})(u_n - u_m) \to 0$, for $m, n \to \infty$. Now,

$$\begin{aligned} (\mathrm{Re}\,\mathfrak{t})(u_n - u_m) &= (\mathrm{Re}\,\mathfrak{t})(u_n) + (\mathrm{Re}\,\mathfrak{t})(u_n, -u_m) + (\mathrm{Re}\,\mathfrak{t})(-u_m, u_n) + (\mathrm{Re}\,\mathfrak{t})(u_m) \\ &= (\mathrm{Re}\,\mathfrak{t})(u_n) + \overline{(\mathrm{Re}\,\mathfrak{t})(-u_m, u_n)} + (\mathrm{Re}\,\mathfrak{t})(-u_m, u_n) + (\mathrm{Re}\,\mathfrak{t})(u_m) \\ &= (\mathrm{Re}\,\mathfrak{t})(u_n) + 2\,\mathrm{Re}[(\mathrm{Re}\,\mathfrak{t})(-u_m, u_n)] + (\mathrm{Re}\,\mathfrak{t})(u_m) \end{aligned}$$

and since all three terms are ≥ 0, then $(\mathrm{Re}\,\mathfrak{t})(u_n) \to 0$, in particular $\{(\mathrm{Re}\,\mathfrak{t})(u_n)\}$ is bounded. From (2.26) we get that

$$|\mathfrak{t}(u_n)| < \epsilon + |(Tu_n|u_m)| \leq \epsilon + \|Tu_n\|\|u_m\|,$$

which implies that

$$|\mathfrak{t}(u_n)| = \lim_{m\to\infty} |\mathfrak{t}(u_n)| < \epsilon + \|Tu_n\| \lim_{m\to\infty} \|u_m\| = \epsilon,$$

so $\mathfrak{t}(u_n) \to 0$. By Prop 2.2.10, $\mathfrak{t}$ is closable. ∎

Since every symmetric operator bounded from below is sectorial, the following is an immediate consequence of Prop. 2.2.19:

Corollary 2.2.20 *A symmetric operator bounded from below is form-closable.*

Proposition 2.2.21 *If T is form-closable, then*

$$\tilde{\mathfrak{t}}(u,v) = (Tu|v), \quad u \in \mathcal{D}(T),\ v \in \mathcal{D}(\tilde{\mathfrak{t}}).$$

Proof: Assume that $\mathfrak{t}$ is form-closable, then the form $\mathfrak{t}$ given by

$$\mathfrak{t}(u,v) = (Tu|v), \quad u,v \in \mathcal{D}(\mathfrak{t}) = \mathcal{D}(T)$$

is closable by Def. 2.2.17, so $\tilde{\mathfrak{t}}$ exists. Now,

$$\begin{cases} \mathcal{D}(\tilde{\mathfrak{t}}) = \{u \in \mathcal{H} \mid \exists \{u_n\} \subseteq \mathcal{D}(\mathfrak{t}) : u_n \xrightarrow{\mathfrak{t}} u\} \\ \tilde{\mathfrak{t}}(u,v) = \lim_{n\to\infty} \mathfrak{t}(u_n,v_n) \quad \text{for some } u_n \xrightarrow{\mathfrak{t}} u,\ v_n \xrightarrow{\mathfrak{t}} v \end{cases}.$$

If $u \in \mathcal{D}(T) = \mathcal{D}(\mathfrak{t})$, put $u_n = u$, for all $n \in \mathbb{N}$. Then for all $v \in \mathcal{D}(\tilde{\mathfrak{t}})$,

$$\tilde{\mathfrak{t}}(u,v) = \lim_{n\to\infty} \mathfrak{t}(u,v_n) = \lim_{n\to\infty} (Tu|v_n), \tag{2.27}$$

for some $\{v_n\} \subseteq \mathcal{D}(\mathfrak{t})$ for which $v_n \xrightarrow{\mathfrak{t}} v$. Since $v_n \xrightarrow{\mathfrak{t}} v$, we also have that $v = \|\cdot\| - \lim_{n\to\infty} v_n$; furthermore, $(Tu|\,\cdot\,)$ is continuous by Cauchy-Schwarz' inequality, so this together with (2.27) gives that

$$\tilde{\mathfrak{t}}(u,v) = (Tu \mid \|\cdot\| - \lim_{n\to\infty} v_n) = (Tu|v).$$

■

Chapter 3

Representation of forms

The primary goal of this chapter is to state and prove what is called the first representation theorem (Th. 3.0.31). It is the main ingredient in the construction of The Friedrichs extension of a sectorial operator.

For bounded forms there is an easier analogue (Prop. 3.0.22), which will be used in the proof of Th. 3.0.31.

Proposition 3.0.22 (The representation theorem for bounded forms) *Let $\mathcal{H}$ be a Hilbert space, and $\mathfrak{t} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ a bounded sesquilinear form on $\mathcal{H}$. Then there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that*

$$\forall u, v \in \mathcal{H} : \mathfrak{t}(u, v) = (u|Sv).$$

In addition: $\|S\| = \|\mathfrak{t}\|$.

Proof: By assumption, the form $\mathfrak{t}$ is bounded, so there is a constant $M \geq 0$ such that $|\mathfrak{t}(u, v)| \leq M\|u\|\|v\|$ for all $u, v \in \mathcal{H}$. Fix $v \in \mathcal{H}$; then the map $u \mapsto \mathfrak{t}(u, v)$ is a bounded linear functional on $\mathcal{H}$ with norm $\leq \|\mathfrak{t}\|\|v\|$. By the representation theorem of Riesz-Frechet, there is a unique $Sv \in \mathcal{H}$ such that $\mathfrak{t}(u, v) = (u|Sv)$ for all $u \in \mathcal{H}$. Also from Riesz-Frechet: $\|\mathfrak{t}(\cdot, v)\| = \|Sv\|$, so

$$\|Sv\| \leq \|\mathfrak{t}\|\|v\|. \tag{3.1}$$

Now, Sv depends linearly on v:

Additivity:

For all $u \in \mathcal{H}$ we have

$$\begin{aligned} (u|S(v_1 + v_2)) &= \mathfrak{t}(u, v_1 + v_2) = \mathfrak{t}(u, v_1) + \mathfrak{t}(u, v_2) = (u|Sv_1) + (u|Sv_2) \\ &= (u|Sv_1 + Sv_2), \end{aligned}$$

which implies that $S(v_1 + v_2) = Sv_1 + Sv_2$.

Homogeneity:

For all $u \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} (u|S(\alpha v)) &= \mathfrak{t}(u, \alpha v) = \overline{\alpha}\mathfrak{t}(u, v) = \overline{\alpha}(u|Sv) = \\ &= (u|\alpha Sv), \end{aligned}$$

which implies that $S(\alpha v) = \alpha S v$. So S is linear. By assumption $\mathfrak{t}$ is bounded, and then $\|\mathfrak{t}\| < \infty$ by Prop. 2.0.13. Then by (3.1), S is bounded. Therefore $S \in \mathcal{B}(\mathcal{H})$, and $\|S\| \leq \|\mathfrak{t}\|$ by (3.1). Now,

$$|\mathfrak{t}(u, v)| = |(u|Sv)| \leq \|u\|\|Sv\| \leq \|u\|\|S\|\|v\|,$$

so $\|\mathfrak{t}\| \leq \|S\|$. Consequently: $\|S\| = \|\mathfrak{t}\|$. ■

Corollary 3.0.23 *Let $\mathcal{H}$ be a Hilbert space, and $\mathfrak{t} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ a bounded sesquilinear form on $\mathcal{H}$. Then there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that*

$$\forall u, v \in \mathcal{H} : \mathfrak{t}(u, v) = (Tu|v).$$

In addition: $\|T\| = \|\mathfrak{t}\|$.

Proof: The map $S \mapsto S^*$ is an involution on $\mathcal{B}(\mathcal{H})$, so in particular $S = S^{**}$. Therefore by Prop. 3.0.22:

$$\forall u, v \in \mathcal{H} : \mathfrak{t}(u, v) = (u|Sv) = (u|S^{**}v) = (S^*u|v) = (Tu|v),$$

where $T := S^*$. Now, $\|T\| = \|S^*\| = \|S\| = \|\mathfrak{t}\|$, where the last equality comes from Prop. 3.0.22. ■

In analogy with $\mathfrak{t}$-convergence for forms, a corresponding notion for operators is defined in the following.

Definition 3.0.24 (T-convergence) *Let T be an operator on $\mathcal{H}$. A sequence $\{u_n\} \subseteq \mathcal{H}$ is said to be T-convergent, if*

(i) $\{u_n\} \subseteq \mathcal{D}(T)$

(ii) $\{u_n\}$ *and* $\{Tu_n\}$ *are Cauchy sequences.*

If in addition to (i) and (ii), $\{u_n\}$ also fulfills

(iii) $u_n \to u \in \mathcal{H}$,

then the sequence $\{u_n\}$ is said to T-converge to u, and this is denoted by $u_n \xrightarrow{T} u$.

Definition 3.0.25 (Closed operator) *An operator T on $\mathcal{H}$ is said to be closed, if*

$$u_n \xrightarrow{T} u \quad \textit{implies} \quad u \in \mathcal{D}(T), \quad Tu = \lim_{n \to \infty} Tu_n,$$

where the limit is in the norm $\|\cdot\|$ on $\mathcal{H}$.

Proposition 3.0.26 *Let T be a closed operator on $\mathcal{H}$, and A a bounded operator on $\mathcal{H}$ such that $\mathcal{D}(T) \subseteq \mathcal{D}(A)$. Then $T + \lambda A$ is closed, for any $\lambda \in \mathbb{C}$.*

Proof: Assume that T is closed, A is bounded, $\mathcal{D}(T) \subseteq \mathcal{D}(A)$, that $\lambda \in \mathbb{C}$ and that $u_n \overset{T+\lambda A}{\longrightarrow} u$. Then $\{u_n\} \subseteq \mathcal{D}(T + \lambda A)$, $\|u_n - u\| \to 0$ and $\{(T + \lambda A)u_n\}$ is Cauchy, i.e. $\|(T + \lambda A)u_n - (T + \lambda A)u_m\| \to 0$ for $m, n \to \infty$. It must be shown that $u \in \mathcal{D}(T + \lambda A)$ and $(T + \lambda A)u = \lim_{n\to\infty}(T + \lambda A)u_n$.

Now,

$$\begin{aligned}
\|Tu_n - Tu_m\| &= \|((T + \lambda A) - \lambda A)(u_n - u_m)\| \\
&\leq \|(T + \lambda A)(u_n - u_m)\| + \|(\lambda A)(u_n - u_m)\| \\
&\leq \|(T + \lambda A)(u_n - u_m)\| + \|\lambda A\|\|u_n - u_m\| \\
&= \|(T + \lambda A)u_n - (T + \lambda A)u_m\| + \|\lambda A\|\|u_n - u_m\| \\
&\to 0,
\end{aligned}$$

as $m, n \to \infty$, so $\{Tu_n\}$ is also Cauchy. Since $\mathcal{D}(T) \subseteq \mathcal{D}(A)$, then $\mathcal{D}(T + \lambda A) = \mathcal{D}(T) \cap \mathcal{D}(\lambda A) = \mathcal{D}(T) \cap \mathcal{D}(A) = \mathcal{D}(T)$, so $\{u_n\} \subseteq \mathcal{D}(T)$. Therefore $u_n \overset{T+\lambda A}{\longrightarrow} u$ implies that $u_n \overset{T}{\to} u$. And T was closed by assumption, so $u \in \mathcal{D}(T)$. And then $u \in \mathcal{D}(T + \lambda A)$.

Now,

$$\begin{aligned}
\|(T + \lambda A)u_n - (T + \lambda A)u\| &= \|(T + \lambda A)(u_n - u)\| = \|T(u_n - u) + (\lambda A)(u_n - u)\| \\
&\leq \|T(u_n - u)\| + \|(\lambda A)(u_n - u)\| \\
&\leq \|Tu_n - Tu\| + \|\lambda A\|\|u_n - u\| \\
&\to 0,
\end{aligned}$$

since T is closed. Consequently: $(T + \lambda A)u = \lim_{n\to\infty}(T + \lambda A)u_n$, so $T + \lambda A$ is closed. ∎

Definition 3.0.27 *An operator T on $\mathcal{H}$ is said to be accretive, if the numerical range of T, $\Theta(T)$, is contained in the right half-plane, i.e. if:*

$$\Theta(T) \subseteq \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}.$$

Definition 3.0.28 *An operator T on $\mathcal{H}$ is said to be m-accretive, if it is accretive and*

$$(T - \zeta)^{-1} \in \mathcal{B}(\mathcal{H}), \quad \|(T - \zeta)^{-1}\| \leq (\operatorname{Re} \zeta)^{-1}, \text{ for all } \zeta \in \mathbb{C} \text{ with } \operatorname{Re} \zeta > 0.$$

An operator T is said to be quasi-m-accretive, if $T + \alpha$ is m-accretive for some $\alpha \in \mathbb{C}$.

Proposition 3.0.29 *An m-accretive operator T is maximal accretive in the sense that it admits no proper accretive extensions.*

Proof: Suppose that T_1 is an accretive extension of T. Then $(T_1 - \zeta)^{-1}$ is an extension of $(T - \zeta)^{-1}$ for $\operatorname{Re} \zeta > 0$, and by definition of an m-accretive operator, $(T - \zeta)^{-1}$ has domain $\mathcal{H}$. Therefore $T_1 = T$. ∎

Definition 3.0.30 *An operator T on $\mathcal{H}$ is said to be m-sectorial, if T is quasi-m-accretive and sectorial.*

Theorem 3.0.31 (The first representation theorem) *Let $\mathfrak{t}$ be a densely defined, closed, sectorial sesquilinear form on $\mathcal{H}$. Then there exists an m-sectorial operator T on $\mathcal{H}$ such that*

(i) $\mathcal{D}(T) \subseteq \mathcal{D}(\mathfrak{t})$ *and*

$$\forall u \in \mathcal{D}(T)\ \forall v \in \mathcal{D}(\mathfrak{t}) : \mathfrak{t}(u, v) = (Tu|v).$$

(ii) $\mathcal{D}(T)$ *is a core of* $\mathfrak{t}$.

(iii) If

$$\forall u \in \mathcal{D}(\mathfrak{t})\ \exists w \in \mathcal{H}\ \forall \textit{core, } \mathcal{D}, \textit{ of } \mathfrak{h}\ \forall v \in \mathcal{D} : \mathfrak{t}(u, v) = (w|v),$$

then $u \in \mathcal{D}(T)$ *and* $Tu = w$.

Remarks: (1) The operator T associated to $\mathfrak{t}$ is uniquely determined. This will be proved later (Cor. 3.0.35). (2) Statement (*iii*) provides a sufficient condition for having membership of the domain of the associated operator. This will become relevant later, when the exact domain of the Hamiltonian of the δ-interaction will be determined.

Proof: It may be assumed without loss of generality that $\mathfrak{t}$ has a vertex $\gamma = 0$, so that $\mathfrak{h} = \operatorname{Re} \mathfrak{t} \geq 0$.

Ad (*i*): Let $\mathcal{H}_{\mathfrak{t}}$ be the associated inner product space. By Prop. 2.1.17, $\mathcal{H}_{\mathfrak{t}}$ is a Hilbert space, since $\mathfrak{t}$ is closed by assumption. Prop. 2.1.16 shows that $\mathfrak{t}$ is bounded on $\mathcal{H}_{\mathfrak{t}}$. And the estimates

$$\begin{aligned} |(\mathfrak{t} + \mathbf{1})(u, v)| &= |\mathfrak{t}(u, v) + (u|v)| \leq |\mathfrak{t}(u, v)| + |(u|v)| \\ &\leq M\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} + \|u\|\|v\| \leq M\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} + \|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} \\ &= (1 + M)\|u\|_{\mathfrak{t}}\|v\|_{\mathfrak{t}} \end{aligned}$$

show that $\mathfrak{t} + \mathbf{1}$ is bounded on $\mathcal{H}_{\mathfrak{t}}$ as well. Put $\mathfrak{t}_1 = \mathfrak{t} + \mathbf{1}$. By Cor. 3.0.23 there is an operator $B \in \mathcal{B}(\mathcal{H}_{\mathfrak{t}})$ such that

$$\mathfrak{t}_1(u, v) = (Bu|v)_{\mathfrak{t}}, \quad u, v \in \mathcal{H}_{\mathfrak{t}}.$$

Since

$$\begin{aligned} \|u\|_{\mathfrak{t}}^2 &= \mathfrak{h}(u) + \|u\|^2 = (\mathfrak{h} + \mathbf{1})(u) = (\operatorname{Re} \mathfrak{t}_1)(u) = \operatorname{Re}(Bu|u)_{\mathfrak{t}} \\ &\leq |(Bu|u)_{\mathfrak{t}}| \leq \|Bu\|_{\mathfrak{t}}\|u\|_{\mathfrak{t}}, \end{aligned}$$

we have: $\|u\|_{\mathfrak{t}} \leq \|Bu\|_{\mathfrak{t}}$ (for $u = 0$ there is nothing to prove). Now, $Bu = 0 \Rightarrow \|Bu\|_{\mathfrak{t}} = 0 \Rightarrow \|u\|_{\mathfrak{t}} = 0 \Rightarrow u = 0$, so B is injective. Therefore B has an inverse on $\mathcal{R}(B)$, and for $u \in \mathcal{D}(B^{-1}) = \mathcal{R}(B)$ we get: $\|B^{-1}u\|_{\mathfrak{t}} \leq \|B(B^{-1})u\|_{\mathfrak{t}} = \|u\|_{\mathfrak{t}}$, so B^{-1} is bounded on $\mathcal{D}(B^{-1}) = \mathcal{R}(B) \subseteq \mathcal{H}_{\mathfrak{t}}$. So $B^{-1} : \mathcal{R}(B) \to \mathcal{D}(B) = \mathcal{H}_{\mathfrak{t}}$ is a homeomorphism. This together with the fact that $\mathcal{H}_{\mathfrak{t}}$ is closed implies that $\mathcal{R}(B) = \mathcal{D}(B^{-1})$ is closed in $\mathcal{H}_{\mathfrak{t}}$. In fact $\mathcal{D}(B^{-1}) = \mathcal{H}_{\mathfrak{t}}$; in order to show this, take $u \in \mathcal{H}_{\mathfrak{t}}$, such that $u \perp_{\mathfrak{t}} \mathcal{D}(B^{-1})$. Then $(z|u)_{\mathfrak{t}} = 0$ for all $z \in \mathcal{D}(B^{-1}) = \mathcal{R}(B)$, in particular: $(u|u)_{\mathfrak{t}} = \|u\|_{\mathfrak{t}}^2 = \operatorname{Re}(Bu|u)_{\mathfrak{t}} = 0$, so $u = 0$, since $\|\cdot\|_{\mathfrak{t}}$ is a norm. This shows that $\mathcal{D}(B^{-1})$ is dense in $\mathcal{H}_{\mathfrak{t}}$; but $\mathcal{D}(B^{-1})$ was closed, so $\mathcal{D}(B^{-1}) = \mathcal{H}_{\mathfrak{t}}$. Therefore $B^{-1} \in \mathcal{B}(\mathcal{H}_{\mathfrak{t}})$. Furthermore: $\|B^{-1}u\|_{\mathfrak{t}} \leq \|u\|_{\mathfrak{t}}$ shows that $\|B^{-1}\| \leq 1$.

Fix $u \in \mathcal{H}$, and consider the conjugate linear functional l_u given by $l_u(v) = (u|v)$, $v \in \mathcal{H}_{\mathfrak{t}}$. Since $|l_u(v)| = |(u|v)| \leq \|u\|\|v\| \leq \|u\|\|v\|_{\mathfrak{t}}$, we see that l_u is a bounded conjugate linear functional on $\mathcal{H}_\mathfrak{t}$ with bound $\leq \|u\|$. The representation theorem of Riesz-Frechet implies that $\exists! u' \in \mathcal{H}_\mathfrak{t}\ \forall v \in \mathcal{H}_\mathfrak{t} : l_u(v) = (u'|v)_\mathfrak{t}$, where $\|u'\|_\mathfrak{t} \leq \|u\|$, since

$$\|u'\|_\mathfrak{t} = \sup_{v\in\mathcal{H}_\mathfrak{t}\setminus\{0\}} \frac{|(u'|v)_\mathfrak{t}|}{\|v\|_\mathfrak{t}} = \sup_{v\in\mathcal{H}_\mathfrak{t}\setminus\{0\}} \frac{|l_u(v)|}{\|v\|_\mathfrak{t}} \leq \sup_{v\in\mathcal{H}\setminus\{0\}} \frac{|l_u(v)|}{\|v\|} = \|u\|.$$

Define the operator A by $Au = B^{-1}u'$. A is a linear operator with $\mathcal{D}(A) = \mathcal{H}$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B^{-1}) = \mathcal{D}(\mathcal{B}) = \mathcal{H}_\mathfrak{t}$. Since $\|Au\|_\mathfrak{t} = \|B^{-1}u'\|_\mathfrak{t} \leq \|B^{-1}\|\|u'\|_\mathfrak{t} \leq \|u\|$, A is bounded on $\mathcal{H}$, so $A \in \mathcal{B}(\mathcal{H})$. By the definition of A we get $(u|v) = l_u(v) = (u'|v)_\mathfrak{t} = (BAu|v)_\mathfrak{t} = \mathfrak{t}_1(Au, v) = \mathfrak{t}(Au, v) + (Au|v)$. This implies that $\mathfrak{t}(Au, v) = (u|v) - (Au|v) = (u - Au|v)$, $u \in \mathcal{H}$, $v \in \mathcal{H}_\mathfrak{t}$. Now, $Au = 0 \Rightarrow (u|v) = \mathfrak{t}(Au|v) + (Au|v) = 0$, for all $v \in \mathcal{H}_\mathfrak{t}$, which implies that $u = 0$, since $\mathcal{H}_\mathfrak{t} = \mathcal{D}(\mathfrak{t})$ is dense in $\mathcal{H}$ by assumption. Therefore A is injective, so it has an inverse $A^{-1} : \mathcal{R}(A) \to \mathcal{D}(A)$. Put $w = Au$; then $u = A^{-1}w$ and denoting by I the identity operator on $\mathcal{R}(A)$ we get

$$\mathfrak{t}(w, v) = (A^{-1}w - AA^{-1}w|v) = ((A^{-1} - I)w|v), \quad w \in \mathcal{R}(A),\ v \in \mathcal{H}_\mathfrak{t}.$$

Define $T = A^{-1} - I$, $\mathcal{D}(T) = \mathcal{D}(A^{-1} - I) = \mathcal{D}(A^{-1}) \cap \mathcal{D}(I) = \mathcal{R}(A) \cap \mathcal{R}(A) = \mathcal{R}(A)$; then

$$\mathfrak{t}(w, v) = (Tw|v), \quad w \in \mathcal{R}(A) = \mathcal{D}(T) \subseteq \mathcal{H}_\mathfrak{t} = \mathcal{D}(\mathfrak{t}),\ v \in \mathcal{H}_\mathfrak{t} = \mathcal{D}(\mathfrak{t}).$$

We know that $A \in \mathcal{B}(\mathcal{H})$, so it is also closed; then A^{-1} is also closed; furthermore I is bounded and $\mathcal{D}(A^{-1}) = \mathcal{R}(A) = \mathcal{D}(I)$, so $T = A^{-1} - I$ is closed by Prop. 3.0.26. Since $\mathfrak{t}(u) = (Tu|u)$, then $\Theta(T) = \Theta(\mathfrak{t})$, and $\mathfrak{t}$ was sectorial by assumption, so T is also sectorial. In addition: $\mathcal{R}(T + I) = \mathcal{R}(A^{-1}) = \mathcal{D}(A) = \mathcal{H}$, so T is m-sectorial.

This proves (i).

