Action Investment Games

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June 7, 2012

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Title:
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Project period:
DAT10, spring semester 2012

Projectgroup:
D102F12

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Circulation: 4
Pages: 31
Finished on 07-06-2012

Synopsis:

We define the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets.

We have given a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We study the decision problem which is the foundation for reasoning about trade-offs and given a comprehensive analysis of complexity bounds for relevant problems within the formalism.

In the worst cases increases the complexity by two levels in the polynomial hierarchy compared to the complexity of the corresponding energy game.

Acknowledgments

I thank the advisors that helped me produce this work. Jiri Srba for always having time and his door open, and Kim G. Larsen for providing helpful insights and encouragement. I thank Jiri Barnat and Jana Tumova from Masaryk University for all the valuable discussions we had when I was in Brno which gave me to the initial idea to this project.

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Chapter 1

Introduction

It is the job of any computer scientist and software engineer to develop software, but software does not always behave correctly or as intended. There can be a lot of reasons for flawed software but by applying formal techniques such as modelling, model-checking and verification it is possible to find bugs, correct errors and prove correctness, even before development process starts. [2]

Embedded software and embedded systems surround us all in our daily lives, making modelling and verification of these systems relevant, not only for our well-being but also for the companies producing these systems as correcting a bug or a flaw in thousands of deployed systems can be extremely costly if not impossible [4].

It is typical for an embedded system to have limited or constrained resources and to interact with uncontrollable or even hostile environments. When adding this to the ever increasing demand on functionality and reliability for the lowest possible price, several interesting problems emerge.

An example of such a system is a simple mobile phone where the constraining resource is the size of the battery. The required functionality could be that the phone must be able to make some number of calls before the battery is empty. It is possible to install different antennas in the phone, a *good* and a *bad*, where the *bad* antenna is less expensive than the *god* antenna, but it consumes more energy when used. Intuitively the battery needs to be larger if the budget limits the phone to have the *bad* antenna, than if the budget was high then the phone could have the *good* antenna.

The formalism for studying this kind of trade-off between a constrained resource of energy and a budget is the main topic of this thesis.

We define and study Action Investment Games (AIG), which are energy games, extended with budgets and prices on actions. An energy game [5] is played by two players on a finite graph, where Player 1 wins if he has a strategy such that the accumulated energy is constrained within an given interval, and Player 2 wins if Player 1 loses.

The extension captures the intuition from the mobile phone example, as Player 1 can make an investment not costing more than his budget and remove behaviour from the resulting energy game. It also adds the possibility of Player 2 to making an investment not costing more than his budget and enable possible behaviour.

We give a comprehensive analysis on the complexity for the decision problem; *is it possible for Player 1 to make an investment costing less than his budget, such that for any Player 2 investment costing less than his budget, Player 1 is the winner of the resulting energy game within a given interval?*

Related work

Energy games that are constrained by either a lower bound or an interval have attracted much attention in recent years. Energy games were introduced with and without time in [5] and generalised with mean-payoff games in [6]. Energy games with multiple weights were introduced in [7]. However, none of these introduce the concept of investments or the trade-offs involved with these.

In [3] a dual-price schema for modal transition systems is studied, this introduces a long-run average cost and a hardware investment cost, it gives a trade-off scenario similar to the one studied in this thesis. However, modal transition systems lack the concept of two players, meaning that there are no uncontrollable states to model an environment and it lack the concept of constraining as it gives a trade-off over a long-run average.

To the best of our knowledge this Theses provides the first study of energy games extended with budgets and prices on actions.

Chapter 2

Preliminaries

We start by defining boolean formulas and the polynomial hierarchy.

2.1 Boolean Formula

Boolean variables $X = \{x_1, x_2, \ldots\}$ are assigned *true* or *false*, by a valuation $\upsilon : X \to \{true,\ false\}$. A boolean formula φ over X consists variables combined with operators $\vee$ (logic OR) , $\wedge$ (logic AND) and $\neg$ (logic NOT). A formula φ is satisfiable if there exists an assignment υ such that φ evaluates to *true* is under υ.

