Computer Based FE Analysis of Reinforced Concrete Walls by the Stringer Method

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Synopsis:

Optimisation of structures to reduce costs and environmental impacts are important issues in the building sector. In order to meet these the computer program `Opti_String` is developed for optimising reinforced concrete walls using the stringer method based on the plasticity theory. The program is formulated by a finite element concept. The Eurocode is implemented in `Opti_String` to ensure practical design demands.

A lower bound method is compared with an upper bound method. The comparison shows that an exact translation for the stringer system exists.

`Opti_String` is capable of optimising an arbitrary geometry and the optimisation is based on the lower bound method. `Opti_String` calculates the ultimate load bearing capacity or design parameters for a given load along with the stress parameters.

By use of a complex structure the influence of practical design restrictions and several load cases are illustrated. Different weighting parameters are applied to force another stress distribution in order to decrease the utilisation of selected stringers.

Preface

This thesis is prepared and compiled as a part of the fourth semester on the M. Sc. in Structural and Civil Engineering at Aalborg University. The period of which this report is written is from the 1st of February 2012 to the 12th of June 2012 under the supervision of professor Lars Damkilde.

Reading Guide

The master thesis consists of two parts; a main report and appendixes. In the main report there are references to the appendixes, where the appertaining calculations and extensional documentation are found.

The files used in the different software e.g. *MATLAB*, are found on the attached CD and a list of the files is found in Appendix A7. The CD also encloses an electronic PDF version of the master thesis. Files that are relevant for a section in the report are placed in a folder with the same name as the section on the CD.

Sources are quoted by the Harvard method of bibliography with the name of the author and year of publication inserted in brackets after the text. Quoted sources from literature, papers, websites and design codes will appear e.g . (Damkilde, 1995).

If the source is placed before the period at the end of a sentence, the source refers only to the sentence, whereas if the source is placed after the period, the source refers to the entire text section.

Figure and table numerations refers to which chapter the desired figure or table is located in. Please note that if a figure or a table is not attached to a source, they are produced by the group.

The bibliography gives extensive information about each source. Since several of the sources are recurrent, the bibliography is not divided into source types. Instead, the sources are sorted alphabetically by notices, under which information about the source type, i.e.; author, title, publisher or editor, year of publication, presentation number, ISBN and URL.

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Nomenclature

Latin upper case letters

$\mathbf{A}$	Matrix of inequality constraints
$A_{c,nec}$	Necessary concrete compression area
A_s	Reinforcement area of one reinforcement bar
A_t	Reinforcement area of a stringer
$\mathbf{C}$	Constraint matrix
$\mathbf{C}_d$	Material constrains. Vector in load optimization. Matrix in material optimization
$\mathbf{E}$	Property matrix linking elements
$F_{c,max}$	Maximum allowable compression strength in stringer
F_t	Tension strength of one stringer
$F_{t,max}$	Maximum allowable tension strength in stringer
$\mathbf{H}$	Global flexibility matrix
$\mathbf{M}$	Material matrix
Q	Load
$\mathbf{R}$	Load vector
$\mathbf{R}_0$	Constant load vector
RAT	Efficiency ratio
$\mathbf{V}$	Displacement vector

Latin lower case letters

$\mathbf{b}$	Vector of inequality constraints
$\mathbf{c}$	Object function regarding load optimisation
$\mathbf{d}$	Design variables
$\mathbf{e}_0$	Zero vector
f_c	Compressive strength of concrete
f_{cd}	Design value of concrete compressive strength
f_y	Yield strength of reinforcement
f_{yd}	Design yield strength of reinforcement
$\mathbf{h}$	Local flexibility matrix
h	Height of shear area
h_s	Height of stringer perpendicular on stringer direction, depends on $A_{c,nec}$
$h_{s,max}$	Maximum allowable stringer height, depends on smallest neighbouring shear area
l	Length
l_b	Anchorage length of reinforcement bar
n	Number of reinforcement bars in a stringer
ns	$2 \cdot n_{stringers} + n_{shear\_areas}$
n_{node}	Number of nodes
n_{shear_areas}	Number of shear areas
$n_{stringers}$	Number of stringers
q	Nodal load vector
$\mathbf{s}$, $\mathbf{s}^+$	Dual values of the design limitations

Latin lower case letters

t	Thickness
v_m	Factor of efficiency caused by the compression from bending effect
v_v	Factor of efficiency for pure shear
var	Number of variables
w	Width of shear area
$\mathbf{w}$	Material weights used in material optimisation
$\mathbf{x}$	Vector of primal variables
x^+	Upper value for stress parameter
x^-	Lower value for stress parameter

Greek lower case letters

$\boldsymbol{\Psi}$	Vector with plastic strain variables

Greek lower case letters

β	Stress parameter
γ	Shear strain
γ_c	Partial factor for normal control class
γ_s	Partial factor for reinforcing steel
ε	Strain
ε_c	Failure strain
$\boldsymbol{\theta}$	Vector of dual variables
λ	Load parameter
λ^+	Upper bound value for the collapse load
σ	Normal stress
τ	Shear stress

Chapter **1**

Introduction

The idea behind this thesis is described which ends up in the thesis statement and the suggested solution. In addition, the problem definition of the thesis is defined. The analysed structures in the project are presented in the end of the chapter.

Cost reduction, optimisation and environmental issues are popular these days and every line of business is affected by these, including the construction sector. Therefore, it is desirable to incorporate these issues in structures e.g. by material consumption and still maintain structural behaviour. Likewise it is profitable to make use of the building materials most optimal when constructing buildings.

Using the issues for saving in connection with knowledge of materials and optimisation it is possible to make use of the materials in a optimum way. This leads to cost savings and, due to material saving, lower environmental impacts of the building.

Material optimisation is obtained in different ways, e.g. by changing the production or by material development. In this project the applied theory for calculation is considered.

Traditional calculations are based on elastic material behaviour where the freedom of choice is limited compared to a calculation using perfect plastic material behaviour. When using perfect plastic material behaviour the distribution of the stresses in the structures can be selected freely as long as certain rules are obeyed.

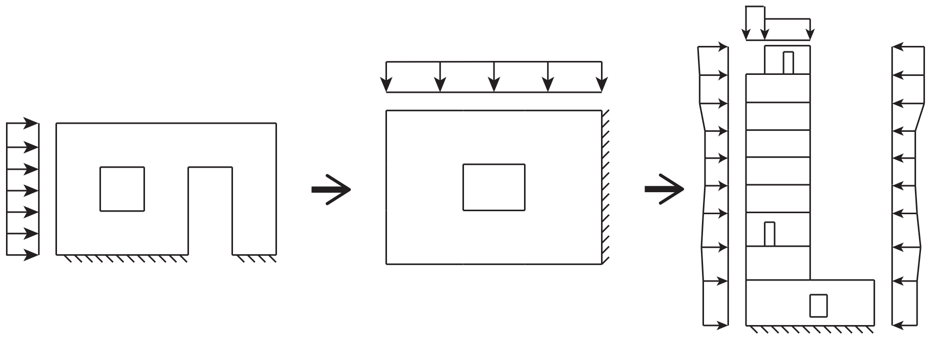
The perfect plastic material behaviour is applied together with the stringer method where the structure is designed using stringers and rectangular shear fields. In order to evaluate the lower bound method it is relevant to describe the upper bound method, which corresponds to the classic yield line method developed by K.W. Johansen (Johansen, 1972).

1.1 Thesis Statement and Problem Definition

Based on the above mentioned the following thesis statement is set up:

How can the stringer method be applied in the design of concrete walls in a practice manner by using a FE program based on perfect plastic material behaviour?

At first the intention of the project is presented by a guiding structure of the developed program, `Opti_String`. In addition, the material model and the appertaining assumptions for the program is described. The project structure is illustrated in Figure 1.1. Four structures has been analysed, cf. section 1.2. The applicability of `Opti_String` is illustrated by structure 3, a real concrete wall where an example of the output list is shown. `Opti_String` and its theory is illustrated by structure 2, an basic example in order to keep a sense of perspective. Simultaneous the input and output for `Opti_String` is still relatively clear.



Presentation of `Opti_String` by structure 3, a real structure.

Verification of `Opti_String` by structure 2, an basic example

Practical use of `Opti_String` by structure 4, a complex structure

Figure 1.1: *Composition of project.*

The practical use of `Opti_String` is proved by an more complex construction element from a patient hotel in Nuuk, Greenland. The building is designed by Grontmij in Aarhus, Denmark. Based on a meeting with the engineers at Grontmij a key element has been selected for further analysis in `Opti_String`by including structural optimisation by use of the stringer method and plasticity theory. The optimisation is performed regarding practical design demands and several load cases.

Optimisation of concrete elements by assuming perfect plastic material behaviour can be achieved through the stringer method. By doing so the stringer method and plastic behaviour needs to be studied careful.

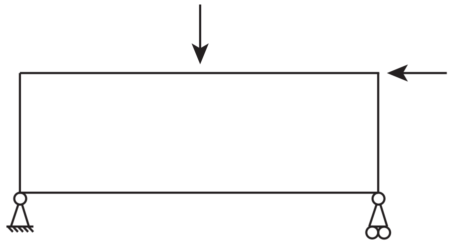
Computational analysis based on the theory is made in `Opti_String` which is written in MATLAB. By this the material optimisation process is automated leading to time saving in the design phase. The program is based on the finite element (FE) concept and the optimisation is done using linear programming (LP). The emphasis of `Opti_String` is concentrated by formulating the lower bound method. The upper bound method is formulated in order to examine whether an exact translation exist between the two methods for the elements in the stringer method. Also the option for analysing a structure subjected to several load cases is possible. LP problems are formulated for both lower and upper bound regarding load and material optimisation which are solved by using the build in function for LP *linprog* from MATLAB.

First, structure 2 will be used for material and load optimisation based on the lower bound method, after which they will be calculated with the upper bound method. The purpose is to compare the load bearing capacity from lower and upper bound method respectively, including the translation from lower to upper bound for the stringer system. Afterwards, practical design will be introduced by linking the material strengths of the different elements together. Several load cases will be introduced on structure 2. Finally, `Opti_String` is used for optimising a real complex structure, structure 4, subjected to several load cases by illustrating its practicability.

An overview of the applied structures in the report is listed in section 1.2. References are made to relevant chapters and sections in the report and appendix as well.

1.2 Structures Analysed by `Opti_String`

Structure 1



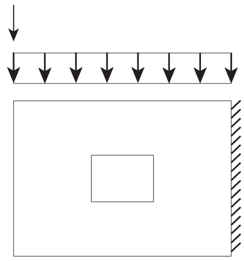
`Opti_String` uses the linear programming algorithm *linprog*. Structure 1 is used as a simple example for illustration the input for *linprog*, cf. Appendix A3.

t = 0.3 m

γ_s = 1.20 f_y = 550 MPa

γ_c = 1.45 f_c = 25 MPa

Structure 2



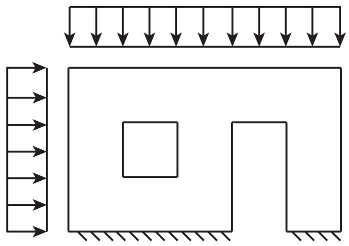
Structure 2 is used for describing the stringer method by hand calculation, cf. chapter 3 and to document `Opti_String` by the lower and upper bound method, cf. chapter 5 and 7, for both load and material optimisation. The structure is used for exemplification of practical design restrictions and several load cases, cf. section 8.1.

t = 0.3 m

γ_s = 1.20 f_y = 550 MPa

γ_c = 1.45 f_c = 25 MPa

Structure 3



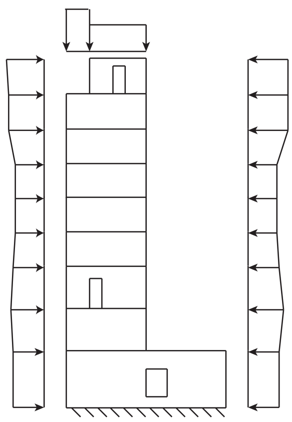
Structure 3 is used as a real structure for the presentation of `Opti_String` cf. chapter 2. The structure is used for exemplification of load combinations and illustrating design demands.

t = 0.3 m

γ_s = 1.20 f_y = 550 MPa

γ_c = 1.45 f_c = 25 MPa

Structure 4



Structure 4 is a key element used for showing the application of `Opti_String` on a realistic complex structure affected by three load cases and subjected to practical design demands.

t = 0.25 m

γ_s = 1.20 f_y = 550 MPa

γ_c = 1.45 f_c = 15 MPa

Chapter 2

Presentation of Opti_String

The guiding structure of the program is explained in the following by describing the steps from the real structure to a calculation model with results. Structure 3 is used for the illustration. In addition, the chapter accounts for the underlying intentions of the applied theory and the appertaining assumptions.

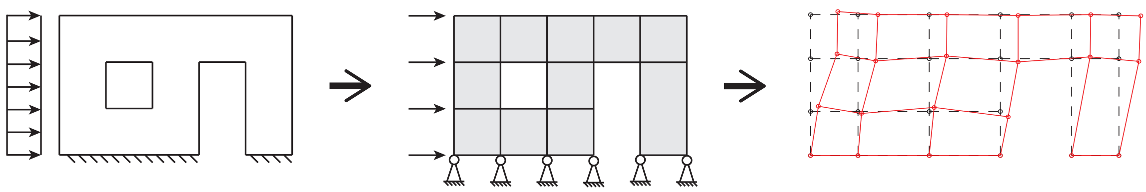
2.1 Program Structure

The purpose is to develop a program, `Opti_String`, for making it easier to apply the stringer method for practical calculations. The process of the program is shown in Figure 2.1.

Figure 2.1*(a)* shows a realistic element having a hole for a window and an opening for a door. The element is exposed a vertical load, load case 2, and is supported along the bottom.

The user must discretise the element manually, cf. Figure 2.1*(b)*. The discretisation must be done according to the stringer method, thus stringers, defined by start and end node, and shear fields, defined by surrounding stringers. Loads and supports may also be specified as well the material parameters. In addition, practical restrictions as stringer lines must be specified. Based on the input the stringer system is optimised using a linear optimisation algorithm.

`Opti_String` is capable to optimise a arbitrary geometry with several load cases regarding the Eurocode.



(a) Structure 3, realistic element. *(b) Stringer system. Shear* fields are marked grey. *(c) Output from* `Opti_String`*,* collapse mechanism.

Figure 2.1: *Process description of* `Opti_String`*.*

After the input `Opti_String` plots the geometry of the stringer system. The plot for structure 3 is shown in Figure 2.2.

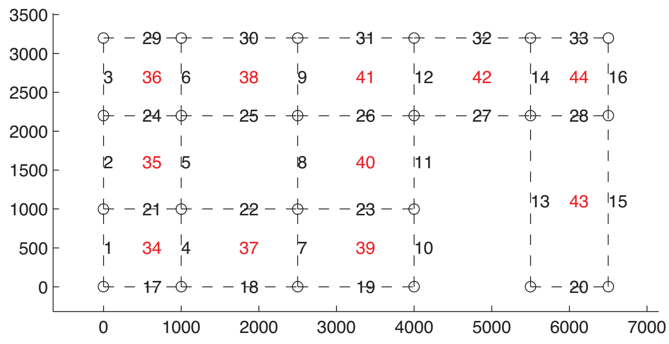


Figure 2.2: *Opti_String: Geometry plot for structure 3*

Load combination one for structure 3 consists of dominating snow load with wind as additional variable load, where combination two is with dominating wind, cf. Figure 2.3. Figure 2.4 shows the support numbers for structure 3.

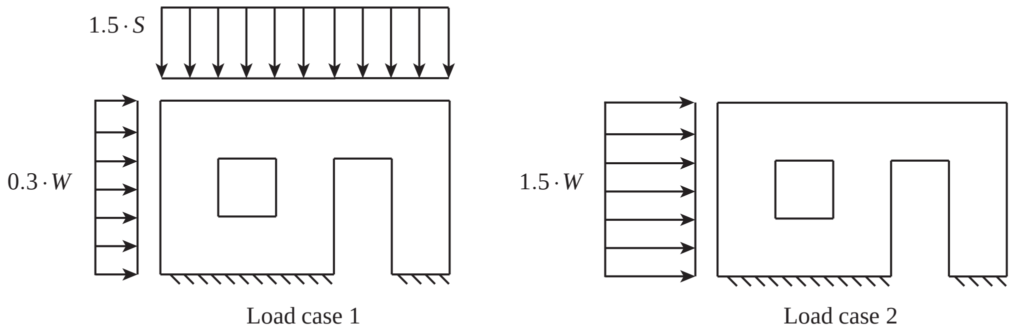


Figure 2.3: *Structure 3 subjected to two load cases.*

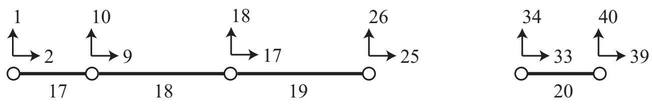


Figure 2.4: *Support numbers for structure 3.*

2.2 Output From Opti_String

Based on the stringer system stringer forces and shear stresses are calculated using linear programming. The calculated values are optimised with regard to load capacity or material parameters according to the applied load. The type of calculation must be specified in the beginning of the calculation. From the optimisation Opti_String creates an output list, cf. Figure 2.6 and 2.7. From the list various information appear, for example load or design parameters and the critical elements of the system are shown as the elements exposed to plastic strains. The list is divided into two because of the length.

The collapse mechanisms of the element from Opti_String is shown in Figure 2.5. The collapse mechanisms only indicates the mode of the collapse as the calculation is done with plastic material behaviour, cf. section 2.3.

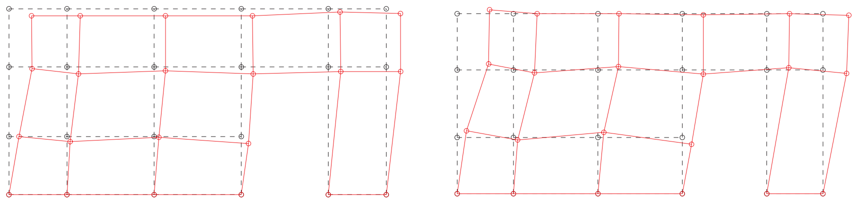


Figure 2.5: *`Opti_String`: Collapse mechanisms for structure 3.*

Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 3

Stringer lines:
1: 1 2 3
2: 4 5 6
3: 7 8 9
4: 10 11 12
5: 13 14
6: 15 16
7: 17 18 19
8: 20
9: 21 22 23
10: 24 25 26 27 28
11: 29 30 31 32 33
12: 34 35 36 37 38 39 40 41 42 43 44

2 load cases

----- Strength parameters and safety factor -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 40

Material optimisation algorithm - 77 design variables
Static independent variables, N = 12

----- Design variables -----
Compression: Tension:
d1_c = 6.825000e+03 d1_t = 1.116608e+04
d2_c = 1.901237e+04 d2_t = 3.383661e+03
d3_c = 2.128009e+04 d3_t = 1.229831e+04
d4_c = 2.576789e+04 d4_t = 0.000000e+00
d5_c = 2.158367e+04 d5_t = 2.484387e+03
d6_c = 1.389660e+04 d6_t = 0.000000e+00
d7_c = 7.178530e+03 d7_t = 9.570597e+03
d8_c = 8.738425e+03 d8_t = 8.010702e+03
d9_c = 1.524600e+04 d9_t = 1.503127e+03
d10_c = 1.663200e+04 d10_t = 0.000000e+00
d11_c = 8.316000e+03 d11_t = 0.000000e+00
d12_c = 4.100000e-02 d12_t = 4.100000e-02

----- Elements exposed to plastic strains -----
Element 1 Element 2
Element 3 Element 3
Element 4 Element 4
Element 4 Element 5
Element 5 Element 6
Element 7 Element 7
Element 7 Element 8
Element 8 Element 9
Element 10 Element 11
Element 12 Element 12
Element 13 Element 13
Element 14 Element 14
Element 15 Element 15
Element 16 Element 16
Element 18 Element 18
Element 19 Element 19
Element 20 Element 20
Element 21 Element 21
Element 22 Element 22
Element 23 Element 23
Element 24 Element 25
Element 26 Element 26
Element 27 Element 28
Element 29 Element 29
Element 30 Element 31
Element 32 Element 32
Element 33 Element 33
Element 34 Element 34
Element 35 Element 35
Element 36 Element 36
Element 37 Element 37
Element 38 Element 38
Element 39 Element 39
Element 40 Element 40
Element 41 Element 41
Element 42 Element 42
Element 43 Element 43
Element 44 Element 44

Figure 2.6: *`Opti_String`: Output list for structure 3 based on material optimisation and lower bound using practical design restrictions for two load cases. First part of list.*

```
----- Reinforcement for stringers [mm^2] -----
A_s(1) = 20.3
A_s(2) = 11.07
A_s(3) = 0
A_s(4) = 6.15
A_s(5) = 6.15
A_s(6) = 6.15
A_s(7) = 22.36
A_s(8) = 22.36
A_s(9) = 0
A_s(10) = 0
A_s(11) = 0
A_s(12) = 0
A_s(13) = 4.52
A_s(14) = 0
A_s(15) = 0
A_s(16) = 0
A_s(17) = 7.18
A_s(18) = 17.4
A_s(19) = 14.56
A_s(20) = 1.64
A_s(21) = 0
A_s(22) = 2.73
A_s(23) = 2.73
A_s(24) = 0
A_s(25) = 0
A_s(26) = 0
A_s(27) = 0
A_s(28) = 0
A_s(29) = 0
A_s(30) = 0
A_s(31) = 0
A_s(32) = 0
A_s(33) = 0

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(34) = 9.228
A_s_x,y(35) = 9.228
A_s_x,y(36) = 0
A_s_x,y(37) = 20.302
A_s_x,y(38) = 6.152
A_s_x,y(39) = 20.302
A_s_x,y(40) = 22.124
A_s_x,y(41) = 1.964
A_s_x,y(42) = 2.016
A_s_x,y(43) = 3.024
A_s_x,y(44) = 0

----- Reactions [kN] -----
Load case 1:
Support 1 = -2.546
Support 2 = 1.084
Support 9 = -2.501
Support 10 = 19.012
Support 17 = -2.091
Support 18 = 21.28
Support 25 = -0.623
Support 26 = 20.967
Support 33 = -0.903
Support 34 = 17.925
Support 39 = -0.76
Support 40 = 13.897
Load case 2:
Support 1 = -10.881
Support 2 = -18.096
Support 9 = -10.695
Support 10 = -3.384
Support 17 = -15.189
Support 18 = -12.298
Support 25 = -8.738
Support 26 = 25.768
Support 33 = -0.506
Support 34 = -2.484
Support 39 = -1.114
Support 40 = 3.565
```

Figure 2.7: *`Opti_String`: Output list for structure 3 based on material optimisation and lower bound using practical design restrictions for two load cases. Second part of list.*

For comparison of the lower bound and the upper bound method two separate codes in MATLAB are developed. The codes used for the comparison are shown in Appendix A7.3 and Appendix A7.4.