Ad *(ii)*: By Prop. 2.2.15 it is enough to show that $\mathcal{D}(T)$ is dense in $\mathcal{H}_\mathfrak{t}$. Since $\mathcal{D}(T) = \mathcal{R}(A)$ and $B : \mathcal{H}_\mathfrak{t} \to \mathcal{H}_\mathfrak{t}$ is a homeomorphism, and denseness is preserved under homeomorphisms, then it suffices to show that $B(\mathcal{R}(A))$ is dense in $\mathcal{H}_\mathfrak{t}$. In the meantime, we have

$$\mathcal{D}(BA) = A^{-1}\mathcal{D}(B) = \{x \in \mathcal{D}(A) = \mathcal{H} \mid Ax \in \mathcal{D}(B)\} = \mathcal{D}(A), \text{ since } \mathcal{R}(A) \subseteq \mathcal{H}_\mathfrak{t} = \mathcal{D}(B),$$

and then

$$\mathcal{R}(BA) = (BA)\mathcal{D}(BA) = (BA)\mathcal{D}(A) = B(A\mathcal{D}(A)) = B(\mathcal{R}(A)).$$

Assume now, that $v \in \mathcal{H}_\mathfrak{t}$ fulfills that $(BAu|v)_\mathfrak{t} = 0$ for all $u \in \mathcal{D}(BA) = \mathcal{D}(A) = \mathcal{H}$; then

$$(u|v) = l_u(v) = (u'|v)_\mathfrak{t} = (BAu, v)_\mathfrak{t} = 0,$$

for all $u \in \mathcal{H}$, so $v = 0$, which implies that $\mathcal{R}(BA) = B(\mathcal{R}(A))$ is dense in $\mathcal{H}_\mathfrak{t}$.

This proves (ii).

Ad *(iii)*: Consider now the adjoint form $\mathfrak{t}^*$. By assumption, the form $\mathfrak{t}$ is closed, sectorial with vertex zero, and densely defined. By definition of $\mathfrak{t}^*$, $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(\mathfrak{t}^*)$, so $\mathfrak{t}^*$ is also densely defined; by Prop. 2.0.23, $\mathfrak{t}^*$ is also sectorial with vertex zero; and by Prop. 2.1.11, $\mathfrak{t}^*$ is also closed. Therefore there is an operator T' associated to $\mathfrak{t}^*$ (by (i)). Now, for any $v \in \mathcal{D}(T') \subseteq \mathcal{D}(\mathfrak{t}^*)$ and $u \in \mathcal{D}(\mathfrak{t}^*) = \mathcal{D}(\mathfrak{t})$ we get

$$\mathfrak{t}^*(u, v) = (T'v|u) \Rightarrow \overline{\mathfrak{t}^*(v, u)} = \overline{(T'v|u)} \Rightarrow \mathfrak{t}(u, v) = (u|T'v).$$

Letting $u \in \mathcal{D}(T) \subseteq \mathcal{D}(\mathfrak{t})$ and $v \in \mathcal{D}(T^{'}) \subseteq \mathcal{D}(\mathfrak{t}^*) = \mathcal{D}(\mathfrak{t})$ we get from above and from (i) that

$$(Tu|v) = (u|T^{'}v), \quad u \in \mathcal{D}(T),\ v \in \mathcal{D}(T^{'}).$$

This implies that $T^{'} \subseteq T^*$. Furthermore, both $T^{'}$ and T^* are m-sectorial, so in particular they are m-accretive by Def. 3.0.30. By Prop. 3.0.29 they are maximal accretive, and therefore $T^{'} = T^*$. Then $(T^{'})^* = T^{**} = T$, where the last equality is true because T is closed[1].

Fix $u \in \mathcal{D}(\mathfrak{t})$ and assume that there is a $w \in \mathcal{H}$ such that for any core $\mathcal{D}$ of $\mathfrak{t}$ one has that

$$\forall v \in \mathcal{D} : \mathfrak{t}(u, v) = (w|v).$$

By Prop. 2.1.16, $\mathfrak{t}$ is continuous on $\mathcal{H}_{\mathfrak{t}} = \mathcal{D}(\mathfrak{t})$. Therefore, if $v \in \mathcal{D}(\mathfrak{t})$ then by Prop. 2.2.15, there is a $\{v_n\} \subseteq \mathcal{D}$ such that $v = \|\cdot\|_{\mathfrak{t}} - \lim_{n\to\infty} v_n$, so now

$$\begin{aligned} \mathfrak{t}(u, v) &= \mathfrak{t}(u, \|\cdot\|_{\mathfrak{t}} - \lim_{n\to\infty} v_n) = \lim_{n\to\infty} \mathfrak{t}(u, v_n) = \lim_{n\to\infty} (w|v_n) = (w|\|\cdot\| - \lim_{n\to\infty} v_n) \\ &= (w|v), \quad v \in \mathcal{D}(\mathfrak{t}). \end{aligned}$$

In particular, if $v \in \mathcal{D}(T^{'})$, then

$$(u|T^{'}v) = \mathfrak{t}(u, v) = (w|v), \quad \text{for all } v \in \mathcal{D}(\mathfrak{t}).$$

This implies that

$$\overline{(u|T^{'}v)} = \overline{(w|v)}, \quad \text{or equivalently:} \quad (T^{'}v|u) = (v|w).$$

By definition of $(T^{'})^*$, $u \in \mathcal{D}((T^{'})^*)$ and $w = (T^{'})^*u$. But it was proved earlier that $(T^{'})^* = T$, so $u \in \mathcal{D}(T)$ and $Tu = w$.

This proves (iii). ■

Corollary 3.0.32 *Let $\mathfrak{t}_0$ be a form defined by: $\mathcal{D}(\mathfrak{t}_0) = \mathcal{D}(T)$, $\mathfrak{t}_0(u, v) = (Tu|v)$, where $u, v \in \mathcal{D}(\mathfrak{t}_0)$ and T is the operator that comes from Th. 3.0.31. Then $\mathfrak{t} = \widetilde{\mathfrak{t}_0}$.*

Proof: Since $\mathcal{D}(\mathfrak{t}) \supseteq \mathcal{D}(T) = \mathcal{D}(\mathfrak{t}_0)$, and $\mathfrak{t}_0(u, v) = (Tu|v) = \mathfrak{t}(u, v)$ on $\mathcal{D}(\mathfrak{t}_0)$, then $\mathfrak{t}_0 = \mathfrak{t}_{|\mathcal{D}(\mathfrak{t}_0)}$. And $\mathcal{D}(\mathfrak{t}_0) = \mathcal{D}(T)$, so $\mathfrak{t}_0 = \mathfrak{t}_{|\mathcal{D}(T)}$. Also, $\mathcal{D}(T)$ is a core of $\mathfrak{t}$, so $\mathfrak{t} = \widetilde{\mathfrak{t}_{|\mathcal{D}(T)}} = \widetilde{\mathfrak{t}_0}$. ■

Corollary 3.0.33 *The numerical range of T is a dense subset of the numerical range of $\mathfrak{t}$, i.e.: $\Theta(T) \subseteq \Theta(\mathfrak{t})$ and $\overline{\Theta(T)} \supseteq \Theta(\mathfrak{t})$.*

Proof: It holds that $\Theta(T) = \Theta(\mathfrak{t}_0) \subseteq \Theta(\widetilde{\mathfrak{t}_0}) = \Theta(\mathfrak{t})$, where the inclusion comes from Prop. 2.2.6, and the last equality from Cor. 3.0.32. This shows that $\Theta(T) \subseteq \Theta(\mathfrak{t})$. By Prop. 2.2.6, $\overline{\Theta(\mathfrak{t}_0)} \supseteq \Theta(\widetilde{\mathfrak{t}_0})$, so now $\overline{\Theta(T)} = \overline{\Theta(\mathfrak{t}_0)} \supseteq \Theta(\widetilde{\mathfrak{t}_0}) = \Theta(\mathfrak{t})$. ■

Corollary 3.0.34 *If S is an operator such that $\mathcal{D}(S) \subseteq \mathcal{D}(\mathfrak{t})$, and such that $\mathfrak{t}(u, v) = (Su|v)$ for all $u \in \mathcal{D}(S)$, and all v belonging to a core of $\mathfrak{t}$, then $S \subseteq T$.*

[1] T is m-sectorial with vertex $\gamma = 0$ by (i), so it is actually m-accretive, and m-accretive operators are closed. See [8], page 201.

Proof: By (*iii*) of Th. 3.0.31, $u \in \mathcal{D}(T)$, and $Su = Tu$, so $S \subseteq T$. ■

Corollary 3.0.35 *The m-sectorial operator T in Th. 3.0.31 is uniquely determined by the condition (i).*

Proof: If S, T are operators fulfilling Th. 3.0.31, they also fulfill Cor. 3.0.34 and then $S \subseteq T$ and $T \subseteq S$. Consequently: $S = T$. ■

Notation 3.0.36 *The operator T constructed from $\mathfrak{t}$ in Th. 3.0.31 will be called the (m-sectorial) operator associated with $\mathfrak{t}$. It will be denoted by $T_{\mathfrak{t}}$.*

Proposition 3.0.37 *If $T = T_{\mathfrak{t}}$, then $T^* = T_{\mathfrak{t}^*}$.*

Proof: In the proof of Th. 3.0.31, T' is defined to be $T_{\mathfrak{t}^*}$, and it is proven there that $T' = T^*$, so $T^* = T_{\mathfrak{t}^*}$. ■

Proposition 3.0.38 *If $\mathfrak{h}$ is a densely defined, symmetric, closed sesquilinear form on $\mathcal{H}$ bounded from below, then $T_{\mathfrak{h}}$ is self-adjoint and bounded from below. Furthermore: $\gamma_{\mathfrak{h}} = \gamma_{T_{\mathfrak{h}}}$.*

Proof: If $\mathfrak{h}$ is symmetric, then $\mathfrak{h} = \mathfrak{h}^*$, whence $T_{\mathfrak{h}} = T_{\mathfrak{h}^*} = T_{\mathfrak{h}}^*$, where the last equality comes from Prop. 3.0.37. From Cor. 3.0.33: $\Theta(T_{\mathfrak{h}}) \subseteq \Theta(\mathfrak{h})$, so $\gamma_{\mathfrak{h}} \leq \gamma_{T_{\mathfrak{h}}}$. In addition, $\overline{\Theta(T_{\mathfrak{h}})} \supseteq \Theta(\mathfrak{h})$, so if $\gamma_{\mathfrak{h}} < \gamma_{T_{\mathfrak{h}}}$, then put $\epsilon := \frac{1}{2}(\gamma_{T_{\mathfrak{h}}} - \gamma_{\mathfrak{h}})$. Now,

$$\gamma_{\mathfrak{h}} < \gamma_{\mathfrak{h}} + \epsilon = \gamma_{\mathfrak{h}} + \frac{1}{2}(\gamma_{T_{\mathfrak{h}}} - \gamma_{\mathfrak{h}}) = \frac{1}{2}(\gamma_{T_{\mathfrak{h}}} + \gamma_{\mathfrak{h}}) < \gamma_{T_{\mathfrak{h}}} \leq x,$$

for all $x \in \Theta(T_{\mathfrak{h}})$. This implies that $|\gamma_{\mathfrak{h}} - x| > |\gamma_{\mathfrak{h}} - (\gamma_{\mathfrak{h}} + \epsilon)| = \epsilon$ for all $x \in \Theta(T_{\mathfrak{h}})$; hence $\overline{\Theta(T_{\mathfrak{h}})} \not\supseteq \Theta(\mathfrak{h})$, which is a contradiction. Consequently: $\gamma_{\mathfrak{h}} = \gamma_{T_{\mathfrak{h}}}$. ■

Proposition 3.0.39 *The map $\mathfrak{t} \mapsto T_{\mathfrak{t}}$ is a bijective correspondence between the set of densely defined, closed, sectorial sesquilinear forms on $\mathcal{H}$, and the set of m-sectorial operators on $\mathcal{H}$. In addition: $\mathfrak{t}$ is bounded if and only if $T_{\mathfrak{t}}$ is bounded; and $\mathfrak{t}$ is symmetric if and only if $T_{\mathfrak{t}}$ is self-adjoint.*

Proof: *The map $\mathfrak{t} \mapsto T_{\mathfrak{t}}$ is injective:* Let $\mathfrak{s}, \mathfrak{t}$ be densely defined, closed, sectorial sesquilinear forms on $\mathcal{H}$. Assume that $T_{\mathfrak{s}} = T_{\mathfrak{t}}$. Then $\mathfrak{s}(u,v) = (T_{\mathfrak{s}}u|v) = (T_{\mathfrak{t}}u|v) = \mathfrak{t}(u,v)$ for all $u \in u \in \mathcal{D}(T_{\mathfrak{s}}) = \mathcal{D}(T_{\mathfrak{t}})$ and all $v \in \mathcal{D}(\mathfrak{s}) = \mathcal{D}(T_{\mathfrak{s}}) = \mathcal{D}(T_{\mathfrak{t}})$. By Cor. 3.0.32, $\mathfrak{t} = \tilde{\mathfrak{s}}$, and since $\mathfrak{s}$ is closed, then $\mathfrak{s} = \tilde{\mathfrak{s}}$ by Prop. 2.2.9. Then $\mathfrak{s} = \mathfrak{t}$. Hence $\mathfrak{t} \mapsto T_{\mathfrak{t}}$ is injective.

The map $\mathfrak{t} \mapsto T_{\mathfrak{t}}$ is surjective: Given an m-sectorial operator T on $\mathcal{H}$. Define as in Cor. 3.0.32 $\mathfrak{t}_0(u,v) = (Tu|v)$, $\mathcal{D}(\mathfrak{t}_0) = \mathcal{D}(T)$. Then $\mathfrak{t}_0$ is densely defined and sectorial. By Prop. 2.2.19 it is closable. Put $\mathfrak{t} = \widetilde{\mathfrak{t}_0}$, and define $T_{\mathfrak{t}}$ according to Th. 3.0.31. By Prop. 3.0.34, $T_{\mathfrak{t}} \supseteq T$, and since both T and $T_{\mathfrak{t}}$ are m-sectorial, $T = T_{\mathfrak{t}}$.

$\mathfrak{t}$ *is bounded if and only if $T_{\mathfrak{t}}$ is bounded:* Assume that T is bounded, then

$$|\mathfrak{t}(u,v)| = |(Tu|v)| \leq \|Tu\|\|v\| \leq \|T\|\|u\|\|v\|,$$

so $\mathfrak{t}$ is bounded.

Assume that $\mathfrak{t}$ is bounded. Then there exists a constant $C \geq 0$ such that for any $v \in \mathcal{D}(T)$ we have

$$|\mathfrak{t}(u,v)| = |(Tu|v)| \leq C\|u\|\|v\|.$$

Now,

$$\|Tu\| = \sup_{v \in \mathcal{H}\setminus\{0\}} \frac{|(Tu|v)|}{\|v\|} \leq C\|u\|,$$

so T is bounded.

$\mathfrak{t}$ *is symmetric if and only if* $T_{\mathfrak{t}}$ *is self-adjoint:* Assume that $\mathfrak{t}$ is symmetric, then $T_{\mathfrak{t}}$ is self-adjoint by Prop. 3.0.38.

Assume that $T_{\mathfrak{t}}$ is self-adjoint, then by Prop. 3.0.37: $T_{\mathfrak{t}} = T_{\mathfrak{t}}^* = T_{\mathfrak{t}^*}$, and since the map $t \mapsto T_{\mathfrak{t}}$ is injective, then $\mathfrak{t} = \mathfrak{t}^*$, so $\mathfrak{t}$ is symmetric. ■

Proposition 3.0.40 *Let $\mathfrak{h}$ be a densely defined, closed, symmetric, lower bounded sesquilinear form on $\mathcal{H}$. Then*

$$\mathcal{D}(T_{\mathfrak{h}}) = \{u \in \mathcal{D}(\mathfrak{h}) \mid \exists C \geq 0\ \forall v \in \mathcal{D}(\mathfrak{h}) : |\mathfrak{h}(u,v)| \leq C\|v\|\}.$$

Remark: The constant C from above is in general dependent on u, but never on v.

Proof: $\subseteq$: Assume that $u \in \mathcal{D}(T_{\mathfrak{h}})$. Then $\mathfrak{h}(u,v) = (T_{\mathfrak{h}}u|v)$, so

$$|\mathfrak{h}(u,v)| = |(T_{\mathfrak{h}}u|v)| \leq \|T_{\mathfrak{h}}u\|\|v\| = C\|v\|.$$

Therefore, $\mathcal{D}(T_{\mathfrak{h}}) \subseteq \{u \in \mathcal{D}(\mathfrak{h}) \mid \exists C \geq 0\ \forall v \in \mathcal{D}(\mathfrak{h}) : |\mathfrak{h}(u,v)| \leq C\|v\|\}$.

$\supseteq$: Assume that $u \in \mathcal{D}(\mathfrak{h})$ such that $\exists C \geq 0\ \forall v \in \mathcal{D}(\mathfrak{h}) : |\mathfrak{h}(u,v)| \leq C\|v\|$. Then by definition, the linear functional $\mathfrak{h}(\cdot,u) : (\mathcal{D}(\mathfrak{h}), \|\cdot\|) \to \mathbb{C}$ is bounded; by [3] Th. 11.4, $\mathfrak{h}(\cdot,u)$ is continuous. It therefore extends uniquely by continuity to the continuous functional

$$\begin{cases} \mathfrak{h}(\cdot,u) : \mathcal{H} \to \mathbb{C} \\ \mathfrak{h}(v,u) = \lim_{n\to\infty} \mathfrak{h}(v_n,u), \quad \text{where} \quad \{v_n\} \subseteq \mathcal{D}(\mathfrak{h}), \quad \lim_{n\to\infty} \|v_n - v\| = 0 \end{cases}.$$

By the representation theorem of Riesz-Frechet,

$$\exists w \in \mathcal{H}\ \forall v \in \mathcal{H} : \mathfrak{h}(v,u) = (v|w).$$

This implies in particular, that

$$\exists w \in \mathcal{H}\ \forall v \in \mathcal{D}(\mathfrak{h}) : \mathfrak{h}^*(u,v) = \mathfrak{h}(u,v) = (w|v),$$

which again implies that

$$\exists w \in \mathcal{H}\ \forall \text{core}, \mathcal{D}, \text{of } \mathfrak{h}\ \forall v \in \mathcal{D} : \mathfrak{h}(u,v) = (w|v).$$

Now, by (iii) of Th. 3.0.31, $u \in \mathcal{D}(T_{\mathfrak{h}})$ and $T_{\mathfrak{h}}u = w$. In particular: $u \in \mathcal{D}(T_{\mathfrak{h}})$. Therefore, $\mathcal{D}(T_{\mathfrak{h}}) \supseteq \{u \in \mathcal{D}(\mathfrak{h}) \mid \exists C \geq 0\ \forall v \in \mathcal{D}(\mathfrak{h}) : |\mathfrak{h}(u,v)| \leq C\|v\|\}$. ■

Chapter 4

The Friedrichs extension

In this chapter, the Friedrichs extension is defined, and some properties of it will be derived. The Friedrichs extension is a canonical m-sectorial extension of a given sectorial operator, and it has various convenient properties. It can be thought of as an extension by closure in a certain sense, and as such, it is the minimal extension of all possible extensions.

In the particular case of extending a symmetric semibounded operator, the Friedrichs extension is self-adjoint. Furthermore, the lower/upper bound is preserved. In the general case of extending sectorial operators, the vertex is preserved.

The process of the extension depends heavily on the first representation theorem (Th. 3.0.31)

Definition 4.0.41 (The Friedrichs extension) *Let T be a densely defined sectorial operator on $\mathcal{H}$, and define the form $\mathfrak{t}$ on $\mathcal{H}$ by: $\mathfrak{t}(u,v) = (Tu|v)$, $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(T)$. The Friedrichs extension of T, denoted by T_F, is defined to be the m-sectorial operator associated to $\tilde{\mathfrak{t}}$, where $\tilde{\mathfrak{t}}$ is the closure of $\mathfrak{t}$, i.e.:*

$$T_F = T_{\tilde{\mathfrak{t}}}.$$

Proposition 4.0.42 *The Friedrichs extension of T is a well-defined extension of T.*

Proof: First, $\mathfrak{t}$ is closable by Prop. 2.2.19, so by Prop. 3.0.39 there is a unique $T_{\tilde{\mathfrak{t}}}$ associated to $\tilde{\mathfrak{t}}$.

Second, $\mathcal{D}(T) = \mathcal{D}(\mathfrak{t})$ is a core of $\tilde{\mathfrak{t}}$, so by Cor. 3.0.34, $T_{\tilde{\mathfrak{t}}} \supseteq T$. ■

Proposition 4.0.43 *If T is a densely defined, sectorial operator, then*

(i) $T_F = T$, if T is m-sectorial.

(ii) The Friedrichs extension of the Friedrichs extension of T is T_F itself, i.e.:

$$(T_F)_F = T_F.$$

Proof: Let T be a densely defined, sectorial operator on $\mathcal{H}$.

Ad *(i)*: An m-sectorial operator has no proper sectorial extensions, so one must have: $T_F = T$.

Ad (*ii*): If T is a densely defined sectorial operator, then the Friedrichs extension T_F of T is m-sectorial by Th. 3.0.31, so by the argument in (i) one has: $(T_F)_F = T_F$. ■

Proposition 4.0.44 *Among all m-sectorial extensions of $T^{'}$ of T, the Friedrichs extension T_F has the smallest form-domain (i.e. the domain of the associated form $\tilde{\mathfrak{t}}$ is contained in the domain of the form associated with any other T').*

Proof: Define $\mathfrak{t}'$ by $\mathfrak{t}'(u,v) = (T^{'}u, v)$, $\mathcal{D}(\mathfrak{t}') = \mathcal{D}(T^{'})$. Then $\widetilde{\mathfrak{t}'}$ is associated with $T^{'}$. Since $T^{'} \supseteq T$, we have $\mathfrak{t}' \supseteq \mathfrak{t}$, whence $\widetilde{\mathfrak{t}'} \supseteq \tilde{\mathfrak{t}}$. Consequently: $\mathcal{D}(\widetilde{\mathfrak{t}'}) \supseteq \mathcal{D}(\tilde{\mathfrak{t}})$. ■

Proposition 4.0.45 *The Friedrichs extension of T is the only m-sectorial extension of T with domain contained in $\mathcal{D}(\tilde{\mathfrak{t}})$.*

Proof: Let $T^{'}$ be any m-sectorial extension of T with $\mathcal{D}(T^{'}) \subseteq \mathcal{D}(\tilde{\mathfrak{t}})$. We need to show that $T^{'} = T_F$. Let $\mathfrak{t}'$ be defined as in Prop. 4.0.44. Then $\mathfrak{t}'(u,v) = (T^{'}u,v) = \tilde{\mathfrak{t}}(u,v)$, $\mathcal{D}(\mathfrak{t}') = \mathcal{D}(T^{'}) \subseteq \mathcal{D}(\tilde{t})$. Since $T^{'} = T_{\widetilde{\mathfrak{t}'}}$, $\tilde{\mathfrak{t}}' \supseteq \tilde{\mathfrak{t}}$ and $\mathcal{D}(T^{'}) \subseteq \mathcal{D}(\tilde{\mathfrak{t}})$, then $T^{'} \subseteq T_F$ by Cor. 3.0.34. Both $T^{'}$ and T_F are m-sectorial, so $T^{'} = T_F$. ■

Part II

Spectral Analysis

Chapter 5

Hamiltonian of the δ-interaction

Consider the form $\mathfrak{h}$ on $L^2(\mathbb{R})$ defined by:

$$\begin{cases} \mathcal{D}(\mathfrak{h}) = \mathcal{H}^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | f' \in L^2(\mathbb{R})\} \\ \mathfrak{h}(f,g) = \int_{\mathbb{R}} f'(x)\overline{g'(x)}dx + \lambda f(0)\overline{g(0)}, \quad \lambda < 0 \end{cases} \tag{5.1}$$

Here the primes "′" are understood to be weak derivatives.

Proposition 5.0.46 *The form defined in* (5.1) *is sesquilinear, densely defined, symmetric and bounded from below.*

Proof: *Sesquilinear:* Linearity in the first variable follows from linearity of the Lebesque integral, and the distributive law of real numbers; conjugate linearity in the second variable follows in the same manner.

Densely defined: $\mathcal{H}^1(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Symmetric: For any $f \in \mathcal{D}(\mathfrak{h})$ one has: $\mathfrak{h}(f) = \mathfrak{h}(f,f) = \int_{\mathbb{R}} |f'(x)|^2 dx + \lambda |f(0)|^2 \in \mathbb{R}$. By Prop. 2.0.7, $\mathfrak{h}$ is symmetric.