2.2 The Polynomial Hierarchy

Definitions and results in this section is based on [1, Chap. 5] and [8, Chap. 17].

Alternating Turing Machines (ATM) are an extension of Turing Machines with existential ($\exists$) and universal ($\forall$) states. Existential ($\exists$) states capture non-determinism and universal ($\forall$) states capture co-non-determinism. The ATM accepts from a existential state ($\exists$) if one successor accepts and from a universal ($\forall$) states if all successors accept. ATMs give a characterisation of the polynomial-time hierarchy (PH).

Definition 1. *Let Σ_k^P be the class of languages L accepted by an ATM that begins in a existential ($\exists$) state, alternates between existential and universal ($\forall$) states at most $k-1$ times.*

Definition 2. *Let Π_k^P be the class of languages L accepted by an ATM that begins in a universal ($\forall$) state, alternates between universal and existential ($\exists$) states at most $k-1$ times.*

Note that for the first level is Σ_1^P =NP and Π_1^P=coNP. It is also clear that $\Sigma_k^P \subseteq \Pi_{k+1}^P$ and $\Pi_k^P \subseteq \Sigma_{k+1}^P$, an illustration of this is shown in Figure 2.1.

Definition 3. *The polynomial hierarchy is defined as $PH = \bigcup_k \Sigma_k^P = \bigcup_k \Pi_k^P$*

Theorem 4. *[8, Prop. 17.1] $PH \subseteq PSPACE$*

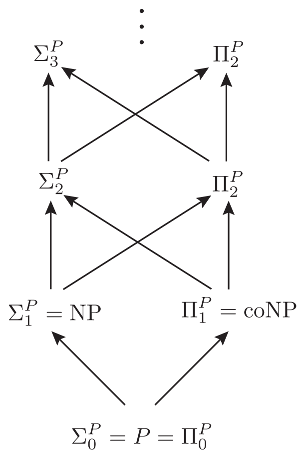


Figure 2.1: Relation between classes in the polynomial time hierarchy

We define quantified boolean formulas which have a finite number of alternations, formally Σ_i^P-SAT

$$\exists \vec{x_1} \forall \vec{x_2} \exists \dots \mathcal{Q} \vec{x_i} \varphi(\vec{x_1}, \vec{x_2}, \dots \vec{x_i})$$

and Π_i^P-SAT

$$\forall \vec{x_1} \exists \vec{x_2} \forall \dots \mathcal{Q} \vec{x_i} \varphi(\vec{x_1}, \vec{x_2}, \dots \vec{x_i})$$

where φ is a boolean formula, each $\vec{x_i}$ is a vector of boolean variables, and $\mathcal{Q}$ is $\exists$ or $\forall$ depending on whether i is odd or even.

QSAT problems is the decision problems whether a quantified boolean formulas is true for false. There are a QSAT problem represented on each level in the polynomial hierarchy.

Theorem 5. *[8, Thm. 17.10] The Σ_i^P-SAT problem is Σ_i^P-complete and the Π_i^P-SAT problems is Π_i^P-complete for all $i \geq 0$*

Chapter 3

Definitions

We defining Weighted Energy Games, runs, plays and strategies for this type of game, these are slight modification of the definitions in [5] and [6]. We then introduce Action Investment Game, which builds on top of Weighted Energy Games by giving each player a set of actions, a budget and introducing a cost for each action.

3.1 Energy Game

Definition 6 (Weighted Energy Game)**.** *A Weighted Energy Game (WEG) is a tuple* $\mathcal{G} = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0)$ *where*

- Q *is a finite state set,*
- Q_1, Q_2 *are a disjoint splitting of* Q, $Q = Q_1 \uplus Q_2$,
- Σ *is a finite set of actions,*
- $\rightarrow \subseteq Q \times \Sigma \times \mathbb{Z} \times Q$ *is a successor relation where* $(q, \sigma, z, q') \in \rightarrow$ *is written as* $q \xrightarrow{\sigma, z} q'$, *and*
- $q_0 \in Q$ *is the initial state.*

A Weighted Energy Game is represented by a graph where each node represents a state, the node is circled if it is in Q_1 and squared if it is in Q_2. Edges represent a transition between states and the edge is labeled with an action name and a weight. Figure 3.1 is an example of a WEG.