The MATLAB code of `Opti_String` and the functions and data files for chapter 8 and chapter 10 are shown in Appendix A7.2.

2.3 Material Models

In general three material models exists, namely the elastic model, elastic-plastic model and plastic mode, cf. Figure 2.8. Traditionally computer programs are based on elastic material behaviour where some includes plastic response by a elastic-plastic material models. Beside the general models each model can be expressed linear or non-linear. In the following only linear models are clarified.

The elastic material behaviour can be expressed by a linear strain-stress curve, cf. Figure 2.8. Hereby the loading and unloading follows the same path and no plastic strains are developed as the strain state never exceeds the yield strain, ε_y.

The linear elastic perfectly plastic model is similar to the elastic model in the elastic range but the strains can exceed the yield strain. Hereafter, the strains increase for a constant value of the stress. Unloading of the material leads to plastic strains if the yield strain is exceeded thus, plastic deformation is introduced.

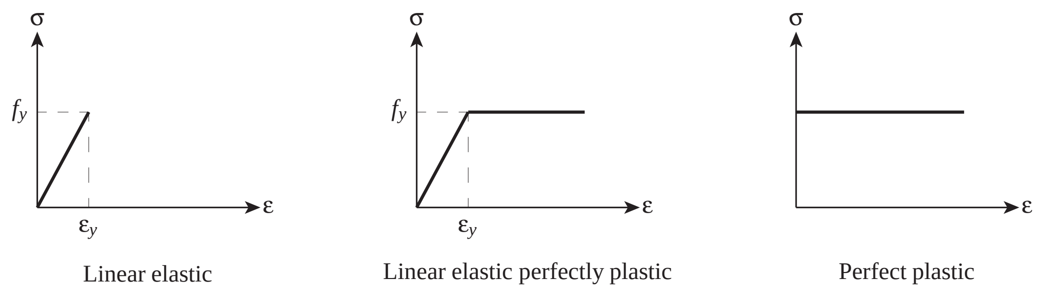


Figure 2.8: *Strain-stress curves for basic material models.*

No elastic deformations exists for the perfect plastic material model, cf. Figure 2.8, thus, no strains are developed until the yield stress of the material is obtained.

The stringer method is based on perfect plastic material behaviour thus, the perfect plastic model is used for the calculations in `Opti_String`. By use of the material model some assumptions for the calculation are made.

Because of perfect plasticity no elastic strains are developed since the material model not contains an elastic range. Thus, because no elastic range exists the superposition principle is not valid. Hereby it is also seen that the assumption of perfect plasticity is better for materials where the failure strains mainly consists of plastic strains.

The theory of perfect plastic material behaviour should only be applied to materials which rightly is assumed having a plastic behaviour. Physical interpretation of plastic behaviour is compared to a ductile behaviour. The ductility is an important property as it allows rearrangement of stresses when the yield strain is exceeded. Reinforced concrete is in general understood as a plastic material because the reinforcement ensures the ductility of the composite material. A disadvantage of plasticity theory is that the ductile behaviour must be documented which is commented in chapter 11.

Application of perfect plastic material behaviour leads to economical structures compared to elastic models because a redistribution of the stresses is allowed. Another advantage of the plastic model is the focus of the collapse mechanisms of the structure and hereby key elements are pointed out. However, for complicated structures many collapse mechanisms might complicate the analysis of the mechanisms. (Jensen and Bonnerup, 2006, section 4.4) The critical collapse mechanism is found in `Opti_String` be means of an optimisation algorithm.

For plasticity calculations two different methods are valid. The methods are either based on a static admissible stress distributions, the lower bound method, or a kinematic admissible mechanism, the upper bound method. Both methods are available in `Opti_String` and the methods are described in detail in chapter 5 and

7, respectively.

The Eurocode (EN 1992-1-1, 2007) is implemented in Opti_String when the demands from the standard can be implemented in the calculations. An example is the determination of reinforcement calculated in the program. The necessary reinforcement is calculated according to the standard and then the user of the program must specify which real reinforcement satisfies the calculated values.

Chapter **3**

Stringer Method

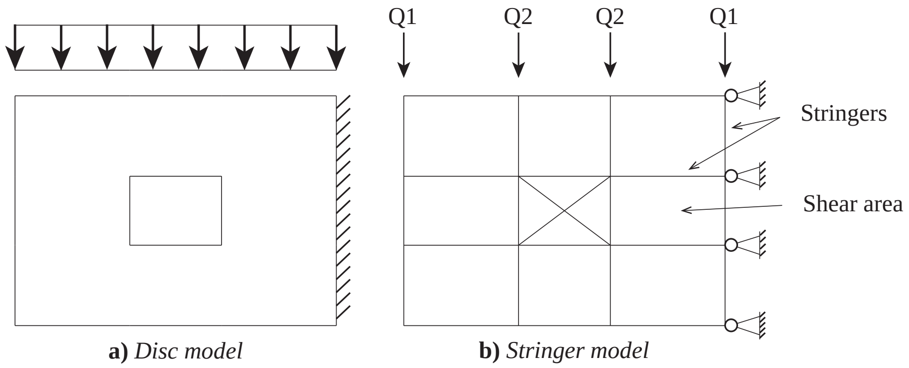
This chapter provides an exposition of the stringer method illustrated by an simple example, structure 2. The stringer method is described in a 2D space for in-plane stresses. In the end of this chapter structure 2 is shown by means of traditional hand calculations.

The stringer method is based on a perfect plastic material behaviour and is either based on the lower bound or upper bound method. The lower bound method seeks a stress field in equilibrium, which do not violate the yield criteria at any point. In addition, it respects all the static boundary conditions for which the load bearing capacity is maximum. In other words, the lower bound method says that the load carrying capacity is at least the largest of all lower bound values. The upper bound method seeks a kinematic admissible collapse mechanism that gives the minimal load barring capacity.

The stringer method is originally based on the lower bound method because a given stress distribution satisfies the equilibrium equations. Moreover, in-plane forces make the problem complex when imagining the collapse mechanism. For a lower bound method all stresses in a given structure are not violating the yield criteria and the solution is said to be on the safe side. This means that the true load carrying capacity is larger or equal to the load for which the necessary dimensions are determined.

3.1 Assumptions for Stringer Method

The principle of the stringer method is to divide the element in bars, so-called stringers and rectangular shear areas stretched by the stringers, cf. Figure 3.1.



a) *Disc model* **b)** *Stringer model*

Figure 3.1: *Disc converted to stringer system.*

A plane stress distribution is simplified by having stringers carrying compression or tension while shear stresses are carried in the areas. It is appropriate to carry shear stress in the areas due to the shear reinforcement. The stringers acting like bars are not suitable for shear stress but on the other hand applicable for tension and compression. The stringers consist of concrete carrying compression and reinforcement bars carrying tension, cf. Figure 3.3. Figure 3.1 illustrates the conversion from disc to stringer system. Loads and reactions are applied as concentrated forces in nodes or as shear stress acting along a stringer. A uniform distributed load is converted to nodal loads. (Damkilde et al., 1994)

Stringers are oriented in the x or y direction, respectively and take both compression and tension. In the conversion from disc to stringer model the normal stresses are equivalent with concentrated normal forces in parallel stringers. For normal stresses in the x-direction, σ_x, the equivalent stresses are illustrated in Figure 3.2. Similar normal stresses in the y-direction are equated with concentrated normal stresses parallel with the y-direction.

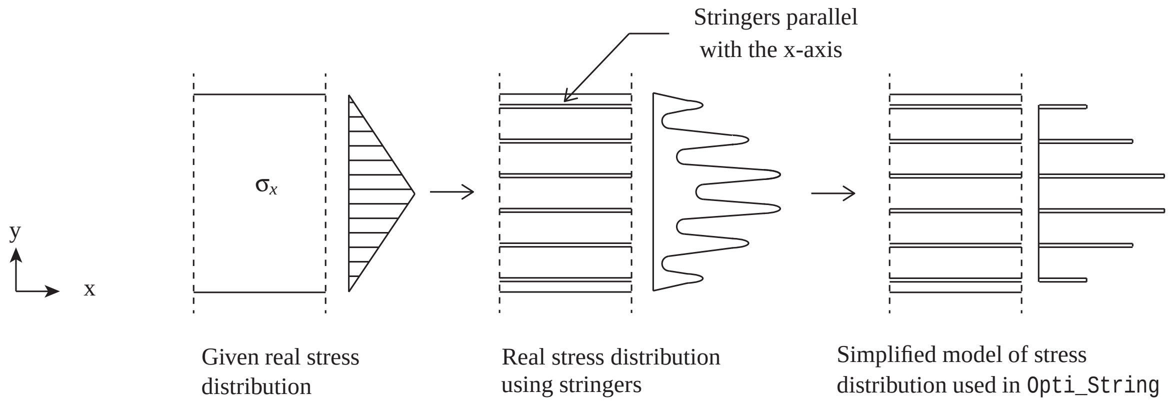


Figure 3.2: *Equivalence between real stress distribution in x-direction and model of stress distribution. After (Dansk Konstruktions- og Betoninstitut, 2011, Appendix A).*

A stringer is defined as a line between to nodes and more stringers in a row makes a stringer line. The concrete and additional reinforcement in the stringer are illustrated in Figure 3.3. Reinforcement bars are located in the centre line of the stringer.

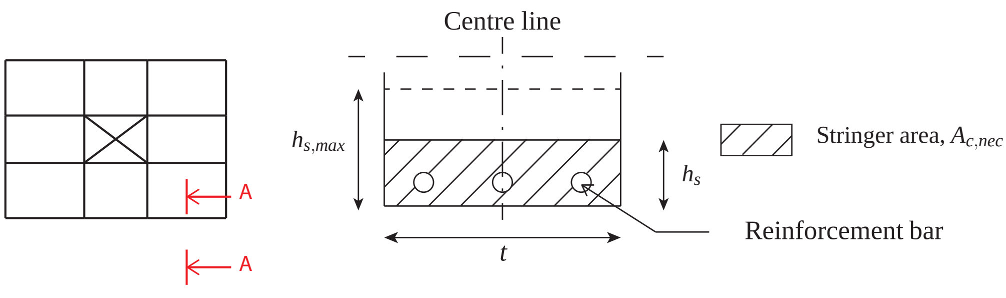


Figure 3.3: *Cross-sectional view A of stringer located at the edge.*

In the stringer only concrete carry compression which results in a necessary concrete compression area, $A_{c,nec}$. To make sure that the stringer compression capacity is acceptable the expression in equation (3.1) must be satisfied, cf. (Jensen, 2008, section 12.3). To ensure equation (3.1) the expression in equation (3.2) can be formulated, regarding (Jensen, 2008).

$$h_s \leq h_{s,max} \tag{3.1}$$

$$F_{c,max} = v_m \cdot f_{cd} \cdot h_s \cdot h_{s,max} \tag{3.2}$$

where

h_s	Height of stringer, depends on $A_{c,nec}$
$h_{s,max}$	Maximum allowable stringer height, depends on smallest neighbouring shear area
$F_{c,max}$	Maximum allowable compression strength in stringer
v_m	Factor of efficiency caused by the compression from bending effect (EN 1992-1-1 DK NA, 2007, 5.102NA)
f_{cd}	Design value of concrete compressive strength

Stringers in tension must be able to transfer shear stress from adjacent areas to the reinforcement bars in the stringers, cf. equation (3.3), and carry the tension in the required reinforcement, cf. equation (3.4).

$$|\tau_A - \tau_B| = \frac{n \cdot A_s \cdot f_{yd}}{l_b \cdot t} \tag{3.3}$$

$$F_t = A_t \cdot f_{yd} \tag{3.4}$$

where

τ_i	Shear stress in adjacent area to the concerned stringer
n	Number of reinforcement bars in stringer
A_s	Reinforcement area of reinforcement bar
l_b	Anchorage length of reinforcement bar
F_t	Tension strength of stringer
A_t	Reinforcement area of stringer

Each stringer line is dimensioned based on the maximum compression and tension forces in the stringers composing a stringer line.

A shear area is located among four stringers and have, due to the orientation of the stringers, a rectangular shape. In the conversion from disc to stringer system the shear stresses are assumed constant for each shear field meaning that the forces in the surrounding stringers varies linear, cf. Figure 3.4. Net reinforcement are placed for carrying the shear stress in the rectangles.

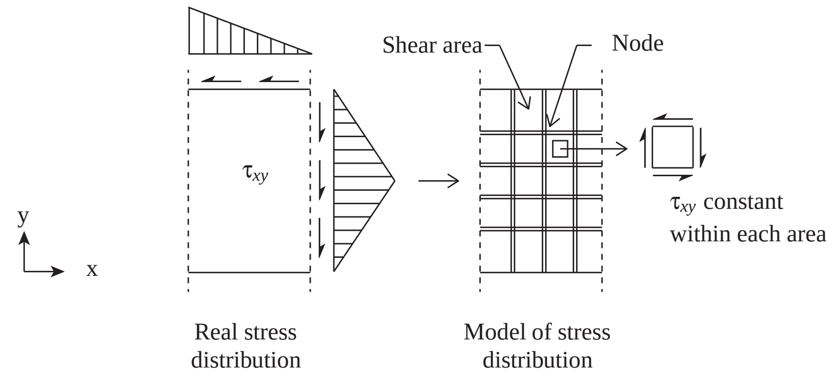


Figure 3.4: *Equivalence between real stress distribution in area and model of stress distribution. (Dansk Konstruktions- og Betoninstitut, 2011, Appendix A)*

Shear stresses may not exceed the expression in equation (3.5) if pure shear is assumed. This means the concrete compression forms the angle θ with the x-axis and yielding is assumed in the reinforcement bars.

$$\tau_{max} \leq \frac{v_v \cdot f_{cd}}{2} \tag{3.5}$$

where

τ_{max}	Maximum allowable shear stress in shear area
v_v	Factor of efficiency for pure shear cf. (EN 1992-1-1 DK NA, 2007, 5.103NA)

Stringers are making continuity in the system by interlink the shear fields. Similar the nodes link the stringers. Two equilibrium equations exist for each node, cf. equation (3.6) and equation (3.7), and one for each stringer, cf. equation (3.8). Figure 3.5 illustrates the continuity condition.

$$\sum F_{x,i} = 0 \tag{3.6}$$

$$\sum F_{y,i} = 0 \tag{3.7}$$

$$S_1 - S_2 = \tau_A - \tau_B \tag{3.8}$$

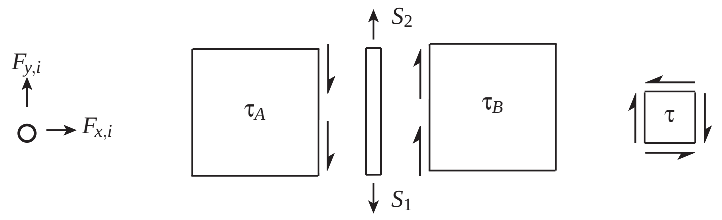


Figure 3.5: *Equilibrium of node affected by forces in x and y direction and stringer affected by two shear areas.*

The stringer system shown in Figure 3.1 is static indeterminate which is general for stringer systems. Through a plastic redistribution of the stresses it is possible to obtain an optimum by means of the nature that ensures accuracy of these stresses.

3.2 Stringer Method by Hand Calculation

With basis in the stringer theory an example follows for structure 2, cf. Figure 3.1. The dimensions and loads for structure 2 are stated in Table 3.1. Because of the thickness the example is representative for a concrete wall.

Thickness	300 mm
Height	1000 mm
Width	1400 mm
q	300 $^{kN}/_{m}$
Q_1	75 kN
Q_2	135 kN

Table 3.1: *Dimensions and loads for structure 2.*

For traditional hand calculation the stringer forces and shear stresses are determined by a stepwise calculation. The steps for the procedure are shown in Appendix A1.

The stringer mesh is placed such that the stringer lines follow the edges both external and around the hole, and in the centre of the loads and supports as well, cf. Figure 3.6. The direction of operational sign is marked with x-y axis.

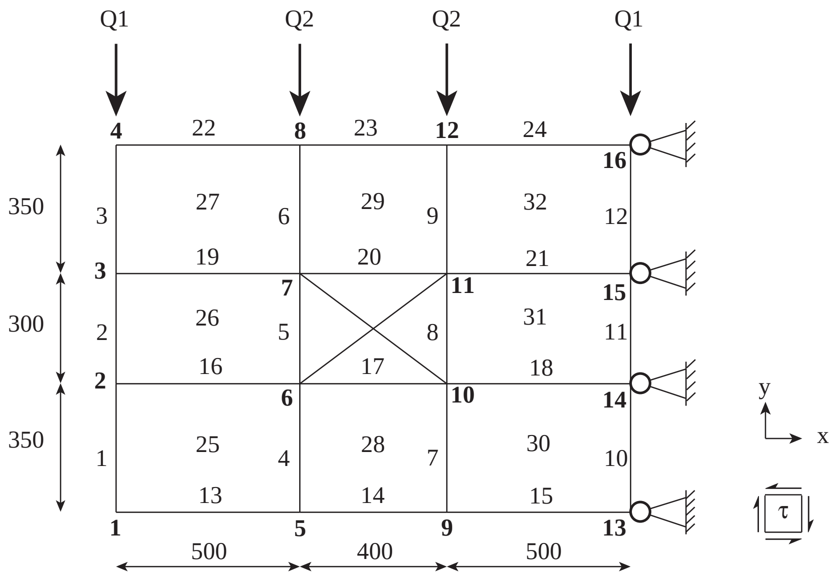


Figure 3.6: *Stringer system for structure 2, nodes are numbered from 1 to 16 with bold, stringers from 1 to 24 and shear areas from 25 to 32.*

The number of static indeterminate, N, is determined by equation (3.9), and for the hand calculation N is interpreted as the number of static indeterminate shear areas.

$$\begin{aligned} \text{Variables:} &\quad 2 \cdot n_{stringers} + n_{shear\_areas} + n_{reactions} \\ \text{Equilibrium equations:} &\quad 2 \cdot n_{nodes} + n_{stringers} \\ \text{Statically indeterminate, } N: &\quad n_{variables} - n_{Equilibrium\_Equations} \end{aligned} \tag{3.9}$$

For structure 2 N is determined as equation (3.10).

$$N = (2 \cdot 24 + 1 \cdot 8 + 1 \cdot 8) - (2 \cdot 16 + 1 \cdot 24) = 8 \tag{3.10}$$

Hereby it is determined that structure 2 is eight times statically indeterminate which lead to following options:

- Stress parameters can in principle be chosen freely
- Free choices for stress parameters leads to a plastic redistribution of the stresses
- Redistribution of stresses facilitate an optimum solution for carrying a given load
- The nature ensure accuracy of these stresses

The equilibrium equations still needs to be satisfied in a local cut. Moreover, free choices of stress parameters may entail static determinate areas which the remaining free choices have to respect.

In this example only three are chosen freely. The shear stress in area 25 and 26 which by vertical projection in a cut through area 25, 26 and 27 gives the shear stress in 27. The third free choice is 28 = 29. After this area 30, 31 and 32 can be determined by horizontal projection.

Detailed calculations are shown in Appendix A1 and on CD, Appendix A7.1. The results are shown in Figure 3.7 for shear areas. After determined shear stresses, stringer forces can be found by free body diagram for each stringer, the results are shown in Figure 3.8 and 3.9.

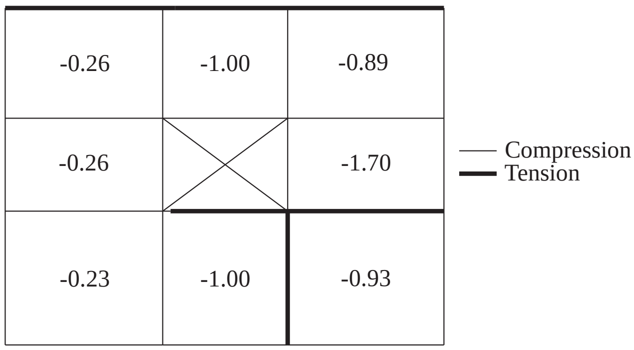


Figure 3.7: *Shear stress in* MPa.

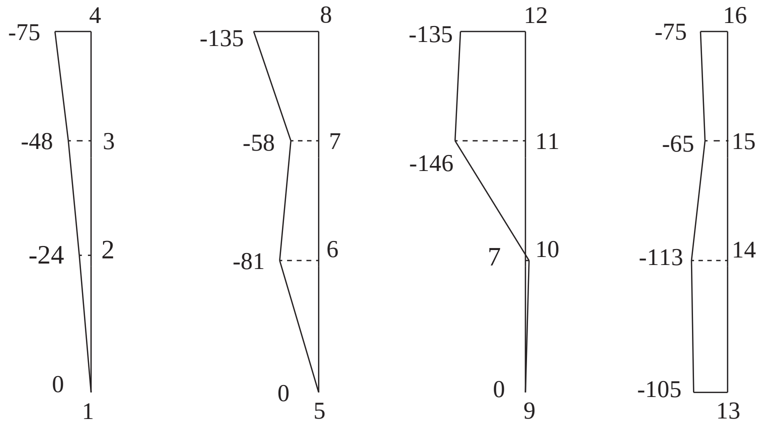


Figure 3.8: Vertical stringer forces in kN.

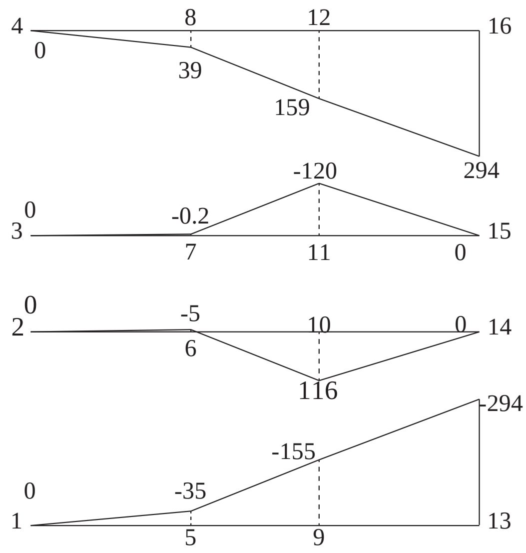


Figure 3.9: Horizontal stringer forces in kN.