Bounded from below: The numerical range of $\mathfrak{h}$ is $\Theta(\mathfrak{h}) = \{\mathfrak{h}(f) \mid f \in \mathcal{D}(\mathfrak{h}),\ \|f\|_{L^2(\mathbb{R})} = 1\}$, so assuming that a given $f \in \mathcal{D}(\mathfrak{h})$ has $\|f\|_{L^2(\mathbb{R})} = 1$, one gets

$$\mathfrak{h}(f) = \int_{\mathbb{R}} |f'(x)|^2 dx + \lambda |f(0)|^2 = \|f'\|_2^2 + \lambda |f(0)|^2. \tag{5.2}$$

From Sobolevs embedding theorem:

$$|f(0)|^2 \leq \|f\|_\infty^2 \leq C^2 \|f\|^2_{\mathcal{H}^1(\mathbb{R})} = C^2(\|f\|_2^2 + \|f'\|_2^2) = C^2(1 + \|f'\|_2^2)$$

and since $\lambda < 0$, then

$$\lambda |f(0)|^2 \geq \lambda C^2 (1 + \|f'\|_2^2),$$

which together with (5.2) gives

$$\mathfrak{h}(f) \geq \|f'\|_2^2 + \lambda C^2(1 + \|f'\|_2^2) = \lambda C^2 + (1 + \lambda C^2)\|f'\|_2^2.$$

If $\lambda \geq -C^{-2}$, then $1 + \lambda C^2 \geq 0$, which implies that $(1 + \lambda C^2)\|f'\|_2^2 \geq 0$; hence $\mathfrak{h}(f) \geq \lambda C^2$. Therefore, $\mathfrak{h}$ is lower bounded with λC^2 as a lower bound. ■

In order to be able to use Th. 3.0.39, it must be shown that $\mathfrak{h}$ is closed.

Proposition 5.0.47 *The form* $\mathfrak{h}$ *defined in* (5.1) *is closed.*

Proof: By Prop. 2.1.11, $\mathfrak{h}$ is closed if and only if $\mathfrak{h} - \lambda C^2$ is closed. First it is shown that $\mathfrak{h} - \lambda C^2$ is closable. Assume that $f_n \overset{\mathfrak{h}-\lambda C^2}{\longrightarrow} 0$, i.e.:

- $\forall n \in \mathbb{N} : f_n \in L^2(\mathbb{R}) \quad \text{and} \quad f_n' \in L^2(\mathbb{R})$
- $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|^2 dx = 0$
- $$\begin{aligned} & \lim_{m,n\to\infty} (\mathfrak{h} - \lambda C^2)(f_n - f_m) \\ = & \lim_{m,n\to\infty} \left\{ \int_{\mathbb{R}} |(f_n - f_m)'(x)|^2 dx + \lambda |(f_n - f_m)(0)|^2 - \lambda C^2 \|f_n - f_m\|^2 \right\} \\ = & \lim_{m,n\to\infty} \int_{\mathbb{R}} |f_n'(x) - f_m'(x)|^2 dx + \lambda \lim_{m,n\to\infty} |f_n(0) - f_m(0)|^2 \\ - & \lambda C^2 \lim_{m,n\to\infty} \|f_n - f_m\|^2 \\ = & \lim_{m,n\to\infty} \|f_n' - f_m'\|^2 + \lambda \lim_{m,n\to\infty} |f_n(0) - f_m(0)|^2 - \lambda C^2 \lim_{m,n\to\infty} \|f_n - f_m\|^2 \\ = & \ 0. \end{aligned}$$

We have the following estimates:

$$|f_n(0) - f_m(0)|^2 \leq \|f_n - f_m\|_\infty^2 \leq C^2(\|f_n - f_m\|^2 + \|f_n' - f_m'\|^2),$$

which implies that

$$\lambda |f_n(0) - f_m(0)|^2 \geq \lambda C^2(\|f_n - f_m\|^2 + \|f_n' - f_m'\|^2). \tag{5.3}$$

Now, using (5.3) we have that

$$\begin{aligned} & \lim_{m,n\to\infty} (\mathfrak{h} - \lambda C^2)(f_n - f_m) \\ \geq & \lim_{m,n\to\infty} \left\{ \|f_n' - f_m'\|^2 - \lambda C^2 \|f_n - f_m\|^2 + \lambda C^2(\|f_n - f_m\|^2 + \|f_n' - f_m\|^2) \right\} \\ = & \lim_{m,n\to\infty} (1 + \lambda C^2) \|f_n' - f_m'\|^2. \end{aligned}$$

Assuming that $\lambda > -\frac{1}{C^2}$, then $1 + \lambda C^2 \geq 0$; so since $\lim_{m,n\to\infty}(\mathfrak{h} - \lambda C^2)(f_n - f_m) = 0$, then $\lim_{m,n\to\infty} \|f_n' - f_m'\| = 0$. Consequently, $\lim_{m,n\to\infty} \|f_n - f_m\|_{\mathcal{H}^1(\mathbb{R})} = 0$. $\mathcal{H}^1(\mathbb{R})$ is complete, so $\exists f \in \mathcal{H}^1(\mathbb{R}) : \|f_n - f\|_{\mathcal{H}^1(\mathbb{R})} \to 0$. This implies that $\|f_n - f\|_2 \to 0$, but $\|f_n\|_2 \to 0$, so $f = 0$ (in $L^2(\mathbb{R})$). Hence $\|f_n'\|_2 \to 0$. Now, $\{f_n\} \subseteq \mathcal{H}^1(\mathbb{R}) \subseteq C(\mathbb{R})$, so $\|f_n\|_{\mathcal{H}^1(\mathbb{R})} \to 0$ implies that $\|f_n\|_\infty \to 0$, and then $|f_n(0)| \to 0$. Hence,

$$\begin{aligned} (\mathfrak{h} - \lambda C^2)(f_n) &= \int_{\mathbb{R}} |f_n'(x)| dx + \lambda |f_n(0)|^2 - \lambda C^2 \|f_n\|^2 \\ &= \|f_n'\|_2^2 + \lambda |f_n(0)|^2 - \lambda C^2 \|f_n\|_2^2 \\ &\to 0. \end{aligned}$$

Consequently, $\mathfrak{h} - \lambda C^2$ is closable by Prop. 2.2.10. Next, it is shown that $\mathfrak{h} - \lambda C^2$ is closed. By Prop. 2.2.9 it is enough to show that $\widetilde{\mathfrak{h} - \lambda C^2} = \mathfrak{h} - \lambda C^2$. Given $f \in \mathcal{D}(\widetilde{\mathfrak{h} - \lambda C^2})$; then $\exists \{f_n\} \subseteq \mathcal{D}(\mathfrak{h} - \lambda C^2) : f_n \overset{\mathfrak{h}-\lambda C^2}{\longrightarrow} f$. It will be shown that $f \in \mathcal{D}(\mathfrak{h} - \lambda C^2) = \mathcal{D}(\mathfrak{h})$. Now, $\|f_n - f\|_2 \to 0$ and $(\mathfrak{h} - \lambda C^2)(f_n - f_m) \to 0$. As shown earlier, $\|f_n' - f_m'\|_2 \to 0$, so

$$\|f_n - f_m\|_{\mathcal{H}^1(\mathbb{R})} = \|f_n - f_m\|_2 + \|f_n' - f_m'\|_2 \to 0.$$

$\mathcal{H}^2(\mathbb{R})$ is complete, so $\exists g \in \mathcal{H}^1(\mathbb{R}) : \|f_n - g\|_{\mathcal{H}^1(\mathbb{R})} \to 0$. This implies that $\|f_n - g\|_2 + \|f_n' - g'\|_2 \to 0$, and in particular: $\|f_n - g\|_2 \to 0$. But $\|f_n - f\|_2 \to 0$, so $f = g \in \mathcal{H}^1(\mathbb{R}) = \mathcal{D}(\mathfrak{h} - \lambda C^2)$. Therefore $\mathfrak{h} - \lambda C^2$ is closed. By Prop. 2.1.11, $\mathfrak{h}$ is closed. ■

Now, since $\mathfrak{h}$ is a sesquilinear, densely defined, closed, symmetric and lower bounded form on the Hilbert space $L^2(\mathbb{R})$, Prop. 3.0.39 applies and there is therefore a self-adjoint operator H on $L^2(\mathbb{R})$ such that

$$\forall f \in \mathcal{D}(H)\ \forall g \in \mathcal{D}(\mathfrak{h}) : \mathfrak{h}(f, g) = (Hf|g).$$

5.1 The domain of the Hamiltonian H

This section is devoted to the derivation of the exact domain of δ-interaction Hamiltonian. Let therefore H be the lower bounded self-adjoint operator associated to $\mathfrak{h}$, which was defined in (5.1). Then by Prop. 3.0.40 we have that

$$\mathcal{D}(H) = \{f \in \mathcal{H}^1(\mathbb{R}) \mid \exists C \geq 0\ \forall g \in \mathcal{H}^1(\mathbb{R}) : |\mathfrak{h}(f, g)| \leq C\|g\|\}.$$

Lemma 5.1.1 *Let $f \in \mathcal{H}^1(\mathbb{R})$ be a function that fulfills that $f \in \mathcal{H}^2(\mathbb{R}\backslash\{0\})$ and $\lim_{\epsilon \searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0)$. Then $f \in \mathcal{D}(H)$.*

Proof: Assume that $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$ satisfies the condition $\lim_{\epsilon \searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0)$. According to Prop. 3.0.40 it is sufficient to prove that $\exists C \geq 0\ \forall g \in \mathcal{H}^1(\mathbb{R}) : |\mathfrak{h}(f, g)| \leq C\|g\|_{L^2(\mathbb{R})}$. Let $g \in \mathcal{H}^1(\mathbb{R})$; then $g, g' \in L^2(\mathbb{R})$. Using Prop. A.1.3 one obtains

$$\begin{aligned}
|\mathfrak{h}(f, g)| &= |-\int_{\mathbb{R}} f'(x)\overline{g'(x)}dx + \lambda f(0)\overline{g(0)}| \\
&= |-\int_{\mathbb{R}\backslash\{0\}} f'(x)\overline{g'(x)}dx + \lambda f(0)\overline{g(0)}| \\
&= |\int_{\mathbb{R}\backslash\{0\}} f''(x)\overline{g(x)}dx - \lambda f(0)\overline{g(0)} + \lambda f(0)\overline{g(0)}| \\
&\leq \|f''\|_{L^2(\mathbb{R}\backslash\{0\})}\|g\|_{L^2(\mathbb{R}\backslash\{0\})} \\
&= C\|g\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, $f \in \mathcal{D}(H)$. ■

Lemma 5.1.2 *Let $f \in \mathcal{H}^1(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi \equiv 1$ on $]-M, M[^c$, $\varphi \equiv 0$ on $[-m, m]$, $0 \leq m < M$ and $0 \leq \varphi \leq 1$. Then $f\varphi \in \mathcal{H}^1(\mathbb{R})$.*

Proof: First, since $f \in \mathcal{H}^1(\mathbb{R})$ by assumption, then in particular $f \in L^2(\mathbb{R})$. Now,

$$\int_{\mathbb{R}} |f(x)\varphi(x)|^2 dx \le \sup_{x\in\mathbb{R}} |\varphi(x)|^2 \int_{\mathbb{R}} |f(x)|^2 dx = C\|f\|^2_{L^2(\mathbb{R})} < \infty,$$

so $f\varphi \in L^2(\mathbb{R})$. Next, it will be shown that $D(f\varphi) \in L^2(\mathbb{R})$. The quantity $D(f\varphi)$ will be computed. Now, assuming that $\psi \in \mathcal{D}(\mathbb{R})$ we get:

$$\begin{aligned}\langle f\varphi, \psi' \rangle &= \int_{\mathbb{R}} f(x)\varphi(x)\psi'(x)dx = \int_{\mathbb{R}} f(x)\varphi(x)\psi'(x)dx = \langle f, \varphi\psi' \rangle \\ &= \langle f, (\varphi\psi)' - \varphi'\psi \rangle = \langle f, (\varphi\psi)' \rangle - \langle f, \varphi'\psi \rangle \\ &= -\langle D(f), \varphi\psi \rangle - \langle f\varphi', \psi \rangle \\ &= -\langle \varphi D(f), \psi \rangle - \langle f\varphi', \psi \rangle \\ &= -\langle \varphi D(f) + f\varphi', \psi \rangle. \end{aligned} \tag{5.4}$$

By definition of the distributional derivative, $\langle D(f\varphi), \psi \rangle = -\langle f\varphi, \psi' \rangle$ for any $\psi \in \mathcal{D}(\mathbb{R})$, so by (5.4), $D(f\varphi) = \varphi D(f) + f\varphi'$. By definition of φ, $\varphi' \in \mathcal{D}(\mathbb{R})$; also, $f \in \mathcal{H}^1(\mathbb{R})$, so $f, D(f) \in L^2(\mathbb{R})$ by definition of $\mathcal{H}^1(\mathbb{R})$. Now,

$$\|\varphi D(f)\|^2_{L^2(\mathbb{R})} \le \sup_{x\in\mathbb{R}} |\varphi(x)|^2 \int_{\mathbb{R}} |D(f)(x)|^2 dx = C\|D(f)\|^2_{L^2(\mathbb{R})} < \infty$$

and

$$\|f\varphi'\|^2_{L^2(\mathbb{R})} \le \sup_{x\in\mathbb{R}} |\varphi'(x)|^2 \int_{\mathbb{R}} |f(x)|^2 dx = C'\|f\|^2_{L^2(\mathbb{R})} < \infty,$$

so $\varphi D(f), f\varphi' \in L^2(\mathbb{R})$, which implies that $D(f\varphi) = \varphi D(f) + f\varphi' \in L^2(\mathbb{R})$. Consequently, $f\varphi \in \mathcal{H}^1(\mathbb{R})$. ■

Lemma 5.1.3 *Let $\mathfrak{h}$ be the form defined in* (5.1) *and H the self-adjoint operator associated to $\mathfrak{h}$. Assume that $f \in \mathcal{D}(H)$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi \equiv 1$ on $]-M, M[^c$, $\varphi \equiv 0$ on $[-m, m]$, $0 \le m < M$, and $0 \le \varphi \le 1$. Then $f\varphi \in \mathcal{H}^2(\mathbb{R})$.*

Proof: By Lemma 5.1.2 it is enough to show that $D^2(f\varphi) \in L^2(\mathbb{R})$. We now compute $D^2(f\varphi)$; let therefore $\psi \in \mathcal{D}(\mathbb{R})$ be given; then

$$\begin{aligned}\langle D^2(f\varphi), \psi \rangle &= -\langle D(f\varphi), \psi' \rangle = -\langle D(f)\varphi + f\varphi', \psi' \rangle \\ &= -\langle D(f)\varphi, \psi' \rangle - \langle f\varphi', \psi' \rangle\end{aligned}$$

By Lemma 5.1.2, $D(f\varphi') = D(f)\varphi' + f\varphi''$. Now, $\varphi', \varphi'' \in \mathcal{D}(\mathbb{R})$ and $f, D(f) \in L^2(\mathbb{R})$ since $f \in \mathcal{H}^1(\mathbb{R})$. Therefore $D(f\varphi') \in L^2(\mathbb{R})$. We now get that

$$\begin{aligned}\langle D^2(f\varphi), \psi \rangle &= -\langle D(f), \varphi\psi' \rangle + \langle D(f\varphi') | \psi \rangle \\ &= -\langle D(f) | (\varphi\psi)' \rangle + \langle D(f), \varphi'\psi \rangle + \langle D(f\varphi'), \psi \rangle.\end{aligned}$$

By definition of φ, $\varphi(0) = 0$, so $\lambda f(0)\overline{(\varphi\psi)(0)} = 0$. Using the definition of $\mathfrak{h}$ we get that

$$\langle D^2(f\varphi), \psi \rangle = -\mathfrak{h}(f, \varphi\psi) + \langle D(f), \varphi'\psi \rangle + \langle D(f\varphi'), \psi \rangle$$

Since $f \in \mathcal{D}(H)$ by assumption, we get from the first representation theorem, Th. 3.0.31, that

$$\begin{aligned} \langle D^2(f\varphi), \psi \rangle &= -\langle Hf, \varphi\psi \rangle + \langle D(f), \varphi'\psi \rangle + \langle D(f\varphi'), \psi \rangle \\ &= -\langle \varphi Hf, \psi \rangle + \langle \varphi' D(f), \psi \rangle + \langle D(f)\varphi' + f\varphi'', \psi \rangle \\ &= \langle -\varphi Hf + 2\varphi' D(f) + f\varphi'', \psi \rangle. \end{aligned} \tag{5.5}$$

Since (5.5) holds for any $\psi \in \mathcal{D}(\mathbb{R})$, we obtain:

$$D^2(f\varphi) = -\varphi Hf + 2\varphi' D(f) + f\varphi''.$$

Now, $\varphi', \varphi'' \in \mathcal{D}(\mathbb{R})$; and $f \in \mathcal{H}^1(\mathbb{R})$, so $f, D(f) \in L^2(\mathbb{R})$. Therefore, $\varphi' D(f), f\varphi'' \in L^2(\mathbb{R})$. Furthermore, $H : \mathcal{D}(H) \to L^2(\mathbb{R})$, so $Hf \in L^2(\mathbb{R})$. Since $0 \leq \varphi \leq 1$, $\varphi Hf \in L^2(\mathbb{R})$. Consequently,

$$D^2(f\varphi) = -\varphi Hf + 2\varphi' D(f) + f\varphi'' \in L^2(\mathbb{R}),$$

so $f\varphi \in \mathcal{H}^2(\mathbb{R})$. ■

Proposition 5.1.4 *The following biimplication holds true:*

$$f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\}) \tag{5.6}$$

if and only if

$$\begin{aligned} &\forall\, m \geq 0\ \forall M > m : \\ &f \in \mathcal{H}^1(\mathbb{R}) \ \wedge \ f''\varphi_{m,M} \in L^2(\mathbb{R}) \ \wedge \ \lim_{M \searrow 0} \int_{\mathbb{R}} |(f''\varphi_{m,M})(x)|^2 dx < \infty. \end{aligned}$$

Proof: $\Rightarrow$: Assume that $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$ and let $m \geq 0$ and $M > m$ be given. Then $f'' \in L^2(\mathbb{R} \setminus \{0\})$, which implies $f'' \in L^2(\mathbb{R} \setminus [-m, m])$. Therefore also $f''\varphi_{m,M} \in L^2(\mathbb{R})$. Now, $|f''\varphi_{m,M}(x)| \leq |f''(x)|$ for all $0 \leq m < M$ and all $x \in \mathbb{R} \setminus \{0\}$ and $f''\varphi_{m,M}(x) \to f''(x)$ as $M \searrow 0$ for all $x \in \mathbb{R} \setminus \{0\}$. The theorem on monotone convergence gives that

$$\lim_{M \searrow 0} \int_{\mathbb{R}} |(f''\varphi_{m,M})(x)|^2 dx = \int_{\mathbb{R} \setminus \{0\}} |f''(x)|^2 dx.$$

The last integral is finite, since $f'' \in L^2(\mathbb{R} \setminus \{0\})$.

$\Leftarrow$: Assume that $f \in \mathcal{H}^1(\mathbb{R})$, that $f''\varphi_{m,M} \in L^2(\mathbb{R})$ for all $0 \leq m < M$ and that the limit $\lim_{M \searrow 0} \int_{\mathbb{R}} |(f''\varphi_{m,M})(x)|^2 dx$ exists and is finite. The same argument as above gives

$$\int_{\mathbb{R} \setminus \{0\}} |f''(x)|^2 dx = \lim_{M \searrow 0} \int_{\mathbb{R}} |(f''\varphi_{m,M})(x)|^2 dx < \infty.$$

Hence $f'' \in L^2(\mathbb{R} \setminus \{0\})$. Since also $f \in \mathcal{H}^1(\mathbb{R})$, one concludes that $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$. ■

Proposition 5.1.5 *Let $f \in \mathcal{H}^1(\mathbb{R})$ and $0 \leq m < M$ be given. Then*

$$D^2(f\varphi_{m,M}) = f''\varphi_{m,M} + 2f'\varphi'_{m,M} + f\varphi''_{m,M},$$

with equality in the sense of distributions.

Furthermore,

$$D^2(f\varphi_{m,M}) \in L^2(\mathbb{R}) \Longleftrightarrow f''\varphi_{m,M} \in L^2(\mathbb{R}).$$

Proof: Let $f \in \mathcal{H}^1(\mathbb{R})$, $\psi \in \mathcal{D}(\mathbb{R})$ and $0 \leq m < M$ be given. Then

$$\begin{aligned}
\langle D^2(f\varphi_{m,M}), \psi\rangle &= \int_{\mathbb{R}} (f\varphi_{m,M})(x)\psi''(x)dx = -\int_{\mathbb{R}} (f\varphi_{m,M})'(x)\psi'(x)dx \\
&= -\int_{\mathbb{R}} (f'\varphi_{m,M} + f\varphi'_{m,M})(x)\psi'(x)dx \\
&= \int_{\mathbb{R}} (f''\varphi_{m,M} + 2f'\varphi'_{m,M} + f\varphi''_{m,M})(x)\psi(x)dx \\
&= \langle f''\varphi_{m,M} + 2f'\varphi'_{m,M} + f\varphi''_{m,M}, \psi\rangle.
\end{aligned}$$

Since ψ was arbitrary,

$$D^2(f\varphi_{m,M}) = f''\varphi_{m,M} + 2f'\varphi'_{m,M} + f\varphi''_{m,M}. \tag{5.7}$$

Now, $\varphi'_{m,M}, \varphi''_{m,M} \in \mathcal{D}(\mathbb{R})$ and $f, f' \in L^2(\mathbb{R})$, so $f'\varphi'_{m,M}, f\varphi''_{m,M} \in L^2(\mathbb{R})$. It is now immediate from (5.7), that $D^2(f\varphi_{m,M}) \in L^2(\mathbb{R})$ if and only if $f''\varphi_{m,M} \in L^2(\mathbb{R})$. ■

Proposition 5.1.6 *Let $f \in \mathcal{D}(H)$ and $0 \leq m < M$ be given. Then $f''\varphi_{m,M} \in L^2(\mathbb{R})$ and*

$$\varphi_{m,M}Hf = -\varphi_{m,M}f''.$$

Proof: Assume that $f \in \mathcal{D}(H)$ and $0 \leq m < M$. Then by Lemma 5.1.3, $D^2(f\varphi_{m,M}) \in L^2(\mathbb{R})$; and then $f''\varphi_{m,M} \in L^2(\mathbb{R})$ by Prop. 5.1.5. Furthermore,

$$D^2(f\varphi_{m,M}) = -\varphi_{m,M}Hf + 2f'\varphi'_{m,M} + f\varphi''_{m,M}. \tag{5.8}$$

By Prop. 5.1.5, $f''\varphi_{m,M} \in L^2(\mathbb{R})$ and

$$D^2(f\varphi_{m,M}) = f''\varphi_{m,M} + 2f'\varphi'_{m,M} + f\varphi''_{m,M}. \tag{5.9}$$

Equating the expressions in (5.8) and (5.9) one obtains:

$$\varphi_{m,M}Hf = -\varphi_{m,M}f''.$$

■

Lemma 5.1.7 *Let $f \in \mathcal{D}(H)$. Then $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$.*

Proof: Assume that $f \in \mathcal{D}(H)$. Then by Prop. 5.1.6, $f''\varphi_{m,M} \in L^2(\mathbb{R})$ and $\varphi_{m,M}Hf = -\varphi_{m,M}f''$ for all $0 \leq m < M$. Now,

$$\begin{aligned}
\lim_{M\searrow 0} \int_{\mathbb{R}} |(f''\varphi_{m,M})(x)|^2 dx &= \lim_{M\searrow 0} \int_{\mathbb{R}} |(\varphi_{m,M}Hf)(x)|^2 dx \\
&\leq \lim_{M\searrow 0} \int_{\mathbb{R}} |(Hf)(x)|^2 dx \\
&= \int_{\mathbb{R}} |(Hf)(x)|^2 dx.
\end{aligned}$$

The last integral is finite, since $Hf \in L^2(\mathbb{R})$. By Prop. 5.1.4, $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$. ■

Proposition 5.1.8 *Let* $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$. *Then*

$$\exists C \geq 0 \ \forall \ 0 < \epsilon < 2 : |f'(\epsilon)| \leq C \wedge |f'(-\epsilon)| \leq C.$$

Proof: It will without loss of generality be proven that $|f'(\epsilon)| < C$ for all $0 < \epsilon < 2$, where $C \geq 0$ is independent of ϵ. Assume that $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$ and let $0 < \epsilon < 2$; then $D^2(f\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}) \in L^2(\mathbb{R})$ by Prop. 5.1.4 and Prop. 5.1.5. Now, Prop. A.0.4 gives that

$$\begin{aligned}
|f'(2) - f'(\epsilon)| &= |(f\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})'(2) - (f\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})'(\epsilon)| \\
&= |\int_\epsilon^2 D^2(f\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)dx| \\
&= |\int_\mathbb{R} (f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + 2f'\varphi'_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + f\varphi''_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)dx| \\
&\leq \int_\epsilon^2 |(f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + 2f'\varphi'_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + f\varphi''_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)|dx \\
&\leq \sqrt{2-\epsilon}\sqrt{\int_\epsilon^2 |(f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + 2f'\varphi'_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} + f\varphi''_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)|^2dx},
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used in the last inequality. The triangle inequality now gives:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}|f'(2) - f'(\epsilon)| \\
\leq &\sqrt{\int_\epsilon^2 |(f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)|^2dx} + \sqrt{\int_\epsilon^2 |2(f'\varphi'_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)|^2dx} + \sqrt{\int_\epsilon^2 |(f\varphi''_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon})(x)|^2dx}.
\end{aligned}$$