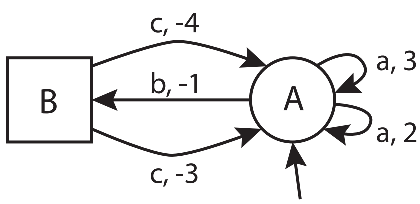


Figure 3.1: Example of a simple energy game

Informally a WEG is a board where an energy game is played on. The energy game is played by moving a token around on the board. The token starts in the initial state q_0. If the token is in a circle state, then Player 1 moves the token to a successor state. Likewise if the token is in a square state then Player 2 moves the token to a successor state. The sequence the token moves is called a run, this is formally defined as follows.

Definition 7 (Run)**.** *A run in $\mathcal{G} = (Q_1, Q_2, \Sigma, \rightarrow, q_0)$ starting from q_0 is a finite or infinite sequence $r = q_0 \xrightarrow{\sigma_0, z_0} q_1 \xrightarrow{\sigma_1, z_1} q_2 \xrightarrow{\sigma_2, z_2} \dots$ where $q_i \in Q$ and $q_i \xrightarrow{\sigma_i, z_i} q_{i+1} \in \rightarrow$ for all i.*

It is possible during run to determine the energy level γ in each step by summing the weight on each transition up to that step in the run. Two runs on the WEG is shown in Figure 3.1 and their energy level through each step is seen in Figure 3.2.

Given a finite run, $r = q_0 \xrightarrow{\sigma_0, z_0} \dots \xrightarrow{\sigma_{n-1}, z_{n-1}} q_n$ let the last state of the run be denoted $Last(r) = q_n$. A finite run r is *maximal* if $Last(r)$ has no successors.

Definition 8 (Valid Run)**.** *A run r is valid in the interval $[a, b]$, $a \in \mathbb{Z}$, $a \leq 0$, and $b \in \mathbb{Z} \cup \{\infty\}$, $b \geq a$ if $r = q_0 \xrightarrow{\sigma_0, z_0} \dots \xrightarrow{\sigma_{n-1}, z_{n-1}} q_n$ is a maximal finite run and*

$$a \leq \sum_{i=0}^{n-1} z_i \leq b \ ,$$

or if $r = q_0 \xrightarrow{\sigma_0, z_0} q_1 \xrightarrow{\sigma_1, z_1} \dots$ is a infinite run and

$$a \leq \sum_{i=0}^{n} z_i \leq b \text{ for all } n \geq 0.$$

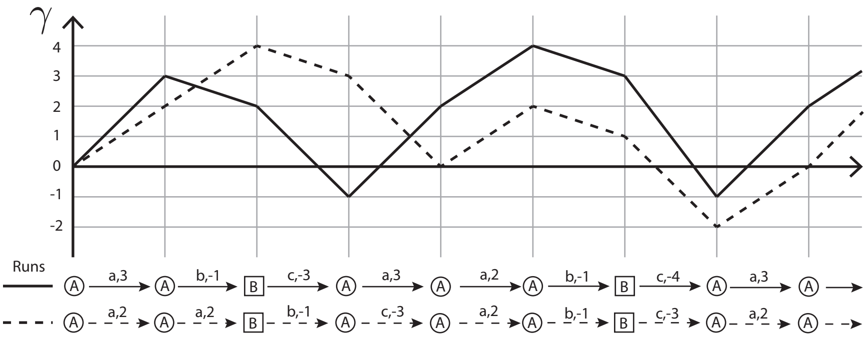


Figure 3.2: Run on the WEG in Figure 3.1

A run r is said to be winning for Player 1 and loosing for Player 2 if it is valid. A Play on a WEG can produce several different runs depending on what strategy each player uses.

A strategy δ for Player i, where $i = \{1, 2\}$, maps a finite non-maximal run r where $Last(r) = q_n$ and $q_n \in Q_i$ to a successor $q_n \xrightarrow{\sigma_n, z_n} q_{n+1}$. We denote a strategy for Player i as δ_i.