The parameters in Table 3.2 are used for the reinforcement and checking the compression capacity.

Safety factor for normal control class	γ_c	1.45
Safety factor for reinforcing steel	γ_s	1.2
Compressive strength of concrete	f_c	25 MPa
Yield strength of reinforcement	f_y	350 MPa

Table 3.2: Material parameters for structure 2.

The compression capacity is maintained in all stringers because the stringer height, h_s, is smaller than the maximum allowable stringer height, $h_{s,max}$, cf. Figure 3.3 and equation (3.1). Tensile stringers and shear areas have the reinforcement shown in Table A1.3 which is determined in Appendix A1.4. Detailed calculations can be found on CD, Appendix A7.1. The results are compared with the output from `Opti_String` in section 8.1.

It is discovered that the calculations is somewhat tedious for even a simple example with all the equilibrium equations. Thus, it is favourable to save time by an automation of the calculations in a computer program, which in addition calculates the optimal stress distribution, and optimises the structure.

Chapter **4**

Mathematical Formulation

A mathematical formulation of the stringer theory is introduced. The formulation describes the local flexibility matrix for the stringer and shear element and and their boundary conditions. Furthermore, the assembling of the flexibility for a global system is shown.

In chapter 3 the stringer method is illustrated by a simple example. The example indicates the number of equations for the simple example and it is seen that for even a small system the number of equations are large and time consuming. Therefore, the stringer method is formulated with a finite element concept, which is implemented in `Opti_String`.

The principle for establish local equilibrium and assembling the local matrices for stringer k and shear area m is explained, cf. Figure 4.1.

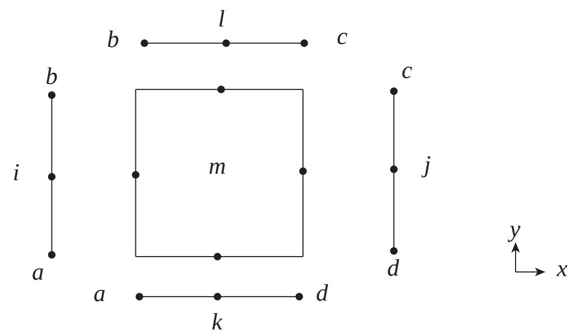


Figure 4.1: *Stringer elements and shear area for illustration of assembling of global flexibility matrix,* ***H****.*

The normal stress varies linear in the stringer as mentioned in section 3.1 thus, two stress parameters, β_a and β_d, are necessary for describing stringer k. The stringer is affected by concentrated nodal loads, or as shear acting on the mid-side of the stringer, cf. Figure 4.2.

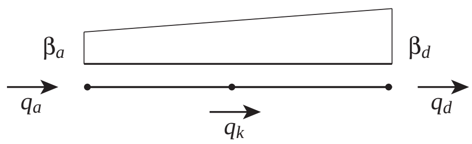


Figure 4.2: *Stringer element k. Right is considered positive.*

The external loads and the stress parameters represents an equilibrium state for the element, cf. equation (4.1).

Notice, the selected positive direction is going to the right.

$$\begin{aligned} q_a &= -\beta_a \\ q_d &= \beta_d \\ q_k &= -\beta_a + \beta_d \end{aligned} \tag{4.1}$$

where

q_i	External nodal force
β_i	Stress parameter

The equilibrium equations are formulated in equation (4.2).

$$\begin{Bmatrix} q_a \\ q_d \\ q_k \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} \beta_a \\ \beta_d \end{Bmatrix} \tag{4.2}$$

Equation (4.2) can be expressed as equation (4.3) using matrix notation.

$$\mathbf{q} = \mathbf{h}\,\boldsymbol{\beta} \tag{4.3}$$

where

$\mathbf{h}$	Local flexibility matrix

A given load leads to a constant shear stress, τ, in a shear areas. This single stress parameter is equivalent to four nodal loads located at the mid-side of each stringer around the shear field, cf. Figure 4.3.

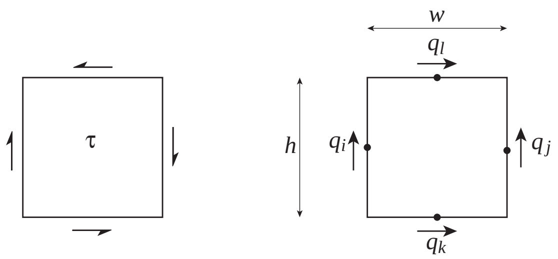


Figure 4.3: *Shear element m with thickness = t.*

The equilibrium equations for shear field m are given in equation (4.4).

$$\begin{aligned} q_i &= h\,\tau\,t \\ q_j &= -h\,\tau\,t \\ q_k &= w\,\tau\,t \\ q_l &= -w\,\tau\,t \end{aligned} \tag{4.4}$$

where

h	Height of shear area
w	Width of shear area
τ	Shear stress
t	Thickness

The equilibrium equations in equation (4.4) are formulated in equation (4.5). The relation can also be given in matrix notation as equation (4.3).

$$\begin{Bmatrix} q_i \\ q_j \\ q_k \\ q_l \end{Bmatrix} = \begin{Bmatrix} h \\ -h \\ w \\ -w \end{Bmatrix} \tau\, t \tag{4.5}$$

To satisfy global equilibrium for a system the sum of internal forces must corresponds to the sum of external forces. This is implemented by an assembling of the local flexibility matrices for stringers and shear areas into the global flexibility matrix, **H**, that ensures equilibrium for the global system. The assembling is done according to (Cook et al., 2002). First, the two equations of the local flexibility matrix for stringer k, which expresses equilibrium in start and end node for the stringer, cf. equation (4.1), are put into the global flexibility matrix. The equations enter in the two rows corresponding to node equilibrium for the nodes in each end of the stringer, cf. Figure 4.4.

Subsequently, the last equation, expressing equilibrium at the mid-side of the stringer, cf. equation (4.1), is put into the global flexibility matrix in the row corresponding to horizontal equilibrium of stringer k. All three equations from the local equilibrium matrix are placed in the two columns describing stringer k, cf. Figure 4.4.

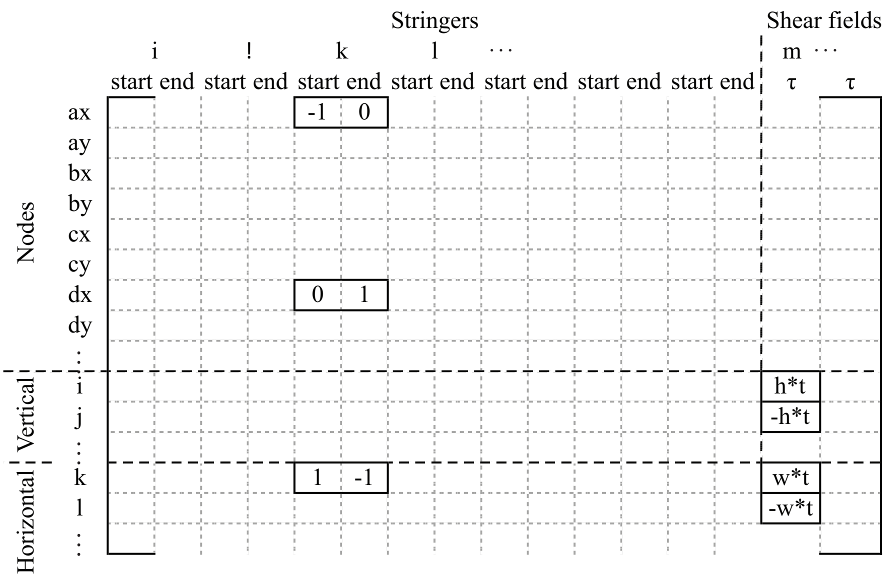


Figure 4.4: *Assembling of global flexibility matrix,* ***H****.*

The four values describing shear field m are arranged in one column describing the shear area, cf. Figure 4.1. The two equations for the nodes along the vertical mid-sides are placed in the rows describing vertical equilibrium for the stringers, i and j, affected by the shear area, cf. Figure 4.4. In the same way the two equations for horizontal mid-side nodes are placed in the rows for horizontal equilibrium of the affected stringers, k and l. The equilibrium relation for a global system is written as equation (4.6) where the left-hand side consists of the global flexibility matrix, **H**, and the stress parameters, $\boldsymbol{\beta}$, corresponding to the internal forces. The right-hand side consists of the external load.

$$\mathbf{H}\,\beta = \lambda\,\mathbf{R} \tag{4.6}$$

where

H	Global flexibility matrix
R	Load
λ	Load parameter

The number of stress parameters is determined from the number of stringers and shear areas. As each stringer is described by two stress parameters and each shear area by one, the total number of stress parameters becomes:

$$n_\beta = 2 \cdot n_{stringers} + n_{shear\_areas}$$

Each row in the global flexibility matrix satisfies either horizontal or vertical equilibrium in a node or equilibrium in a stringer. A simple example showing the assembling of the global flexibility matrix is illustrated in appendix A3 for structure 1, cf. section 1.2.

After the global flexibility matrix is established the rows containing supports are removed. These rows are redundant for solving the system. Moreover, the reactions can be found as the stress parameters sharing node and direction with the relevant reaction. It is also possible to formulate restrictions for the reactions, this practical feature is commented in chapter 11.

For statically determined systems the number of stress parameters are the same as the equilibrium equations, while for indeterminate systems the number of stress parameters is larger than the number of equilibrium equations. This introduces free variables which can be selected freely to optimise the system, e.g. to maximise the load bearing capacity or minimise the material requirements. This optimisation is done by means of LP, cf. chapter 6. The application of the mathematical formulation is used in `Opti_String` and the implementation is shown in chapter 9.

Chapter 5

Lower Bound Method

This chapter describes the lower bound method. The theory is described according to the stringer method and the mathematical formulation in chapter 4. LP problems are formulated for both load and material optimisation. After the description of the theory examples of both load and material optimisation are shown.

The LP problem formulated for both load and material optimisation are restricted by;

- Equilibrium which must be satisfied
- Yield criteria may not be violated

By using the build in function *linprog* in MATLAB, which is a linear programming algorithm, the LP problems are solved.

5.1 Load Optimisation

The objective of load optimisation using the lower bound method is to maximise the load to find the ultimate load bearing capacity of the structure. The applied loads are multiplied with a load parameter, λ. The lower bound method determines the optimal load parameter, λ and the optimal stress parameters, β as a by-product.

The two above mentioned restrictions are described for each stringer and shear area. Hereby it is possible to define individual limits for each element in accordance with current standards. The above mentioned problem is mathematical described in equation (5.1) (Damkilde, 1995, equation (15)). Detailed examples of the matrix layout are shown in appendix A3.

$$\begin{aligned} \text{maximise}: &\quad \lambda \\ \text{restrictions}: &\quad -\mathbf{H}\,\boldsymbol{\beta} + \lambda\,\mathbf{R} = -\mathbf{R}_0 \\ &\quad \mathbf{C}\,\boldsymbol{\beta} \leq \mathbf{C}_d \end{aligned} \tag{5.1}$$

where

λ	Load parameter
$\mathbf{H}$	Flexibility matrix
$\boldsymbol{\beta}$	Vector containing stress parameters and load parameter
$\mathbf{R}$	Load vector
$\mathbf{R}_0$	Constant load vector, e.g. self weight
$\mathbf{C}$	Constraint matrix
$\mathbf{C}_d$	Material constraint matrix containing strength values

λ is maximised by using the object function, **c**, in equation (5.2).

$$\mathbf{c}^T\mathbf{x} = \begin{Bmatrix} 0 \\ \vdots \\ 0_{ns} \\ 1 \end{Bmatrix}^T \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 & \dots & 0_{ns} & 1 \end{Bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ \lambda \end{Bmatrix} \tag{5.2}$$

where

$\mathbf{c}$	Object function regarding load optimisation
$\mathbf{x}$	Vector with variables to be determined containing $\boldsymbol{\beta}$, and λ
ns	$2 \cdot n_{stringers} + n_{shear\_areas}$

c is made as a zero vector with the last values set to one and the length of the vector corresponds to the number of columns in the flexibility matrix **H**. **x** contains the load parameter, λ, which the purpose is to maximise, cf. equation (5.2). In order to do so **c** is multiplied with -1 since *linprog* minimises the object function, cf. equation (A3.1).

The first restriction in equation (5.1) ensures equilibrium is satisfied in the whole structure. The expression can be rewritten as equation (5.3). The matrix on the left hand side with both **H** and **R** is the matrix for linear equality constraints *linprog* needs, cf. appendix A3. The number of rows and columns for **H** are: $n_{rows} = 2 \cdot n_{nodes} + n_{stringers}$ and $n_{columns} = 2 \cdot n_{stringers} + n_{shear\_areas}$, cf. chapter 4.

$$\begin{bmatrix} -\mathbf{H} & \mathbf{R} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta} \\ \lambda \end{Bmatrix} = -\mathbf{R}_0 \tag{5.3}$$

The solution provides stress parameters and the load parameter as primary values. The load parameter indicates, when multiplied with the load, the maximum load for the system. The stress parameters are within the defined limits stated in the material constraint matrix, **C**, by rewritten equation (5.4) to equation (5.5).

$$-N_y^- \leq \beta_i \leq N_y^+ \tag{5.4}$$

$$\begin{aligned} -\beta_i &\leq -N_y^- \\ \beta_i &\leq N_y^+ \end{aligned} \tag{5.5}$$

where

N_y^-	Negative yield strength
N_y^+	Positive yield strength

Hereby the yield criteria for all elements are stated in equation (5.6) where $n_{rows} = 4 \cdot n_{stringers} + 2 \cdot n_{shear\_areas}$ and $n_{columns} = 2 \cdot n_{stringers} + n_{shear\_areas}$. An example of **C** and $\mathbf{C}_d$ is illustrated in Appendix Figure A3.3. The yield strength for the stringers are given by equation (3.2) and equation (3.4) for compression and tension respectively. The yield strength for the shear areas are given by equation (3.5) for both compression and tension. An example is shown for structure 1in Appendix A3.3.

$$\mathbf{C}\,\boldsymbol{\beta} \leq \mathbf{C}_d \tag{5.6}$$

Beside the primary values the function calculates the dual values, shadow prices, for the LP problem. The dual values based on the inequalities indicates elements exposed to plastic strains, in either tension or compression. The displacements for the collapse mechanism is interpreted using the dual values of the equalities. Only the shape of the collapse mechanism is found, thus the deformations are unknown as the calculation is done with perfect plastic material behaviour.

5.2 Material Optimisation

The LP problem can be used for minimising the material, e.g. by weight or cost, of the structure. Thus, the problem is described as equation (5.7) (Damkilde, 1995, equation (24)). This is the general formulation for material optimisation where the strength values are taken into account by the design variables **d**. Opposite to load optimisation $\mathbf{C}_d$ is a matrix which ensure $\mathbf{d}_{compression}$ and $\mathbf{d}_{tension}$ for each element instead of strength values. It is possible to take strength values into account as restrictions, which is done in chapter 8.2 and is omitted here because of clarity.

$$\begin{aligned} \text{minimise}: \quad & \mathbf{w}^T\,\mathbf{d} \\ \text{restrictions}: \quad & \mathbf{H}\,\boldsymbol{\beta} = \mathbf{R} \\ & \mathbf{C}\,\boldsymbol{\beta} - \mathbf{C}_d\,\mathbf{d} \leq \mathbf{C}_0 \\ & \mathbf{d} \geq 0 \end{aligned} \tag{5.7}$$

where

w	Object function given as weighting parameters for material
d	Design variables
$\mathbf{C}_d$	Material constraint matrix

The design variables are minimised by the object function in equation (5.8). Instead of the load parameter, λ, weighting parameters, **w**, are included in the object function, **c**. By these variables it is possible to group and weight part of the structure and hereby have an influence of e.g. the design of the structure according to material use or the way the structure supports the load. When minimising $\mathbf{c}^T\mathbf{x}$ the optimal design parameters, **d**, are determined. The main purpose is to minimise **d** why the stress parameters are a by-product.

$$\mathbf{c}^T\mathbf{x} = \begin{Bmatrix} 0 \\ \vdots \\ 0_{ns} \\ w_1 \\ \vdots \\ w_{ns} \end{Bmatrix}^T \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ d_1 \\ \vdots \\ d_{ns} \end{Bmatrix} = \{\ 0 \quad \dots \quad 0_{ns} \quad w_1 \quad \dots \quad w_{ns}\ \} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{ns} \\ d_1 \\ \vdots \\ d_{ns} \end{Bmatrix} \tag{5.8}$$

where

c	Object function with weighting parameters **w**
x	Vector with variables to be determined containing β and **d**

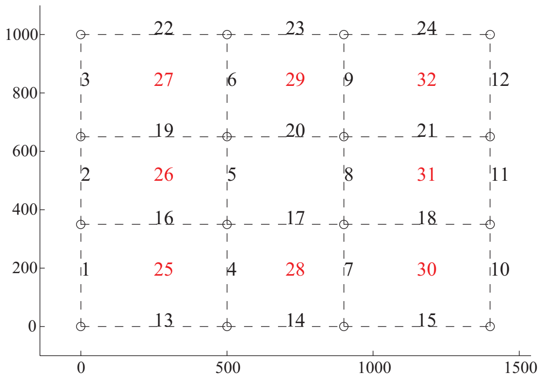
The second restriction in equation (5.7) is rewritten as equation (5.9), where the matrix with both $\mathbf{C}$ and $\mathbf{C}_d$ is the matrix for liniear inequality constraints *linprog* needs, cf. appendix A3. The size of $\mathbf{C}_d$ is similar to $\mathbf{C}$ which is illustrated in Appendix Figure A3.4. If same strengths for a group are desired the $\mathbf{C}_d$ matrix can be modified, for instance if all stringers must have same compression and tension strength and all shear areas the same shear strength. This is illustrated for structure 1 in Appendix Figure A3.5. It is not advisable to change $\mathbf{C}_d$, instead extra restrictions should be introduced, cf. chapter 8.1.

$$\begin{bmatrix} \mathbf{C} & -\mathbf{C}_d \end{bmatrix} \begin{Bmatrix} \beta \\ \mathbf{d} \end{Bmatrix} \leq \mathbf{C}_0 \tag{5.9}$$

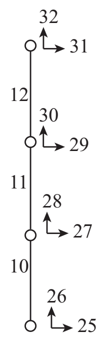
5.3 Example - Structure 2

For validating `Opti_String` structure 2 is calculated based on material and load optimisation respectively. The purpose of this example is to show the basic principle of material and load optimisation. For clarity this example is done with a modified $\mathbf{C}_d$ matrix resulting in three design variables, **d**, instead of 56, which is illustrated in Appendix Figure A3.5.

First, the calculation is made with material optimisation and the calculated design parameters will be used as basis for a load optimisation for the same example. Geometry, loads and boundary conditions for the structure are the same as used in section 3.2. The self weight is neglected, and the structure is only affected by the four nodal loads. By typing in the node coordinates, stringers, shear areas etc. into the data file for structure 2 the following stringer system is obtained, cf. Figure 5.1.



(a) Geometry generated by `Opti_String`. Stringers are assigned 1-31 and shear fields 32-42.



(b) Support numbers.

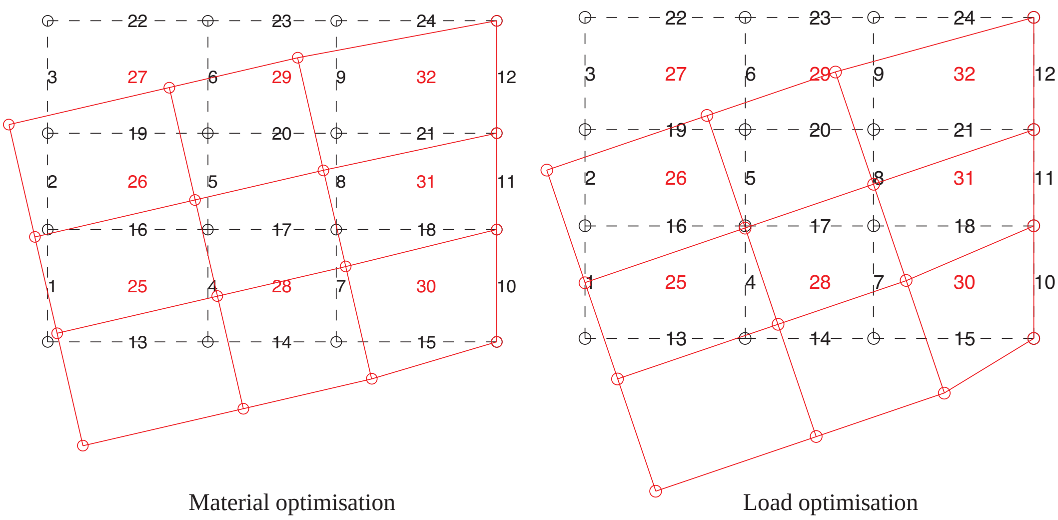
Figure 5.1: *`Opti_String`: Geometry plot for structure 2.*

The calculated design variables from material optimisation are based on the geometry and the material restrictions stated in the Eurocode. `Opti_String` automatically determine the material strengths and provides the matrix `mat`, shown in Figure 5.2. The working procedure of `Opti_String` for generating this list is described in section 9.1.

		mat	
stringers	1	481,03	350,00
	2	481,03	350,00
	3	481,03	350,00
	4	384,83	350,00
	5	481,03	350,00
	6	384,83	350,00
	7	384,83	350,00
	8	481,03	350,00
	9	384,83	350,00
	10	481,03	350,00
	11	481,03	350,00
	12	481,03	350,00
	13	336,72	350,00
	14	336,72	350,00
	15	336,72	350,00
	16	288,62	350,00
stringers	17	336,72	350,00
	18	288,62	350,00
	19	288,62	350,00
	20	336,72	350,00
	21	288,62	350,00
	22	336,72	350,00
	23	336,72	350,00
	24	336,72	350,00
shear areas	25	4,96	4,96
	26	4,96	4,96
	27	4,96	4,96
	28	4,96	4,96
	29	4,96	4,96
	30	4,96	4,96
	31	4,96	4,96
	32	4,96	4,96

Figure 5.2: *`Opti_String`: Matrix `mat.m` containing material parameters for structure 2. based on geometry and restrictions stated in the Eurocode.*

The collapse mode for structure 2 is shown in Figure 5.3 for both material and load optimisation. The calculated displacements from the dual equalities produces the collapse mechanism. This is due to the link between the primal and dual variables in LP, cf. equation (6.4). An example of the dual equalities which `Opti_String` handle is shown for structure 1 in Appendix Figure A5.2.