Now, $\varphi'_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}, \varphi''_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon} \equiv 0$ on $[\epsilon, 2]$, so

$$|f'(2) - f'(\epsilon)| \leq \sqrt{2}\sqrt{\int_\epsilon^2 |f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}|^2dx},$$

which implies that

$$\begin{aligned}
|f'(2) - f'(\epsilon)| &\leq \sqrt{2}\lim_{\epsilon\searrow 0}\sqrt{\int_\epsilon^2 |f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}|^2dx} \\
&= \sqrt{2\lim_{\epsilon\searrow 0}\int_\epsilon^2 |f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}|^2dx}.
\end{aligned}$$

Finally,

$$|f'(\epsilon)| \leq |f'(2)| + \sqrt{2\lim_{\epsilon\searrow 0}\int_\epsilon^2 |f''\varphi_{\frac{1}{4}\epsilon,\frac{1}{2}\epsilon}|^2dx} < \infty,$$

since the limit under the square root is finite by Prop. 5.1.4. ■

Lemma 5.1.9 *Let $\{\varphi_{\eta,\epsilon} \mid 0 < \eta < \epsilon\}$ be a family of functions satisfying the properties described in Lemma B.0.13. Then*

$$\mathcal{D}' - \lim_{\eta \searrow 0} D\varphi_{\epsilon-\eta,\epsilon} = -\delta_{-\epsilon} + \delta_\epsilon.$$

Proof: First, make the identification $L^1_{\mathrm{loc}}(\mathbb{R}) \ni \varphi_{\epsilon-\eta,\epsilon} \mapsto \Lambda_{\varphi_{\epsilon-\eta,\epsilon}} \in \mathcal{D}'(\mathbb{R})$ and let $\psi \in \mathcal{D}(\mathbb{R})$ be given; then, denoting by H the Heaviside function,

$$\begin{aligned}
& \Lambda_{\varphi_{\epsilon-\eta,\epsilon}}\psi \\
= & \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x)\psi(x)dx \\
= & \int_{-\infty}^{-\epsilon} \psi(x)dx + \left(\int_{-\epsilon}^{-\epsilon+\eta} + \int_{\epsilon-\eta}^{\epsilon}\right) \varphi_{\epsilon-\eta,\epsilon}(x)\psi(x)dx + \int_{\epsilon}^{\infty} \psi(x)dx \\
= & \int_{\mathbb{R}} H(-\epsilon - x)\psi(x)dx + \left(\int_{-\epsilon}^{-\epsilon+\eta} + \int_{\epsilon-\eta}^{\epsilon}\right) \varphi_{\epsilon-\eta,\epsilon}(x)\psi(x)dx + \int_{\mathbb{R}} H(x-\epsilon)\psi(x)dx.
\end{aligned}$$

Write

$$\begin{aligned}
\left|\int_{\epsilon-\eta}^{\epsilon} \varphi_{\epsilon-\eta,\epsilon}(x)\psi(x)dx\right| &= \left|\int_{\mathbb{R}} \chi_{[\epsilon-\eta,\epsilon]}(x)\varphi_{\epsilon-\eta,\epsilon}(x)\psi(x)dx\right| \\
&\leq \sup_{x\in\mathbb{R}} |\psi(x)| \cdot \mathrm{vol}([\epsilon-\eta,\epsilon]). \qquad (5.10)
\end{aligned}$$

Now, since $\mathrm{vol}([\epsilon-\eta,\epsilon]) \searrow 0$ as $\eta \searrow 0$, then by (5.10), $\int_{\epsilon-\eta}^{\epsilon} \varphi_{\epsilon-\eta,\epsilon}\psi \searrow 0$ as $\eta \searrow 0$. Similarly, $\int_{-\epsilon}^{-\epsilon+\eta} \varphi_{\epsilon-\eta,\epsilon}\psi \searrow 0$ as $\eta \searrow 0$. Consequently,

$$\begin{aligned}
\lim_{\eta\searrow 0} \Lambda_{\varphi_{\epsilon-\eta,\epsilon}}\psi &= \int_{\mathbb{R}} H(-\epsilon - x)\psi(x)dx + \int_{\mathbb{R}} H(x-\epsilon)\psi(x)dx \\
&= \Lambda_{H(-\epsilon-\cdot)}\psi + \Lambda_{H(\cdot-\epsilon)}\psi.
\end{aligned}$$

Since this holds for every test function $\psi \in \mathcal{D}(\mathbb{R})$, then

$$\mathcal{D}' - \lim_{\eta\searrow 0} \Lambda_{\varphi_{\epsilon-\eta}} = \Lambda_{H(-\epsilon-\cdot)} + \Lambda_{H(\cdot-\epsilon)}.$$

By [14], Th. 6.17,

$$\mathcal{D}' - \lim_{\eta\searrow 0} \Lambda'_{\varphi_{\epsilon-\eta,\epsilon}} = \Lambda'_{H(-\epsilon-\cdot)} + \Lambda'_{H(\cdot-\epsilon)} = -\delta_{-\epsilon} + \delta_\epsilon.$$

■

Lemma 5.1.10 *Assume that $f \in \mathcal{D}(H)$. Then*

$$\lim_{\epsilon\searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0).$$

Proof: Assume that $f \in \mathcal{D}(H)$, $g \in \mathcal{H}^1(\mathbb{R})$ and that $\{\varphi_{\epsilon-\eta,\epsilon} \mid 0 < \eta < \epsilon\}$ is a family of functions satisfying the properties described in Lemma B.0.13. Then

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g'(x)}dx &= (\varphi_{\epsilon-\eta,\epsilon} f' | g') \\
&= (D(\varphi_{\epsilon-\eta,\epsilon} f) - \varphi'_{\epsilon-\eta,\epsilon} f | g') \\
&= -(D^2(\varphi_{\epsilon-\eta,\epsilon} f) - D(\varphi'_{\epsilon-\eta,\epsilon} f) | g) \\
&= (-f\varphi''_{\epsilon-\eta,\epsilon} - 2f'\varphi'_{\epsilon-\eta,\epsilon} + \varphi_{\epsilon-\eta,\epsilon}(Hf) + \varphi''_{\epsilon-\eta,\epsilon} f + f'\varphi'_{\epsilon-\eta,\epsilon} | g) \\
&= (\varphi_{\epsilon-\eta,\epsilon}(Hf) - f'\varphi'_{\epsilon-\eta,\epsilon} | g) \\
&= \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x)(Hf)(x)\overline{g(x)}dx - \int_{\mathbb{R}} \varphi'_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g(x)}dx. \quad (5.11)
\end{aligned}$$

Now, Lemma 5.1.9 gives that

$$\begin{aligned}
\lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi'_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g(x)}dx &= (-\delta_{-\epsilon} + \delta_{\epsilon})(f'\overline{g}) \\
&= f'(\epsilon)\overline{g(\epsilon)} - f'(-\epsilon)\overline{g(-\epsilon)}. \quad (5.12)
\end{aligned}$$

It should be noted that the δ-functional is usually defined on $\mathcal{D}(\mathbb{R})$, but can be extended to $C(\mathbb{R})$ sacrificing continuity though.

Since $\varphi_{\epsilon-\eta,\epsilon} \searrow \chi_{]-\epsilon,\epsilon[^c}$ as $\eta \searrow 0$, then the monotone convergence theorem gives that the limit $\lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(Hf)\overline{g}$ exists and

$$\lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x)(Hf)(x)\overline{g(x)}dx = \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)(Hf)(x)\overline{g(x)}dx. \quad (5.13)$$

Now, (5.11), (5.12) and (5.13) implies that

$$\begin{aligned}
&\lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g'(x)}dx \\
= &\lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x)(Hf)(x)\overline{g(x)}dx - \lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi'_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g(x)}dx \\
= &\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)(Hf)(x)\overline{g(x)}dx + f'(-\epsilon)\overline{g(-\epsilon)} - f'(\epsilon)\overline{g(\epsilon)}.
\end{aligned}$$

Since, $\chi_{]-\epsilon,\epsilon[^c}(Hf)\overline{g} \nearrow (Hf)\overline{g} \in L^1(\mathbb{R})$ as $\epsilon \searrow 0$, then by the monotone convergence theorem, $\lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \chi_{]-\epsilon,\epsilon[^c}(Hf)\overline{g}$ exists and

$$\begin{aligned}
\lim_{\epsilon \searrow 0} \lim_{\eta \searrow 0} \int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x)(Hf)(x)\overline{g(x)}dx &= \lim_{\epsilon \searrow 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)(Hf)(x)\overline{g(x)}dx \\
&= \int_{\mathbb{R}} (Hf)(x)\overline{g(x)}dx \\
&= (Hf|g).
\end{aligned}$$

Now, on the one hand,

$$\begin{aligned}
\lim_{\epsilon\searrow 0}\lim_{\eta\searrow 0}\int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g'(x)}dx &= \lim_{\epsilon\searrow 0}\int_{\mathbb{R}} \chi_{]-\epsilon,\epsilon[^c}(x) f'(x)\overline{g'(x)}dx \\
&= \int_{\mathbb{R}} f'(x)\overline{g'(x)}dx \\
&= (f'|g') \\
&= (Hf|g) - \lambda f(0)\overline{g(0)}. \qquad (5.14)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\lim_{\eta\searrow 0}\int_{\mathbb{R}} \varphi_{\epsilon-\eta,\epsilon}(x) f'(x)\overline{g'(x)}dx \\
=\ & \lim_{\epsilon\searrow 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)(Hf)(x)\overline{g(x)}dx + \lim_{\epsilon\searrow 0}\left(f'(-\epsilon)\overline{g(-\epsilon)} - f'(\epsilon)\overline{g(\epsilon)}\right) \\
=\ & (Hf|g) + \lim_{\epsilon\searrow 0}\left(f'(-\epsilon)\overline{g(-\epsilon)} - f'(\epsilon)\overline{g(\epsilon)}\right). \qquad (5.15)
\end{aligned}$$

The expressions, (5.14) and (5.15) are equal, so

$$\lim_{\epsilon\searrow 0}\left(f'(\epsilon)\overline{g(\epsilon)} - f'(-\epsilon)\overline{g(-\epsilon)}\right) = \lambda f(0)\overline{g(0)}. \qquad (5.16)$$

Also,

$$\begin{aligned}
[f'(\epsilon) - f'(-\epsilon)]\overline{g(0)} &= f'(\epsilon)\left(\overline{g(0)} - \overline{g(\epsilon)}\right) - f'(-\epsilon)\left(\overline{g(0)} - \overline{g(-\epsilon)}\right) \\
&+ f'(\epsilon)\overline{g(\epsilon)} - f'(-\epsilon)\overline{g(-\epsilon)}
\end{aligned}$$

Since g is continuous, $\overline{g(0)} = \lim_{\epsilon\searrow 0}\overline{g(\pm\epsilon)}$. Furthermore, for some $C \geq 0$, $|f'(\pm\epsilon)| \leq C$ for all $0 < \epsilon < 1$, by Prop. 5.1.8. Consequently,

$$\lim_{\epsilon\searrow 0}\left([f'(\epsilon) - f'(-\epsilon)]\overline{g(0)}\right) = \lim_{\epsilon\searrow 0}\left(f'(\epsilon)\overline{g(\epsilon)} - f'(-\epsilon)\overline{g(-\epsilon)}\right) = \lambda f(0)\overline{g(0)}. \qquad (5.17)$$

The equality in (5.17) holds for all $g \in \mathcal{H}^1(\mathbb{R})$. Therefore,

$$\lim_{\epsilon\searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0).$$

■

Now, finally, we arrive at the main result in this chapter:

Theorem 5.1.11 *The domain of the associated Hamiltonian H is given by*

$$\mathcal{D}(H) = \{f \in \mathcal{H}^1(\mathbb{R}) \mid f \in \mathcal{H}^2(\mathbb{R}\setminus\{0\}),\quad \lim_{\epsilon\searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0)\}.$$

Proof: $\subseteq$: Fulfilled by Lemma 5.1.7 and Lemma 5.1.10.

$\supseteq$: Fulfilled by Lemma 5.1.1. ■

In addition, $f' \in C(\mathbb{R} \setminus \{0\})$; in fact f' satisfies a Hölder condition.

Definition 5.1.12 *Let $\alpha \in \in]0,1]$; the set*

$$C^{0,\alpha}(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid \exists C \geq 0 \; \forall x \in \mathbb{R} \; \forall h \in \mathbb{R} : |f(x+h) - f(x)| \leq C|h|^{\alpha}\}$$

is called the Hölder space on $\mathbb{R}$ of order α.

Lemma 5.1.13 $\forall \epsilon \in]\frac{1}{2}, 1[\, : \mathcal{H}^1(\mathbb{R}) \subseteq C^{0,1-\epsilon}(\mathbb{R})$.

Remark: The elements of $\mathcal{H}^1(\mathbb{R})$ are equivalence classes of functions, so the inclusion is supposed to be interpreted as follows: any member, $[f]$ say, of $\mathcal{H}^1(\mathbb{R})$ has a representative that is equal almost everywhere to a function belonging to $C^{0,1-\epsilon}(\mathbb{R})$.

Proof: Let $\epsilon \in]\frac{1}{2}, 1[$ and $f \in \mathcal{H}^1(\mathbb{R})$ be given. Then $f, f' \in L^2(\mathbb{R})$, where f' is considered as the distributional derivative of f. First, by the inversion theorem, one can write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \hat{f}(k) dk,$$

with equality for almost all $x \in \mathbb{R}$ (with respect to the Lebesgue measure on $\mathbb{R}$). Since $f \in \mathcal{H}^1(\mathbb{R})$, and that $\mathcal{H}^1(\mathbb{R}) \subseteq C(\mathbb{R})$ by Sobolevs lemma (see e.g. [11], Th. IX.24, p. 52) we have that the equality holds for all $x \in \mathbb{R}$. For $h \neq 0$ one gets

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h^{1-\epsilon}} &= \frac{1}{h^{1-\epsilon}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} (e^{ikh} - 1) \hat{f}(k) dk \\
&= \frac{1}{h^{1-\epsilon}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} (e^{ik\frac{h}{2}} (e^{ik\frac{h}{2}} - e^{-ik\frac{h}{2}})) \hat{f}(k) dk \\
&= \frac{1}{h^{1-\epsilon}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} e^{ik\frac{h}{2}} 2i \sin(k\frac{h}{2}) \hat{f}(k) dk \\
&= \frac{1}{h^{1-\epsilon}} \frac{2i}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ik(x+\frac{h}{2})} \sin(k\frac{h}{2}) \hat{f}(k) dk.
\end{aligned}$$

Now,

$$|\sin(k\frac{h}{2})| \leq 1 \quad \text{and} \quad |\sin(k\frac{h}{2})| \leq |k\frac{h}{2}|,$$

so one has that

$$\begin{aligned}
|\sin(k\frac{h}{2})| &= |\sin(k\frac{h}{2})|^{\epsilon} |\sin(k\frac{h}{2})|^{1-\epsilon} \leq 1^{\epsilon} |k\frac{h}{2}|^{1-\epsilon} = (|k||\frac{h}{2}|)^{1-\epsilon} \\
&= (\frac{1}{2}|k|)^{1-\epsilon} |h|^{1-\epsilon}. \qquad (5.18)
\end{aligned}$$

$$\begin{aligned}
\frac{|f(x+h)-f(x)|}{|h|^{1-\epsilon}} &= \frac{1}{|h|^{1-\epsilon}}\frac{2}{\sqrt{2\pi}}\left|\int_{\mathbb{R}} e^{ik(x+\frac{h}{2})}\sin(k\frac{h}{2})\hat{f}(k)dk\right| \\
&\leq \frac{1}{|h|^{1-\epsilon}}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}|\sin(k\frac{h}{2})||\hat{f}(k)|dk \\
&\leq \frac{1}{|h|^{1-\epsilon}}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}(\frac{1}{2}|k|)^{1-\epsilon}|h|^{1-\epsilon}|\hat{f}(k)|dk, \quad \text{using} \quad (5.18) \\
&\leq 2^{\epsilon-1}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}(1+|k|^2)^{\frac{1-\epsilon}{2}}|\hat{f}(k)|dk \\
&= 2^{\epsilon-1}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}(1+|k|^2)^{\frac{1}{2}}|\hat{f}(k)|(1+|k|^2)^{-\frac{\epsilon}{2}}dk. \qquad (5.19)
\end{aligned}$$

For $\epsilon > \frac{1}{2}$ we have

$$(1+|k|^2)^{-\frac{\epsilon}{2}} \in L^2(\mathbb{R}). \qquad (5.20)$$

For $f \in \mathcal{H}^1(\mathbb{R})$ we have by [11], Prop. 1, p. 51, that

$$(1+|k|^2)^{\frac{1}{2}}\hat{f}(k) \in L^2(\mathbb{R}). \qquad (5.21)$$

By (5.20), (5.21) and Hölders inequality we get that

$$(1+|k|^2)^{\frac{1}{2}}|\hat{f}(k)|(1+|k|^2)^{-\frac{\epsilon}{2}} \in L^1(\mathbb{R}).$$

Therefore,

$$\frac{|f(x+h)-f(x)|}{|h|^{1-\epsilon}} \leq C.$$

Consequently, $f \in C^{0,1-\epsilon}(\mathbb{R})$. ■

Chapter 6

The eigenvalue problem

Having constructed the Hamiltonian of the δ-interaction in chapter 5 we are now ready to compute the spectrum of H. However, only the discrete spectrum will be computed. The essential spectrum of H is $[0, \infty[$, but this not be proved.

6.1 Construction of the resolvent

The purpose of this section is to show that for a spinless particle moving on a string and subjected to a δ-potential, there is exactly one bound state.

The free Hamiltonian is

$$H_0 = -\frac{d^2}{dx^2}, \quad \mathcal{D}(H_0) = H^2(\mathbb{R}).$$

This operator is self-adjoint, which will be shown in the sequel. First it is shown, that it is symmetric.

Symmetric: Let $f, g \in \mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R})$; then by using Prop. A.1.2, one gets

$$\begin{aligned}(H_0 f|g) &= -\int_{\mathbb{R}} f''(x)\overline{g(x)}dx = \int_{\mathbb{R}} f'(x)\overline{g'(x)}dx \\ &= -\int_{\mathbb{R}} f(x)\overline{g''(x)}dx = (f|H_0 g),\end{aligned}$$

and so H_0 is symmetric.

To show that H_0 is actually self-adjoint, it will be shown that H_0 unitarily equivalent to a multiplication operator. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function that is finite almost everywhere. Define now

$$\begin{cases}\mathcal{D}(M_f) = \{\varphi \in L^2(\mathbb{R}, \mathcal{B}, dx) \mid f\varphi \in L^2(\mathbb{R}, \mathcal{B}, dx)\} \\ M_f = f\varphi\end{cases}.$$

Consider now the following 'truncations' of M_f: Let $n \in \mathbb{N}$ and consider $(X_n, \mathcal{B}_n, d\mu_n)$, where $X_n = [-n, n]$, $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}(X_n)$ the subsets of X_n that are Borel sets, and $d\mu_n$ is the Lebesgue measure on $\mathbb{R}$ restricted to $[-n, n]$. For each $n \in \mathbb{N}$, the trippel $(X_n, \mathcal{B}_n, d\mu_n)$ is a finite measure space. Let the functions $f_n : X_n \to \mathbb{R}$ be given by $f_n = f_{|X_n}$. Define the operators

$$\begin{cases} \mathcal{D}(M_{f_n}) = \{\varphi \in L^2(X_n, \mathcal{B}_n, d\mu_n) \mid f_n\varphi \in L^2(X_n, \mathcal{B}_n, d\mu_n)\} \\ M_{f_n} = f_n\varphi \end{cases} .$$

By [10] Prop. 1, p. 259, these operators are self-adjoint.

Lemma 6.1.1 *With the notation as above, we have the inclusion*

$$\bigcup_{n=1}^{\infty} \mathcal{G}(M_{f_n}) \subseteq \mathcal{G}(M_f).$$

Proof: Assume that $x \in \bigcup_{n=1}^{\infty} \mathcal{G}(M_{f_n})$; then $\exists N \in \mathbb{N} : x \in \mathcal{G}(M_{f_N})$. Now, $\exists \varphi \in \mathcal{D}(M_{f_N}) : x = (\varphi, M_{f_N}\varphi)$. Define the map $i_N : L^2(X_N, \mathcal{B}_N, d\mu_N) \to L^2(\mathbb{R}, \mathcal{B}, dx)$ by $i_N(\varphi) = \tilde{\varphi}$, where

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & \text{for} \quad x \in X_N \\ 0, & \text{for} \quad x \in \mathbb{R} \setminus X_N \end{cases} .$$

Then i_N is clearly linear; and since $\int_{\mathbb{R}} |f\tilde{\varphi}|^2 dx = \int_{X_N} |f_N\varphi|^2 dx < \infty$, then i_N is an isometry, so $f\tilde{\varphi} \in L^2(\mathbb{R}, \mathcal{B}, dx)$. Therefore $\tilde{\varphi} \in \mathcal{D}(M_f)$. Consequently: $(\tilde{\varphi}, M_f\tilde{\varphi}) \in \mathcal{G}(M_f)$. Identifying $(\varphi, M_{f_N}\varphi)$ with $(i_N(\varphi), M_f i_N(\varphi)) = (\tilde{\varphi}, M_f\tilde{\varphi})$, we get $x \in \mathcal{G}(M_f)$. ■

In fact, the sequence of graphs $\{\mathcal{G}(M_{f_n})\}_{n\in\mathbb{N}}$ is totally ordered with $\mathcal{G}(M_{f_1}) \subset \cdots \subset \mathcal{G}(M_{f_n}) \subset \cdots \subset \mathcal{G}(M_f)$, but this will not be proven.

Lemma 6.1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be Borel-measurable and finite almost everywhere. Then the operator*

$$\begin{cases} \mathcal{D}(M_f) = \{\varphi \in L^2(\mathbb{R}, \mathcal{B}, dx) \mid f\varphi \in L^2(\mathbb{R}, \mathcal{B}, dx)\} \\ M_f\varphi = f\varphi \end{cases}$$

is self-adjoint.

Proof: M_f is symmetric, since f is real-valued, so $\mathcal{G}(M_f) \subseteq \mathcal{G}(M_f^*)$. Put $L^2(\mathbb{R}, \mathcal{B}, dx) = \mathcal{H}$ and define $V : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ by $V(a, b) = (-b, a)$, for $(a, b) \in \mathcal{H} \times \mathcal{H}$. Now,

$$\begin{aligned} \mathcal{G}(M_f^*) &= [V\mathcal{G}(M_f)]^{\perp}, \quad \text{by [14], Th.13.8} \\ &\subseteq [V\bigcup_{n=1}^{\infty} \mathcal{G}(M_{f_n})]^{\perp}, \quad \text{by Lemma 6.1.1} \\ &= [\bigcup_{n=1}^{\infty} V\mathcal{G}(M_{f_n})]^{\perp} \subseteq [V\mathcal{G}(M_{f_1})]^{\perp} \\ &\subseteq \bigcup_{n=1}^{\infty} [V\mathcal{G}(M_{f_n})]^{\perp} = \bigcup_{n=1}^{\infty} \mathcal{G}(M_{f_n}^*) \\ &= \bigcup_{n=1}^{\infty} \mathcal{G}(M_{f_n}), \quad \text{since} \quad M_{f_n} = M_{f_n}^* \\ &\subseteq \mathcal{G}(M_f). \end{aligned}$$

Therefore $\mathcal{G}(M_f^*) \subseteq \mathcal{G}(M_f)$, and since also $\mathcal{G}(M_f) \subseteq \mathcal{G}(M_f^*)$, then $\mathcal{G}(M_f) = \mathcal{G}(M_f^*)$. Hence $M_f = M_f^*$. ■

Theorem 6.1.3 *The free Hamiltonian* $H_0 = -\frac{d^2}{dx^2}$, $\mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R})$ *is self-adjoint.*

Proof: Put $f(\xi) = \xi^2$, $\xi \in \mathbb{R}$. Then according to Lemma 6.1.2, the multiplication operator M_f is self-adjoint on $\mathcal{D}(M_f) = \{\varphi \in L^2(\mathbb{R}) \mid f\varphi \in L^2(\mathbb{R})\}$. It will proven that M_f is unitarily equivalent to H_0, making H_0 self-adjoint as well.