A play on a WEG using strategy δ_i produces either a infinite run $q_0 \xrightarrow{\sigma_0,z_0} \dots q_n \xrightarrow{\sigma_n,z_n} q_{n+1} \dots$ where for any n, if $q_n \in Q_i$ then $q_n \xrightarrow{\sigma_n,z_n} q_{n+1} = \delta_i(q_0 \xrightarrow{\sigma_0,z_0} \dots q_n)$. Or it produces a maximal finite run $q_0 \xrightarrow{\sigma_0,z_0} \dots \xrightarrow{\sigma_{n-1},z_{n-1}} q_n$ where for $0 \leq m < n$ if $q_m \in Q_i$ then $q_m \xrightarrow{\sigma_m,z_m} q_{m+1} = \delta_i(q_0 \xrightarrow{\sigma_0,z_0} \dots q_m)$.

We now define the *interval bound problem*, which is the question whether Player 1 wins or looses a given WEG.

Interval bound problem:
Given a WEG $\mathcal{G}$ and an interval $[a, b]$, does there exist a strategy for Player 1, δ_1, such that any play on $\mathcal{G}$ using the strategy δ_1 produces a run valid in the interval $[a, b]$?

Player 1 wins and Player 2 looses a WEG $\mathcal{G}$ in the interval $[a, b]$ if the answer to the interval bound problem for $\mathcal{G}$ in the interval $[a, b]$ is positive.

Example 9. *Let us now turn to the example WEG in Figure 3.1 and consider this game with the interval $[-2, 4]$. We want to find the winner of the game, Player 1 wins if we can find a strategy which is a solution to the* interval bound problem *for $[-2, 4]$.*

A strategy is a function which take a non-maximal run and gives the next move. But in this game we do not need the full run to describe the next move for Player 1, we only need the current energy level γ and the current location.

The strategy covers behaviour in the state A as this is the only state in Q_1. The strategy is as follows:

- *If $\gamma \geq 3$ do $A \xrightarrow{b,-1} B$*
- *If $\gamma = 2$ do $A \xrightarrow{a,2} A$*
- *If $\gamma < 2$ do $A \xrightarrow{a,3} A$*

The solid run on Figure 3.2 is a run that can occur if Player 1 plays according to this strategy. It is possible to see that any play where Player 1 plays according to this strategy produces a valid run.

We now argue that this is true for the lower bound in the interval, let $\gamma = 3$ and the play has reached state A, then Player 1 takes the move to state B changing γ to 2, then if Player 2 takes the worst possible transition $B \xrightarrow{c,-4} A$ do $\gamma = -2$ and the game will continue from A. The strategy tells us that Player 1 does only take transitions from A to B if the $\gamma \geq 3$, therefore we know that the lower bound has not been breached.

This strategy yields a positive solution to the interval bound problem, *and therefore Player 1 is the winner of the WEG in Figure 3.1 in the interval $[-2, 4]$.*

There are a less strict version of the *interval bound problem* called the *lower-bound* *problem* which was studied in [5]. In this, a *lower-bound problem* is an *interval bound problem* for $[a, \infty]$.

Energy games in [5] have an initial energy level on c and the *interval bound problem* is solved for the interval $[0, b]$. In our definition the initial energy level is always 0, and the *interval bound problem* is solved for $[a, b]$ where $a \in \mathbb{Z}$, $a \leq 0$. The two definitions are similar as our definition can model an initial energy level on c by changing the interval to $[a-c, b-c]$, and their definition can model an interval $[a, b]$ by setting the initial energy to $|a|$ and the interval to $[0, b+|a|]$.

As these two definitions are similar can we use the complexity results from [5].