Material optimisation Load optimisation

Figure 5.3: *`Opti_String`: Collapse mechanism for structure 2of material and load optimisation respectively.*

Area 25 to 28 rotates like a rigid body which satisfy the fact that non of these areas and surrounding stringers are exposed to plastic strains. This is evident in Figure 5.4 and 5.5 showing output lists from `Opti_String`. According to Figure 5.3 failure occur in stringer 15, 18, 21 and 24 and area 31 for material optimisation, which are in accordance with the fact that plastic strains appear in these elements according to the list from `Opti_String`. Failure is expected in the horizontal stringers connected to support 25 and 31 since the moment from the load is largest here. The dual inequalities are interpreted as plastic strains, an example of these are shown in Appendix Figure A5.2 for structure 1.

```
Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 2
1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c  = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 100

Material optimisation algorithm - 3 design variables
Static independent variables, N = 8

----- Design parameters -----
d1 = 2.261538e+05
d2 = 2.261538e+05
d3 = 1.710020e+00

----- Maximum stringer forces [kN] -----
Stringer 1 = 21.239
Stringer 2 = -96.244
Stringer 3 = -96.244
Stringer 4 = -83.203
Stringer 5 = -83.203
Stringer 6 = -135
Stringer 7 = -68.011
Stringer 8 = -221.912
Stringer 9 = -221.912
Stringer 10 = 66.126
Stringer 11 = -78.045
Stringer 12 = 31.82
Stringer 13 = 30.341
Stringer 14 = -40.475
Stringer 15 = -226.154
Stringer 16 = -226.146
Stringer 17 = -226.146
Stringer 18 = -226.154
Stringer 19 = 226.154
Stringer 20 = 226.154
Stringer 21 = 226.154
Stringer 22 = -30.349
Stringer 23 = 138.835
Stringer 24 = 226.154

----- Shear stresses [MPa] -----
25 = 0.202
26 = 1.305
27 = 0.202
28 = 0.59
29 = 1.41
30 = 1.238
31 = 1.71
32 = 0.582

----- Elements exposed to plastic strains -----
Element 15
Element 18
Element 21
Element 24
Element 31

----- Reactions [kN] -----
Load case 1:
Support 25 = -226.154
Support 26 = 63.849
Support 27 = -226.154
Support 28 = 144.171
Support 29 = 226.154
Support 30 = 105.16
Support 31 = 226.154
Support 32 = 106.82

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 46.34
A_s(2) = 46.34
A_s(3) = 0
A_s(4) = 0
A_s(5) = 74.79
A_s(6) = 74.79
A_s(7) = 0
A_s(8) = 0
A_s(9) = 0
A_s(10) = 144.27
A_s(11) = 165.51
A_s(12) = 69.43
A_s(13) = 66.2
A_s(14) = 66.2
A_s(15) = 0
A_s(16) = 0
A_s(17) = 0
A_s(18) = 0
A_s(19) = 493.43
A_s(20) = 493.43
A_s(21) = 493.43
A_s(22) = 0
A_s(23) = 302.91
A_s(24) = 493.43

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 132.397
A_s_x,y(26) = 854.421
A_s_x,y(27) = 132.431
A_s_x,y(28) = 386.271
A_s_x,y(29) = 922.82
A_s_x,y(30) = 810.233
A_s_x,y(31) = 1119.29
A_s_x,y(32) = 381.029
```

Figure 5.4: *`Opti_String`: Output list for structure 2, material optimisation.*

Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 2
1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 100

Load optimisation algorithm - 1 design variables
Static independent variables, N = 8

----- Load parameter -----
Lambda = 1.000000

----- Maximum stringer forces [kN] -----
Stringer 1 = -21.239
Stringer 2 = 96.244
Stringer 3 = 96.244
Stringer 4 = 134.822
Stringer 5 = 134.822
Stringer 6 = 135
Stringer 7 = -24.903
Stringer 8 = 128.999
Stringer 9 = 135
Stringer 10 = 44.34
Stringer 11 = 76.951
Stringer 12 = 51.209
Stringer 13 = -30.341
Stringer 14 = 99.468
Stringer 15 = 226.154
Stringer 16 = 226.146
Stringer 17 = 226.146
Stringer 18 = 226.154
Stringer 19 = -226.154
Stringer 20 = -226.154
Stringer 21 = -226.154
Stringer 22 = 30.349
Stringer 23 = -79.842
Stringer 24 = -226.154

----- Shear stresses [MPa] -----
25 = 0.202
26 = 1.305
27 = 0.202
28 = 1.082
29 = 0.918
30 = 0.845
31 = 1.71
32 = 0.975

----- Elements exposed to plastic strains -----
Element 15
Element 18
Element 21
Element 24

----- Reactions [kN] -----
Load case 1:
Support 25 = -226.154
Support 26 = 44.34
Support 27 = -226.154
Support 28 = 121.291
Support 29 = 226.154
Support 30 = 128.16
Support 31 = 226.154
Support 32 = 126.209

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 0
A_s(2) = 209.99
A_s(3) = 209.99
A_s(4) = 294.16
A_s(5) = 294.16
A_s(6) = 294.55
A_s(7) = 0
A_s(8) = 281.45
A_s(9) = 294.55
A_s(10) = 96.74
A_s(11) = 167.89
A_s(12) = 111.73
A_s(13) = 0
A_s(14) = 217.02
A_s(15) = 493.43
A_s(16) = 493.41
A_s(17) = 493.41
A_s(18) = 493.43
A_s(19) = 0
A_s(20) = 0
A_s(21) = 0
A_s(22) = 66.22
A_s(23) = 66.22
A_s(24) = 0

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 132.397
A_s_x,y(26) = 854.421
A_s_x,y(27) = 132.432
A_s_x,y(28) = 708.049
A_s_x,y(29) = 601.041
A_s_x,y(30) = 552.81
A_s_x,y(31) = 1119.29
A_s_x,y(32) = 638.452

Figure 5.5: `Opti_String`*: Output list for structure 2, load optimisation.*

The stringer forces, shear stresses, load variable λ and design variables $\mathbf{d}$ are interpreted from the primal variables. An example for the primal variables is shown in Appendix Figure A5.1. The design parameters, $\mathbf{d}$, from equation (5.8) are listed from d1 to d3 under Design parameters in the output list, cf. Figure 5.4. The design parameters are shown in Table 5.1. The design parameters are based on the largest occurring values in the elements when minimising the material use in the structure. It is seen from Table 5.1 that the elements are not utilised to the limit since the design parameters are lower than the values stated in Figure 5.2

$F_{c,max}$	d1	226.15 kN
$F_{t,max}$	d2	226.15 kN
τ_{max}	d3	1.71 MPa

Table 5.1: *Design parameters for structure 2 using the lower bound method.*

By using the three design parameters from material optimisation, load optimisation is performed on the stringer system. The loads applied are the same values as for material optimisation.

Load optimisation leads to a load parameter, λ, of 1, cf. Figure 5.5. The value is expected as the applied design parameters are based on material optimisation thus, the design parameters calculated using material optimisation reflects the optimum values for the given load.

The maximum shear stress appears in shear area 31 for both material and load optimisation. The number of stringers which must be reinforced is larger for load optimisation.

Compared to the stringer forces calculated in section 3.2 based on traditional hand calculations the forces are lower. The maximum stringer force is 226.15 kN which is a reduction of 23%. It is noticed that the optimal shear stress determined by `Opti_String` is similar to the one determined by the hand calculations, cf. Figure 3.9.

The reactions calculated for both material and load optimisation are controlled by summation of horizontal and vertical loads and reactions, respectively.

$$\Sigma P_x = 0\,\text{kN}$$
$$\Sigma R_x = 0\,\text{kN}$$
$$\Sigma P_y = -420\,\text{kN}$$
$$\Sigma R_y = 420\,\text{kN}$$

Chapter **6**

Linear Programming

The principle of LP is described in this chapter. LP is illustrated in connection with the stringer method and the meaning of primal and dual variables are explained.

The formulation of the equations for the stringer system is described in chapter 4. Based on this formulation the system is optimised, either regarding stress distribution or kinematics, cf. chapter 5 and 7, respectively. LP is originally a mathematical method for economical optimisation where resources could not be negative. Modern systems does not care and let $\mathbf{x}$ take both positive and negative values.

In particular LP is an optimisation of a linear object function subjected to restrictions consisting of inequalities which may be linearised if necessary. By use of LP in stringer method the inequalities include constraints such as material strength and the equalities are the equilibrium conditions.

If a LP problem is formulated using the lower bound method, cf. equation (5.1), the primal variables are static, i.e. optimal stresses, $\boldsymbol{\beta}$, and optimal load parameter, λ, cf. equation (6.3). The general formulation is written as equation (6.1) where the object function, $\mathbf{c}$, is a vector of known coefficients to be optimised multiplied with a vector of variables, $\mathbf{x}$, to be determined. This optimisation is subjected to a number of restrictions collected in matrices and vectors where $\mathbf{A}$ is a known matrix of coefficients containing inequality constraints. $\mathbf{b}$ is a vector with known coefficients.

$$\begin{aligned} \text{maximise}: \quad & \mathbf{c}^T\,\mathbf{x} \\ \text{restrictions}: \quad & \mathbf{A}\,\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \tag{6.1}$$

where

$\mathbf{c}$	Object function
$\mathbf{x}$	Vector of primal variables
$\mathbf{A}$	Matrix for inequality constraints
$\mathbf{b}$	Vector of linear inequality constraints

According to the stringer method equation (6.1) is expanded to equation (6.2) in order to account for stresses which can both be negative and positive. The stresses are included in $\mathbf{x}$.

$$\begin{aligned} \text{maximise :} \quad & \mathbf{c}^T \mathbf{x} \\ \text{restrictions :} \quad & \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{Bmatrix} \geq \mathbf{b} \\ & \mathbf{x} = \mathbf{x}^+ - \mathbf{x}^- \end{aligned} \tag{6.2}$$

For each primal LP problem a dual problem can be formulated and the optimum solution for the dual problem is the optimum solution to the primal problem (Damkilde, 1995). For equation (6.1) the primal variables are expressed in equation (6.3) and the dual variables in equation (6.4).

$$\text{Primal variables :} \quad \mathbf{x} = \begin{Bmatrix} \boldsymbol{\beta} \\ var \end{Bmatrix} \tag{6.3}$$

$$\text{Dual variables (shadow price) :} \quad \boldsymbol{\theta} = \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} \tag{6.4}$$

where

$\boldsymbol{\beta}$	Stress parameters
var	Variables, load or design parameter
$\boldsymbol{\Psi}$	Plastic strains
$\mathbf{V}$	Displacements for nodes and stringers

If the primal problem is a lower bound, e.g. equation (5.1), the dual problem for this equivalent is the upper bound problem formulated in equation (7.3) (Damkilde, 1995). Thus, besides the stress distribution found from the primal problem, the dual values are kinematic, which provides the plastic strains, $\boldsymbol{\Psi}$, and displacements, $\mathbf{V}$, for the system. The dual variables from equation (6.1) is the primal variables in equation (6.5) when the optimisation is based on the upper bound method e.g. a kinematic admissible collapse mechanism. The primal variables for the upper bound are shown in equation (6.7) and the dual values in equation (6.8).

$$\begin{aligned} \text{minimaize :} \quad & \boldsymbol{\theta}^T \mathbf{b} \\ \text{restrictions :} \quad & \boldsymbol{\theta}^T \mathbf{A} \leq \mathbf{c}^T \\ & \boldsymbol{\theta} \geq 0 \end{aligned} \tag{6.5}$$

The general formulation in equation (6.5) is expanded to equation (6.6) in order to account for displacements,

V, which both can be negative and positive.

$$\begin{aligned}
\text{maximise}: &\quad \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} \mathbf{b} \\
\text{restrictions}: &\quad \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} \geq \mathbf{c}^T \\
&\quad \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V}^+ \\ \mathbf{V}^- \end{Bmatrix} \geq 0
\end{aligned} \tag{6.6}$$

$$\text{Primal variables}: \quad \boldsymbol{\theta} = \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{V} \end{Bmatrix} \tag{6.7}$$

$$\text{Dual variables (shadow price)} \ : \quad \mathbf{x} = \begin{Bmatrix} \beta \\ var \end{Bmatrix} \tag{6.8}$$

where

$\boldsymbol{\theta}$	Primal variables
$\mathbf{x}$	Dual variables

Obviously this property is also valid when formulating the primal problem from the upper bound method thus, the primal values consists of plastic strains and displacements while the dual values represents the stress distribution. The shadow price indicates the profit of changing a restriction. If a restriction is changed and do not limit the optimum, the shadow price becomes zero. If the optimum on the other hand is limited by a restriction then one will pay extra in order to obtain a larger load or smaller material parameters, which is interpreted as plastic strains for an element.

If a LP problem is formulated using the upper bound method, cf. equation (7.3), the load parameter, λ, is found by the value of the object function. For material optimisation the design variables are interpreted as the dual variables of the inequalities.

The inequality $\mathbf{A} \cdot \mathbf{x} \geq \mathbf{b}$ in equation (6.1) defines the feasible region for the linear optimisation by n convex polyhedrals. n is the number of restrictions in the inequality. An example of a feasible region is illustrated in Figure 6.1.

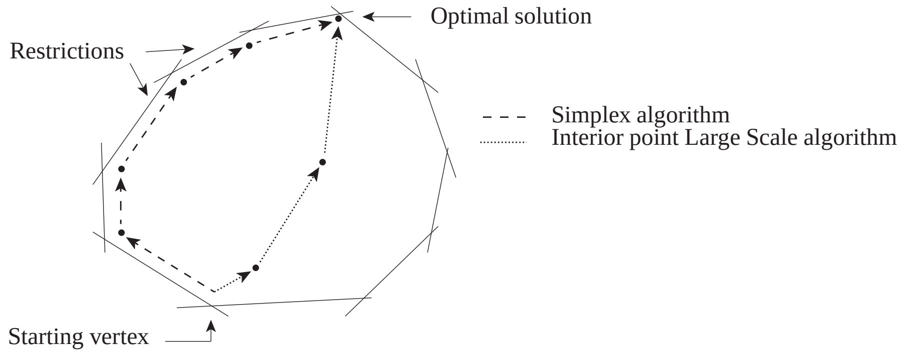


Figure 6.1: *Illustration of iteration procedure for simplex and interior point algorithm respectively.*

The LP problems are solved using the *lingprog* function in MATLAB. *lingprog* always minimise, why the object function needs to be multiplied by -1 in equation (6.1) for carry out the maximisation.

For the solution of the LP problem one of two algorithms has to be chosen. The Simplex algorithm is the simplest of the two algorithms. The solution space is a n-dimensional space limited by restrictions. Simplex tends to find a extreme solution as it follows one edge and hereafter the next edge. Thereby, the order of definition of elements can affect the result.

The other algorithm, Large-Scale, is an interior point algorithm. Opposite to Simplex every steps is based on all the restrictions. This means that every step is costly but less steps are needed for finding the optimum. In addition, Large-Scale do not tends to find an extreme optimum but instead find an balanced optimum.

The algorithms can be illustrated by a mountain climber reaching the top where; simplex is limited by a fogged weather and therefore only take small steps every time and interior-point see the top from the beginning and walk in the correct direction all time. The algorithms are illustrated in Figure 6.1. Every line of demarcation in the feasible region corresponds to a restriction in the inequalities.

The two algorithms have been compared for structure 1in appendix A3, and based on the simple comparison the Large-Scale algorithm is chosen for all the following calculations, since all restrictions are included in the optimisation.

Chapter **7**

Upper Bound Method

This chapter describes the upper bound method. The theory is described according to the stringer method and the mathematical formulation from chapter 4. After the description of the theory structure 2 is shown for both load and material optimisation. A comparison with the results from lower bound is made in the end of the chapter

For practical use the upper bound method is somewhat more complicated because this require knowledge within the mechanics and collapse modes. The upper bound formulation in equation (7.3) is a direct mathematical translation from the lower bound in equation (5.7). This is possible due to the primal and dual link between upper and lower bound method from chapter 6.

The upper bound method seeks:

- A kinematic admissible collapse mechanism
- Equilibrium between internal and external work

The procedure for upper bound method is to calculate the external and internal work, and hereby determine the collapse load λ^+ as the ratio between these two. This is illustrated in equation (7.1)

$$\begin{gathered} A_{internal} = A_{external} \\ \mathbf{C}_d^T\ \mathbf{\Psi} = \lambda\, \mathbf{R}_0^T\ \mathbf{V} + \mathbf{R}^T\ \mathbf{V} \end{gathered} \tag{7.1}$$

where

$A_{internal}$	Internal plastic work
$A_{external}$	External work from the real collapse load
λ	Upper bound value for the collapse load
$\mathbf{C}_d$	Vector with material constraint, strength values
$\mathbf{\Psi}$	Vector with plastic strain variables
$\mathbf{R}_0$	Constant load vector
$\mathbf{V}$	Vector with displacements for each node
$\mathbf{R}$	Load vector

By isolating λ in equation (7.1) it is possible to express the collapse load, cf. equation (7.2). Hereby λ^+ is expressed by a geometric feasible collapse mechanism.

$$\lambda^{+} = \frac{\mathbf{C}_d^T \mathbf{\Psi} - \mathbf{R}_0^T \mathbf{V}}{\mathbf{R}^T \cdot V} \tag{7.2}$$

7.1 Load Optimisation

Opposite to the lower bound method the variables to be determined are displacements in each node, collected in the vector $\mathbf{V}$, and plastic strains which can illustrate positive or negative yield strength, collected in the vector $\mathbf{\Psi}$.

Since LP are only able to work with linear problems equation (7.2) is linearised by minimising the strains and displacements caused by the load, in this case minimising the numerator and keep the denominator as a constant of one, $\mathbf{R}^T\ \mathbf{V} = 1$. The external load is kept constant in order to make the optimisation easier. Equation (7.3) describes the LP problem for finding an upper value of the load bearing capacity of the structure (Damkilde, 1995, equation (18)).

$$\begin{aligned} \text{minimise}: \quad & \mathbf{C}_d^T\ \mathbf{\Psi} - \mathbf{R}_0^T\ \mathbf{V} \\ \text{restrictions}: \quad & -\mathbf{H}^T\ \mathbf{V} + \mathbf{C}^T\ \mathbf{\Psi} = 0 \\ & \mathbf{R}^T\ \mathbf{V} = 1 \\ & \mathbf{\Psi} \geq 0 \end{aligned} \tag{7.3}$$

By rewriting the numerator the object function for *linprog* is shown to the left which is multiplied with the variables to be minimised, cf. equation (7.4). The object function for structure 1 is exemplified in Appendix Figure A3.7 where $\mathbf{C}_d$ and $\mathbf{R}_0$ are illustrated.

$$\left\{ \begin{array}{cc} \mathbf{C}_d^T & -\mathbf{R}_0^T \end{array} \right\} \left\{ \begin{array}{c} \mathbf{\Psi} \\ \mathbf{V} \end{array} \right\} \tag{7.4}$$

The first restriction in equation (7.3), $-\mathbf{H}^T\ \mathbf{V} + \mathbf{C}^T\ \mathbf{\Psi} = 0$, ensures compatibility between the plastic strains, $\mathbf{\Psi}$, and the displacements, $\mathbf{V}$, which is written as equation (7.5). $\mathbf{H}$ and $\mathbf{C}$ is known from the lower bound method, cf. chapter 5, and is illustrated for structure 1 in Appendix Figure A3.2 and A3.3. By multiplying $-\mathbf{H}^T$ with $\mathbf{V}$ the following is obtained; $n_{displacements} = n_{equality\_equations}$.

$$\left[\begin{array}{cc} \mathbf{C}^T & -\mathbf{H}^T \end{array} \right] \left\{ \begin{array}{c} \mathbf{\Psi} \\ \mathbf{V} \end{array} \right\} = 0 \tag{7.5}$$

By identifying the input in equation (7.3) a physical interpretation is obtained. For each element compatibility is satisfied in the upper bound method. This means that the collapse mechanism is compatible with the physical conditions of the structure and material. Compatibility is obtained by considering all strains as plastic strains. In equation (7.4) and equation (7.5) compatibility is ensured by setting displacements equal to plastic strains. The plastic strains, $\mathbf{\Psi}$, in equation (7.4) cover both strains for stringers, ε, and areas, γ, and similar for the displacements, cf. equation (7.6).

$$\mathbf{\Psi} = \left\{ \begin{array}{c} \varepsilon \\ \gamma \end{array} \right\} \qquad \mathbf{V} = \left\{ \begin{array}{c} \mathbf{v}_{stringers} \\ \mathbf{v}_{areas} \end{array} \right\} \tag{7.6}$$

Compatibility is satisfied if the difference between displacements, Δl, corresponds to the difference in plastic strains, $\Delta\varepsilon$. Compatibility for stringer,k, is ensured by equation (7.7) and illustrated in Figure 7.1.

$$\Delta\varepsilon_{k1} = \frac{(v_a - v_k)}{l} \qquad (7.7)$$
$$\Delta\varepsilon_{k2} = \frac{(v_b - v_k)}{l}$$

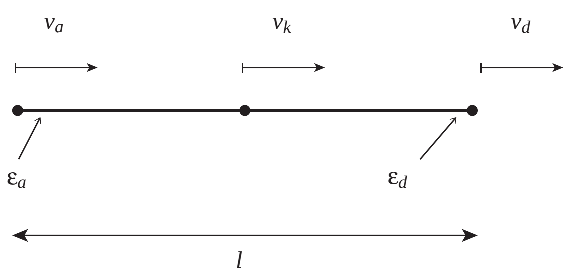


Figure 7.1: *Compatibility between displacements and strains for stringer k.*

Compatibility for shear areas between displacements, Δv, and strains, γ, is expressed in equation (7.8). Figure 7.2 illustrates how different strains results in different gradients for the shear areas. Together the displacements v_i to v_l represent a rotation.

$$\gamma_m = \frac{v_l - v_k}{h} + \frac{v_j - v_i}{w} \qquad (7.8)$$

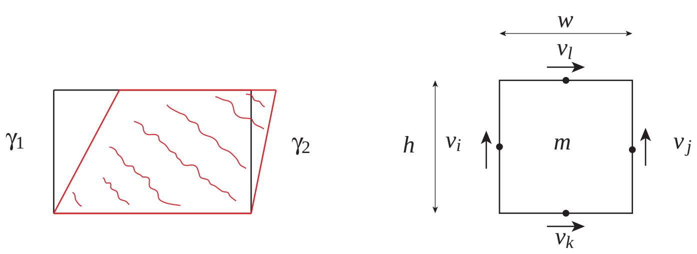


Figure 7.2: *Compatibility between displacements and strains for area m.*

Figure 7.3 illustrates the assembling of equation (7.5). In the assembling stringers are expressed by equation (7.7) and shear areasby equation (7.8). It is illustrated how compatibility is satisfied when displacements result in strains. Hereby a geometric band exist which connect strains with a displacement variation.