First,

$$\begin{aligned} \mathcal{D}(\mathcal{F}^{-1}M_f\mathcal{F}) &= \mathcal{F}^{-1}M_f^{-1}\mathcal{D}(\mathcal{F}^{-1}) \\ &= \mathcal{F}^{-1}M_f^{-1}L^2(\mathbb{R}) = \mathcal{F}^{-1}\mathcal{D}(M_f) \\ &= \{\varphi \in L^2(\mathbb{R}) \mid \hat{\varphi} \in \mathcal{D}(M_f)\} \\ &= \{\varphi \in L^2(\mathbb{R}) \mid f\hat{\varphi} \in L^2(\mathbb{R})\} \\ &= \mathcal{D}(M_f\mathcal{F}). \end{aligned}$$

Assume that $\varphi \in \mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R})$; then $H_0\varphi = -\varphi'' \in L^2(\mathbb{R})$. This implies that $\widehat{-\varphi''} \in L^2(\mathbb{R})$. By [14] Th. 7.15, $\mathcal{F}P(-i\frac{d}{dx})u = P\hat{u}$ for any tempered distribution u, and any polynomial P. In particular, it holds for any function in $L^2(\mathbb{R})$. Therefore $(\cdot)^2\hat{\varphi} = f\hat{\varphi} \in L^2(\mathbb{R})$, so $\varphi \in \mathcal{D}(M_f\mathcal{F}) = \mathcal{D}(\mathcal{F}^{-1}M_f\mathcal{F})$. Now,

$$\mathcal{F}H_0\varphi = \mathcal{F}(-i\frac{d}{dx})^2\varphi = M_{(\cdot)^2}\mathcal{F}\varphi \Longrightarrow H_0\varphi = \mathcal{F}^{-1}M_{(\cdot)^2}\mathcal{F}\varphi.$$

Hence, $H_0 \subseteq \mathcal{F}^{-1}M_f\mathcal{F}$.

Assume now, that $\varphi \in \mathcal{D}(\mathcal{F}^{-1}M_f\mathcal{F}) = \mathcal{D}(M_f\mathcal{F})$; then $f\hat{\varphi} = (\cdot)^2\hat{\varphi} \in L^2(\mathbb{R})$. As before, this implies that $\widehat{-\varphi''} \in L^2(\mathbb{R})$; applying the inverse Fourier transform yields that $-\varphi'' \in L^2(\mathbb{R})$. Therefore $\varphi \in \mathcal{H}^2(\mathbb{R})$; hence $\mathcal{D}(\mathcal{F}^{-1}M_f\mathcal{F}) \subseteq \mathcal{H}^2(\mathbb{R}) = \mathcal{D}(H_0)$. Consequently, $H_0 = \mathcal{F}^{-1}M_f\mathcal{F}$. ■

Next, we prove that H_0 is lower bounded. Let $f \in \mathcal{D}(H_0)$; then

$$\begin{aligned} (H_0 f|f) &= -\int_{\mathbb{R}} f''(x)\overline{f(x)}dx = \int_{\mathbb{R}} f'(x)\overline{f'(x)}dx \\ &= \int_{\mathbb{R}} |f'(x)|^2 dx = \|f'\|_2^2 \\ &\geq 0, \end{aligned}$$

so $\Theta(H_0) \subseteq [0, \infty[$, which means that H_0 is lower bounded with zero as a lower bound.

Now, since H_0 is self-adjoint by Th. 6.1.3 and $\Theta(H_0) \subseteq [0, \infty[$, then $\sigma(H_0) \subseteq [0, \infty[$ by [14] Th. 13.31. In particular: $z \in \rho(H_0)$ the resolvent set of H_0, whenever $z < 0$. This means that $(H_0 - z)^{-1}$ exists, $\mathrm{Ran}(H_0 - z) = L^2(\mathbb{R})$ and$(H_0 - z)^{-1} \in \mathcal{B}(L^2(\mathbb{R}))$.

At this point, the Green's function for H_0 is computed; it is called the free Green's function. The purpose is to obtain an explicit expression for the free resolvent $(H_0 - z)^{-1}$; more precisely, $(H_0 - z)^{-1}$ will be expressed as an integral operator with the free Green's function as kernel. This will again be used later to calculate the resolvent $(H - z)^{-1}$.

$$\begin{aligned} [R_{H_0}(z)\psi](x) &= [(H_0 - z)^{-1}\psi](x) = [\mathcal{F}^{-1}M_{((\cdot)^2 - z)^{-1}}\mathcal{F}\psi](x) \\ &= \frac{1}{\sqrt{2\pi}}[\mathcal{F}^{-1}\big(((\cdot)^2 - z)^{-1}\big) * \mathcal{F}^{-1}\hat{\psi}](x). \end{aligned}$$

Now,

$$\mathcal{F}^{-1}\big(((\cdot)^2-z)^{-1}\big)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{1}{k^2-z}e^{ikx}dk.$$

This inverse Fourier transform will computed using calculus of residues as follows: Put

$$f(k)=\frac{e^{ikx}}{k^2-z}=\frac{e^{ikx}}{(k+\sqrt{z})(k-\sqrt{z})};$$

The numbers $\pm\sqrt{z}$ are simple poles. Then the residues of f at $\pm\sqrt{z}$ are:

$$\mathrm{Res}(f,\sqrt{z})=\frac{e^{i\sqrt{z}x}}{(k^2-z)'(\sqrt{z})}=\frac{e^{i\sqrt{z}x}}{2\sqrt{z}}$$

and

$$\mathrm{Res}(f,-\sqrt{z})=\frac{e^{-i\sqrt{z}x}}{(k^2-z)'(-\sqrt{z})}=\frac{e^{-i\sqrt{z}x}}{-2\sqrt{z}}.$$

Assume that $x>0$; then

$$\begin{aligned}\lim_{R\to\infty}\max_{\substack{|k|=R\\ \mathrm{Im}\,k\geq 0}}|f(k)| &= \lim_{R\to\infty}\max_{0\leq\theta\leq\pi}\left|\frac{1}{R^2e^{2i\theta}-z}\right|\leq\lim_{R\to\infty}\max_{0\leq\theta\leq\pi}\left|\frac{1}{R^2-|z|}\right|\\ &= \lim_{R\to\infty}\frac{1}{|R^2-|z||}=0.\end{aligned}$$

Then [15] Th. 1, p. 303 gives that

$$PV\int_{\mathbb{R}}\frac{1}{k^2-z}e^{ikx}dk=2\pi i\mathrm{Res}(f,\sqrt{z})=\frac{\pi i}{\sqrt{z}}e^{i\sqrt{z}x}$$

Similarly: for $x<0$, $\lim_{R\to\infty}\max\{|f(k)|\mid |k|=R,\ \mathrm{Im}\,k\leq 0\}=0$, so again, [15] Th.2, p. 304 gives that

$$PV\int_{\mathbb{R}}\frac{1}{k^2-z}e^{ikx}dk=-2\pi i\mathrm{Res}(f,-\sqrt{z})=\frac{\pi i}{\sqrt{z}}e^{-i\sqrt{z}x}.$$

Hence

$$\mathcal{F}^{-1}\big(((\cdot)^2-z)^{-1}\big)(x)=\frac{1}{\sqrt{2\pi}}\frac{\pi i}{\sqrt{z}}e^{i\sqrt{z}|x|}=\frac{i\sqrt{\pi}}{\sqrt{2}\sqrt{z}}e^{i\sqrt{z}|x|}.$$

Now,

$$\begin{aligned}[(H_0-z)^{-1}\psi](x) &= \frac{1}{\sqrt{2\pi}}[\mathcal{F}^{-1}\big(((\cdot)^2-z)^{-1}\big)*\mathcal{F}^{-1}\hat{\psi}](x)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{i\sqrt{\pi}}{\sqrt{2}\sqrt{z}}e^{i\sqrt{z}|\cdot|}*\psi\right)(x)\\ &= \int_{\mathbb{R}}\frac{i}{2\sqrt{z}}e^{i\sqrt{z}|x-y|}\psi(y)dy\\ &:= \int_{\mathbb{R}}G(x,y;z)\psi(y)dy,\end{aligned}$$

where the function G in the two variables x and y given by

$$G(x,y;z)=\frac{i}{2\sqrt{z}}e^{i\sqrt{z}|x-y|}$$

is the free Green's function.

We now introduce the auxiliary operator, τ, given by

$$\begin{cases} \tau : \mathcal{H}^1(\mathbb{R}) \to \mathbb{C} \\ \tau f = f(0) \end{cases}.$$

Now, τ is linear, and $|\tau f| = |f(0)| \leq \|f\|_\infty \leq C\|f\|_{\mathcal{H}^1(\mathbb{R})}$, where the last inequality comes from Sobolevs embedding theorem. The operator τ is thus seen to be bounded; specifically: $\tau \in \mathcal{B}(\mathcal{H}^1(\mathbb{R}), \mathbb{C})$. Therefore it has an adjoint operator $\tau^* \in \mathcal{B}(\mathbb{C}, \mathcal{H}^{-1}(\mathbb{R}))$ that satisfies

$$(\tau f|\alpha)_{\mathbb{C}} = \langle f, \tau^*\alpha\rangle.$$

Here the brackets $\langle\cdot,\cdot\rangle$ are duality brackets.

Let $\tilde{f}, \tilde{g} \in L^2(\mathbb{R})$ and define

$$\mathfrak{h}_\lambda(\tilde{f}, \tilde{g}) = \lambda\langle\tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}, \tau(H_0 - z)^{-\frac{1}{2}}\tilde{g}\rangle_{\mathbb{C}}.$$

The form $\mathfrak{h}_\lambda$ is bounded which is seen as follows:

$$\begin{aligned} |\mathfrak{h}_\lambda(\tilde{f}, \tilde{g})| &= |\lambda||\langle\tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}, \tau(H_0 - z)^{-\frac{1}{2}}\tilde{g}\rangle_{\mathbb{C}}| \\ &\leq |\lambda||\tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}||\tau(H_0 - z)^{-\frac{1}{2}}\tilde{g}| \\ &= |\lambda||[(H_0 - z)^{-\frac{1}{2}}\tilde{f}](0)||[(H_0 - z)^{-\frac{1}{2}}\tilde{g}](0)| \\ &\leq |\lambda|\|(H_0 - z)^{-\frac{1}{2}}\tilde{f}\|_\infty\|(H_0 - z)^{-\frac{1}{2}}\tilde{g}\|_\infty \\ &\leq |\lambda|C\|(H_0 - z)^{-\frac{1}{2}}\tilde{f}\|_{\mathcal{H}^1(\mathbb{R})}C\|(H_0 - z)^{-\frac{1}{2}}\tilde{g}\|_{\mathcal{H}^1(\mathbb{R})} \\ &\leq |\lambda|C\|(H_0 - z)^{-\frac{1}{2}}\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))}\|\tilde{f}\|_2 C\|(H_0 - z)^{-\frac{1}{2}}\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))}\|\tilde{g}\|_2 \\ &= \text{const}(z)\|\tilde{f}\|_2\|\tilde{g}\|_2. \end{aligned}$$

By the representation theorem for bounded forms, or rather the corollary to it, Cor. 3.0.23, there is an operator $V_\lambda(z) \in \mathcal{B}(L^2(\mathbb{R}))$ such that

$$\forall\tilde{f}, \tilde{g} \in L^2(\mathbb{R}) : \mathfrak{h}_\lambda(\tilde{f}, \tilde{g}) = (V_\lambda(z)\tilde{f}|\tilde{g}).$$

Now, for any $\tilde{f} \in L^2(\mathbb{R})$,

$$\begin{aligned} \mathfrak{h}_\lambda(\tilde{f}, \tilde{f}) &= \lambda\langle\tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}, \tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}\rangle_{\mathbb{C}} \\ &= (\lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\tilde{f}|\tilde{f}) \\ &= (V_\lambda(z)\tilde{f}|\tilde{f}). \end{aligned}$$

By the corollary to Th. 12.7 in [14], $V_\lambda(z) = \lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}$.

The next proposition says that for sufficiently negative z, the operator $V_\lambda(z)$ has norm less than one. This is convenient in the construction of the resolvent of the Hamiltonian H.

Proposition 6.1.4 $\forall\lambda \leq 0\ \exists z < 0 : \|V_\lambda(z)\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))} < 1.$

Proof: Assume first, that $\lambda = 0$; then put e.g. $z = -\frac{1}{2}$. Now, $\|V_0(-\frac{1}{2})\| = \|\mathbf{0}\| = 0 < 1$. Suppose next, that $\lambda < 0$ and that $z < 0$; then

$$\begin{aligned}\|V_\lambda(z)\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))} &= \|\lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))} \\ &= |\lambda|\|[\tau(H_0 - z)^{-\frac{1}{2}}]^*[\tau(H_0 - z)^{-\frac{1}{2}}]\|_{\mathcal{B}(L^2(\mathbb{R}),\mathcal{H}^1(\mathbb{R}))} \\ &\leq |\lambda|\|\tau(H_0 - z)^{-\frac{1}{2}}\|^2_{\mathcal{B}(L^2(\mathbb{R}),\mathbb{C})}. \end{aligned} \tag{6.1}$$

Now,

$$\begin{aligned}|\lambda|\|\tau(H_0 - z)^{-\frac{1}{2}}\|^2_{\mathcal{B}(L^2(\mathbb{R}),\mathbb{C})} &= |\lambda| \sup\{|\tau(H_0 - z)^{-\frac{1}{2}}f| \mid \|f\|_{L^2(\mathbb{R})} = 1\}^2 \\ &= |\lambda| \sup\{|[(H_0 - z)^{-\frac{1}{2}}f](0)|^2 \mid \|f\|_{L^2(\mathbb{R})} = 1\} \\ &\leq |\lambda| \sup\{\|(H_0 - z)^{-\frac{1}{2}}f\|^2_\infty \mid \|f\|_{L^2(\mathbb{R})} = 1\}. \end{aligned} \tag{6.2}$$

By the Sobolev embedding theorem,

$$\begin{aligned} & |\lambda| \sup\{\|(H_0 - z)^{-\frac{1}{2}}f\|^2_\infty \mid \|f\|_{L^2(\mathbb{R})} = 1\} \\ \leq\ & |\lambda| \sup\{C\|(H_0 - z)^{-\frac{1}{2}}f\|^2_{\mathcal{H}^s(\mathbb{R})} \mid \|f\|_{L^2(\mathbb{R})} = 1\}, \quad \text{where} \quad \frac{1}{2} < s < 1 \\ =\ & |\lambda| C \sup\{\|\langle\cdot\rangle^s \mathcal{F}(H_0 - z)^{-\frac{1}{2}}f\|^2_{L^2(\mathbb{R})} \mid \|f\|_{L^2(\mathbb{R})} = 1\} \\ =\ & |\lambda| C \sup\{\|\langle k\rangle^s (k^2 - z)^{-\frac{1}{2}}\hat{f}(k)\|^2_{L^2(\mathbb{R})} \mid \|f\|_{L^2(\mathbb{R})} = 1\}, \end{aligned} \tag{6.3}$$

where $\langle k\rangle := (1 + |k|^2)^{\frac{1}{2}}$. We now have that

$$\begin{aligned}|\lambda| C \sup_{\|f\|_{L^2(\mathbb{R})}=1} \|\langle k\rangle^s (k^2 - z)^{-\frac{1}{2}}\hat{f}(k)\|^2_{L^2(\mathbb{R})} &\leq |\lambda| C \sup_{\|f\|_{L^2(\mathbb{R})}=1} \sup_{k\in\mathbb{R}} \frac{\langle k\rangle^{2s}}{k^2 - z}\|\hat{f}\|_{L^2(\mathbb{R})} \\ &= |\lambda| C \sup_{k\in\mathbb{R}} \frac{(k^2 + 1)^s}{k^2 - z}. \end{aligned} \tag{6.4}$$

Consider now the function

$$L_z(k) = \frac{(k^2 + 1)^s}{k^2 - z}.$$

It can be shown that if $z \leq -\frac{1}{s}$, then the derivative of L_z, L'_z, has three distinct zeros:

$$k_1 = 0, \ k_2 = \sqrt{\frac{1 + sz}{s - 1}}, \ k_3 = -\sqrt{\frac{1 + sz}{s - 1}}.$$

Computing the second derivative of L_z and evaluating yields that

$$L''_z(k_1) > 0, \ L''_z(k_2) < 0, \ L''_z(k_3) < 0.$$

Therefore, $k_1 = 0$ is a local minimum (the only one), and k_2 and k_3 are local maxima (the only two). Now,

$$L_z(k_2) = L_z(k_3) = L_z\left(\pm\sqrt{\frac{1 + sz}{s - 1}}\right) = \frac{\left(\frac{1+sz}{s-1} + 1\right)^s}{\frac{1+sz}{s-1} - z}.$$

Hence

$$\sup_{k\in\mathbb{R}} \frac{(k^2+1)^s}{k^2-z} = \frac{\left(\frac{1+sz}{s-1}+1\right)^s}{\frac{1+sz}{s-1}-z} =: \frac{f(z)}{g(z)}.$$

Now,

$$\frac{f'(z)}{g'(z)} = \frac{s\left(\frac{1+sz}{s-1}+1\right)^{s-1}\frac{s}{s-1}}{\frac{s}{s-1}-1} = s^2\left(\frac{1+sz}{s-1}+1\right)^{s-1},$$

and then

$$\lim_{z\to-\infty}\frac{f'(z)}{g'(z)} = 0.$$

Furthermore,

$$\lim_{z\to-\infty} g(z) = \lim_{z\to-\infty}\left(\frac{1+sz}{s-1}-z\right) = \infty.$$

L'Hôspital's rule now applies (see e.g. [12], Th. 5.13) and one obtains:

$$\lim_{z\to-\infty}\sup_{k\in\mathbb{R}}\frac{(k^2+1)^s}{k^2-z} = \lim_{z\to-\infty}\frac{f(z)}{g(z)} = \lim_{z\to-\infty}\frac{f'(z)}{g'(z)} = 0.$$

So for the number $C^{-1}|\lambda|^{-1}$, there is a number $z_0 = z_0(C,\lambda)$ such that for any $z \le z_0$ one has that

$$\sup_{k\in\mathbb{R}}\frac{(k^2+1)^s}{k^2-z} < C^{-1}|\lambda|^{-1};$$

and this implies that

$$|\lambda| C \sup_{k\in\mathbb{R}}\frac{(k^2+1)^s}{k^2-z} < |\lambda| C C^{-1}|\lambda|^{-1} = 1. \tag{6.5}$$

By (6.1), (6.2), (6.3), (6.4) and (6.5), one has for sufficiently negative z that

$$\|V_\lambda(z)\| < 1.$$

■

A priori one has that

$$H - z : \mathcal{D}(H) \to {}^2(\mathbb{R})$$
$$H_0 - z + \lambda\tau^*\tau : \mathcal{H}^2(\mathbb{R}) \to \mathcal{H}^{-1}(\mathbb{R}).$$

We rewrite as follows:

$$\begin{aligned} H_0 - z + \lambda\tau^*\tau &= (H_0-z)^{\frac12}[\mathbf{1} + \lambda(H_0-z)^{-\frac12}\tau^*\tau(H_0-z)^{-\frac12}](H_0-z)^{\frac12} \\ &= (H_0-z)^{\frac12}[\mathbf{1}+V_\lambda(z)](H_0-z)^{\frac12} \end{aligned}$$

Now, $(H_0-z)^{\frac12}$ is invertible for $z<0$, and for z sufficiently negative, $\|V_\lambda(z)\| < 1$ which implies that $\mathbf{1}+V_\lambda(z)$ is invertible and can be expressed as a Neumann series:

$$[\mathbf{1}+V_\lambda(z)]^{-1} = \sum_{n=0}^{\infty}(-1)^n V_\lambda(z)^n,$$

the series being absolutely convergent in the operator norm. Therefore the operator $H_0 - z + \lambda\tau^*\tau$ is invertible with

$$\begin{aligned} R(z) &:= (H_0 - z + \lambda\tau^*\tau)^{-1} \\ &= (H_0 - z)^{-\frac{1}{2}}[\mathbf{1} + V_\lambda(z)]^{-1}(H_0 - z)^{-\frac{1}{2}} \\ &= (H_0 - z)^{-\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n V_\lambda(z)^n (H_0 - z)^{-\frac{1}{2}} \\ &= (H_0 - z)^{-\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n \left(\lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\right)^n (H_0 - z)^{-\frac{1}{2}} \\ &= (H_0 - z)^{-1} \\ &+ (H_0 - z)^{-\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n \left(\lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\right)^n (H_0 - z)^{-\frac{1}{2}}. \end{aligned} \tag{6.6}$$

One can prove by induction (the proof is left out here) that for any $n \in \mathbb{N}$,

$$\begin{aligned} &(H_0 - z)^{-\frac{1}{2}}(-1)^n\lambda^n \left((H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\right)^n (H_0 - z)^{-\frac{1}{2}} \\ = \ &(-1)^n\lambda^n(H_0 - z)^{-1}\tau^*[\tau(H_0 - z)^{-1}\tau^*]^{n-1}\tau(H_0 - z)^{-1}. \end{aligned}$$

Inserting this into (6.6), one gets

$$\begin{aligned} R(z) &= (H_0 - z)^{-1} \\ &+ \sum_{n=1}^{\infty}(-1)^n\lambda^n(H_0 - z)^{-1}\tau^*[\tau(H_0 - z)^{-1}\tau^*]^{n-1}\tau(H_0 - z)^{-1}. \end{aligned} \tag{6.7}$$

In order to proceed we take a look at the operator $\tau(H_0 - z)^{-1}\tau^* : \mathbb{C} \to \mathbb{C}$. Given an arbitrary $\alpha \in \mathbb{C}$; then by using that $\mathcal{F}(\alpha\delta) = \mathcal{F}(\tau^*\alpha) = \alpha(2\pi)^{-\frac{1}{2}}$ we get that

$$\begin{aligned} (H_0 - z)^{-1}(\tau^*\alpha) &= \mathcal{F}^{-1}\frac{1}{k^2 - z}\mathcal{F}(\tau^*\alpha) \\ &= \alpha(2\pi)^{-\frac{1}{2}}\mathcal{F}^{-1}\left(\frac{1}{k^2 - z}\right) \\ &= \alpha(2\pi)^{-\frac{1}{2}}\frac{i\sqrt{\pi}}{\sqrt{2}\sqrt{\pi}}e^{i\sqrt{z}|\cdot|} \\ &= \alpha\frac{i}{2\sqrt{z}}e^{i\sqrt{z}|\cdot|} \\ &= G(\cdot, 0; z)\alpha, \end{aligned}$$

where G is the free Green's function computed earlier. This implies that

$$[\tau(H_0 - z)^{-1}\tau^*](\alpha) = G(0, 0; z)\alpha.$$

Since the above computations hold for an arbitrary $\alpha \in \mathbb{C}$, we have

$$\tau(H_0 - z)^{-1}\tau^* = G(0, 0; z).$$

Substituting this into (6.7) we get

$$\begin{aligned} R(z) &= (H_0 - z)^{-1} + \sum_{n=1}^{\infty} (-1)^n \lambda^n (H_0 - z)^{-1} \tau^* [G(0,0;z)]^{n-1} \tau (H_0 - z)^{-1} \\ &= (H_0 - z)^{-1} \\ &- (H_0 - z)^{-1} \tau^* \left(\lambda \sum_{n=1}^{\infty} [-\lambda G(0,0;z)]^{n-1} \right) \tau (H_0 - z)^{-1} \end{aligned} \tag{6.8}$$

If $z < -\frac{|\lambda|^2}{16\pi^2}$, then

$$|G(0,0;z)| = \left| \frac{i}{4\pi\sqrt{z}} \right| = \frac{1}{4\pi} \left| \frac{1}{\sqrt{z}} \right| < \frac{1}{|\lambda|}.$$

This implies that

$$|[-\lambda G(0,0;z)]^{n-1}| < 1.$$

The series in (6.8) is then the geometric series:

$$\sum_{n=1}^{\infty} [-\lambda G(0,0;z)]^{n-1} = \frac{1}{1 + \lambda G(0,0;z)}.$$

Now, we obtain an expression for $R(z)$:

$$R(z) = (H_0 - z)^{-1} - \frac{\lambda}{1 + \lambda G(0,0;z)} (H_0 - z)^{-1} \tau^* \tau (H_0 - z)^{-1}. \tag{6.9}$$

This operator-valued function has a singularity, whenever $1 + \lambda G(0,0;z)$ has a zero.

$$1 + \lambda G(0,0;z) = 0 \Longleftrightarrow G(0,0;z) = \frac{i}{2\sqrt{z}} = -\frac{1}{\lambda},$$

which is equivalent to

$$z = -\frac{\lambda^2}{4}.$$

The next step in calculating $(H - z)^{-1}$ is to show that whenever z is negative enough, then $(H - z)R(z) = I_{L^2(\mathbb{R})}$ and $R(z)(H - z) = I_{\mathcal{D}(H)}$, which means that $R(z) = (H - z)^{-1}$. To do so, the following lemma is needed.