3.2 Action Investment Game

Definition 10 (Action Investment Game)**.** *An action investment game (AIG) is a tuple, $\mathcal{A}_\mathcal{G} = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0, \Sigma_1, \Sigma_2, actCost, B_1, B_2)$ where,*

- $(Q, Q_1, Q_2, \Sigma, \rightarrow, q_0)$ *is a WEG,*
- $\Sigma_1, \Sigma_2 \subseteq \Sigma$ *are disjoint action sets* $\Sigma_1 \cap \Sigma_2 = \emptyset$
- *actCost is a function* $actCost : \Sigma_1 \cup \Sigma_2 \rightarrow \mathbb{N}_0$ *and*
- $B_1, B_2 \in \mathbb{N}$ *are two budgets.*

An investment is a subset of actions $I \subseteq \Sigma$. An investment for Player i is denoted I_i, and $I_i \subseteq \Sigma_i$. The cost of an investment is the sum of the cost of the actions in the investment $invCost(I) = \sum_{\sigma \in I} actCost(\sigma)$.

The AIG problem for an interval:
Given AIG $\mathcal{A}_\mathcal{G}$, and an interval $[a, b]$ does there exist an initial investment $I_1 \subseteq \Sigma_1$ where $invCost(I_1) \leq B_1$, such that for all possible initial investment $I_2 \subseteq \Sigma_2$ where $invCost(I_2) \leq B_2$, Player 1 wins the WEG $\mathcal{G}' = (Q, Q_1, Q_2, \Sigma', \rightarrow', q_0)$ in the interval $[a, b]$, where $\Sigma' = (\Sigma \setminus I_1) \cap ((\Sigma \setminus \Sigma_2) \cup I_2)$ and $\rightarrow' = \rightarrow \cap (Q \times \Sigma' \times \mathbb{Z} \times Q)$?

Player 1 is wins and Player 2 loses the AIG $\mathcal{A}_\mathcal{G}$ for the interval $[a, b]$ if the answer to the AIG problem for $\mathcal{A}_\mathcal{G}$ and the interval $[a, b]$ is positive. Meaning that Player 1 wins the AIG in the interval $[a, b]$ if he can find an investment I_1 such that for any Player 2 investment I_2, Player 1 has a strategy to win the resulting WEG in the interval $[a, b]$.

An action in $I_1 \subseteq \Sigma_1$ is disabled from the resulting WEG, meaning that it is not possible for either player to take any transitions labeled with that action in the resulting WEG. Informally Player 1 can see his investment as a guarantee which ensures that some action can not happen.

An action in $I_2 \subseteq \Sigma_2$ is enabled in the resulting WEG, meaning that it possible for either player to take any transitions labeled with that action in the resulting WEG.

Informally Player 2 enables actions and thereby possible behaviour in the resulting WEG by his investment.

It is clear that for an AIG where $\Sigma_1 = \emptyset$ and $\Sigma_2 = \emptyset$, the *AIG problem for an interval* is an *interval bound problem*, as neither player can make an investment.

It is also clear that for a AIG $\mathcal{A}_{\mathcal{G}} = (Q, Q_1, Q_2, \Sigma, \rightarrow, q_0, \Sigma_1, \Sigma_2, actCost, B_1, B_2)$ where $B_1 = 0$ and $B_2 = 0$ the *AIG problem for an interval* can be solved by solving a single *interval bound problem* for $\mathcal{G}' = (Q, Q_1, Q_2, \Sigma', \rightarrow', q_0)$ where $\Sigma' = \Sigma \setminus \Sigma_2$ and $\rightarrow' = \rightarrow \cap (Q \times \Sigma' \times \mathbb{Z} \times Q)$, as the only investment either player can make is the empty investment.

Chapter 4

Motivating Example

This is a motivational example of a network relay station. The network relay station contains a battery, a solar panel to charge the battery, and an antenna from which it can receive signals. It is clear that the amount of energy charged to the battery depends on the weather. There are two types of weather conditions in this study case, *mild* and *extreme*. If the weather is *mild*, and the station needs to charge there is added 2 or 3 energy units to the battery, if the weather is *extreme* then there is added 1 or 6 energy units to the battery.

When the network relay receives a signal, the signal can be *good*, *normal*, or *bad*. It requires more energy to decode the signal if it is *bad* than if it is *normal* or *good*, and less energy if it is *good* than *normal*. Furthermore the weather also influences how much energy is needed to receive and decode the signal. It is possible to make a *test* of the system, in a *test* no energy is added to or consumed from the battery.