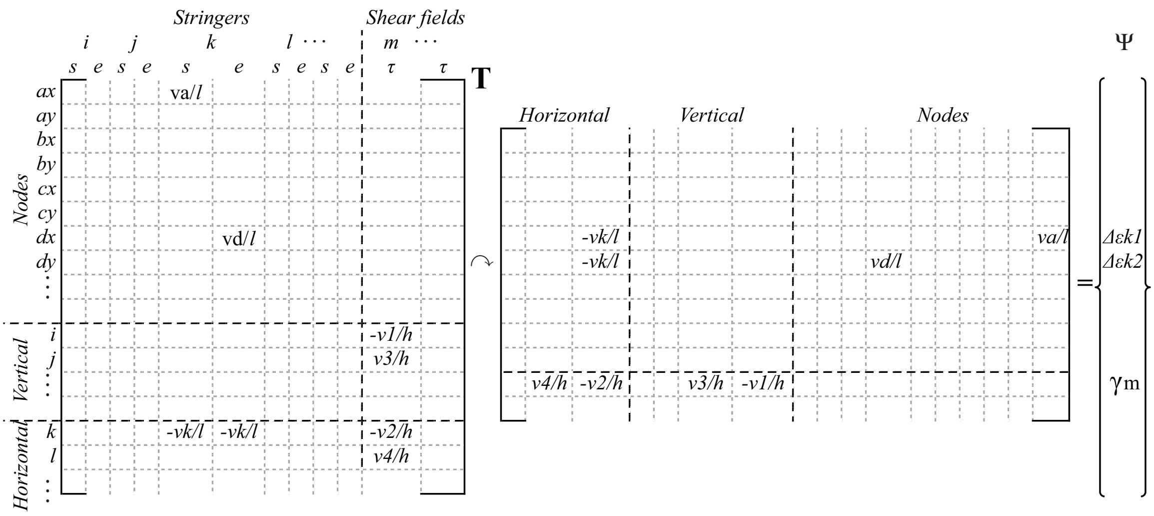


Figure 7.3: *Assembling of flexibility matrix,* **H** *for stringer, k, and area, m. s = start node, e = end node.*

The restrictions consist of two equalities and one inequality that ensure the plastic strains are positive defined. The built-in function in MATLAB *linprog*, is not capable of solving an equation system with two equalities thus, the second equality is formulated as two inequalities, $-\mathbf{R}^T\mathbf{V} \leq -1$ and $\mathbf{R}^T\mathbf{V} \leq 1$, cf. equation (7.9). A more detailed exposition solving the linear programming problem using *linprog* is made in Appendix Figure A3.8. Compatibility is also ensured by keeping the external work constant.

The matrix on the left hand side is the matrix for *linprog* describing the linear inequality constraints, **A**, where the right hand side is the vector *linprog* needs as linear inequality constraints.

$$\begin{bmatrix} \mathbf{0} & -\mathbf{R}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} \begin{Bmatrix} \mathbf{\Psi} \\ \mathbf{V} \end{Bmatrix} \leq \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{7.9}$$

The dimensions of **A** are shown below. The first parenthesis represents the plastic strains and the last the displacements.

$$\begin{aligned} n_{rows} &= 2 \\ n_{rows} &= (4 \cdot n_{stringers} + 2 \cdot n_{shear\_areas}) + (2 \cdot n_{nodes} - n_{supports} + n_{stringers}) \end{aligned}$$

Except of the new variables, $\mathbf{\Psi}$ and **V**, all the vectors and matrices are known from the lower bound method in chapter 5. This is due to the relation between primal and dual problems in LP, cf. chapter 6.

By use og *linprog* `Opti_String` calculates primal and dual variables. Solution of equation (7.3) provides primary values containing the plastic strains, equalling the collapse mode of the system and the displacements of the nodes in the collapse mechanism. The dual problem contains the stringer and shear forces which are interpreted as the shadow prices of the equalities while the load parameter are found by the value of the object function.

7.2 Material Optimisation

The upper bound method for material optimisation, equation (7.10) comes from a translation of the lower bound method. This is possible due to the connection between primal and dual variables in LP, cf. chapter 6.

$$\begin{aligned} \text{maximise}: \quad & \mathbf{R}^T\ \mathbf{V} - \mathbf{C}_0^T\ \mathbf{\Psi} \\ \text{restrictions}: \quad & \mathbf{H}^T\ \mathbf{V} - \mathbf{C}^T\ \mathbf{\Psi} = 0 \\ & \mathbf{C}_d^T\ \mathbf{\Psi} \leq \mathbf{w} \\ & \mathbf{\Psi} \geq 0 \end{aligned} \tag{7.10}$$

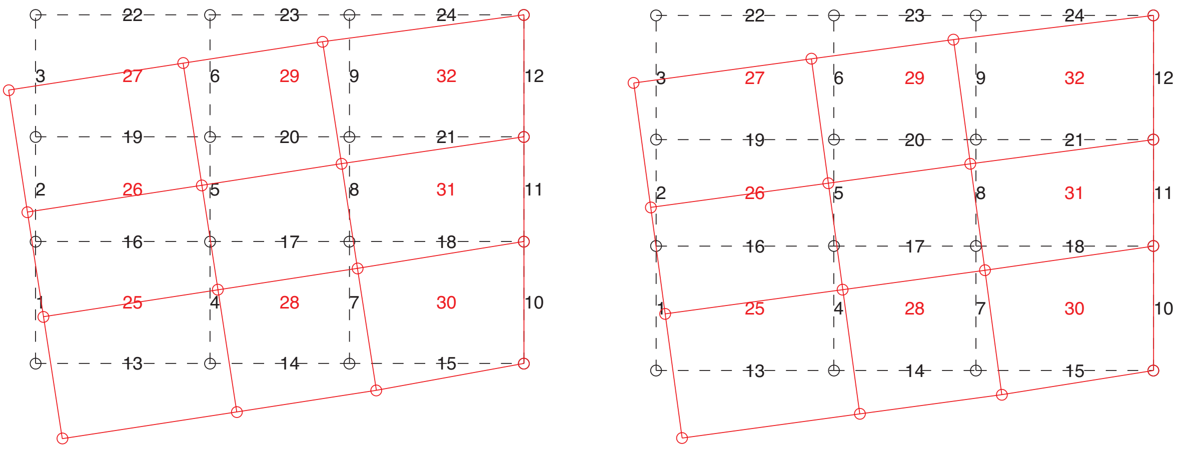
After the translation the intention is to maximise the external work, $\mathbf{R}^T\ \mathbf{V}$, minus the internal plastic work, $\mathbf{C}_0\ \mathbf{\Psi}$, which is constant and depends on the weighting factor. The external load is known why the external plastic work are kept constant. The internal plastic work is unknown for which the intention is to maximise in order to obtain optimal design variables for the given load.

Material optimisation provides similar to load optimisation plastic strains and displacements as primal variables. In addition, the dual variables consists of the weighting parameter, $\mathbf{w}$, and the stringer forces and shear stresses. Thus the upper bound variable due not enter explicit in the formulation the restrictions for the internal plastic work provides the shadow prices where the design parameters are found.

7.3 Example - Structure 2

Structure 2 is used for illustrating `Opti_String` when calculations are based on the upper bound method. The procedure in this example is similar to section 5.3 for lower bound. As for lower bound this example is performed with a modified $\mathbf{C}_d$ matrix for clarity and the possibility to compare the results.

First the optimal design variables $\mathbf{d}$ are determined by material optimisation after which the load optimisation is based on these design variables which should lead to the load parameter $\lambda = 1$. The geometry plot is equal to the one from section 5.3. The primal variables provides now the plastic strains in the elements and the displacements of the nodes and stringers as well, cf. equation (6.7). These are used for plotting the collapse mechanism shown in Figure 7.4.



Material optimisation Load optimisation

Figure 7.4: *`Opti_String`: Collapse mechanism for structure 2based on material and load optimisation.*

The two collapse modes are similar which are substantiated by the fact that material and load optimisation provides plastic strains in the same elements, except area 31. This appear in the two output lists from `Opti_String` shown in Figure 7.5 for material optimisation and 7.6 for load optimisation.

```
Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Upper bound method

Structure 2
1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c  = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 40

Material optimisation algorithm - 3 design variables
Static independent variables, N = 8

----- Design parameters -----
d1 = 2.261538e+05
d2 = 2.261538e+05
d3 = 1.710000e+00

----- Maximum stringer forces [kN] -----
Stringer 1 = 21.242
Stringer 2 = -96.242
Stringer 3 = -96.242
Stringer 4 = -149.5
Stringer 5 = -149.5
Stringer 6 = -135
Stringer 7 = 51.314
Stringer 8 = -102.586
Stringer 9 = -135
Stringer 10 = -38.733
Stringer 11 = -76.998
Stringer 12 = -57.521
Stringer 13 = 30.346
Stringer 14 = -116.234
Stringer 15 = -226.154
Stringer 16 = -226.154
Stringer 17 = -226.154
Stringer 18 = -226.154
Stringer 19 = 226.154
Stringer 20 = 226.154
Stringer 21 = 226.154
Stringer 22 = -30.346
Stringer 23 = 63.074
Stringer 24 = 226.154

----- Shear stresses [MPa] -----
25 = 0.202
26 = 1.305
27 = 0.202
28 = 1.222
29 = 0.778
30 = 0.733
31 = 1.71
32 = 1.087

----- Elements exposed to plastic strains -----
Element 15
Element 18
Element 21
Element 24
Element 31

----- Reactions [kN] -----
Load case 1:
Support 25 = -226.154
Support 26 = 38.733
Support 27 = -226.154
Support 28 = 115.209
Support 29 = 226.154
Support 30 = 134.422
Support 31 = 226.154
Support 32 = 131.635

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 46.35
A_s(2) = 46.35
A_s(3) = 0
A_s(4) = 0
A_s(5) = 0
A_s(6) = 0
A_s(7) = 111.96
A_s(8) = 111.96
A_s(9) = 0
A_s(10) = 83.37
A_s(11) = 167.79
A_s(12) = 123.57
A_s(13) = 66.21
A_s(14) = 66.21
A_s(15) = 0
A_s(16) = 0
A_s(17) = 0
A_s(18) = 0
A_s(19) = 493.43
A_s(20) = 493.43
A_s(21) = 493.43
A_s(22) = 0
A_s(23) = 137.62
A_s(24) = 493.43

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 132.42
A_s_x,y(26) = 854.434
A_s_x,y(27) = 132.42
A_s_x,y(28) = 799.529
A_s_x,y(29) = 509.562
A_s_x,y(30) = 479.65
A_s_x,y(31) = 1119.27
A_s_x,y(32) = 711.623
```

Figure 7.5: *`Opti_String`: Output list for structure 2 based on material optimisation.*

As mentioned for lower bound it is expected that plastic strains occur in the horizontal elements nearest the fixed support, which is the case.

Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Upper bound method

Structure 2
1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 10

Load optimisation algorithm - 1 design variables
Static independent variables, N = 8

----- Load parameter -----
Lambda = 1.000000

----- Maximum stringer forces [kN] -----
Stringer 1 = 21.242
Stringer 2 = -96.242
Stringer 3 = -96.242
Stringer 4 = -133.11
Stringer 5 = -133.11
Stringer 6 = -135
Stringer 7 = 21.811
Stringer 8 = -132.089
Stringer 9 = -135
Stringer 10 = 45.028
Stringer 11 = -76.95
Stringer 12 = -50.522
Stringer 13 = 30.346
Stringer 14 = -97.502
Stringer 15 = -226.154
Stringer 16 = -226.154
Stringer 17 = -226.154
Stringer 18 = -226.154
Stringer 19 = 226.154
Stringer 20 = 226.154
Stringer 21 = 226.154
Stringer 22 = -30.346
Stringer 23 = 81.805
Stringer 24 = 226.154

----- Shear stresses [MPa] -----
25 = 0.202
26 = 1.305
27 = 0.202
28 = 1.065
29 = 0.935
30 = 0.858
31 = 1.71
32 = 0.962

----- Elements exposed to plastic strains -----
Element 15
Element 18
Element 21
Element 24

----- Reactions [kN] -----
Load case 1:
Support 25 = -226.154
Support 26 = 45.028
Support 27 = -226.154
Support 28 = 121.978
Support 29 = 226.154
Support 30 = 127.472
Support 31 = 226.154
Support 32 = 125.522

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 46.35
A_s(2) = 46.35
A_s(3) = 0
A_s(4) = 0
A_s(5) = 0
A_s(6) = 0
A_s(7) = 47.59
A_s(8) = 47.59
A_s(9) = 0
A_s(10) = 98.24
A_s(11) = 167.89
A_s(12) = 110.23
A_s(13) = 66.21
A_s(14) = 66.21
A_s(15) = 0
A_s(16) = 0
A_s(17) = 0
A_s(18) = 0
A_s(19) = 493.43
A_s(20) = 493.43
A_s(21) = 493.43
A_s(22) = 0
A_s(23) = 178.48
A_s(24) = 493.43

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 132.42
A_s_x,y(26) = 854.433
A_s_x,y(27) = 132.419
A_s_x,y(28) = 697.356
A_s_x,y(29) = 611.735
A_s_x,y(30) = 561.388
A_s_x,y(31) = 1119.27
A_s_x,y(32) = 629.884

Figure 7.6: *`Opti_String`: Output list for structure 2 based on load optimisation.*

The calculated design parameters are interpreted as Table 7.1 and is in accordance with the example for lower bound. These are used as material strengths in the $\mathbf{C}_d$ vector from 7.4 when performing load optimisation.

$F_{c,max}$	d1	226.15 kN
$F_{t,max}$	d2	226.15 kN
τ_{max}	d3	1.71 MPa

Table 7.1: *Design parameters for structure 2 using the upper bound method.*

This results in the output list shown in Figure 7.6. By use of the determined design parameters the load parameter is calculated to a value of one. Thus, the design parameters represents the optimal material strengths for carrying the applied load.

When `Opti_String` is using *linprog* and performing load or material optimisation regarding upper bound method some values, e.g. dimensions, must be scaled for executing the optimisation algorithm. The values to be scaled depends on the optimisation method. If the concerned values are not scaled *linprog* displays a termination message saying that; "the residuals, duality gab or total relative error has stalled the calculation". After the calculation is executed the output values needs to be converted backwards in order to obtain the requested units of the output.

7.4 Comparison of Lower and Upper Bound

The calculated design parameters; d1, d2 and d3 are identical to the design parameters determined in the lower bound material optimisation, cf. the output list in Figure 5.4. The collapse mode is also identically between the lower and upper bound method. This means that an exact solution exists for the stringersand shear areas when the calculations are based on the upper bound method.

The translation for stringers was expected to be exact since they are similar to bars and beams for which an exact translation exist.

An exact translation from lower to upper bound method exist for shear areas as the opposite to plates where the translation is a numerical approximation. This means that the duality described for the general LP problem in chapter 6 is fulfilled. Hereby it is proved that an upper bound formulation provides the matching lower bound formulation by a direct mathematically translation.

The amount of elements where plastic strains occur is similar for lower and upper bound. In addition area 31 is exposed to plastic strains when performing material optimisation for both lower and upper bound method. For this case it can be concluded that the material optimisation seeks to utilise more elements to the yield limit which are expressed in the number of elements where plastic strains occur.

Chapter **8**

Design Demands of Structures

Practical conditions regarding material optimisation is formulated and implemented in `Opti_String`*. This is first done by linking material properties and implement material limits and secondly by illustrating several load combinations for a stringer system. Through the chapter examples are made to illustrate the theory. The practical material optimisation is based on the lower bound method.*

The previous use of the stringer method is only based on the mathematical optimisation and the restrictions associated with LP and the stringer method. In order to improve the practicality of the stringer method different design demands are introduced. Examples of design demands are:

- Same amount of reinforcement in a stringer line based on the largest occurring force
- Same shear reinforcement in all areas
- Differentiate stringer line or shear areas for large structures
- Material strengths regarding Eurocodes
- Optimisation with regard to several load cases

The principle of these design restrictions are illustrated in Figure 8.1. Two load cases are added which result in tension for the illustrated stringer line. Without any design demands the reinforcement is designed according to the stringer force in each stringer. By introducing design demand these ensure the same amount of reinforcement in the stringer line and hereby the practicability is improved. Thus, the stringer line is designed according to the largest occurring force.

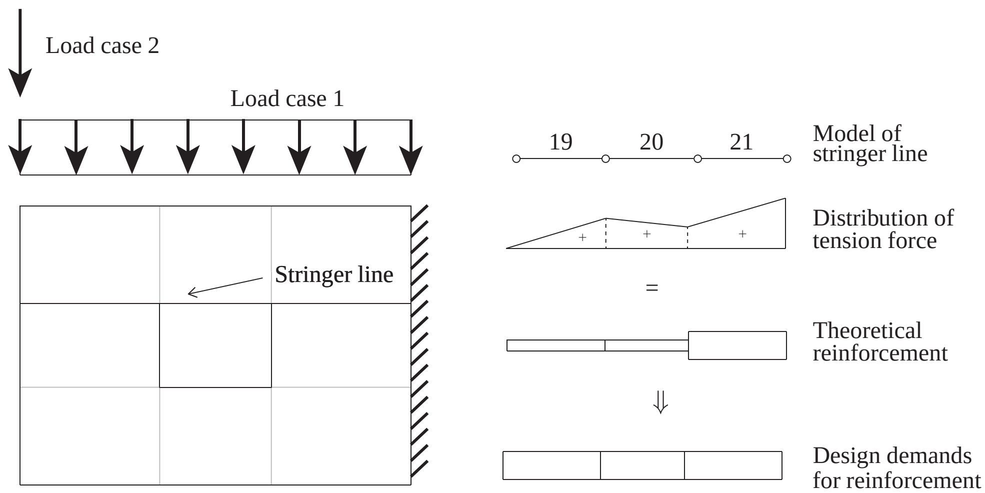


Figure 8.1: *Principle of design demands.*

To ensure that a structure can be made using standard material restrictions, material parameters are added to the optimisation.

8.1 Practical Design Restrictions

Two sorts of practical design restrictions are presented and implemented in `Opti_String`. First, each sort is explained and shown theoretical and afterwards an example using both design restrictions are presented.

Linking Design Variables

For construction of a structure it is obvious that some elements must be made using the same material parameters, for example tension strength. This is pertinent for each stringer line and for all shear areas as they often are made using the same amount of reinforcement. Therefore, the design variables are linked in order to provide the same results for each stringer line and shear area. The principle for linking elements is shown in equation (8.1) for a stringer line consisting of three stringers, illustrated in Figure 8.2.

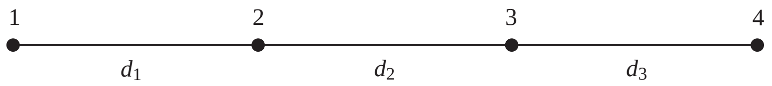


Figure 8.2: *Stringer line of three stringers and their design variables, d.*

$$d_1 = d_2 = d_3 \quad \Rightarrow \quad \begin{matrix} d_1 - d_2 = 0 \\ d_1 - d_3 = 0 \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{d} = \mathbf{0} \tag{8.1}$$

The linking of design variables is formulated using matrix notation in equation (8.2) where the links between elements are expressed in the property matrix, **E**.

$$\mathbf{E}\,\mathbf{d} = \mathbf{e}_0 \tag{8.2}$$

where

$\mathbf{e}_0$	Zero vector

The equations are applied as extra rows to the equalities in equation (5.7) which afterwards are formulated by equation (8.3). Notice, adding up extra equations do not chance the object function. Equation (8.3) is illustrated in Appendix Figure A4.2 for structure 1.

$$\begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{Bmatrix} \beta \\ \mathbf{d} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{e}_0 \end{Bmatrix} \tag{8.3}$$

Implementation of Material Limits in Material Optimisation

Implementation of material limits are not possible in the material optimisation problem stated in equation (5.7) thus, the design variables for the structure is only based on the applied load and weighting parameters. However, it is possible to include material limits by adding up extra equations to the inequalities and hereby set up limits for the material, for example by stringer forces and shear stresses. By this feature the individual elements are affected by forces or stresses less or equal to the limits set up for the element when performing material optimisation. For illustrating material limits, m_i, for stringer i these are formulated in equation (8.4).

$$\begin{aligned} -N_y^- \leq m_i \leq N_y^+ \\ -m_i \leq -N_y^- \\ m_i \leq N_y^+ \end{aligned} \tag{8.4}$$

Using matrix notation the inequalities for the material limits can be expressed as equation (8.5) where the material matrix, **M**, links elements to material parameters.

$$-\mathbf{M}\,\mathbf{d} \leq -\mathbf{m}_0 \tag{8.5}$$

where

$\mathbf{m}_0$	Material parameters

The extra inequalities are added up to the inequalities from equation (5.7) which afterwards is formulated by equation (8.6). Equation (8.6) is illustrated in Appendix Figure A4.3.

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{Bmatrix} \beta \\ \mathbf{d} \end{Bmatrix} - \begin{Bmatrix} \mathbf{C}_d \\ \mathbf{0} \end{Bmatrix} \mathbf{d} \leq \begin{Bmatrix} \mathbf{C}_0 \\ -\mathbf{m}_0 \end{Bmatrix} \tag{8.6}$$

Example - Structure 2

Structure 2 is calculated using the above mentioned design restrictions for verifying `Opti_String` performing practical design restrictions. Stringer lines are made according to Table 8.1. Notice, all shear areas are combined in two design variables, $d_{9\,c}$ and $d_{9\,t}$.