Lemma 6.1.5 *Let* $z \in]-\infty, 0[$ *be a real number, negative enough to make* $\mathbf{1} + V_\lambda(z)$ *invertible. Then*

$$\text{Ran}\left((H_0 - z)^{-\frac{1}{2}} [\mathbf{1} + V_\lambda(z)]^{-1} (H_0 - z)^{-\frac{1}{2}} \right) \subseteq \mathcal{D}(H).$$

Proof: Put $R(z) := (H_0 - z)^{-\frac{1}{2}} [\mathbf{1} + V_\lambda(z)]^{-1} (H_0 - z)^{-\frac{1}{2}}$; then being a product of everywhere defined bounded operators on $L^2(\mathbb{R})$, $R(z)$ itself is everywhere defined and bounded. Now, given $f \in \text{Ran}R(z)$; then there is an $\tilde{f} \in L^2(\mathbb{R})$ such that $f = R(z)\tilde{f}$. Let $f, g \in L^2(\mathbb{R})$ be given and define

$$\begin{aligned} \mathfrak{h}_z(f,g) &:= \mathfrak{h}(f,g) - z(f|g) \\ &= (f'|g') + \lambda \langle f, \tau^* \tau g \rangle - z(f|g), \end{aligned}$$

where $\mathcal{D}(\mathfrak{h}_z) = \mathcal{D}(\mathfrak{h})$. We now get that

$$\begin{aligned}
\mathfrak{h}_z(f,g) &= \int_{\mathbb{R}} k\hat{f}(k)\overline{k\hat{g}(k)}dk - z(\hat{f}|\hat{g}) + \lambda\langle f, \tau^*\tau g\rangle - z(f|g) \\
&= \int_{\mathbb{R}} (k^2 - z)\hat{f}(k)\overline{\hat{g}(k)}dk + \lambda\langle f, \tau^*\tau g\rangle - z(f|g) \\
&= \int_{\mathbb{R}} \sqrt{k^2 - z}\hat{f}(k)\overline{\sqrt{k^2 - z}\hat{g}(k)}dk + \lambda\langle f, \tau^*\tau g\rangle - z(f|g).
\end{aligned}$$

The functional calculus version of the spectral theorem of unbounded operators now gives that

$$\begin{aligned}
\mathfrak{h}_z(f,g) &= \int_{\mathbb{R}} [(H_0 - z)^{\frac{1}{2}}f](x)\overline{[(H_0 - z)^{\frac{1}{2}}g](x)}dx \\
&= ((H_0 - z)^{\frac{1}{2}}f|(H_0 - z)^{\frac{1}{2}}g) + \lambda\langle f, \tau^*\tau g\rangle.
\end{aligned}$$

Put $\Phi := (H_0 - z)^{\frac{1}{2}}f$ and $\Psi := (H_0 - z)^{\frac{1}{2}}g$. Then,

$$\begin{aligned}
\mathfrak{h}_z(f,g) &= (\Phi|\Psi) + \lambda\langle (H_0 - z)^{-\frac{1}{2}}\Phi, \tau^*\tau(H_0 - z)^{-\frac{1}{2}}\Psi\rangle \\
&= (\Phi|\Psi) + \lambda([\tau^*\tau(H_0 - z)^{-\frac{1}{2}}]^*(H_0 - z)^{-\frac{1}{2}}\Phi|\Psi) \\
&= (\Phi|\Psi) + (\lambda(H_0 - z)^{-\frac{1}{2}}\tau^*\tau(H_0 - z)^{-\frac{1}{2}}\Phi|\Psi) \\
&= (\Phi|\Psi) + (V_\lambda(z)\Phi|\Psi) \\
&= ([\mathbf{1} + V_\lambda(z)]\Phi|\Psi) \\
&= ((H_0 - z)^{-\frac{1}{2}}\tilde{f}|\Psi) \\
&= (\tilde{f}|(H_0 - z)^{-\frac{1}{2}}\Psi) \\
&= (\tilde{f}|g).
\end{aligned}$$

The above computations lead to

$$\begin{aligned}
|\mathfrak{h}(f,g)| &= |\mathfrak{h}_z(f,g) + z(f|g)| = |(\tilde{f} + zf)|g)| \leq \|\tilde{f} + zf\|_2\|g\|_2 \\
&= \text{const}(z,f)\|g\|_2. \qquad (6.10)
\end{aligned}$$

Since (6.10) holds for any $g \in L^2(\mathbb{R})$, then by Prop. 3.0.40, $f \in \mathcal{D}(H)$. ■

Next, it will be shown that $(H - z)^{-1} = R(z)$ for z negative enough.

Proposition 6.1.6 *Let $z \in\ ] - \infty, 0[$ be a real number, negative enough to make $\mathbf{1} + V_\lambda(z)$ invertible. With $R(z)$ as in Lemma 6.1.5, then*

$$(H - z)^{-1} = R(z).$$

Proof: Given $\tilde{f} \in L^2(\mathbb{R})$. Since $\mathcal{D}(\mathfrak{h})$ is dense in $L^2(\mathbb{R})$ and

$$\mathfrak{h}_z(f,g) = ((H - z)f|g) = (\tilde{f}|g)$$

for all $g \in L^2(\mathbb{R})$, then $(H - z)f = \tilde{f}$. But $f = R(z)\tilde{f}$, so $(H - z)R(z)\tilde{f} = \tilde{f}$. This implies that

$$(H - z)R(z) = I_{L^2(\mathbb{R})}.$$

The proof that $R(z)$ is also a left inverse is left out. ■

Now, analytic continuation will be used to extend $(H-z)^{-1}$ to the biggest open set in $\mathbb{C}$ in which $(H-z)^{-1}$ is norm-analytic.

Theorem 6.1.7 *The resolvent of H is given by*

$$(H-z)^{-1} = (H_0-z)^{-1} - \frac{\lambda}{1+\lambda G(0,0;z)}(H_0-z)^{-1}\tau^*\tau(H_0-z)^{-1},$$

where $z \in \mathbb{C} \setminus (\{-\frac{\lambda^2}{4}\} \cup [0,\infty[)$.

Proof: It is proven that the operator-valued functions $(H-z)^{-1}$ and $R(z)$ coincide, whenever z is negative enough, that is on $]-\infty, z_0]$ for some $z_0 < 0$. Given $f, g \in L^2(\mathbb{R})$, then the complex-valued functions $z \mapsto ((H-z)^{-1}f|g)$ and $z \mapsto (R(z)f|g)$ also coincide on $]-\infty, z_0]$. The set $]-\infty, z_0]$ has an accumulation point in $\mathbb{C}$, and then by [2], Th. 16.26, $((H-\cdot)^{-1}f|g)$ and $(R(\cdot)f|g)$ coincide on any open set in $\mathbb{C}$ on which they are both analytic. They are both analytic on $\mathbb{C} \setminus (\{-\frac{\lambda^2}{4}\} \cup [0,\infty[)$.

Now, given $z \in \mathbb{C} \setminus (\{-\frac{\lambda^2}{4}\} \cup [0,\infty[)$, then $((H-z)^{-1}f|g) = (R(z)f|g)$ for all $f, g \in L^2(\mathbb{R})$. By the corollary to Th. 12.7 in [14], then $(H-z)^{-1} = R(z)$. ■

6.2 Locating the discrete spectrum

In this section, it will be proven that $-\frac{\lambda^2}{4}$ is a discrete eigenvalue of multiplicity one using Riesz projections, and the corresponding eigenfunction will be computed. Due to lack of time, the essential spectrum of H will not be determined.

Definition 6.2.1 (Admissible contour) *Let A be a closed operator on $\mathcal{H}$, with spectrum $\sigma(A)$. Assume that λ_0 is an isolated point of $\sigma(A)$. A closed contour Γ is said to be an admissible contour for A and λ_0, if the closed convex envelope of Γ, $\overline{co}(\Gamma)$ fulfills that*

$$\overline{co}(\Gamma) \cap \sigma(A) = \{\lambda_0\}.$$

Definition 6.2.2 (Riesz integral) *Let A be a closed operator on $\mathcal{H}$, λ_0 an isolated point of $\sigma(A)$ and Γ an admissible contour of A and λ_0. The Riesz integral for A and λ_0 is defined to be the operator-valued integral*

$$P_A(\{\lambda_0\}) = \frac{1}{2\pi i}\oint_\Gamma (A-z)^{-1}dz, \tag{6.11}$$

the contour integral being evaluated in the counter clockwise direction.

Denoting by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbb{R})$, the operator given by the Riesz integral in (6.11) is the unique operator that fulfills that

$$\forall \psi, \varphi \in L^2(\mathbb{R}) : \left\langle \left(\frac{1}{2\pi i}\oint_\Gamma (A-z)^{-1}dz\right)\psi, \varphi\right\rangle = \frac{1}{2\pi i}\oint_\Gamma \langle (A-z)^{-1}\psi, \varphi\rangle dz.$$

Now, the main result in this chapter can be stated:

Theorem 6.2.3 *Let H be the self-adjoint operator that is associated to the symmetric, lower bounded, closed sesquilinear form $\mathfrak{h}$ defined in (5.1). Then $-\frac{\lambda^2}{4}$ is the only discrete eigenvalue of H. Its multiplicity equals one.*

In particular: the δ-interaction Hamiltonian has exactly one bound state.

Proof: Put $E := -\frac{\lambda^2}{4}$. Assuming that $\sigma(H) \subseteq \{E\} \cup [0, \infty[$, where $\sigma_{\text{ess}}(H) = [0, \infty[$, then put $\Gamma := \partial B(E, \frac{1}{2}|E|)$, the boundary of the disc centered at E with radius $\frac{1}{2}|E|$. This closed contour, Γ, is an admissible contour for E and H. We now compute the Riesz integral for H and E. Write

$$\begin{aligned} & \langle (H-z)^{-1}\psi, \varphi\rangle \\ = \; & \langle (H_0-z)^{-1}\psi, \varphi\rangle - \left\langle \frac{\lambda}{1+\lambda G(0,0;z)}(H_0-z)^{-1}\tau^*\tau(H_0-z)^{-1}\psi, \varphi \right\rangle . \end{aligned} \tag{6.12}$$

Now,

$$\begin{aligned} \langle (H_0-z)^{-1}\psi, \varphi\rangle &= \left\langle \int_{\mathbb{R}} G(x,y;z)\psi(y)dy, \varphi \right\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} G(x,y;z)\psi(y)dy\overline{\psi(x)}dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} G(x,y;z)\psi(y)\overline{\psi(x)}dydx. \end{aligned} \tag{6.13}$$

The second term in (6.12) is treated as follows:

$$\begin{aligned} & \left\langle \frac{\lambda}{1+\lambda G(0,0;z)}(H_0-z)^{-1}\tau^*\tau(H_0-z)^{-1}\psi, \varphi \right\rangle \\ = \; & \frac{\lambda}{1+\lambda G(0,0;z)} \left\langle (H_0-z)^{-1}\tau^*\tau(H_0-z)^{-1}\psi, \varphi \right\rangle \\ = \; & \frac{\lambda}{1+\lambda G(0,0;z)} \left\langle \tau(H_0-z)^{-1}\psi, \tau(H_0-\overline{z})^{-1}\varphi \right\rangle_{\mathbb{C}} \\ = \; & \frac{\lambda}{1+\lambda G(0,0;z)} \int_{\mathbb{R}} G(0,y;z)\psi(y)dy \cdot \overline{\int_{\mathbb{R}} G(0,x;\overline{z})\varphi(x)dx}. \end{aligned} \tag{6.14}$$

Since $G(x,0;z) = \overline{G(0,x;\overline{z})}$, we get that (6.14) equals

$$\begin{aligned} & \frac{\lambda}{1+\lambda G(0,0;z)} \int_{\mathbb{R}}\int_{\mathbb{R}} G(0,y;z)\psi(y)\overline{G(0,x;\overline{z})\varphi(x)}dxdy \\ = \; & \frac{\lambda}{1+\lambda G(0,0;z)} \int_{\mathbb{R}}\int_{\mathbb{R}} G(0,y;z)\psi(y)G(x,0;z)\overline{\varphi(x)}dxdy. \end{aligned} \tag{6.15}$$

Next, the contour integral of (6.13),

$$\frac{1}{2\pi i}\oint_{\Gamma}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x,y;z)\psi(y)\overline{\psi(x)}dydx = 0,$$

since the function $\langle (H-\cdot)^{-1}\psi, \varphi\rangle$ is analytic in the region enclosed by Γ. The contour integral of (6.15),

$$\frac{1}{2\pi i}\oint_{\Gamma}\left\{ \frac{\lambda}{1+\lambda G(0,0;z)} \int_{\mathbb{R}}\int_{\mathbb{R}} G(0,y;z)\psi(y)G(x,0;z)\overline{\varphi(x)}dxdy \right\} dz \tag{6.16}$$

can be computed by use of the calculus of residues. Denote by f, the integrand in (6.16). Put

$$\begin{aligned} g(z) &:= -\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} G(0,y;z)\psi(y)G(x,0;z)\overline{\varphi(x)}dydx, \\ h(z) &:= 1 + \lambda G(0,0;z) \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{Res}(f,E) &= \frac{g(E)}{h'(E)} = \frac{-\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} G(0,y;E)\psi(y)G(x,0;E)\overline{\varphi(x)}dydx}{-\frac{i\lambda}{4}E^{-\frac{3}{2}}} \\ &= -4E^{\frac{3}{2}} i \int_{\mathbb{R}} \int_{\mathbb{R}} G(0,y;E)\psi(y)G(x,0;E)\overline{\varphi(x)}dydx. \end{aligned}$$

The integral in (6.16) then equals

$$2\pi i \mathrm{Res}(f,E) = 8E^{\frac{3}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} G(0,y;E)\psi(y)G(x,0;E)\overline{\varphi(x)}dydx. \tag{6.17}$$

The expression in (6.17) can be re-expressed using Diracs bra-ket notation:

$$8E^{\frac{3}{2}} |G(0,\cdot;E)\rangle\langle G(0,\cdot;E)|,$$

which is seen to be a rank one operator. Hence $E = -\frac{\lambda^2}{4}$ is the only discrete eigenvalue of H. Consequently, H has exactly one bound state. ∎

Appendix A

Prerequisites

Proposition A.0.4 *Let $f \in \mathcal{H}^1(\mathbb{R})$. Then*

$$f(b) - f(a) = \int_a^b (Df)(x)dx. \tag{A.1}$$

Proof: First, the integral is convergent, since $Df \in L^2(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R})$ and $[a,b]$ is compact. Therefore, $Df \in L^1([a,b])$. Since $L^1([a,b]) \subseteq \mathcal{D}'([a,b])$, we get from [9] p. 87, that Df is equal almost everywhere to a function that is absolutely continuous on $[a,b]$. By [13] Th. 7.18, (A.1) is true. ■

A.1 Integration by parts in Sobolev spaces

Lemma A.1.1 *Let $a < b$. Assume that $f \in \mathcal{H}^2([a,b])$ and $g \in \mathcal{H}^1([a,b])$. Then usual integration by parts still holds true:*

$$\int_a^b f'(x)g'(x)dx = -\int_a^b f''(x)g(x)dx + [f'(x)g(x)]_a^b.$$

Due to lack of time, the proof of Lemma A.1.1 is left out.

Proposition A.1.2 *Assume that $f \in \mathcal{H}^2(\mathbb{R})$ and $g \in \mathcal{H}^1(\mathbb{R})$. Then usual integration by parts still holds true with vanishing boundary terms:*

$$\int_{\mathbb{R}} f'(x)g'(x)dx = -\int_{\mathbb{R}} f''(x)g(x)dx.$$

Proof: Truncate the function $f'g'$ with $\chi_{[-n,n]}f'g'$; then for almost any $x \in \mathbb{R}$, the sequence $\{\chi_{[-n,n]}(x)f'(x)g'(x)\}_{n\in\mathbb{N}}$ is monotically increasing with $\chi_{[-n,n]}(x)f'(x)g'(x) \nearrow f'(x)g'(x)$ as $n \to \infty$. By the monotone convergence theorem, $\lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]}f'g'$ exists and

$$\begin{aligned}\int_{\mathbb{R}} f'(x)g'(x)dx &= \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]}(x)f'(x)g'(x)dx \\ &= \lim_{n\to\infty} \left(-\int_{-n}^{n} f''(x)g(x)dx + [f'(x)g(x)]_{-n}^{n} \right) \end{aligned} \tag{A.2}$$

Similarly, $\chi_{[-n,n]}(x)f''(x)g(x) \nearrow f''(x)g(x)$ as $n \to \infty$, and $f''g \in L^1(\mathbb{R})$ since $f \in \mathcal{H}^2(\mathbb{R})$ and $g \in \mathcal{H}^1(\mathbb{R})$. Again, by the monotone convergence theorem, $\lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]}(x) f_n''(x) g_n(x) dx$ exists and

$$\int_{\mathbb{R}} f''(x)g(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]}(x) f_n''(x) g_n(x) dx = \lim_{n\to\infty} \int_{-n}^{n} f''(x)g(x)dx. \tag{A.3}$$

Sobolevs embedding theorem ([9], Th. 7, p. 93) gives that $f', g \in C_\infty(\mathbb{R})$, the continuous functions on $\mathbb{R}$ that vanish at infinity. Therefore also $f'g \in C_\infty(\mathbb{R})$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} [f'(x)g(x)]_{-n}^{n} &= \lim_{n\to\infty} \big(f'(n)g(n) - f'(-n)g(-n)\big) \\ &= \lim_{n\to\infty} f'(n)g(n) - \lim_{n\to\infty} f'(-n)g(-n) \\ &= 0. \end{aligned} \tag{A.4}$$

Now, (A.2), (A.3) and (A.4) imply that

$$\begin{aligned}\int_{\mathbb{R}} f'(x)g'(x)dx &= -\lim_{n\to\infty} \int_{-n}^{n} f''(x)g(x)dx + \lim_{n\to\infty} [f'(x)g(x)]_{-n}^{n} \\ &= -\int_{\mathbb{R}} f''(x)g(x)dx.\end{aligned}$$

■

Proposition A.1.3 *Assume that* $f \in \mathcal{H}^1(\mathbb{R}) \cap \mathcal{H}^2(\mathbb{R} \setminus \{0\})$, $\lim_{\epsilon \searrow 0}[f'(\epsilon) - f'(-\epsilon)] = \lambda f(0)$, *and* $g \in \mathcal{H}^1(\mathbb{R})$. *Then usual integration by parts still holds true:*

$$\int_{\mathbb{R}\setminus\{0\}} f'(x)\overline{g'(x)}dx = -\int_{\mathbb{R}\setminus\{0\}} f''(x)\overline{g(x)}dx - \lambda f(0)\overline{g(0)}.$$

Proof: The function $f'\overline{g'}$ is truncated as $\chi_{[-n,-\frac{1}{n}]\cup[\frac{1}{n},n]} f'\overline{g'}$, $n \in \mathbb{N}$. Then we have that $\chi_{[-n,-\frac{1}{n}]\cup[\frac{1}{n},n]}(x) f'(x)\overline{g'(x)} \nearrow f'(x)\overline{g'(x)}$ for all $x \in \mathbb{R} \setminus \{0\}$. The monotone convergence theorem implies that $\lim_{n\to\infty} \int_{\mathbb{R}\setminus\{0\}} \chi_{[-n-\frac{1}{n}]\cup[\frac{1}{n},n]}(x) f'(x)\overline{g'(x)}dx$ exists and

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} f'(x)\overline{g'(x)}dx &= \lim_{n\to\infty} \int_{\mathbb{R}\setminus\{0\}} \chi_{[-n,-\frac{1}{n}]\cup[\frac{1}{n},n]}(x) f'(x)\overline{g'(x)}dx \\ &= \lim_{n\to\infty} \left(\int_{-n}^{-\frac{1}{n}} f'(x)\overline{g'(x)}dx + \int_{\frac{1}{n}}^{n} f'(x)\overline{g'(x)}dx \right).\end{aligned} \tag{A.5}$$

Using Prop. A.1.1 we get

$$\begin{aligned}\int_{\frac{1}{n}}^{n} f'(x)\overline{g'(x)}dx &= -\int_{\frac{1}{n}}^{n} f''(x)\overline{g(x)}dx + [f'(x)\overline{g(x)}]_{\frac{1}{n}}^{n} \\ &= -\int_{\frac{1}{n}}^{n} f''(x)\overline{g(x)}dx + f'(n)\overline{g(n)} - f'(n^{-1})\overline{g(n^{-1})} \\ &= -\int_{\frac{1}{n}}^{n} f''(x)\overline{g(x)}dx + (f\varphi_{\frac{1}{4},\frac{1}{2}})'(n)\overline{g(n)} - f'(n^{-1})\overline{g(n^{-1})}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
\int_{-n}^{-\frac{1}{n}} f'(x)\overline{g'(x)}dx &= -\int_{-n}^{-\frac{1}{n}} f''(x)\overline{g(x)}dx + [f'(x)\overline{g(x)}]_{-n}^{-\frac{1}{n}} \\
&= -\int_{-n}^{-\frac{1}{n}} f''(x)\overline{g(x)}dx + f'(-n^{-1})\overline{g(-n^{-1})} - f'(-n)\overline{g(-n)} \\
&= -\int_{-n}^{-\frac{1}{n}} f''(x)\overline{g(x)}dx + f'(-n^{-1})\overline{g(-n^{-1})} - (f\varphi_{\frac{1}{4},\frac{1}{2}})'(-n)\overline{g(-n)}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\int_{-n}^{-\frac{1}{n}} f'(x)\overline{g'(x)}dx + \int_{\frac{1}{n}}^{n} f'(x)\overline{g'(x)}dx \\
= &\int_{-n}^{-\frac{1}{n}} f''(x)\overline{g(x)}dx + f'(-n^{-1})\overline{g(-n^{-1})} - (f\varphi_{\frac{1}{4},\frac{1}{2}})'(-n)\overline{g(-n)} \\
- &\int_{\frac{1}{n}}^{n} f''(x)\overline{g(x)}dx + (f\varphi_{\frac{1}{4},\frac{1}{2}})'(n)\overline{g(n)} - f'(n^{-1})\overline{g(n^{-1})} \\
= &-\int_{-n}^{-\frac{1}{n}} f''(x)\overline{g(x)}dx - \int_{\frac{1}{n}}^{n} f''(x)\overline{g(x)}dx \\
+ &\ (f\varphi_{\frac{1}{4},\frac{1}{2}})'(n)\overline{g(n)} - (f\varphi_{\frac{1}{4},\frac{1}{2}})'(-n)\overline{g(-n)} \qquad (A.6) \\
+ &\ f'(-n^{-1})\overline{g(-n^{-1})} - f'(n^{-1})\overline{g(n^{-1})} \qquad (A.7)
\end{aligned}
$$

The boundary terms in (A.6) corresponding to $\pm\infty$ vanish as follows: since $f \in \mathcal{H}^2(\mathbb{R} \setminus \{0\})$, then $f\varphi_{\frac{1}{4},\frac{1}{2}} \in \mathcal{H}^2(\mathbb{R})$, and then by Sobolevs embedding theorem $f\varphi_{\frac{1}{4},\frac{1}{2}} \in C_\infty(\mathbb{R})$, the continuous functions on $\mathbb{R}$ that vanish at infinity. Furthermore, $(f\varphi_{\frac{1}{4},\frac{1}{2}})(x) = f(x)$ for all $|x| \geq \frac{1}{2}$; this implies that $(f\varphi_{\frac{1}{4},\frac{1}{2}})'(n) = f'(n)$ for all $n \in \mathbb{Z} \setminus \{0\}$. Consequently,

$$\lim_{n\to\pm\infty} (f\varphi_{\frac{1}{4},\frac{1}{2}})'(n) = \lim_{n\to\pm\infty} f'(n) = 0. \qquad (A.8)$$

Also, $g \in \mathcal{H}^1(\mathbb{R}) \subseteq C_\infty(\mathbb{R})$, so $\lim_{n\to\pm\infty} \overline{g(n)} = \lim_{n\to\pm\infty} g(n) = 0$; this together with (A.8) gives that

$$\lim_{n\to\pm\infty} (f\varphi_{\frac{1}{4},\frac{1}{2}})'(n)\overline{g(n)} = \lim_{n\to\pm\infty} f'(n)\overline{g(n)} = 0. \qquad (A.9)$$

The behaviour of the boundary terms (A.7) in the origin is determined in the following. Write

$$
\begin{aligned}
f'(-n^{-1})\overline{g(-n^{-1})} - f'(n^{-1})\overline{g(n^{-1})} &= f'(n^{-1})[\overline{g(0)} - \overline{g(n^{-1})}] \\
&- f'(-n^{-1})[\overline{g(0)} - \overline{g(-n^{-1})}] \\
&- \overline{g(0)}[f'(n^{-1}) - f'(-n^{-1})]. \qquad (A.10)
\end{aligned}
$$

By Prop. 5.1.8, $\{f'(n^{-1})\}_{n\in\mathbb{N}}$ and $\{f'(-n^{-1})\}_{n\in\mathbb{N}}$ are bounded. Since g is continuous, then $\overline{g(0)} = \lim_{n\to\pm\infty} \overline{g(n^{-1})}$. Consequently,

$$\lim_{n\to\infty} f'(n^{-1})[\overline{g(0)} - \overline{g(n^{-1})}] = 0 \quad \text{and} \quad \lim_{n\to\infty} f'(-n^{-1})[\overline{g(0)} - \overline{g(-n^{-1})}] = 0. \qquad (A.11)$$

Now, by assumption $\lim_{\epsilon\searrow 0}[f'(\epsilon)-f'(-\epsilon)]=\lambda f(0)$, so

$$\lim_{n\to\infty}\overline{g(0)}[f'(n^{-1})-f'(-n^{-1})]=\lambda f(0)\overline{g(0)}. \tag{A.12}$$

By (A.10), (A.11) and (A.12), one arrives at

$$\lim_{n\to\infty}[f'(-n^{-1})\overline{g(-n^{-1})}-f'(n^{-1})\overline{g(n^{-1})}]=-\lambda f(0)\overline{g(0)}.$$

Since $\chi_{[\frac{1}{n},n]}(x)f''(x)\overline{g(x)}\nearrow f''(x)\overline{g(x)}$ for all $x\in\,]0,\infty[$ and $f''\overline{g}\in L^1(]0,\infty[)$, then the limit $\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}f''(x)\overline{g(x)}dx$ exists and

$$\int_{]0,\infty[}f''(x)\overline{g(x)}dx=\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}f''(x)\overline{g(x)}dx.$$

Similarly,

$$\int_{]-\infty,0[}f''(x)\overline{g(x)}dx=\lim_{n\to\infty}\int_{-n}^{-\frac{1}{n}}f''(x)\overline{g(x)}dx.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}}f'(x)\overline{g'(x)}dx &= \lim_{n\to\infty}\left(\int_{-n}^{-\frac{1}{n}}f'(x)\overline{g'(x)}dx+\int_{\frac{1}{n}}^{n}f'(x)\overline{g'(x)}dx\right)\\ &= -\int_{\mathbb{R}\setminus\{0\}}f''(x)\overline{g(x)}dx-\lambda f(0)\overline{g(0)}.\end{aligned}$$

■

Appendix B

Existence of non-trivial testfunctions

This is devoted to the explicit constrution of a testfunction on the real axis. A testfunction on $\mathbb{R}$ is a function that fulfills that it is smooth (i.e. arbitrarily often differentiable in the classical sense), and has compact support.