On the station is it possible to install a better antenna and thereby remove the *bad* or *normal* signal, a better antenna naturally comes at a cost.

Figure 4.1 shows the AIG for the network relay station, where $\Sigma_1 = \{bad, normal, good\}$, $\Sigma_2 = \{test, mild, extreme\}$ and the *actCost* function is given in Table 4.1.

$\sigma \in \Sigma_1$	*bad*	*normal*	*good*
$actCost(\sigma)$	4	2	1

$\sigma \in \Sigma_2$	*test*	*mild*	*extreme*
$actCost(\sigma)$	1	3	5

Table 4.1: The *actCost* function for the AIG in Figure 4.1

The intuition is that if Player 1 has a high budget B_1 then can he buy a better antenna and ensure a better signal, this corresponds to Player 1 including the action *bad* in his investment, this implies that the station works with a smaller battery. If Player 2 has a high budget B_2 he can require that the station must work in extreme weather conditions, and the station needs a larger battery to work.

We now investigate two instances of the relay station to show this intuition.

1. For the budgets $B_1 = 6$, $B_2 = 3$ and the interval $[0, 4]$ Player 1 wins, but if the budget for Player 1 is lowered to $B_1 = 5$ then Player 1 loses. The reason is that for $B_1 = 6$ Player 1 can make the investment $I_1 = \{bad, normal\}$, and

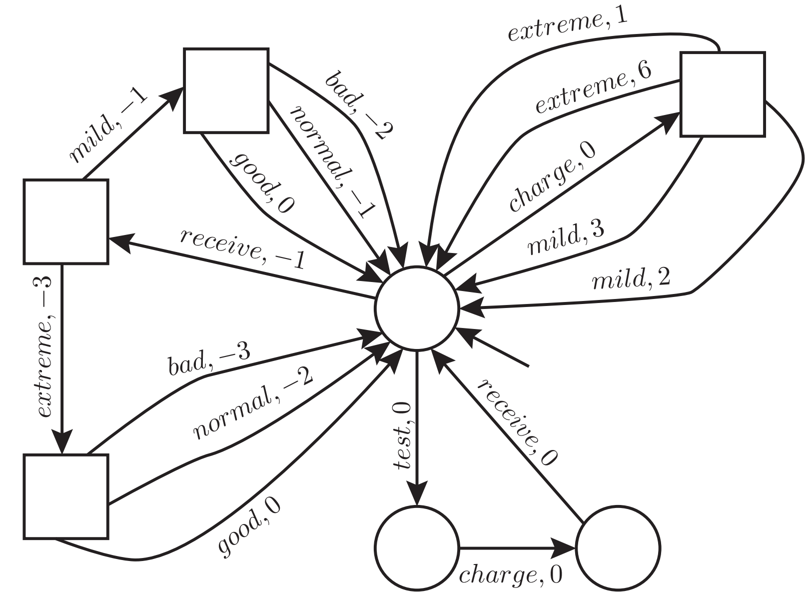


Figure 4.1: Motivating example of a network relay station

ensure that the station only receives good signals, this is not possible if $B_1 = 5$ as $actCost(\{bad, normal\}) = 6$.

2. For the budgets $B_1 = 4$, $B_2 = 4$ and the interval $[0, 6]$ Player 1 wins, but if the budget for Player 2 B_2 is increased to $B_2 = 5$ then Player 1 loses. It is possible for Player 1 to win $B_1 = 4$ and $B_2 = 5$ but the interval needs to be $[0, 11]$, the reason is that the *extreme* actions could be enabled in the resulting WEG.

CHAPTER 5

Complexity

In this chapter we find the complexity of solving the AIG problem for an interval. We give results under different budget restrictions and for intervals with and without no upper bound. The chapter begins by introducing three gadgets, and defines how these can be linked, this is used when finding complexity bounds in the end of the chapter.

5.1 Gadgets for complexity bounds

Gadgets are basically constructors for AIGs. They are used later when satisfiability (SAT) problems are reduced to AIG problems for an interval.