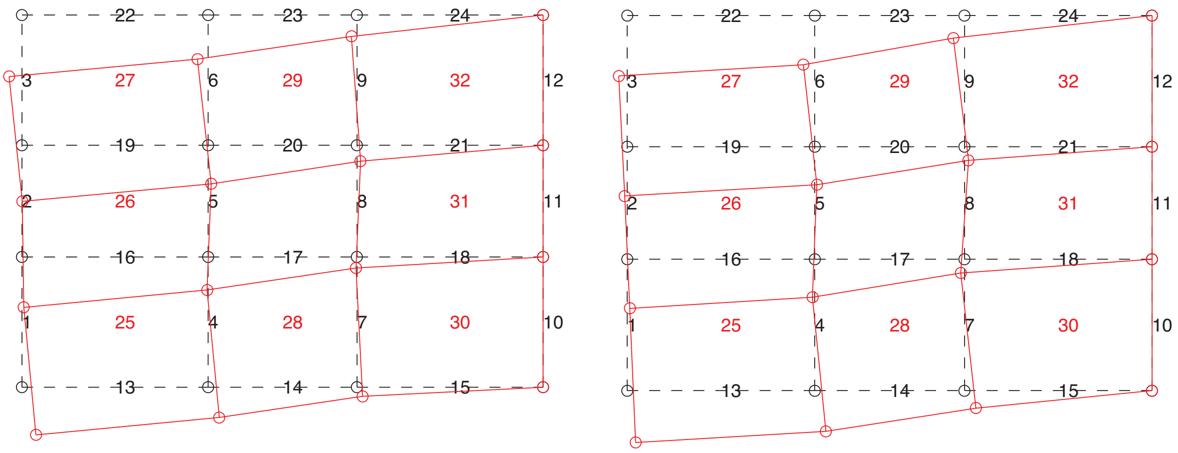
Stringer line	Design variable	Elements
1	d_{1c}, d_{1t}	1-2-3
2	d_{2c}, d_{2t}	4-5-6
3	d_{3c}, d_{3t}	7-8-9
4	d_{4c}, d_{4t}	10-11-12
5	d_{5c}, d_{5t}	13-14-15
6	d_{6c}, d_{6t}	16-17-18
7	d_{7c}, d_{7t}	19-20-21
8	d_{8c}, d_{8t}	22-23-24
9	d_{9c}, d_{9t}	25-26-27-28-29-30-31-32

Table 8.1: *Stringer lines for structure 2 using practical design restrictions.*

For evaluating the optimisation regarding **E** og **M** structure 2 has been calculated without these restrictions which results in two optimisations cases:

- Case 1: Material optimisation
- Case 2: Material optimisation regarding practical design restrictions

The plots of the collapse modes from `Opti_String` are shown in Figure 8.3 for case 1 and 2, respectively.



a) Case 1: Material optimisation

b) Case 2: Material optimisation regarding **M** og **E**

Figure 8.3: *`Opti_String`: Collapse mechanism for structure 2 without and with **M** and **E** as practical design restrictions.*

The matching output lists from `Opti_String` are shown in Figure 8.4 and 8.5. The design parameters for case 1 are listed for each element, and for case 2 the parameters are shown for the stinger lines and the appurtenant stringers are listed in Table 8.1. All shear areas are represented by d_{9c} and d_{9t}. The values are equal because the areas are set to carry the same shear stress in both positive and negative. The design parameters for stringers indicate the stringer lines with tension, which also can be seen in the list showing the stringer reinforcement. The necessary reinforcement for all stringers are shown in order to enhance where tension occur, i.e. stringers that require reinforcement. It must be noticed that the reinforcement in a stringer line must be designed for the largest amount of reinforcement in the appurtenant stringers, for example stringer line 8 for both cases must be designed using the reinforcement calculated for stringer 24, $A_{s(24)}$.

A comparison of the necessary reinforcement indicates that case 2 are the most expensive to perform. This is caused by the extra added restrictions in the inequalities which *linprog* is limited by when performing the linear optimisation.

Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 2

1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 15

Material optimisation algorithm - 56 design variables
Static independent variables, N = 8

----- Design variables -----
d1_c = 6.806897e+03 d1_t = 0.000000e+00
d2_c = 7.500000e+04 d2_t = 0.000000e+00
d3_c = 7.500000e+04 d3_t = 0.000000e+00
d4_c = 1.037448e+05 d4_t = 0.000000e+00
d5_c = 1.037448e+05 d5_t = 0.000000e+00
d6_c = 1.350000e+05 d6_t = 0.000000e+00
d7_c = 0.000000e+00 d7_t = 0.000000e+00
d8_c = 1.082069e+05 d8_t = 0.000000e+00
d9_c = 1.350000e+05 d9_t = 0.000000e+00
d10_c = 5.553846e+04 d10_t = 5.501327e+04
d11_c = 5.953724e+04 d11_t = 4.866965e+04
d12_c = 6.787285e+04 d12_t = 5.836853e+04
d13_c = 9.724138e+03 d13_t = 0.000000e+00
d14_c = 1.360690e+05 d14_t = 0.000000e+00
d15_c = 2.940000e+05 d15_t = 0.000000e+00
d16_c = 1.039310e+05 d16_t = 0.000000e+00
d17_c = 1.039310e+05 d17_t = 2.241379e+04
d18_c = 0.000000e+00 d18_t = 2.241379e+04
d19_c = 0.000000e+00 d19_t = 1.136552e+05
d20_c = 0.000000e+00 d20_t = 1.136552e+05
d21_c = 0.000000e+00 d21_t = 0.000000e+00
d22_c = 0.000000e+00 d22_t = 0.000000e+00
d23_c = 0.000000e+00 d23_t = 1.136552e+05
d24_c = 0.000000e+00 d24_t = 2.940000e+05
d25_s = 6.500000e-02 d26_s = 7.580000e-01
d27_s = 7.580000e-01 d28_s = 0.000000e+00
d29_s = 0.000000e+00 d30_s = 1.053000e+00
d31_s = 1.053000e+00 d32_s = 9.470000e-01

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 0
A_s(2) = 0
A_s(3) = 0
A_s(4) = 0
A_s(5) = 0
A_s(6) = 0
A_s(7) = 0
A_s(8) = 0
A_s(9) = 0
A_s(10) = 120.03
A_s(11) = 106.19
A_s(12) = 127.35
A_s(13) = 0
A_s(14) = 0
A_s(15) = 0
A_s(16) = 0
A_s(17) = 48.9
A_s(18) = 48.9
A_s(19) = 247.97
A_s(20) = 247.97
A_s(21) = 0
A_s(22) = 0
A_s(23) = 247.97
A_s(24) = 641.45

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 42.433
A_s_x,y(26) = 495.95
A_s_x,y(27) = 0
A_s_x,y(28) = 689.154
A_s_x,y(29) = 619.937
A_s_x,y(30) = 689.154
A_s_x,y(31) = 786.959
A_s_x,y(32) = 786.959

----- Reactions [kN] -----
Load case 1:
Support 25 = -294
Support 26 = 55.618
Support 27 = -0
Support 28 = 114.557
Support 29 = -0
Support 30 = 116.546
Support 31 = 294
Support 32 = 133.278

Figure 8.4: `Opti_String`*: Output list for structure 2Ċase 1. Elements exposed to plastic strains are illustrated in Figure 8.6.*

Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 2

Stringer lines:
1: 1 2 3
2: 4 5 6
3: 7 8 9
4: 10 11 12
5: 13 14 15
6: 16 17 18
7: 19 20 21
8: 22 23 24
9: 25 26 27 28 29 30 31 32

1 load case

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 40

Material optimisation algorithm - 56 design variables
Static independent variables, N = 8

----- Design variables -----

d1_c = 7.500000e+04	d1_t = 0.000000e+00
d2_c = 1.350000e+05	d2_t = 0.000000e+00
d3_c = 1.350000e+05	d3_t = 0.000000e+00
d4_c = 4.432913e+04	d4_t = 9.067087e+04
d5_c = 2.715191e+05	d5_t = 0.000000e+00
d6_c = 1.200153e+05	d6_t = 0.000000e+00
d7_c = 0.000000e+00	d7_t = 1.199847e+05
d8_c = 0.000000e+00	d8_t = 2.714809e+05
d9_c = 1.500000e+00	d9_t = 1.500000e+00

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 0
A_s(2) = 0
A_s(3) = 0
A_s(4) = 0
A_s(5) = 0
A_s(6) = 0
A_s(7) = 0
A_s(8) = 0
A_s(9) = 0
A_s(10) = 165.92
A_s(11) = 197.83
A_s(12) = 166.56
A_s(13) = 0
A_s(14) = 0
A_s(15) = 0
A_s(16) = 0
A_s(17) = 0
A_s(18) = 0
A_s(19) = 261.78
A_s(20) = 261.78
A_s(21) = 163.68
A_s(22) = 3.31
A_s(23) = 265.09
A_s(24) = 592.32

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 6.479
A_s_x,y(26) = 530.182
A_s_x,y(27) = 6.612
A_s_x,y(28) = 654.629
A_s_x,y(29) = 654.462
A_s_x,y(30) = 654.629
A_s_x,y(31) = 981.818
A_s_x,y(32) = 654.462

----- Reactions [kN] -----
Load case 1:
Support 25 = -271.519
Support 26 = 28.965
Support 27 = -74.981
Support 28 = 120.377
Support 29 = 75.019
Support 30 = 119.316
Support 31 = 271.481
Support 32 = 151.342

Figure 8.5: *`Opti_String`: Output list for structure 2Ċase 2 using practical design restrictions. Elements exposed to plastic strains are illustrated in Figure 8.6.*

The list with elements exposed to plastic strains are illustrated in Figure 8.6. It is seen that some of the elements occurs twice which is a result of both compression and tension strains in each end of the element. This is especially the case for the elements around the hole due to the shear that must be transferred. The vertical stringers connected to the supports only occur once in the list for case 2 since they are only exposed to tension or compression. Tensile strains exist for element 15 and 18 while compression tension occur in element 21 and 24. These strains are caused by the maximum moment.

----- Elements exposed to plastic strains -----

Element 1	Element 1	Element 18	Element 18
Element 2	Element 3	Element 19	Element 19
Element 3	Element 4	Element 20	Element 20
Element 4	Element 5	Element 21	Element 21
Element 6	Element 7	Element 22	Element 22
Element 7	Element 8	Element 23	Element 23
Element 8	Element 9	Element 24	Element 25
Element 10	Element 10	Element 26	Element 27
Element 11	Element 11	Element 28	Element 29
Element 12	Element 12	Element 30	Element 31
Element 13	Element 13	Element 32	
Element 14	Element 15		
Element 16	Element 16		
Element 17	Element 17		

a) Case 1

----- Elements exposed to plastic strains -----

Element 1	Element 3	Element 27	Element 28
Element 4	Element 6	Element 29	Element 31
Element 7	Element 7	Element 31	Element 32
Element 8	Element 8		
Element 9	Element 9		
Element 11	Element 11		
Element 13	Element 15		
Element 16	Element 16		
Element 17	Element 17		
Element 18	Element 19		
Element 19	Element 20		
Element 20	Element 21		
Element 22	Element 24		
Element 25	Element 26		

b) Case 2

Figure 8.6: *Elements exposed to plastic strains for case 1 and 2.*

The occasion for plastic strains in stringer 3 and 6 can be found in their stringer forces and their adjacent shear areas. From the lists for both cases it is seen that the shear stresses in area 29 and 32 are significant larger than the shear stress in area 27. This stress distribution induce stringer 3 and 6 to carry the load which results in the plastic strains.

With the linking of the design variables for all shear areas in case 2 the system may give a low priority to the strengths in area 28 and 29, as an increase would concern all areas. Instead the strength in the stringers increase. This can be controlled by the weighting parameters which indicate the priority of the elements. Compared with case 1, where the shear areas are not linked, the system increase the strength in the two shear areas because they do not affect the strengths of the other areas. This results in a collapse acting more like a rigid body rotation but still a shade of shear failure occur. The extra restrictions explain why plastic strains do not occur in exact same elements.

By comparing case two with the hand calculations in section 3.2 the cost reductions is 14 % when the amount of reinforcement decrees from 44 kg to 38 kg, cf. Appendix Table A1.3 and A4.1.

8.2 Load Combinations

Structures are always subjected to several load cases and a material optimisation problem with one load case does only in few cases give the optimum solution. Several load cases can be managed in the material optimisation for either lower bound or upper bound method.

Load Combinations Formulated Using Lower Bound Method

The LP problem formulated in equation (5.7) for material optimisation based on lower bound method can take load cases in to consideration by multiplying the input matrices and vectors by the number of load cases, n. Hereby, the size of the flexibility, $\mathbf{H}$, and constraint matrix, $\mathbf{C}$, are multiplied with n which results in $\boldsymbol{\beta} \cdot n$ variables.

This means that the LP problem is growing with the size n^2. The stress parameters are independent of each other and the mutual variables for the different load cases are the design variables.

By having two load cases, **I** and **II**, equation (5.7) is rewritten to equation (8.7) for material optimisation based on the lower bound method. Notice, all vectors and matrices increase to double size. Only the load vector, **R**, changes for each load case.

$$
\begin{aligned}
\text{minimise :} \quad & \mathbf{w}^T \ \mathbf{d} \\
\text{restrictions :} \quad & \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta}_I \\ \boldsymbol{\beta}_{II} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_I \\ \mathbf{R}_{II} \end{Bmatrix} \\
& \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta}_I \\ \boldsymbol{\beta}_{II} \end{Bmatrix} - \begin{Bmatrix} \mathbf{C}_d \\ \mathbf{C}_d \end{Bmatrix} \mathbf{d} \leq \begin{Bmatrix} \mathbf{C}_0 \\ \mathbf{C}_0 \end{Bmatrix} \\
& \mathbf{d} \geq 0
\end{aligned}
\tag{8.7}
$$

Implementing of the practical design restrictions from section 8.1 leads to an expanding of the expression. For two load cases with **M** and **E** the expression is illustrated in Appendix Figure A4.4.

Example - Structure 2

In this example structure 2 is subjected to two load cases, cf. Figure 8.7. Furthermore, the linking of materials and strength parameters will be taken in to account by the matrices **E** and **M**.

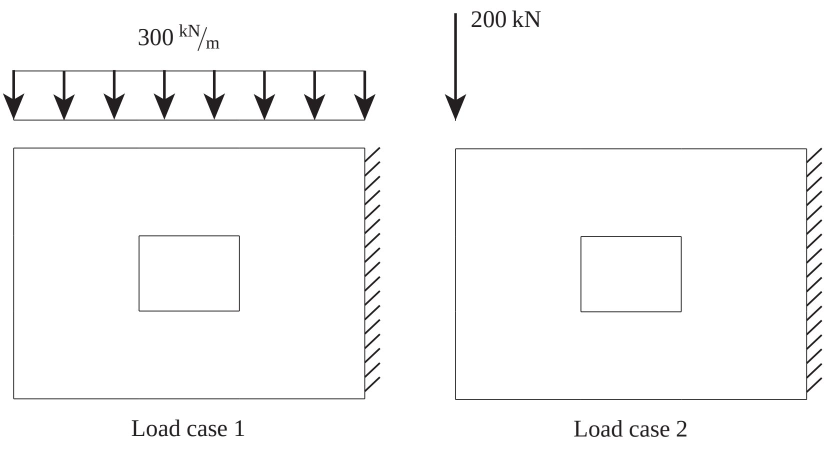
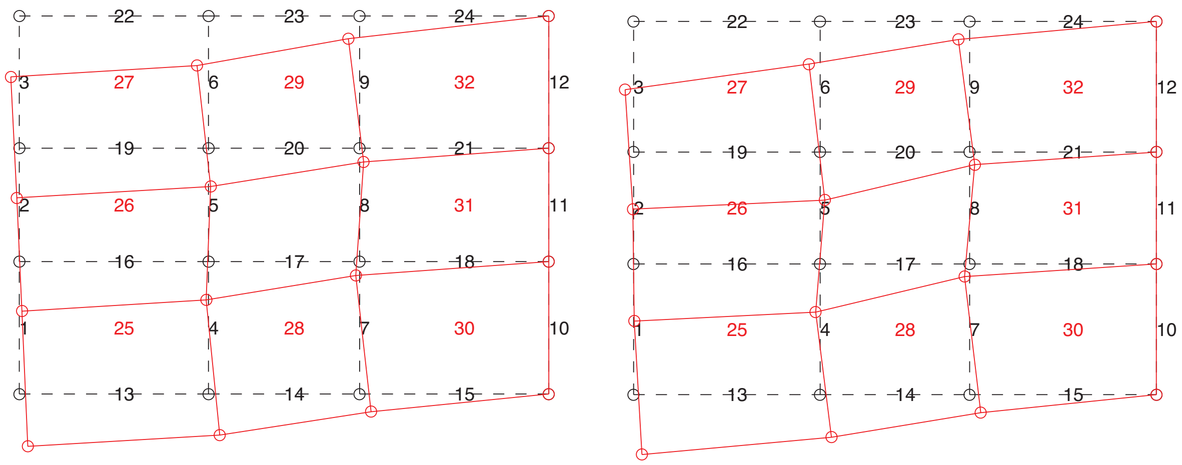


Figure 8.7: *Structure 2 subjected to two load cases.*

Load case 1 consist of the uniform load from the previous examples where case 2 is a concentrated nodal force.

The generated geometry plot from `Opti_String` is illustrated in Figure 5.1. The two collapse mechanisms are illustrated in Figure 8.8 for load case 1 and 2, respectively.



Load case 1　　Load case 2

Figure 8.8: *`Opti_String`: Collapse mechanism for structure 2. Load case 1 and 2.*

The two collapse modes differs because load case 1 affects the structure more by bending compared to load case 2 where the point load more results in shear effect. This explains why the middle part of the structure deforms different for load case 2 compared to load case 1 since the middle part is weak for shear. However, shear is still affecting the collapse mode of load case 1 and in similar way is bending affecting the collapse mode for load case 2. It appears from the collapse mode for load case 2 that the point load affects stringer 3 and area 27 significant.

The output list from `Opti_String` is shown in Figure 8.9. From the list of reinforcement in the stringers it is evident in which stringers tension occur. Each load case which leads to a moment that must be taken by the horizontal stringers at the supports by compression and tension, which is expected.

From the dual inequalities the elements exposed to plastic strains are found. The two load cases leads to plastic strains in 29 out of 32 elements where some of the elements are exposed to plastic strains in both compression and tension. Only stringer 2, 14 and 23 do not experience plastic strains. By comparing with the example where only one load case is added previous in this chapter it is concluded that adding load cases to structure 2 the number of elements exposed to plastic strains increase as load case 2 loads the structure in another way.

The design variables are listed for each stringer line. The variables are based on the two load cases thus, the maximum design variables for each stringer line is shown. For studying which load case results in which design variables each load case must be calculated separately.

The design variables shown in the list reflect the maximum design variables for the two load cases.

The reactions from Figure 8.9 shows the reactions for each load case and for a design of supports the maximum reaction forces must be used.

```
Opti_String
Daniel Refer and Flemming Højbjerre Sørensen
Last modified 10/6 2012
Lower bound method

Structure 2

Stringer lines:
1: 1 2 3
2: 4 5 6
3: 7 8 9
4: 10 11 12
5: 13 14 15
6: 16 17 18
7: 19 20 21
8: 22 23 24
9: 25 26 27 28 29 30 31 32

2 load cases

----- Strength parameters and safety factors -----
f_s = 550 MPa
f_c = 25 MPa
gamma_s = 1.2
gamma_c = 1.45

Scale of deformation = 15

Material optimisation algorithm - 56 design variables
Static independent variables, N = 8

----- Design variables -----
Compression:          Tension:
d1_c = 2.000000e+05   d1_t = 0.000000e+00
d2_c = 1.350000e+05   d2_t = 5.636082e+04
d3_c = 1.350000e+05   d3_t = 3.202177e+04
d4_c = 4.177346e+04   d4_t = 9.322654e+04
d5_c = 2.752381e+05   d5_t = 0.000000e+00
d6_c = 1.194796e+05   d6_t = 0.000000e+00
d7_c = 0.000000e+00   d7_t = 1.205204e+05
d8_c = 0.000000e+00   d8_t = 2.752381e+05
d9_c = 1.500000e+00   d9_t = 1.500000e+00

----- Elements exposed to plastic strains -----
Stringers:
Element 1     Element 3
Element 4     Element 4
Element 5     Element 5
Element 6     Element 6
Element 7     Element 7
Element 8     Element 8
Element 9     Element 9
Element 10    Element 10
Element 11    Element 11
Element 12    Element 12
Element 13    Element 15
Element 16    Element 16
Element 17    Element 17
Element 18    Element 19
Element 19    Element 20
Element 20    Element 21
Element 22    Element 24
Shear areas:
Element 25    Element 26
Element 27    Element 28
Element 29    Element 30
Element 31    Element 32

----- Reinforcement for stringers [mm^2] -----
A_s(1) = 0
A_s(2) = 0
A_s(3) = 0
A_s(4) = 0
A_s(5) = 122.97
A_s(6) = 122.97
A_s(7) = 69.87
A_s(8) = 69.87
A_s(9) = 0
A_s(10) = 170.75
A_s(11) = 203.4
A_s(12) = 171.25
A_s(13) = 0
A_s(14) = 0
A_s(15) = 0
A_s(16) = 0
A_s(17) = 0
A_s(18) = 0
A_s(19) = 262.95
A_s(20) = 262.95
A_s(21) = 162.22
A_s(22) = 140.11
A_s(23) = 392.73
A_s(24) = 600.52

----- Reinforcement for shear areas [mm^2/m] -----
A_s_x,y(25) = 293.294
A_s_x,y(26) = 785.455
A_s_x,y(27) = 280.212
A_s_x,y(28) = 651.707
A_s_x,y(29) = 657.384
A_s_x,y(30) = 651.707
A_s_x,y(31) = 981.818
A_s_x,y(32) = 657.384

----- Reactions [kN] -----
Load case 1:
Support 25 = -270.85
Support 26 = 26.283
Support 27 = -75.65
Support 28 = 120.035
Support 29 = 74.35
Support 30 = 120.191
Support 31 = 272.15
Support 32 = 153.491
Load case 2:
Support 25 = -275.238
Support 26 = 24.904
Support 27 = -15.873
Support 28 = 66.667
Support 29 = 15.873
Support 30 = 66.667
Support 31 = 275.238
Support 32 = 41.763
```

Figure 8.9: *`Opti_String`: Output list for structure 2, material optimisation, practical design restrictions and two load cases.*

Load Combinations Formulated Using Upper Bound Method

For the upper bound method the material optimisation problem for two load cases, **I** and **II**, is formulated as equation (8.8). The expression is made by a mathematical conversion of equation (8.7) to the dual problem (Damkilde, 1995, page 35-36). In the conversion the practical design restrictions from section 8.1 are included. The expression is only shown to be aware of the existence and is not implemented in `Opti_String`. No physical interpretation of the expression is made.

$$\begin{aligned}
\text{maximise}: \quad & (\mathbf{R}^{I})^{T}\,\mathbf{V}^{I} + (\mathbf{R}^{II})^{T}\,\mathbf{V}^{II} - \mathbf{C}_0^{T}\,\Psi^{I} - \mathbf{C}_0^{T}\,\Psi^{II} + \mathbf{m}_0^{T}\,\mathbf{s} + \mathbf{e}_0^{T}\,\mathbf{s}^{+} \\
\text{restrictions}: \quad & \begin{bmatrix} \mathbf{H}^{T} & 0 \\ 0 & \mathbf{H}^{T} \end{bmatrix} \begin{Bmatrix} \mathbf{V}_{I} \\ \mathbf{V}_{II} \end{Bmatrix} - \begin{bmatrix} \mathbf{C}^{T} & 0 \\ 0 & \mathbf{C}^{T} \end{bmatrix} \begin{Bmatrix} \Psi^{I} \\ \Psi^{II} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\
& \mathbf{C}_d^{T}\,\Psi^{I} + \mathbf{C}_d^{T}\,\Psi^{II} + \mathbf{M}^{T}\,\mathbf{s} + \mathbf{E}^{T}\,\mathbf{s}^{+} \le \mathbf{w} \\
& \mathbf{s}^{+} \ge 0
\end{aligned} \tag{8.8}$$

where

$\mathbf{s}$, $\mathbf{s}^{+}$	Dual values of the design limitations

The dual variables of the restrictions of the internal plastic work provides the design variables, **d**. A physical interpretation of the dual variables of the design limitations, **s** and $\mathbf{s}^{+}$, are not possible, cf. (Damkilde, 1995, page 36).