Lemma B.0.4 *Define the function* $f :]-1, 1[\to \mathbb{R}$ *by*

$$f(x) = \frac{1}{x^2 - 1}.$$

Then the n'th derivative of f exists and is given by

$$f^{(n)}(x) = \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n (x+1)^{-\alpha}(x-1)^{-\beta}. \tag{B.1}$$

In particular:

$$f \in \bigcap_{n=0}^{\infty} C^n(]-1,1[) = C^\infty(]-1,1[).$$

Proof: The proof is by induction: For $n = 1$, one gets:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\frac{1}{x^2-1} = \frac{d}{dx}\left((x+1)^{-1}(x-1)^{-1}\right) \\ &= 1!\cdot(-1)^1\left[(x+1)^{-2}(x-1)^{-1} + (x+1)^{-1}(x-1)^{-2}\right] \\ &= 1!\cdot(-1)^1 \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=1+2}} (x+1)^{-\alpha}(x-1)^{-\beta}. \end{aligned}$$

Assume that (B.1) is true for a given $n \in \mathbb{N}$. Now,

$$
\begin{aligned}
f^{(n+1)}(x) &= \frac{d}{dx}\left[n!(-1)^n \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} (x+1)^{-\alpha}(x-1)^{-\beta} \right] \\
&= n!(-1)^n \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} \frac{d}{dx}(x+1)^{-\alpha}(x-1)^{-\beta} \\
&= n!(-1)^n \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} [(-\alpha)(x+1)^{-\alpha-1}(x-1)^{-\beta} + (x+1)^{-\alpha}(-\beta)(x-1)^{-\beta-1}].
\end{aligned}
$$

Shifting index one gets that

$$
\begin{aligned}
f^{(n+1)}(x) &= n!(-1)^n \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+3}} (-\alpha+1)(x+1)^{-\alpha}(x-1)^{-\beta} \\
&+ n!(-1)^n \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+3}} (x+1)^{-\alpha}(-\beta+1)(x-1)^{-\beta} \\
&= n!(-1)^n(-\alpha-\beta+2) \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+3}} (x+1)^{-\alpha}(x-1)^{-\beta} \\
&= n!(-1)^{n+1}(\alpha+\beta-2) \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+3}} (x+1)^{-\alpha}(x-1)^{-\beta} \\
&= \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=(n+1)+2}} (n+1)!(-1)^{n+1}(x+1)^{-\alpha}(x-1)^{-\beta},
\end{aligned}
$$

which is (B.1) when $n := n+1$. By induction, (B.1) is true for all $n \in \mathbb{N}$. ■

Definition B.0.5 *Let $E \subseteq \mathbb{R}$, $f, g : E \to \mathbb{R}$ and $a \in \mathbb{R}$ be an accumulation point of E. We say that f is $\mathcal{O}(g)$ as $x \to a$, if*

$$\exists \delta > 0\ \exists C > 0\ \forall x \in E \cap (B(a,\delta) \setminus \{a\}) : |f(x)| \le C|g(x)|.$$

We say that f is $\mathcal{O}(g)$ as $x \nearrow a$, if

$$\exists x_0 \in E \cap]-\infty, a[\ \exists C > 0\ \forall x \in E \cap]x_0, a[\ : |f(x)| \le C|g(x)|.$$

Notation B.0.6 *The notations $f \in \mathcal{O}(g)$, or $f(x) \in \mathcal{O}(g(x))$ will sometimes be used to express that f is $\mathcal{O}(g)$.*

Proposition B.0.7 *Let $E \subseteq \mathbb{R}$, $f_1, f_2, g : E \to \mathbb{R}$ and $a \in \mathbb{R}$ be an accumulation point of E. Then if*

$$\begin{cases} f_1 \in \mathcal{O}(g) & as \quad x \nearrow a \\ f_2 \in \mathcal{O}(g) & as \quad x \nearrow a \end{cases},$$

then the function $\max\{f_1, f_2\} : E \to \mathbb{R}$ *defined by* $\max\{f_1, f_2\}(x) := \max\{f_1(x), f_2(x)\}$ *fulfills that*

$$\max\{f_1, f_2\} \in \mathcal{O}(g) \quad as \quad x \nearrow a.$$

Proof: Assume that $f_1, f_2 \in \mathcal{O}(g)$ as $x \nearrow a$; then for $i = 1, 2$ we have by Def. B.0.5 that

$$\exists x_i \in E \cap\,]-\infty, a[\ \exists C_i \geq 0\ \forall x \in E \cap\,]x_i, a[\ : |f_i(x)| \leq C_i|g(x)|.$$

Put $x_0 := \max\{x_1, x_2\}$ and $C := \max\{C_1, C_2\}$; then

$$\forall x \in E \cap\ ]x_0, a[\ : |f_1(x)| \leq C|g(x)|\ \wedge\ |f_2(x)| \leq C|g(x)|,$$

which implies

$$\forall x \in E \cap\ ]x_0, a[\ : \max\{f_1(x), f_2(x)\} \leq C|g(x)|.$$

Consequently, $\max\{f_1, f_2\} \in \mathcal{O}(g)$ as $x \nearrow a$. ∎

Definition B.0.8 *Let* $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}$ *and* $a \in \mathbb{R}$, *an accumulation point of* E, *be given. Then the limit superior of* f *as* $x \to a$ *is defined as*

$$\limsup_{x \to a} f(x) = \lim_{\epsilon \searrow 0} \sup\{f(x) \mid x \in E \cap (B(a, \epsilon) \setminus \{a\})\}.$$

Proposition B.0.9 *Let* $E \subseteq \mathbb{R}$, $f, g : E \to \mathbb{R}$ *and* $a \in E$ *be given. In order to have that* $f(x)$ *is* $\mathcal{O}(g(x))$ *as* $x \nearrow a$, *it is sufficient that*

$$\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} < \infty.$$

Proof: The proposition will be proven by contraposition. Assume therefore that $f(x)$ is *not* $\mathcal{O}(g(x))$ as $x \nearrow a$; then

$$\forall x_0 \in\,]-\infty, a[\ \forall C > 0\ \exists x = x(x_0, C) \in E \cap\,]x_0, a[\ : |f(x)| > C|g(x)|.$$

Now, given $x_0 \in\,]-\infty, a[$ and $C > 0$ we get

$$\begin{aligned}
\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} &= \lim_{\epsilon \searrow 0} \sup \left\{ \frac{|f(x)|}{|g(x)|} \mid x \in E \cap\,]a - \epsilon, a[\right\} \\
&= \lim_{x_0 \nearrow a} \sup \left\{ \frac{|f(x)|}{|g(x)|} \mid x \in E \cap\,]x_0, a[\right\} \\
&> \lim_{x_0 \nearrow a} C \\
&= C.
\end{aligned}$$

Since $\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} > C$ for the given C, which was arbitrary, then it must be larger than any real number. Consequently:

$$\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} = \infty.$$

∎

The next lemma will be used to make some estimates in the proof of Lemma B.0.11, which says that there exist non-trivial smooth functions with compact support.

Lemma B.0.10 *Let f be the function defined in Lemma B.0.4. If $k, n \in \mathbb{N}$, $k \leq n$, then*

$$|f^{(k)}(x)| \quad \text{is} \quad \mathcal{O}(|f^{(n)}(x)|) \quad \text{as} \quad x \nearrow 1.$$

Proof: If $k = n$, there is nothing to prove, so assume that $k, n \in \mathbb{N}$ with $k < n$. Then

$$\begin{aligned}
\frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} &= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=k+2}} k!(-1)^k(x+1)^{-\alpha}(x-1)^{-\beta}|}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{-\alpha}(x-1)^{-\beta}|} \\
&= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=k+2}} k!(-1)^k(x+1)^{-\alpha}(x-1)^{-\beta}||(x+1)^{n+1}(x-1)^{n+1}|}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{-\alpha}(x-1)^{-\beta}||(x+1)^{n+1}(x-1)^{n+1}|} \\
&= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=k+2}} k!(-1)^k(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|} \\
&\leq \frac{\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=k+2}} k!|x+1|^{n+1-\alpha}|x-1|^{n+1-\beta}}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|} \\
&\leq \frac{2^n k! \sum_{\beta=1}^{k+1} |x-1|^{n+1-\beta}}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|} \\
&= \frac{2^n k! \sum_{\beta=1}^{k+1} |x-1|^{n+1-\beta}}{|n!(-1)^n(x+1)^n + \sum_{\substack{2\leq\alpha\leq n+1\\ 1\leq\beta\leq n,\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|}
\end{aligned} \tag{B.2}$$

The expression in the denominator in (B.2) is a polynomial, call it p, of degree n that has n roots counted with multiplicity according to the Fundamental Theorem of Algebra. Enumerate them as $z_1, \ldots, z_n$. Since $x = 1$ is a root in $p(x) - n!(-1)^n(x+1)^n$ and $(1+1)^n \neq 0$ one cannot have that $x = 1$ is a root in $p(x)$. Consider the set

$$Z = [0, 1[\cap \bigcup_{k=1}^{n} \{z_k\}.$$

Put

$$A := \begin{cases} \max Z, & \text{for} \quad Z \neq \emptyset \\ 0, & \text{for} \quad Z = \emptyset \end{cases} \quad ;$$

If $Z = \emptyset$, then there exists a number $M > 0$ such that $p(x) \geq M$ for all $x \in \,]0, 1[$. If on the other hand $Z \neq \emptyset$, then now, there is a number M' such that $p(x) \geq M'$ for all $x \in \ [x_0, 1[$, whenever $x_0 \in \,]A, 1[$. Eitherway, one has from (B.2)

$$\frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} \leq M^{-1} 2^n k! \sum_{\beta=1}^{k+1} |x-1|^{n+1-\beta}, \tag{B.3}$$

for all $x \in [x_0, 1[$. In (B.3) one has that as $x \nearrow 1$, then $|x-1| \searrow 0$; all the exponents $\{n+1-\beta \mid \beta = 1, \ldots, k+1\}$ are non-negative, so one has readily that the function on the right-hand side of (B.3) is decreasing on $[x_0, 1[$. It therefore attains its maximum at x_0. Now,

$$\frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} \leq M^{-1} 2^n k! \sum_{\beta=1}^{k+1} |x_0 - 1|^{n+1-\beta}, \tag{B.4}$$

whenever $x \in [x_0, 1[$, $x_0 \in]A, 1[$. Now,

$$\begin{aligned}
\limsup_{x \to 1} \frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} &:= \lim_{\epsilon \searrow 0} \sup \left\{ \frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} \mid x \in]1-\epsilon, 1[,\ f^{(n)}(x) \neq 0 \right\} \\
&= \lim_{x_0 \nearrow 1} \sup \left\{ \frac{|f^{(k)}(x)|}{|f^{(n)}(x)|} \mid x \in]x_0, 1[,\ f^{(n)}(x) \neq 0 \right\} \\
&\leq M^{-1} 2^n k! \lim_{x_0 \nearrow 1} \sum_{\beta=1}^{k+1} |x_0 - 1|^{n+1-\beta} \\
&= 0 \\
&< \infty.
\end{aligned}$$

Therefore, according to Prop. B.0.9 we have that $|f^{(k)}(x)|$ is $\mathcal{O}(|f^{(n)}(x)|)$ as $x \nearrow 1$. ■

Lemma B.0.11 *The function $j : \mathbb{R} \to \mathbb{R}$ defined by*

$$j(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right), & \textit{for} \quad |x| < 1 \\ 0, & \textit{otherwise} \end{cases},$$

has the following three properties: $j \geq 0$, $\operatorname{supp} j = [-1, 1]$ and $j \in C^\infty(\mathbb{R})$.

Proof: $j \geq 0$: Obvious.

$\operatorname{supp} j = [-1, 1]$: Immediate from the definition of j.

$j \in C^\infty(\mathbb{R})$: For any $n \in \mathbb{N}$, the n'th derivative of j restricted to $[-1, 1]^c$ is equal to 0. Write $f(x) = (x+1)^{-1}(x-1)^{-1}$, for $|x| < 1$; then $j_{|]-1,1[} = \exp \circ f$. Now, $\exp, f \in C^\infty(]-1, 1[)$. For any $n \in \mathbb{N}$, the n'th derivative of j restricted to $]-1, 1[$ exists and is given by Faà di Brunos formula, Bell polynomial form as

$$\begin{aligned}
j^{(n)}(x) &= (\exp \circ f)^{(n)}(x) \\
&= \sum_{k=1}^{n} (\exp^{(k)} \circ f)(x) \mathbf{B}_{n,k}[f'(x), \ldots, f^{(n-k+1)}(x)] \\
&= (\exp \circ f)(x) \sum_{k=1}^{n} \mathbf{B}_{n,k}[f'(x), \ldots, f^{(n-k+1)}(x)] \\
&= j(x) \sum_{k=1}^{n} \mathbf{B}_{n,k}[f'(x), \ldots, f^{(n-k+1)}(x)].
\end{aligned} \tag{B.5}$$

Consequently, j is smooth on $]-1, 1[$.

The only thing left in showing smoothness of j is to show that j is smooth on the boundary of $\operatorname{supp} j$, $\partial(\operatorname{supp} j)$. Without loss of generality it will be shown that j is smooth at $x = 1$. This will be done by induction.

First, since

$$\lim_{x \nearrow 1} \exp\left(\frac{1}{x^2-1}\right) = 0 \quad \text{and} \quad \lim_{x \searrow 1} 0 = 0,$$

one has

$$j(1) = 0 = \lim_{x \to 1} j(x),$$

so j is continuous at $x = 1$ with $j(1) = 0$.

Next, suppose that the induction hypothesis is true for some $n \in \mathbb{N}$, i.e. given some $n \in \mathbb{N}$, then $j^{(n-1)}(1)$ exists, $j^{(n-1)}(1) = 0$ and $j^{(n-1)}$ is continuous at $x = 1$. We want to show that $j^{(n)}(1)$ exists, $j^{(n)}(1) = 0$ and $j^{(n)}$ is continuous at $x = 1$.

Consider the limit

$$\lim_{x \to 1} j^{(n)}(x). \tag{B.6}$$

We want to show that it exists, and is equal to $j^{(n)}(1) = 0$. In order to do so, the corresponding one-sided limits will be computed. The right-hand limit is obtained as follows: given $\epsilon > 0$, put $\delta(\epsilon) = c > 0$; then, since $j^{(n)}(x) = 0$ for any $x \in [-1,1]^c$,

$$|j^{(n)}(x) - 0| = |j^{(n)}(x)| < \epsilon, \quad \text{whenever} \quad x \in]1, 1+\delta[.$$

This means that

$$\lim_{x \searrow 1} j^{(n)}(x) = 0. \tag{B.7}$$

The left-hand limit will be treated in several steps. By (B.5) it can be written as

$$\lim_{x \nearrow 1} j^{(n)}(x) = \lim_{x \nearrow 1} j(x) \sum_{k=1}^{n} \mathbf{B}_{n,k}[f'(x), \ldots, f^{(n-k+1)}(x)]. \tag{B.8}$$

It will now be shown that $\lim_{x \nearrow 1} j(x) f^{(n)}(x) = 0$. For that purpose, let $0 < h < 1$ be given; then

$$\begin{aligned} j(1-h)f^{(n)}(1-h) &\leq j(1-h)|f^{(n)}(1-h)| \\ &\leq j(1-h) \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ \alpha+\beta = n+2}} n!(2-h)^{-\alpha} h^{-\beta} \\ &\leq j(1-h) h^{-n-1} \sum_{\alpha=1}^{n+1} n!(2-h)^{-\alpha} \\ &\leq j(1-h) h^{-n-1} (n+1)! \ . \end{aligned}$$

By definition of j we get

$$\begin{aligned} \exp\left(\frac{1}{(1-h)^2-1}\right) h^{-n-1}(n+1)! &= \exp\left(\frac{1}{h^2-2h}\right) h^{-n-1}(n+1)! \\ &\leq \exp\left(-\frac{1}{2h}\right) h^{-n-1}(n+1)!. \end{aligned}$$

The change of variables $h \mapsto (2t)^{-1}$ gives

$$\begin{aligned}
\exp\left(-\frac{1}{2h}\right) h^{-n-1}(n+1)! &= \exp\left(-\frac{1}{2(2t)^{-1}}\right)\left(\frac{1}{2t}\right)^{-n-1}(n+1)! \\
&= e^{-t}\left(\frac{1}{2t}\right)^{-n-1}(n+1)! \\
&= 2^{n+1}(n+1)!\cdot e^{-t}t^{n+1}.
\end{aligned}$$

Now, $h \searrow 0$ if and only if $t \to \infty$, so [12], Th. 8.6 yields that

$$\begin{aligned}
\lim_{x\nearrow 1} j(x)|f^{(n)}(x)| &= \lim_{h\searrow 0} j(1-h)|f^{(n)}(1-h)| \\
&\leq 2^{n+1}(n+1)! \lim_{t\to\infty} e^{-t}t^{n+1} = 0.
\end{aligned}$$

Since $j \geq 0$ on $[-1,1]$, this implies that

$$\lim_{x\nearrow 1} j(x)|f^{(n)}(x)| = 0. \tag{B.9}$$

Note that in showing (B.9), the induction hypothesis was not used; it will be used later. Consequently, (B.9) is true for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned}
|j^{(n)}(x)| &= j(x)\left|\sum_{k=1}^{n} \mathbf{B}[f'(x),\dots,f^{(n-k+1)}(x)]\right| \\
&\leq j(x)\sum_{k=1}^{n} |\mathbf{B}[f'(x),\dots,f^{(n-k+1)}(x)]| \qquad \text{(B.10)}
\end{aligned}$$

The Bell polynomials are given by

$$\begin{aligned}
&\mathbf{B}_{n,k}[f'(x),\dots,f^{(n-k+1)}(x)] \\
= &\sum_{(j_1,\dots,j_{n-k+1})\in S} \frac{n!}{\prod_{m=1}^{n-k+1} j_m!(m!)^{j_m}} \prod_{m=1}^{n-k+1} (f^{(m)}(x))^{j_m},
\end{aligned}$$

where $S = \{(j_1,\dots,j_{n-k+1}) \in \mathbb{N}_0^{n-k+1} \mid \sum_{m=1}^{n-k+1} j_m = k, \quad \sum_{m=1}^{n-k+1} m j_m = n\}$. Now, Lemma B.0.10 will be exploited. Since $m \in \{1,\dots,n-k+1\}$ we get that $|f^{(m)}(x)|$ is $\mathcal{O}(|f^{(n-k+1)}(x)|)$ as $x \nearrow 1$. Therefore,

$$\exists x_0 \in]0,1[\ \exists C_m > 0\ \forall x \in]x_0,1[: |f^{(m)}(x)| \leq C_m|f^{(n-k+1)}(x)|.$$

This formula implies that for all $x \in]x_0,1[$ one has

$$\begin{aligned}
|\mathbf{B}_{n,k}[f'(x),\dots,f^{(n-k+1)}(x)]| &\leq \text{const}\cdot \sum_{(j_1,\dots,j_{n-k+1})\in S} \prod_{m=1}^{n-k+1} |f^{(m)}(x)|^{j_m} \\
&\leq \text{const}\cdot \sum_{(j_1,\dots,j_{n-k+1})\in S} \prod_{m=1}^{n-k+1} (C_m|f^{(n-k+1)}(x)|)^{j_m} \\
&= \text{const}\cdot \text{card}(S)\cdot |f^{(n-k+1)}(x)|^k \cdot \prod_{m=1}^{n-k+1} C_m^{j_m} \\
&= \text{const}\cdot |f^{(n-k+1)}(x)|^k. \qquad \text{(B.11)}
\end{aligned}$$

Above, the constants card(S) and $\prod_{m=1}^{n-k+1} C_m^{j_m}$ were absorbed into const. From (B.10), (B.11) and Lemma B.0.10, we get for any $x \in \,]x_0, 1[$ that

$$
\begin{aligned}
|j^{(n)}(x)| &\leq j(x) \cdot \text{const} \cdot \sum_{k=1}^{n} |f^{(n-k+1)}(x)|^k \\
&\leq j(x) \cdot \text{const} \cdot \sum_{k=1}^{n} (\tilde{C}_k |f^{(n)}(x)|)^k \\
&\leq j(x) \cdot \text{const} \cdot n \cdot \max_{1 \leq k \leq n} \{\tilde{C}_k^k\} \cdot \begin{cases} |f^{(n)}(x)|, & \text{for} \quad |f^{(n)}(x)| < 1 \\ |f^{(n)}(x)|^n, & \text{for} \quad |f^{(n)}(x)| \geq 1 \end{cases} \\
&\leq \text{const} \cdot j(x) \cdot \begin{cases} |f^{(n)}(x)|, & \text{for} \quad |f^{(n)}(x)| < 1 \\ |f^{(n)}(x)|^n, & \text{for} \quad |f^{(n)}(x)| \geq 1 \end{cases} \\
&\leq \text{const} \cdot j(x) \cdot \max\{|f^{(n)}(x)|, |f^{(n)}(x)|^n\}. \qquad \text{(B.12)}
\end{aligned}
$$

Above, the constants n and $\max_{1 \leq k \leq n}\{\tilde{C}_k^k\}$ were absorbed into const. In the following it will be proven, that for some $N \in \mathbb{N}$, $\max\{|f^{(n)(x)}|, |f^{(n)}(x)|^n\}$ is $\mathcal{O}(|f^{(N)}(x)|)$ as $x \nearrow 1$. In order to do so we prove that $|f^{(n)}(x)|$ and $|f^{(n)}(x)|^n$ are $\mathcal{O}(|f^{(N)}(x)|)$ as $x \nearrow 1$. In other words it is needed, that

$$
\exists x_1 \in \,]0, 1[\; \exists C_1 > 0 \; \forall x \in \,]x_1, 1[\, : |f^{(n)}(x)| \leq C_1 |f^{(N)}(x)| \qquad \text{(B.13)}
$$

and

$$
\exists x_2 \in \,]0, 1[\; \exists C_2 > 0 \; \forall x \in \,]x_2, 1[\, : |f^{(n)}(x)|^n \leq C_2 |f^{(N)}(x)|. \qquad \text{(B.14)}
$$

For $n = 1$, put $N = 1$; then (B.13) and (B.14) are true. For $n > 1$, put $N = n^2 + n - 1$; then $N > n$, so (B.13) is true by Lemma B.0.10. Therefore, $|f^{(n)}(x)|$ is $\mathcal{O}(|f^{(N)}(x)|)$ as $x \nearrow 1$.