In a gadget boolean variables are represented by actions. The boolean x is represented by two actions x and x'. The idea is that, a player has to choose his investment such that either the action x or x' is present in the resulting WEG, but not both. When the action x is present the boolean variable x assignment is true, $v(x) = true$, and when x' is present $v(x) = false$. This creates a map from investments to the assignment of booleans.

Definition 11 (Valid investment)**.** *An investment $I_i \subseteq \Sigma_i$ is valid for Player i, where $i = \{1, 2\}$, if for all $x \in \Sigma_i$ either x or x' is in I_i, but not both $(x \in I_i \vee x' \in I_i) \wedge \neg(x \in I_i \wedge x' \in I_i)$.*

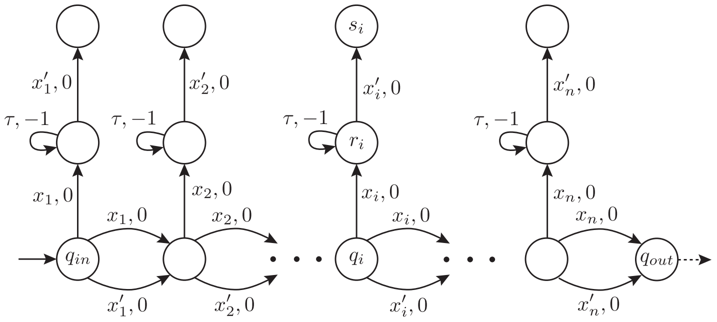
It is clear that each player needs a sufficient budget to make a valid investment. This is defined as follows.

Definition 12 (Sufficient Budget)**.** *The budget B_i is sufficient for Player i, where $i = \{1, 2\}$, if $actcost(I_i) \leq B_i$ for any valid investments $I_i \subseteq \Sigma_i$.*

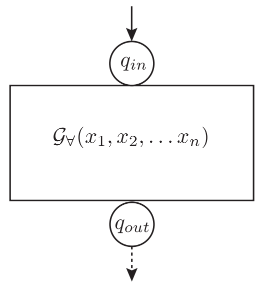
The two gadgets which are presented in the two following sections, have the property that a player needs to choose a valid investment or he risks loosing.

5.1.1 Gadget $\mathcal{G}_\forall(\vec{x})$

The purpose of the gadget $\mathcal{G}_\forall(\vec{x})$ is to construct an AIG, $\mathcal{A}_\forall$, from a vector of booleans $\vec{x} = (x_1 \dots x_n)$. The property is that Player 2 needs to choose his investment as a valid investment, or Player 1 has a strategy to win any play. In addition if Player 2 choose a valid investment, any play starting from q_{in} reaches q_{out} or it is loosing for Player 1. Let $\Sigma_2 = \{x_1, \dots x_n, x'_1 \dots x'_n\}$ such that it is possible for Player 2 to make a valid investment. The construction of $\mathcal{A}_\forall$ and representation of the gadget is shown on Figure 5.1.



(a) Construction



(b) Representation

Figure 5.1: Gadget 2

Lemma 13.

(a) If I_2 is a valid investment for Player 2, then any play starting from q_{in} is either loosing for Player 1 or reaches q_{out}.

(b) If I_2 is a valid investment for Player 2, then Player 1 has a strategy such that any play from q_{in} reaches q_{out}.

(c) If I_2 is not a valid investment for Player 2, then Player 1 has a strategy to win any play starting from q_{in}.

Proof. *(a)* Let I_2 be a valid investment for Player 2. We evaluate how a play can evolve in the first step from q_{in}. I_2 is a valid investment therefore either $x_1 \in I_2$ or $x'_1 \in I_2$. We evaluate the outcome for each case. If $x_1 \in I_2$, Player 1 has two choices, a x_1 transition *up* and a x_1 transition *right*. If Player 1 takes the transition *up* he loses, as he will be in an infinite negative loop as $x'_1 \notin I_2$. If Player 1 takes the transition *right* the play is one step closer to q_{end}. If $x'_1 \in I_2$ Player 1 can only take a transition *right* and the play is one step closer to q_{end}.

Now we know how any play can evolve from q_{in} to be either loosing for Player 1 or one step closer to q_{end}.