Chapter 9

Working Procedure of Opti_String

This chapter covers the main structure of `Opti_String` *and describe the working procedure. Through the chapter references from different functions will be made to the relevant equations.*

The idea of the structure in `Opti_String` is to have a relative short and clear main file which include a number of functions and sub-functions, cf. Figure 9.3 for an illustration of the program structure. The main file and appurtenant functions are to be found on the CD, cf. Appendix A7.2.

For the main file four choices must be made:

- Type of calculation; load or material optimisation
- Import of data file
- Implementing of practical design restrictions
- Scale factor for plot of collapse mode

The overall structure of `Opti_String` is divided in three parts which individually is described in the following.

9.1 Input of Data

Geometry, loads, material parameters, thickness, supports etc. are specified in a separate data file, `data_i.m`, and imported to the main file. `data_i.m` also contains the weighting factors if `Opti_String` is performing material optimisation.

In the data file the user must specify the geometry by typing in node coordinates, stringer nodes, shear areas, which stringers encircle the different shear areas and which elements having identical strength values e.g. by defining stringer lines.

The following units are used:

$$
\begin{aligned}
\text{Length} &= \text{mm} \\
\text{Force} &= \text{N} \\
\text{Stress} &= \text{MPa}
\end{aligned}
$$

Material Strengths, `mat.m`

After specifying the safety factor and strength values for concrete and reinforcement, respectively, the matrix `mat.m` is generated with strength values for the different elements regarding Eurocodes. An example is shown in Figure 9.1 for structure 1. `mat.m` is a $n \cdot 2$ matrix where:

$$
\begin{aligned}
n_{rows} &= n_{stringers} + n_{shear\_areas} \\
n_{columns} &= 2
\end{aligned}
$$

$$\begin{array}{lrc}
 & & \text{mat} \\
\begin{array}{l} \\ \\ \\ \text{stringers} \\ \\ \\ \\ \text{shear} \\ \text{areas} \end{array} & \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \left[\begin{array}{rr} 2886{,}21 & 350{,}00 \\ 2886{,}21 & 350{,}00 \\ 2886{,}21 & 350{,}00 \\ 1924{,}14 & 350{,}00 \\ 1924{,}14 & 350{,}00 \\ 1924{,}14 & 350{,}00 \\ 1924{,}14 & 350{,}00 \\ \hline 4{,}96 & 4{,}96 \\ 4{,}96 & 4{,}96 \end{array}\right]
\end{array}$$

Figure 9.1: *`Opti_String`: Matrix `mat.m` containing material parameters for structure 1. The limits are in kN for stringers and MPa for shear areas.*

The first column represents the negative yield strength and the second column the positive yield strength. The values from `mat.m` can be equivalent to $\mathbf{C}_d$ when performing load optimisation, cf. Figure A3.3 and $\mathbf{m}_0$ when performing material optimisation by means of practical design demands, cf. Figure A4.3. Furthermore `mat.m` can be used fro calculating the efficient ratio.

The maximum allowable compression strength in a stringer, $F_{c,max}$, depends on the stringer height and consequently also the smallest neighbouring shear area, cf. equation (3.2). This is handled in the function `upper_limit.m` by looping over each stringer for finding the smallest height of the neighbour shear area.

The maximum allowable shear stress in the areas, τ_{max}, cf. equation (3.5) and Appendix A2 are generated in the function `Tau.m` with regard to the plastic concrete compression. $N_{c,max}$, F_t and τ_{max} are assembled in `mat.m` for each stringer and shear area.

9.2 Generation of Matrices and Vectors

A number of matrices and vectors are generated before LP is performed. The matrices and vectors needed for *linprog* are shown in Appendix A3 for structure 1.

Object Function, c

The object function, **c**, is a vector generated by `object_function.m` based on the number of stringers, shear areas and the type of calculation. **c** is assembled by a zero vector and extra rows depending on the optimisation, where:

$$\begin{aligned}
n_{zeroes} &= 2 \cdot n_{stringers} + n_{shear_areas} = n_{\beta} \\
n_{exstra_load} &= 1 \text{ corresponding to } \lambda \\
n_{exstra_material} &= 2 \cdot n_{stringers} + n_{shear_areas}
\end{aligned}$$

In case of material optimisation the weighting factors specified in `data_i.m` are put in continuation of the zero vector. The object function for load and material optimisation is illustrated in equation (5.2) and equation (5.8), respectively.

Flexibility Matrix, H

The flexibility matrix, **H** is assembled in the function `flex.m` depending on whether it is load or material optimisation to be performed. For material optimisation a number of zero columns are added corresponding to the number of weighting parameters in the object function. In `flex.m` design variables are linked if the optimisation is restricted by practical design demands in the form of the property matrix **E**, cf. equation (8.3).

The flexibility matrix for a single stringer is first set up, cf. equation (4.2). After this the function `assem.m` assembling the global flexibility matrix, cf. equation (4.6) where:

$$\begin{aligned} n_{rows} &= 2 \cdot n_{nodes} + n_{stringers} \\ n_{columns} &= 2 \cdot n_{stringers} + n_{shear\_areas} \end{aligned}$$

H is assembled by first looping over all stringers for node and stringer equilibrium and afterwards over shear areas for equilibrium of these. When assembling the stringers regarding shear stress the stringer length is taken in to account by the function `LengthOfStringers.m`. An example of a **H** matrix is shown in Appendix A3.2 for structure 1. After the matrix is established rows containing a support are removed. The same is done for the load vector to ensure the loads still are applied in the correct nodes.

Constraint Matrix, C

The constraint matrix, **C**, is generated by `inequalities.m` where:

$$\begin{aligned} n_{rows} &= 4 \cdot n_{stringers} + 2 \cdot n_{shear\_areas} \\ n_{columns} &= 2 \cdot n_{stringers} + n_{shear\_areas} + n_{variants} \end{aligned}$$

C is assembled by several for loops over $n_{stringers}$ and n_{shear_areas}. Four rows for each stringer represents lower and upper values for start and end of a stringer respectively where shear areas is described by two rows a lower and a upper value.

The difference between load and material optimisation for inequalities is expressed in $\mathbf{C}_d$. $\mathbf{C}_d$ is a vector for load optimisation and is generated in `variable_limit.m` where the values from `mat.m` are importet, cf. Figure 9.3. The length corresponds to n_{rows} in **C**.

For material optimisation $\mathbf{C}_d$ is a matrix generated in `inequalities.m` where:

$$\begin{aligned} n_{rows} &= 4 \cdot n_{stringers} + 2 \cdot n_{shear\_areas} \\ n_{columns} &= n_{variants} \end{aligned}$$

Load Combinations

The number of load cases are taken in to account by the function `load_combinations.m`. A for loop runs over the number of load cases for correct position of the matrices. This is important because the LP problem growing with the size n^2, where n is the number of load cases. cf. section 8.2. All the matrices is unchanged except the load vector **R** which changes for every load case.

9.3 Post Processing

Depending on whether the calculations in `Opti_String` are based on the lower or upper bound method the primal and dual results are interpreted different.

Lower Bound Method

When `Opti_String` uses *linprog* two vectors are generated, one with primal and one with dual variables. The vector **x** in chapter 6 and chapter 5 corresponds to the primal values containing the stress parameters $\boldsymbol{\beta}$ and the variables; λ for load optimisation and design parameters, **d**, for material optimisation, cf. Figure 9.2 a). The following are interpreted:

$$\begin{aligned}
\beta_{stringer} &= \text{Normal force in stringer [kN]} \\
\beta_{areas} &= \text{Shear stress in area [MPa]} \\
\lambda &= \text{Load parameter} \\
d &= \text{Design variables}
\end{aligned}$$

Regarding the primal variables for lower bound following are interpreted:

$$n_{rows} = 2 \cdot n_{stringers} + n_{shear\_areas} + n_{variants}$$

The dual variables contains first inequalities representing the plastic strains and therefore show which elements are exposed to these. Figure 9.2 b) illustrated the vector *ShadowPrice* generated by `Opti_String`. More over does the dual variables contain displacements for nodes and stringers, respectively. These are used in the function `draw_collapse.m` for plotting the collapse mode. Regarding the dual variables for lower bound following are interpreted:

$$\begin{aligned}
\text{Dual inequalities: } n_{rows} &= 4 \cdot n_{stringers} + 2 \cdot n_{shear_areas} \\
\text{Dual equalities: } n_{rows} &= (2 \cdot n_{nodes} - n_{supports}) + n_{stringers}
\end{aligned}$$

All the lists from `Opti_String` are generated in `print_results_LB.m`. If material optimisation is performed and three weighting factors exist, then they represents two design parameters for stringers and one for shear areas

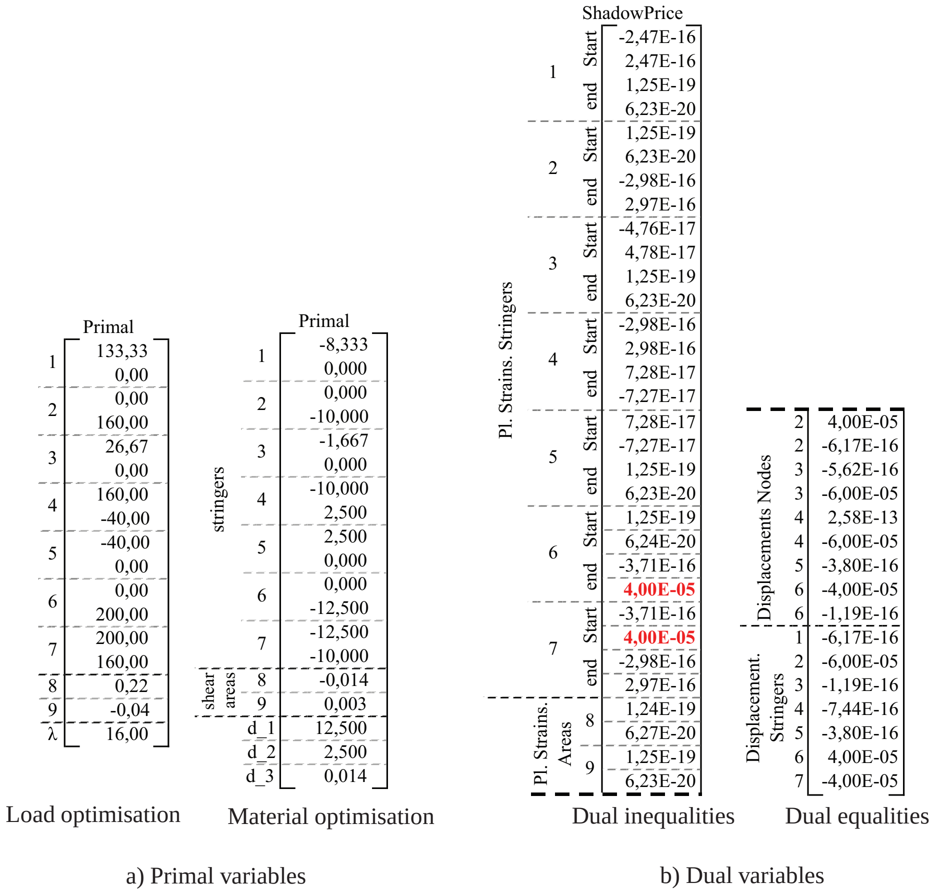


Figure 9.2: *`Opti_String`: Primal and dual variables for structure 1. The elements with plastic strains are marked with bold, read. The vectors are split in two because of the length.*

If material optimisation is performed regarding practical design restrictions, cf. chapter 8.2, then design variables exist for each variant, e.g. each stringer line and all the shear areas as a group.

The $\beta_{stringer}$ values are used in the function `stringer_shear_result.m` to determine maximum and minimum forces in stringer lines and β_{shear_areas} as stresses in areas. A for loop over $n_{stringers}$ is first established and afterwards over n_{areas}. The variables, such as λ or $\mathbf{d}$, are generated in `var_print_LB.m`.

The elements exposed to plastic strains are listed by the function `plastic_strains.m` where a for loop runs from one to the number of elements with a shadow price larger than zero.

The reactions are used as a control parameters by comparing these with the applied load. Reactions are determined in the function `reactions.m` where a for loop runs over the length of the vector `Support` given in the `data_i.m`.

The necessary reinforcement in the areas are calculated in `shear_reinforcement.m` and runs over a for loop by the number of shear areas.

Upper Bound Method

The output from `Opti_String` are treated opposite regarding the link between lower and upper bound in LP, cf. chapter 6. This means that the primal values gives the collapse mechanism and elements exposed to plastic strains. Similar does the dual values give the stress parameters for stringers and areas respectively.

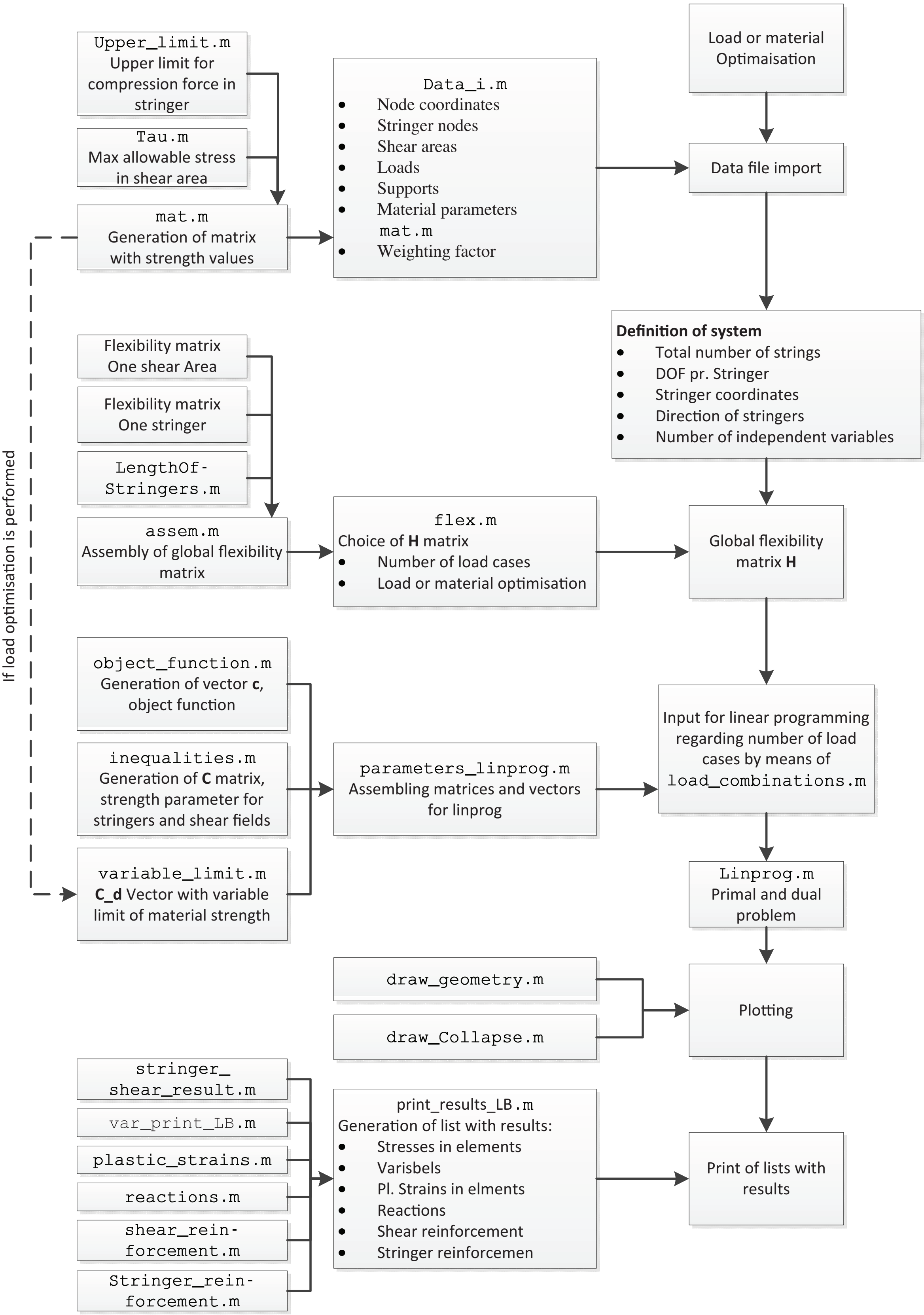


Figure 9.3: *Composition of `Opti_String` in MATLAB.*

Chapter 10

Application of `Opti_String` for Complex Structure

This chapter illustrates the practicability of `Opti_String` by analysing a real complex structure. The calculations are based on the lower bound with linking of elements and several load cases, cf. chapter 8. Based on a meeting with the engineers at Grontmij in Århus a key element is chosen. Thus, structure 4 is analysed in this chapter subjected to three load cases.

The building is a hotel for patients connected to the Queen Ingrid Hospital in Nuuk, Greenland. The hotel is designed by Grontmij in Århus and is a eight-storey building including one floor for basement. A two-storey administration building connects the hotel with the hospital, cf. Figure 10.1.

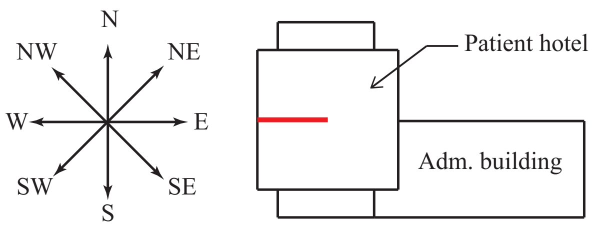


Figure 10.1: *Orientation of patient hotel. Structure 4 is marked with red.*

A 3D view of the main structure is shown in Figure 10.2. The model is from Robot Structural Analysis (Robot) and made by the engineers at Grontmij (Grontmij A/S Aarhus, 2011). Figure 10.3 shows the middle wall which is one of the key elements in the structure. This wall is calculated in `Opti_String` and are referred to as structure 4, cf. section 1.2.

The applied safety factors and material strengths are listed in Table 10.1.

Safety factor for reinforced concrete normal control class	γ_c	1.45
Safety factor for reinforcing steel	γ_s	1.2
Compressive strength of concrete	f_c	15 MPa
Yield strength of reinforcement	f_y	550 MPa

Table 10.1: *Safety factors and material parameters for structure 4.*

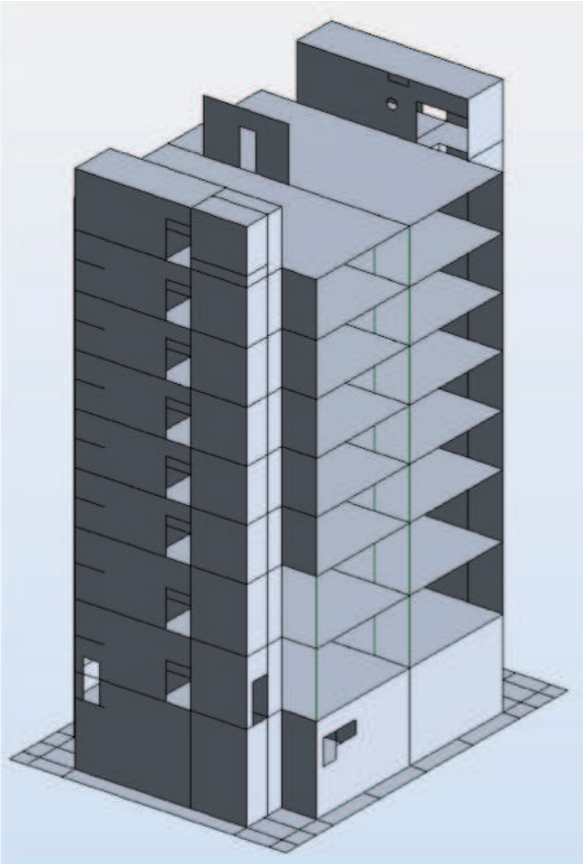


Figure 10.2: *3D view of the patient hotel from south east. Model from Robot, cf. (Grontmij A/S Aarhus, 2011).*

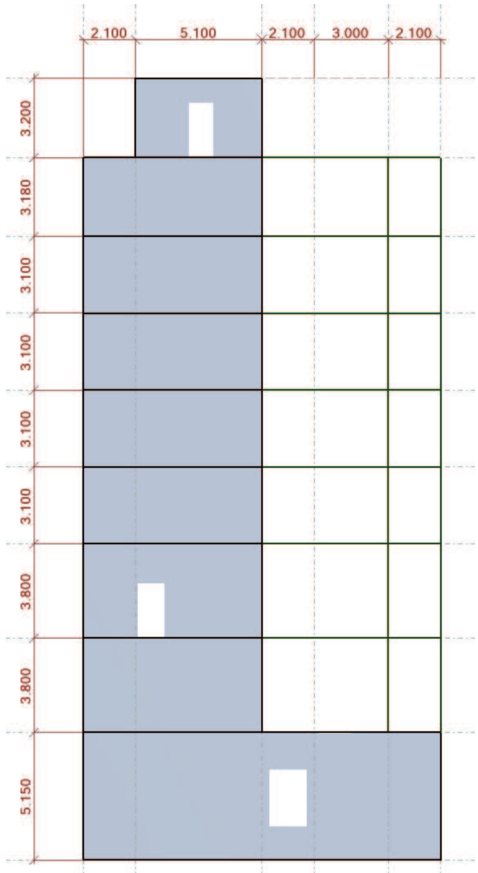


Figure 10.3: *Construction drawing from Robot showing structure 4.*

Structure 4 is shown in Figure 10.3 with dimensions according to drawings from Revit Structure made by Grontmij.