In order to show formula (B.14), it is sufficient according to Prop. B.0.9 to show that

$$
\begin{aligned}
\limsup_{x \to 1} \frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} &:= \limsup_{\epsilon \searrow 0} \left\{ \frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} \;\middle|\; x \in \,]1-\epsilon, 1[, \; f^{(N)}(x) \neq 0 \right\} \\
&< \infty.
\end{aligned}
$$

Now,

$$\begin{aligned}
\frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} &= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{-\alpha}(x-1)^{-\beta}|^n}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{-\alpha}(x-1)^{-\beta}|} \\
&= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{-\alpha}(x-1)^{-\beta}|^n|(x+1)^{N+1}(x-1)^{N+1}|}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{-\alpha}(x-1)^{-\beta}||(x+1)^{N+1}(x-1)^{N+1}|} \\
&= \frac{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!(-1)^n(x+1)^{n+1-\alpha}(x-1)^{n+1-\beta}|^n}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{N+1-\alpha}(x-1)^{N+1-\beta}|} \\
&\le \frac{\left(\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=n+2}} n!|x+1|^{n+1-\alpha}|x-1|^{n+1-\beta}\right)^n}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{N+1-\alpha}(x-1)^{N+1-\beta}|} \\
&\le \frac{\left(\sum_{\beta=1}^{n+1} n!2^n|x-1|^{n+1-\beta}\right)^n}{|\sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{N+1-\alpha}(x-1)^{N+1-\beta}|} \qquad \text{(B.15)}
\end{aligned}$$

In the expression (B.15), the term that corresponds to $\alpha = 1$ is singled out, and one gets that the expression (B.15) is equal to

$$\frac{n!^n 2^{n^2}\left(\sum_{\beta=1}^{n+1}|x-1|^{n+1-\beta}\right)^n}{|N!(-1)^N(x+1)^N + \sum_{\substack{2\le\alpha\le N+1\\ 1\le\beta\le N,\ \alpha+\beta=N+2}} N!(-1)^N(x+1)^{N+1-\alpha}(x-1)^{N+1-\beta}|}. \qquad \text{(B.16)}$$

The expression in the denominator in (B.16) is a polynomial, call it p, of degree N that has N roots counted with multiplicity according to The Fundamental Theorem of Algebra. Enumerate them as $z_1, \ldots, z_N$. Since $x = 1$ is a root in $p(x) - N!(-1)^N(x+1)^N$ and $N!(-1)^N(1+1)^N \neq 0$ one cannot have that $x = 1$ is a root in $p(x)$. Consider the set

$$Z = [0,1[\ \cap \bigcup_{k=1}^{N}\{z_k\}.$$

Put

$$A := \begin{cases} \max Z, & \text{for} \quad Z \neq \emptyset \\ 0, & \text{for} \quad Z = \emptyset \end{cases} \quad ;$$

if $Z = \emptyset$, then there exists a number $M > 0$ such that $p(x) \geq M$ for all $x \in\ ]0,1[$. If on the other hand $Z \neq \emptyset$, then now, there is a number M' such that $p(x) \geq M'$ for all $x \in\ ]x_0, 1[$, whenever $x_0 > A$. Eitherway, one has from (B.15) and (B.16) that

$$\frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} \leq M^{-1} n!^n 2^{n^2}\left(\sum_{\beta=1}^{n+1}|x-1|^{n+1-\beta}\right)^n. \qquad \text{(B.17)}$$

In (B.17) one has that as $x \nearrow 1$, then $|1-x| \searrow 0$; all the exponents $\{n+1-\beta \mid \beta = 1, \ldots, n+1\}$ are non-negative, so one has readily that the function on the right-hand side of (B.17) is decreasing on $]0,1[$. Now,

$$\frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} \leq M^{-1} n!^n 2^{n^2} \left(\sum_{\beta=1}^{n+1} |x_0 - 1|^{n+1-\beta} \right)^n, \tag{B.18}$$

whenever $x \in]x_0, 1[$, $x_0 > A$. Now,

$$\begin{aligned}
\limsup_{x\to 1} \frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} &:= \lim_{\epsilon \searrow 0} \sup \left\{ \frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} \mid x \in]1-\epsilon, 1[,\ f^{(N)}(x) \neq 0 \right\} \\
&= \lim_{x_0 \nearrow 1} \sup \left\{ \frac{|f^{(n)}(x)|^n}{|f^{(N)}(x)|} \mid x \in]x_0, 1[,\ f^{(N)}(x) \neq 0 \right\} \\
&\leq M^{-1} n!^n 2^{n^2} \lim_{x_0 \nearrow 1} \left(\sum_{\beta=1}^{n+1} |x_0 - 1|^{n+1-\beta} \right)^n \\
&= M^{-1} n!^n 2^{n^2} \\
&< \infty.
\end{aligned}$$

Therefore, according to Prop. B.0.9, formula (B.14) is true. This means that $|f^{(n)}(x)|^n$ is $\mathcal{O}(|f^{(N)}(x)|)$ as $x \nearrow 1$. Now, Prop. B.0.7 gives that $\max\{|f^{(n)}(x)|, |f^{(n)}(x)|^n\}$ is $\mathcal{O}(|f^{(N)}(x)|)$ as $x \nearrow 1$. This together with (B.12) imply that

$$\begin{aligned}
\lim_{x\nearrow 1} |j^{(n)}(x)| &\leq \text{const} \cdot \lim_{x\nearrow 1} j(x) \max\{|f^{(n)}(x)|, |f^{(n)}(x)|^n\} \\
&\leq \text{const} \cdot \lim_{x\nearrow 1} j(x) |f^{(n^2+n-1)}(x)|.
\end{aligned}$$

Since $\lim_{x\nearrow 1} j(x)|f^{(n)}(x)| = 0$ for all $n \in \mathbb{N}$, one has in particular that

$$\lim_{x\nearrow 1} j^{(n)}(x) = 0. \tag{B.19}$$

Now, (B.7) and (B.19) imply that

$$\lim_{x\to 1} j^{(n)}(x) = 0. \tag{B.20}$$

Now that it is proven that the limit (B.6) exists and equals zero, the only thing left is to show that $j^{(n)}(1)$ exists and equals zero. The left-hand side derivatives of j are by definition given by

$$j^{(n)}(1-) = \lim_{x\nearrow 1} \frac{j^{(n-1)}(x) - j^{(n-1)}(1)}{x-1}.$$

At this point, the induction hypothesis will be used: since $j^{(n-1)}$ is continuous at $x = 1$ by induction hypothesis, and $j^{(n-1)}$ is differentiable on $]-1,1[$, then in particular it must be differentiable on $]x,1[$ and continuous on $[x,1]$ for every $x \in]0,1[$. The mean value theorem now gives that there is a $c_x \in]x,1[$ such that

$$j^{(n-1)}(x) - j^{(n-1)}(1) = j^{(n)}(c_x)(x-1),$$

or equivalently,

$$\frac{j^{(n-1)}(x) - j^{(n-1)}(1)}{x-1} = j^{(n)}(c_x).$$

This implies that

$$\lim_{x \nearrow 1} \frac{j^{(n-1)}(x) - j^{(n-1)}(1)}{x-1} = \lim_{x \nearrow 1} j^{(n)}(c_x),$$

provided that the limits exist. Since $\lim_{x \nearrow 1} j^{(n)}(c_x) = \lim_{x \nearrow 1} j^{(n)}(x) = 0$, we infer that

$$j^{(n)}(1-) = \lim_{x \nearrow 1} j^{(n)}(c_x) = 0. \tag{B.21}$$

The same argument applies to the right-hand derivative, and we get that

$$j^{(n)}(1+) = 0. \tag{B.22}$$

The formulas (B.21) and (B.22) imply that $j^{(n)}(1)$ exists with

$$j^{(n)}(1) = j^{(n)}(1-) = j^{(n)}(1+) = 0. \tag{B.23}$$

Now, (B.20) and (B.23) imply that

$$j^{(n)}(1) = \lim_{x \to 1} j^{(n)}(x),$$

which says that $j^{(n)}$ is continuous at $x = 1$. It is now proven that j is continuous at $x = 1$ with $j(1) = 0$, and that if it is the case that $j^{(n-1)}(1)$ exists, $j^{(n-1)}(1) = 0$, and $j^{(n-1)}$ is continuous at $x = 1$, then it is also the case that $j^{(n)}(1)$ exists, $j^{(n)}(1) = 0$ and $j^{(n)}$ is continuous at $x = 1$. By induction,

$$\forall n \in \mathbb{N} : \; j^{(n)}(1) \text{ exists, } j^{(n)}(1) = 0 \quad \text{and} \quad j^{(n)} \text{ is continuous at } x = 1.$$

Consequently, j is smooth at $x = 1$. ■

Lemma B.0.12 *Let j be as in Lemma B.0.11. Define for $\epsilon > 0$, $j_\epsilon : \mathbb{R} \to \mathbb{R}$ by*

$$j_\epsilon(x) = \|j\|_{L^1(\mathbb{R})}^{-1} \epsilon^{-1} j(\epsilon^{-1} x).$$

Then $j_\epsilon \geq 0$, $j_\epsilon \in C^\infty(\mathbb{R})$, $\operatorname{supp} j_\epsilon = [-\epsilon, \epsilon]$ and $\int_{\mathbb{R}} j_\epsilon(x)dx = 1$.

Proof: $j_\epsilon \geq 0$: Obvious.

$j_\epsilon \in C^\infty(\mathbb{R})$: Since j is smooth by Lemma B.0.11, one has that

$$j_\epsilon^{(n)}(x) = \frac{d^n}{dx^n}\left(\|j\|_{L^1(\mathbb{R})}^{-1} \epsilon^{-1} j(\epsilon^{-1} x)\right) = \|j\|_{L^1(\mathbb{R})}^{-1} \epsilon^{-1-n} j^{(n)}(\epsilon^{-1} x),$$

so j_ϵ is smooth.

$\operatorname{supp} j_\epsilon = [-\epsilon, \epsilon]$: Given $x \in [-\epsilon, \epsilon]^c$, then $\epsilon^{-1} x \in [-1, 1]^c = (\operatorname{supp} j)^c$, which is open, so there exists a $\delta > 0$ such that $B(\epsilon^{-1}x, \delta) \subseteq (\operatorname{supp} j)^c$; therefore $j(B(\epsilon^{-1}x, \delta)) = \|j\|_{L^1(\mathbb{R})}^{-1} \epsilon^{-1} \cdot j(B(\epsilon^{-1}x, \delta)) = j_\epsilon(B(x, \epsilon\delta)) = 0$, so $x \in (\operatorname{supp} j_\epsilon)^c$. Given $x \in (\operatorname{supp} j_\epsilon)^c$; then there is a $\delta > 0$ such that $B(x, \delta) \subseteq (\operatorname{supp} j_\epsilon)^c$; therefore $j_\epsilon(B(x, \delta)) = 0$, or equivalently: $\|j\|_{L^1(\mathbb{R})}^{-1} \epsilon^{-1} \cdot$

$j(\epsilon^{-1}B(x,\delta)) = j(B(\epsilon^{-1}x, \epsilon^{-1}\delta)) = 0$. This implies that $\epsilon^{-1}x \in (\operatorname{supp} j)^c = [-1,1]^c$. Hence $x \in [-\epsilon, \epsilon]^c$.

$\int_{\mathbb{R}} j_\epsilon(x)dx = 1$: We have that

$$\begin{aligned}
\int_{\mathbb{R}} j_\epsilon(x)dx &= \int_{\mathbb{R}} \|j\|^{-1}_{L^1(\mathbb{R})}\epsilon^{-1} j(\epsilon^{-1}x)dx = \|j\|^{-1}_{L^1(\mathbb{R})}\epsilon^{-1}\int_{\mathbb{R}} j(\epsilon^{-1}x)dx \\
&= \|j\|^{-1}_{L^1(\mathbb{R})}\epsilon^{-1}\int_{\mathbb{R}} j(y)\epsilon dy \\
&= 1.
\end{aligned}$$

■

A function j_ϵ defined as above is called a 'mollifier', and ϵ is a corresponding 'radius of mollification'. A sequence of mollifiers $\{j_n\}_{n\geq 0}$ is called a 'regularizing sequence', or an 'approximate identity'. The reason for name approximate identity is that as $\epsilon \searrow 0$, then $f * j_\epsilon \to f$ in $L^1(\mathbb{R})$. Embedding $L^1(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$, one has that $\delta = \mathcal{D}' - \lim_{\epsilon\searrow 0} j_\epsilon$, and $f * \delta = f$, where δ, the delta distribution, plays the role of the identity element in the algebra $(\mathcal{D}'(\mathbb{R}), *)$.

Lemma B.0.13 *Let $M > m \geq 0$. There exists a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ on $]-M, M[^c$, $\varphi \equiv 0$ on $[-m, m]$, and $0 \leq \varphi \leq 1$.*

Proof: Denoting by $\chi_{]-\frac{1}{2}(M+m),\frac{1}{2}(M+m)[^c}$ the characteristic function of $]-\frac{1}{2}(M+m), \frac{1}{2}(M+m)[^c$, we then define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
\varphi(x) &= \left(j_{\frac{1}{2}(M-m)} * \chi_{]-\frac{1}{2}(M+m),\frac{1}{2}(M+m)[^c}\right)(x) \\
&= \int_{\mathbb{R}} j_{\frac{1}{2}(M-m)}(x-t)\chi_{]-\frac{1}{2}(M+m),\frac{1}{2}(M+m)[^c}(t)dt \\
&= \int_{\mathbb{R}} \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} j\left(\frac{x-t}{\frac{1}{2}(M-m)}\right)\chi_{]-\frac{1}{2}(M+m),\frac{1}{2}(M+m)[^c}(t)dt \\
&= \int_{-\infty}^{-\frac{1}{2}(M+m)} \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} j\left(\frac{x-t}{\frac{1}{2}(M-m)}\right)dt \qquad \text{(B.24)} \\
&+ \int_{\frac{1}{2}(M+m)}^{\infty} \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} j\left(\frac{x-t}{\frac{1}{2}(M-m)}\right)dt. \qquad \text{(B.25)}
\end{aligned}$$

$\varphi \equiv 0$ on $[-m, m]$: Given $x_0 \in [-m, m]$; assume first, that $x_0 \in]-\infty, m]$; then

$$\begin{aligned}
\frac{x_0 - t}{\frac{1}{2}(M-m)} &\leq \frac{m-t}{\frac{1}{2}(M-m)} \leq \frac{m - \frac{1}{2}(M+m)}{\frac{1}{2}(M-m)} \\
&= \frac{\frac{1}{2}m - \frac{1}{2}M}{\frac{1}{2}(M-m)} = \frac{-\frac{1}{2}(M-m)}{\frac{1}{2}(M-m)} \\
&= -1,
\end{aligned}$$

when $t \geq \frac{1}{2}(M+m)$. Therefore, $j(\frac{x_0-t}{\frac{1}{2}(M-m)}) = 0$ when $t \geq \frac{1}{2}(M+m)$, since $\operatorname{supp} j = [-1, 1]$.

Consequently, the integral in (B.25) is equal to zero. Assume now, that $x_0 \in [-m, \infty[$; then

$$\begin{aligned}\frac{x_0 - t}{\frac{1}{2}(M-m)} &\geq \frac{-m-t}{\frac{1}{2}(M-m)} = \frac{-m+(-t)}{\frac{1}{2}(M-m)} \\ &\geq \frac{-m+\frac{1}{2}(M+m)}{\frac{1}{2}(M-m)} = \frac{\frac{1}{2}(M-m)}{\frac{1}{2}(M-m)} \\ &= 1,\end{aligned}$$

when $t \leq -\frac{1}{2}(M+m)$. Therefore, $j(\frac{x_0-t}{\frac{1}{2}(M-m)}) = 0$ when $t \leq -\frac{1}{2}(M+m)$, since $\operatorname{supp} j = [-1, 1]$. Consequently, the integral in (B.24) is equal to zero. It is now shown that $x_0 \in]-\infty, m] \cap [-m, \infty[= [-m, m]$ implies that $\varphi(x_0) = 0$, so $\varphi \equiv 0$ on $[-m, m]$.

$\varphi \equiv 1$ on $]-M, M[^c$: Assume without loss of generality, that $x_0 \in [M, \infty[$. Then

$$\frac{x_0-t}{\frac{1}{2}(M-m)} \geq \frac{M-t}{\frac{1}{2}(M-m)} = \frac{M+(-t)}{\frac{1}{2}(M-m)} \geq \frac{M+\frac{1}{2}(M+m)}{\frac{1}{2}(M-m)} \geq 1,$$

when $t \leq -\frac{1}{2}(M+m)$. Therefore, $j(\frac{x_0-t}{\frac{1}{2}(M-m)}) = 0$ when $t \leq -\frac{1}{2}(M+m)$, since $\operatorname{supp} j = [-1, 1]$. Consequently, the integral in (B.24) is equal to zero. In order to compute the integral (B.25), a change of variables is convenient. Take $g : \mathbb{R} \to \mathbb{R}$, $g(t) = \frac{x_0-t}{\frac{1}{2}(M-m)}$ with Jacobian $J_g(t) = \frac{-1}{\frac{1}{2}(M-m)}$. With $T = [\frac{1}{2}(M+m), \infty[$, one has $g(T) = \left]-\infty, \frac{x_0-\frac{1}{2}(M+m)}{\frac{1}{2}(M-m)}\right]$. Now the assumption $x_0 \geq M$ implies that $g(T) \supseteq]-\infty, 1]$. The integral (B.25) can now be calculated: integral (B.25) is equal to

$$\begin{aligned}&\int_{\frac{1}{2}(M+m)}^{\infty} \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} j\left(\frac{x_0-t}{\frac{1}{2}(M-m)}\right) dt \\ =& \int_T \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} |J_g|^{-1} (j \circ g)(t) |J_g| dt \\ =& \frac{\|j\|^{-1}_{L^1(\mathbb{R})}}{\frac{1}{2}(M-m)} |J_g|^{-1} \int_{g(T)} j(y) dy \\ =& \|j\|^{-1}_{L^1(\mathbb{R})} \int_{-\infty}^{1} j(y) dy, \quad \text{since} \quad \operatorname{supp} j = [-1,1] \\ =& \|j\|^{-1}_{L^1(\mathbb{R})} \int_{\mathbb{R}} |j(y)| dy, \quad \text{since} \quad \operatorname{supp} j = [-1,1] \quad \text{and} \quad j \geq 0 \\ =& 1,\end{aligned}$$

so $\varphi(x_0) = 1$. Consequently: $\varphi \equiv 1$ on $]-M, M[^c$.

$0 \leq \varphi \leq 1$: Assume without loss of generality that $x_0 \in]m, M[$. Now, putting $T_1 =]-\infty, -\frac{1}{2}(M+m)]$, one has that $g(T_1) = \left[\frac{x_0+\frac{1}{2}(M+m)}{\frac{1}{2}(M-m)}, \infty\right[$; the assumption $x_0 > m$ now gives $g(T_1) \subset \left[\frac{\frac{1}{2}(M+3m)}{\frac{1}{2}(M-m)}, \infty\right[\subsetneq [1, \infty[$, since $\frac{M+3m}{M-m} > 1$. Similarly, putting $T_2 = [\frac{1}{2}(M+m), \infty[$ one gets $g(T_2) = \left]-\infty, \frac{x_0 - \frac{1}{2}(M+m)}{\frac{1}{2}(M-m)}\right]$; the assumption $x_0 < M$ now gives that $g(T_2) \subsetneq]-\infty, 1]$.

$$\begin{aligned}
\varphi(x_0) &= \int_{-\infty}^{-\frac{1}{2}(M+m)} \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} j\left(\frac{x_0 - t}{\frac{1}{2}(M-m)}\right) dt \\
&+ \int_{\frac{1}{2}(M+m)}^{\infty} \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} j\left(\frac{x_0 - t}{\frac{1}{2}(M-m)}\right) dt \\
&= \int_{T_1} \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} |J_g|^{-1} (j \circ g)(t) |J_g| dt \\
&+ \int_{T_2} \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} |J_g|^{-1} (j \circ g)(t) |J_g| dt \\
&= \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} |J_g|^{-1} \left(\int_{g(T_1)} j(y) dy + \int_{g(T_2)} j(y) dy \right) \\
&= 0 + \frac{\|j\|_{L^1(\mathbb{R})}^{-1}}{\frac{1}{2}(M-m)} |J_g|^{-1} \int_{g(T_2)} j(y) dy \\
&< 1.
\end{aligned}$$

Furthermore, $\varphi \geq 0$, since $j \geq 0$. Therefore $0 \leq \varphi \leq 1$.

$\varphi \in C^\infty(\mathbb{R})$: Clearly, $\chi_{]-\frac{1}{2}(M+m), \frac{1}{2}(M+m)[^c} \in L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$; by Lemma B.0.12, $j_{\frac{1}{2}(M-m)} \in \mathcal{D}(\mathbb{R})$. By [14], Th. 6.30, $j_{\frac{1}{2}(M-m)} * \chi_{]-\frac{1}{2}(M+m), \frac{1}{2}(M+m)[^c} \in C^\infty(\mathbb{R})$ with

$$D^n(j_{\frac{1}{2}(M-m)} * \chi_{]-\frac{1}{2}(M+m), \frac{1}{2}(M+m)[^c}) = (D^n j_{\frac{1}{2}(M-m)}) * \chi_{]-\frac{1}{2}(M+m), \frac{1}{2}(M+m)[^c}.$$

■

Notation B.0.14 *The function φ constructed in the proof of Lemma B.0.13 will occasionally be denoted by $\varphi_{m,M}$.*

Appendix C

List of symbols

Symbol	Description
	Sesquilinear forms
$\mathfrak{s}, \mathfrak{t}$	Sesquilinear forms
$\mathcal{D}(\mathfrak{t})$	The domain of $\mathfrak{t}$
$\operatorname{Re}\mathfrak{t}$	The real part of $\mathfrak{t}$
$\operatorname{Im}\mathfrak{t}$	The imaginary part of $\mathfrak{t}$
$\mathfrak{t}^*$	The adjoint form of $\mathfrak{t}$
$\|\mathfrak{t}\|$	The norm of $\mathfrak{t}$
$\gamma_{\mathfrak{h}}$	The lower bound of the symmetric form $\mathfrak{t}$
$\Theta(\mathfrak{t})$	The numerical range of $\mathfrak{t}$
$u_n \xrightarrow{\mathfrak{t}} u$	The sequence $\{u_n\}$ is $\mathfrak{t}$-convergent to u
$(\cdot\|\cdot)_{\mathfrak{h}}$	The inner product $\mathfrak{h} + \mathbf{1}$
$\|\cdot\|_{\mathfrak{t}}$	The norm induced by $(\cdot\|\cdot)_{\mathfrak{h}}$
$\tilde{\mathfrak{t}}$	The closure of $\mathfrak{t}$
	Function spaces
$\mathcal{H}$	A Hilbert space
$\mathcal{B}(\mathcal{H})$	The C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$
$\mathcal{B}(X, Y)$	The algebra of bounded operators from between the Hilbert spaces X and Y
$\mathcal{D}(\mathbb{R})$	Test functions; smooth functions defined on $\mathbb{R}$ with compact support
$\mathcal{D}'(\mathbb{R})$	Distributions; continuous linear functionals on $\mathcal{D}(\mathbb{R})$
$\mathcal{H}^m(\mathbb{R})$	The L^2-based Sobolev space of order m
$L^1_{\mathrm{loc}}(\mathbb{R})$	The vector space of locally integrable functions defined on $\mathbb{R}$
$C_\infty(\mathbb{R})$	Continuous functions on $\mathbb{R}$ that vanish at infinity

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