L1	Wind from west	W_W
L2	Wind from east	W_E
L3	Snow on roof and balcony	S
L4	Self weight	G
L5	Payload	N
LC1	L1 + L4	$W_W + G$
LC2	L2 + L4	$W_E + G$
LC3	L3 + L4 + L5	$S + G + N$

Table 10.2: *Loads and load combinations for design criteria for structure 4.*

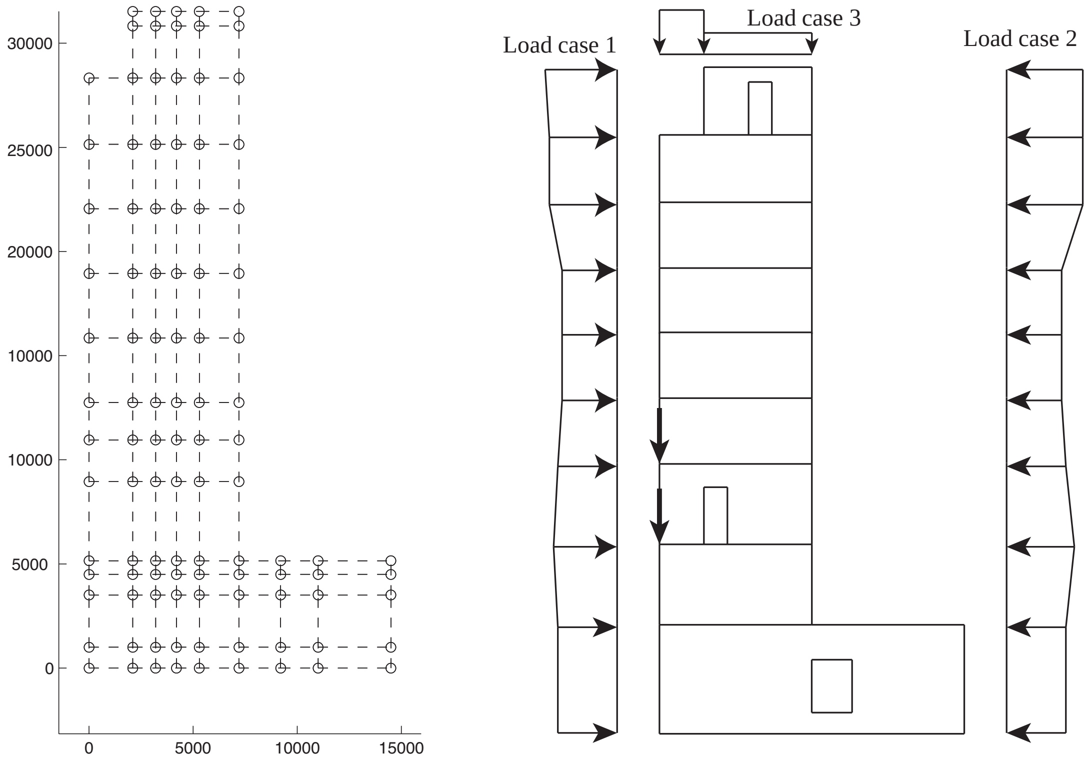
The Robot model is subjected to 60 load cases and with basis in these three critical load cases is chosen and listed in Table 10.2 with the load combinations as well, cf. (Grontmij A/S Aarhus, 2011).

Structure 4 is designed for these three load combinations, $LC1$, $LC2$ and $LC3$, when performing practical material optimisation, cf. chapter 8. The three load cases are illustrated in Figure 10.4.

When wind load is applied the forces are transmitted from the façade through the construction joint thus, the wind load are converted to point loads instead of a uniform distributed line load.

For all three combinations the self weight of structure 4 is added. The payload is combined with the snow load due to the use phase, while it is omitted in the combinations of wind for illustrated a critical situation during the erection.

The combination with snow takes snow accumulation on the roof into account and in addition two balconies are connected at the two construction joints in structure 4 whereupon snow accumulation also is taken into account.



a) `Opti_String`: Stringer system

b) Load cases

Figure 10.4: *Load case 1 to 3 and stringer system for structure 4.*

The generated geometry plot from `Opti_String` is illustrated in Figure 10.4. A detailed figure with numbers for stringers and shear areas is illustrated in Appendix Figure A6.2.

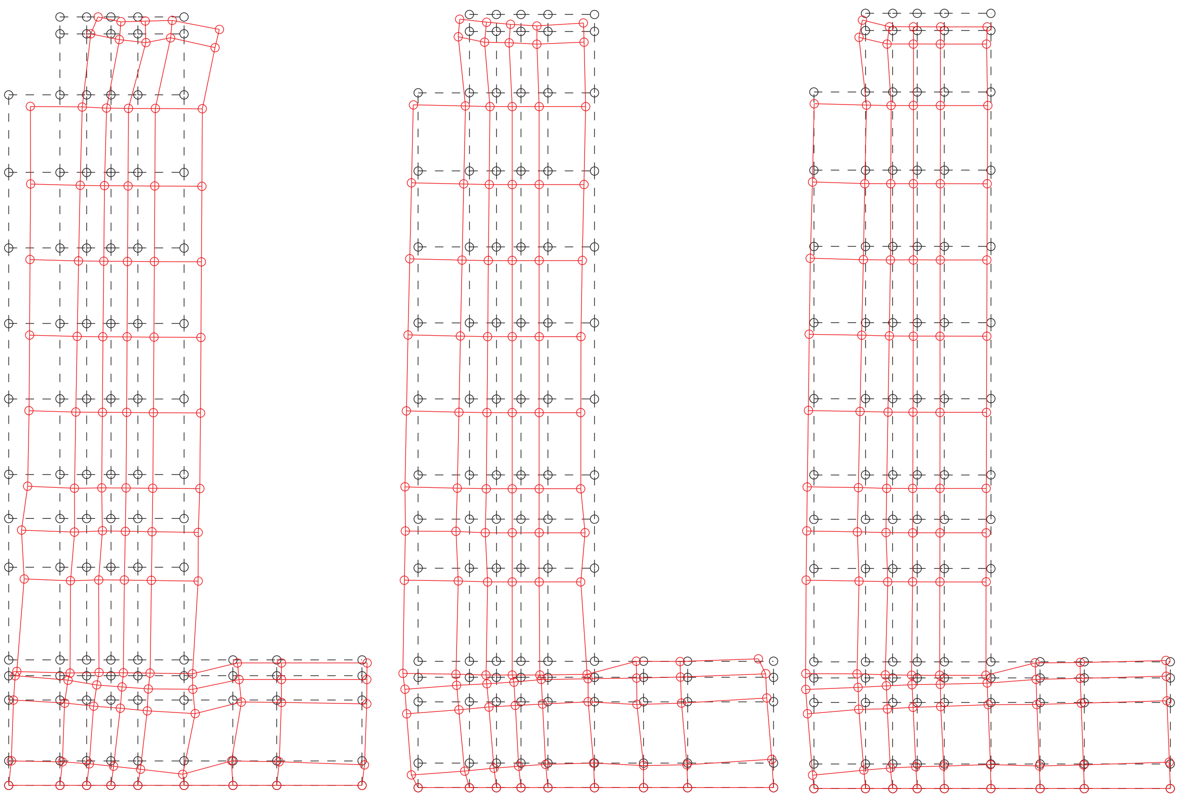
The purpose of this calculation is to evaluate the meaning of practical design demands and several load cases. Therefore the normal generated output list from `Opti_String` is omitted due to the length.

----- Design variables -----

Compression:	Tension:		
d1_c = 9.999469e+05	d1_t = 1.376800e+01	d13_c = 2.213440e+02	d13_t = 3.039266e+03
d2_c = 8.998019e+04	d2_t = 5.690888e+04	d14_c = 7.469644e+04	d14_t = 4.469284e+03
d3_c = 2.853214e+05	d3_t = 3.709800e+01	d15_c = 1.600285e+04	d15_t = 1.379431e+04
d4_c = 3.624236e+05	d4_t = 4.257400e+01	d16_c = 4.469700e+01	d16_t = 1.354540e+04
d5_c = 6.002793e+05	d5_t = 5.149100e+01	d17_c = 1.400287e+04	d17_t = 1.471820e+02
d6_c = 9.998623e+05	d6_t = 1.550000e+01	d18_c = 1.300335e+04	d18_t = 6.689000e+01
d7_c = 6.423332e+04	d7_t = 2.009776e+03	d19_c = 1.300507e+04	d19_t = 6.178300e+01
d8_c = 1.161534e+04	d8_t = 9.459440e+02	d20_c = 1.301018e+04	d20_t = 6.614400e+01
d9_c = 1.737913e+04	d9_t = 5.050694e+03	d21_c = 2.800747e+04	d21_t = 2.994800e+01
d10_c = 1.406789e+04	d10_t = 1.898183e+04	d22_c = 5.349577e+04	d22_t = 2.239200e+01
d11_c = 1.269917e+04	d11_t = 6.531344e+03	d23_c = 4.444923e+03	d23_t = 1.324943e+04
d12_c = 4.473017e+04	d12_t = 2.990493e+04	d24_c = 2.700348e+04	d24_t = 3.746900e+01
		d25_c = 2.463000e+00	d25_t = 2.463000e+00

Figure 10.5: *Opti_String: Design parameters for structure 4.*

When Opti_String performs material optimisation for the three load combinations a collapse plot is shown for each combination, cf. Figure 10.6. As expected tension occur in all the vertical stringer lines which is illustrated in Figure 10.5 for the design parameters. Furthermore, it is seen that all the vertical stringers are exposed to compression which must be due to load combination 3.



Load case 1 Load case 2 Load case 3

Figure 10.6: *Opti_String: Collapse mechanisms for structure 4 illustrating LC1, LC2 and LC3.*

Influence of Material Limits and Linking Elements

The influence of introducing material limits by matrix **M** and linking elements by matrix **E** is presented in section 8.1. The meaning of these extra restrictions is analysed for structure 4. Table 10.3 illustrates the importance of practical demands regarding the material use. The results are shown for load combination one.

By introducing extra restrictions the necessary reinforcement increase, which is in in agreement with section 8.1 where extra restrictions increase the price of the structure.

The significant increase of reinforcement may be found in the size of the structure, which contributes to long stringer lines which must be reinforced. This is especially the case for the shear areas which must be equally due to the same design parameter.

It is clear that dividing stringer lines into sub-stringer lines and grouping shear areas are obvious. This will properly lead to a decrease of the total reinforcement amount.

	Reinforcement
No practical demands	1369 kg
Use of practical demands	3226 kg

Table 10.3: *Necessary reinforcement for by means of practical demands.*

Importance of Several Load Cases

By including several load cases in the optimisation it is expected that the total amount of reinforcement increase, which is illustrated in Table 10.4. Adding extra load cases will lead to other stress distributions for which extra reinforcement are needed.

	Reinforcement
1 load case	3226 kg
3 load cases	4015 kg

Table 10.4: *Necessary reinforcement for one and three load cases.*

Weighting of Stringers

The efficiency ratios, RAT, are calculated for the stringers in structure 4. An evaluation of these indicates that stringer line 3, 4 and 5, cf. Figure 10.7, are high utilised in their lower parts, cf. Table 10.5. In the optimisation all the elements are weighed with the same factors, cf. Table 10.5, thus, no considerations of the price of the elements are incorporated.

Two initiatives are introduced to decrease the efficiency ratio. First, each of the three stringer lines are split up in two, for example 3_{lower} and 3_{upper}, in order to decrease the design variables of the upper parts of the stringer lines to save material costs. By grouping of sub-stringer lines within a continuous stringer line the connection must ensure the forces to be transmitted. This is most critical in tension where sufficient additional reinforcement must ensure a transferable connection.

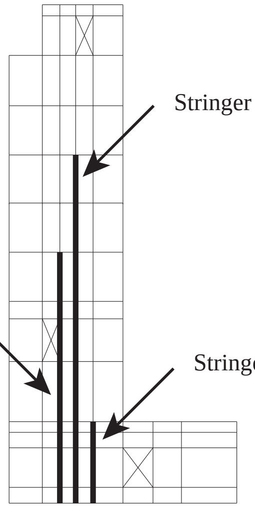


Figure 10.7: *Lower parts of stringer line 3, 4 and 5 are marked.*

Stringer line	Stringer	RAT [%]	Weighting factor [-]	RAT [%]	Weighting factor [-]
	27	99.93	1	39.92	1.1
	28	99.93	1	39.92	1.1
	29	99.93	1	39.92	1.1
	30	99.93	1	39.42	1.1
3_{lower}	31	92.64	1	32.63	1.1
	32	79.33	1	28.80	1.1
	33	85.84	1	33.07	1.1
	34	75.99	1	29.62	1.1
	41	99.97	1	40.88	1.15
	42	99.97	1	40.88	1.15
	43	99.97	1	40.88	1.15
	44	99.97	1	40.88	1.15
	45	92.68	1	33.59	1.15
4_{lower}	46	92.67	1	33.59	1.15
	47	96.43	1	37.43	1.15
	48	92.61	1	33.59	1.15
	49	85.53	1	30.62	1.15
	50	72.37	1	27.65	1.15
	55	73.51	1	41.61	2.75
	56	73.51	1	41.61	2.75
5_{lower}	57	73.51	1	41.61	2.75
	58	73.51	1	41.61	2.75

Table 10.5: *Efficiency ratios of lower parts of stringer line 3, 4 and 5 before and after weighting.*

Secondly, the weighting factors of the lower parts of the three stringer lines are increased, cf. Table 10.5. An increasement of the factors indicates that the price of each stringer increases and, as the target of the optimisation is to minimise the materials, the system decrease the utilisation of the mentioned stringer lines. The efficiency ratios after increase of the weighting factors are shown in Table 10.5.

By increasing the weighting factors of the three stringer lines the forces in the structure finds another way to the supports. This leads to increased utilisation of other stringers thus, incorporating of weighting factors is an iteration process until acceptable efficiency ratios are achieved for all elements.

The use of weighting factors must not lead to a state where the target is to reach a specific efficiency ratio, for example 80%. If this is the case the material limits specified in the vector containing material parameters, $\mathbf{m}_0$, in equation (8.6) should be decreased to the requested efficiency ratio.

Node coordinates, stringers, shear areas etc. are specified in a data file for structure 4. For a structure of the size of structure 4 the data entry is time-consuming and chance of typing errors present. To avoid this a code for importing a geometry file and convert it into a data file must be developed.

Node coordinates and stringers can fairly easy be exported from a CAD based program, e.g. AutoCAD. The difficult part of converting the geometry file to data is to describe the connection between different elements, e.g. that four combined stringers in a square defines a shear area or a hole in the structure.

Chapter **11**

Comments On Practical Applications

The practical applicability of `Opti_String` and the stringer method is commented. Some of the topics mentioned in the following are based on considerations and discussions through out the project.

The stringer method is an efficient tool for optimisation of structures for both load and material optimisation.

Load optimisation is useful for finding the maximum load bearing capacity for reinforced concrete structures. This ability is well suited in situations where the design of a structure is fixed and the load bearing capacity must be known for example in refurbishment of buildings where a structure is exposed to an increased load.

The material optimisation is useful in design processes, especially in the beginning of the design phase. By specifying the position of stringers and shear areas the method finds the optimum design of the structure for a given load and regarding restrictions of both material and practical kind. The practical restrictions regarding design parameters are especially applicable for shear areas as all the areas then must be reinforced with respect to the same stress according to practice.

`Opti_String` is based on the stringer method formulated in a FE concept and afterwards linear optimisation algorithm finds an optimum solution for ultimate limit state (ULS). This optimisation is not suitable for a hand calculation using the stringer method.

The plasticity theory provides an economic advantage compared to an elastic calculation because the plasticity theory permits development of collapse mechanisms. When using a plastic stress distribution a plastic mode of operation of the structure must be ensured by development of sufficient plasticity in the reinforcement. Thus, the yield in the reinforcement occurs to a certain extend before other failure conditions affect the ductile failure. Eurocode sets up requirements that must be satisfied to ensure ductile behaviour, cf. (EN 1992-1-1 DK NA, 2007, pp. 15-16). The following demands are taken in to account in `Opti_String`:

- Ductile behaviour of the reinforcement ensures establishment of the expected collapse mode cf. (EN 1992-1-1, 2007, Table C.1)
- No stress increase after yield stress which is obtained by an idealised strain-stress curve

With regard to mode of operation further demands must be respected and can if possible be implemented in a further development of `Opti_String`:

- Satisfying the expression $\frac{1}{3}A_{sE} \leq A_{sP} \leq 3A_{sE}$ for the plastic reinforcement area, A_{sP}, in proportion to the elastic reinforcement area, A_{sE}
- The minimum requirement of reinforcement must be obeyed and the reinforcement must yield in failure, e.i. a normally reinforcement ratio
- The plastic stress distribution may not differ significant from an elastic distribution for a cracked cross-section (EN 1992-1-1 DK NA, 2007, p. 16)

The requirements ensure sufficient rotation capacity, which is assumed in the plasticity theory, where large stress redistributions occur.

By compliance of the above mentioned requirements both ULS and SLS are satisfied.

Because the program uses plasticity theory the size of the deformation of the structure is unknown thus, the requirement from the standard regarding for example deflection in SLS can not be controlled. This is ensured with the above mentioned demands.

The width of the stringers is designed according to current requirements and the maximum allowed shear stress in `Opti_String` is limited to satisfy the requirement in (EN 1992-1-1 DK NA, 2007, pp. 16 - 17). To ensure the deformation capacity for bending the concrete compression strength is calculated using a factor of efficiency according to (EN 1992-1-1 DK NA, 2007, 5.102NA).

The reactions calculated in `Opti_String` are only based on the stringer forces. The only restrictions on these forces are the material limits specified in the data file thus, no limits are set up for the supports. In reality the strength of supports are restricted by soil conditions, type of foundation, space, surrounding elements etc. All these factors may be implemented in the calculation by adding one stringer with appropriate limits for each support which only is connected to the node of the support. Adding stringers at supports improve the practical applicability of `Opti_String` but more restrictions lead to a less optimum solution, cf. chapter 8. Thus, the engineer must judge the number of practical restrictions in proportion to a optimum solution.

In the stringer method loads are converted and applied as concentrated point loads in either end nodes or mid-side nodes. Thus, the distance between the nodes affects the size of the point load.

However, many point loads, corresponding to a fine stringer net, are not preferable because it entails deviation from the principle of concentrated reinforcement in the structure. In design situations the engineer must carefully judge the conversion of line loads to point loads in order to meet the principle of concentrated reinforcement, which must be interpreted as a contribution to the practical applicability of `Opti_String`, and at the same time avoid large concentrated loads.

Interaction among programs for different professions is used extensively in reality. Thus, models in the design phase are often made in commercial programs by the architect, for example Autoesk Revit, and afterwards imported into the calculation software, for example Robot Structural Analysis.

In `Opti_String` the geometry must be defined by a data file which is imported into `Opti_String` after which the calculation is performed. Because `Opti_String` imports a separate data file it is prepared for enhancement involving interaction with other BIM models. For example importing a CAD based model by making a syntax for converting the output file from another program into the format of `Opti_String`.

Horizontal and vertical stringers combined with rectangular shear areas are possible in `Opti_String`. The design of some structures can be improved by use of stringers not horizontal or vertical. An example is a console where it is preferable to take the tension in the cantilever part of the structure by a diagonal stringer. Thus, a refinement of `Opti_String` must involve handling of diagonal stringers which leads to introduction of triangular shear areas. Figure 11.1 illustrate the principle of strut and tie which is used for practical design problems by

(Schlaich and Schäfer, 1991). The basic principles can be used when introducing diagonal stringers. A useful formulation of a plastic triangular element is described, cf. (Sloan, 1988).

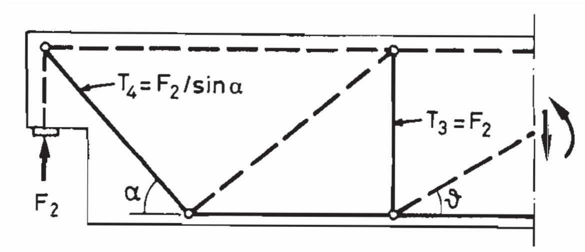


Figure 11.1: Strut and tie model for a console beam. Ties are marked with dash line. (Schlaich and Schäfer, 1991)

For some structures it may be preferable to use more than one design variable for a stringer line or shear areas. This can be present for large structures in order to save material costs. In case of a change in the amount of the reinforcement, for example in a stringer line, additional reinforcement must ensure transfer of forces between the reinforcement.

The weighting parameters from material optimisation can be used for manage which elements to be utilised more than others. If a specific efficiency ratio is desired one must influence the individual elements by change the material vector $\mathbf{m}_0$.

Chapter **12**

Conclusion

Opti_String is a finite element program developed in MATLAB based on the lower bound method of the plasticity theory. Opti_String is capable of optimising arbitrary concrete walls regarding the stringer method and includes following features:

- Optimisation of the load regarding given material strengths
- Optimisation of material consumption subjected to a specific load
- Practical material optimisation regarding linking of elements and several load cases.

The application of the stringer method leads to a number of optimisation options which induce additional optimisation potentials by implementing the theory by using a FE concept. The optimisation features are:

- Optimisation due to statically indeterminate stress parameters which in principle can be chosen freely
- Free choices for stress parameters lead to a plastic redistribution of the stresses
- Redistribution of stresses facilitates an optimum solution for carrying a given load
- The nature ensures accuracy of these stresses

Opti_String is verified by calculating a simple structure for both lower and upper bound and the practical optimisation as well. The upper bound calculations are based on a discretisation of the physical model and provides the exact same optimised results as the lower bound calculations. This indicates that an exact translation from the lower bound method to the upper bound method and opposite exists for the stringers and shear elements in the stringer method.

Opti_String could have been made by taking the upper bound method as starting point. Although, with this knowledge, it is still preferred that Opti_String uses the lower bound method because it is easier to imagine the principle of lower bound, especially for several load cases.

The optimisation of a complex structure by introducing practical design demands and several load cases results in a less optimal solution. The explanation is found in the extra restrictions which are added when linking elements and specifying material strengths. The results of these extra bands, which are impose to the optimisation, are seen in the necessary reinforcement area which increase with the amount of extra restrictions.

Opti_String is general formulated which makes it possible for an experienced programmer to write a code for linking arbitrary geometry from CAD to the data file in Opti_String. Subsequent it will be obvious to generate a code for returning stringer geometry, including stringer height to a calculation based CAD program such as Robot Structural Analysis 2011. The purpose is to verify the given structure regarding the extra design demands such as minimum reinforcement ect.

Alternativ Opti_String can be written in another format, in this way it can become a part of a program package for a commercial calculation program.

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