There are n steps from q_{in} to q_{out} and if the argument is repeated to cover all n steps, then it gives that any play from q_{in} either is loosing for Player 1 or reaches q_{out}.

(b) Let I_2 be a valid investment for Player 2. Now we need to find a strategy for Player 1 which ensures that any play from q_{in} reaches q_{out}. The Player 1 strategy is to take either the transition x_j or x'_j *right* for $0 < j \leq n$, this is possible as I_2 is valid.

Any play from q_{in} where Player 1 use this strategy reaches q_{out}.

(c) Let I_2 be not a valid investment for Player 2. This means that there is a smallest i such that $\{x_i, x'_i\} \in I_2$ or $\{x_i, x'_i\} \notin I_2$. Now we need to find a winning strategy for Player 1. The Player 1 strategy from q_{in} to q_i is to take a transition *right*, this is possible as $x_j \in I_2$ or $x'_j \in I_2$ for $1 \leq j < i$. From q_i there are two cases, if $\{x_i, x'_i\} \in I_2$ or if $\{x_i, x'_i\} \notin I_2$. If $\{x_i, x'_i\} \in I_2$ then the strategy is to take two transition *up*, $q_i \xrightarrow{x_i,0} r_i$ followed by $r_i \xrightarrow{x'_i,0} s_i$ to s_i, this is winning for Player 1 as s_i does not have any successors. If $\{x_i, x'_i\} \notin I_2$ then Player 1 wins as q_i does not have any successors.

Any play from q_{in} where Player 1 use this strategy is Player 1 winning, as the resulting run always is a valid maximal run to q_i or s_i. □

5.1.2 Gadget $\mathcal{G}_\exists(\vec{x})$

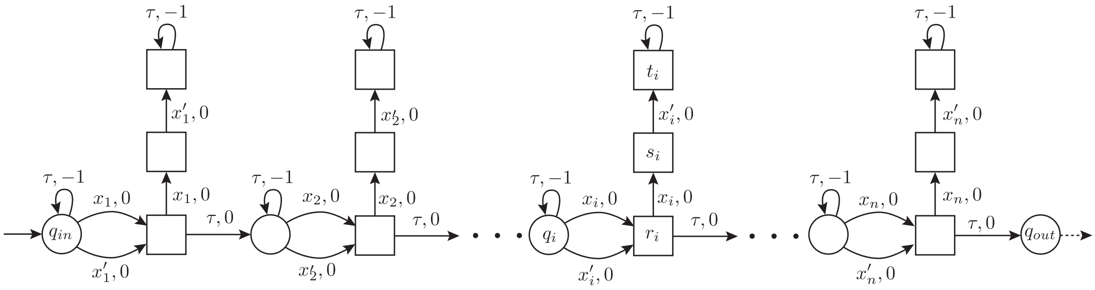
The purpose of the gadget $\mathcal{G}_\exists(\vec{x})$ is to construct an AIG, $\mathcal{A}_\exists$, from a vector of booleans $\vec{x} = (x_1 \ldots x_n)$. The property is that Player 1 needs to choose his investments as a valid investment, or Player 2 has a strategy to win any play. Let $\Sigma_1 = \{x_1, \ldots x_n, x'_1 \ldots x'_n\}$ such that it is possible for Player 1 to make a valid investment. The constructed and representation of the gadget is shown in Figure 5.2.

Lemma 14.

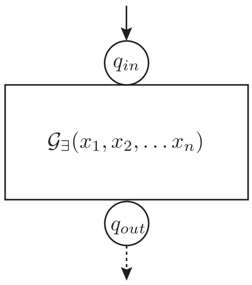
(a) If I_1 is a valid investment for Player 1, then Player 1 has a strategy such that any play from q_{in} either reaches q_{out} or is loosing for Player 2.

(b) If I_1 is a valid investment for Player 1, then Player 2 has a strategy such that any play from q_{in} either reaches q_{out} or is loosing for Player 1.

(c) If I_1 is not a valid investment for Player 1, then Player 2 has a strategy to win any play starting from q_{in}.



(a) Construction



(b) Representation

Figure 5.2: Gadget 1

Proof. (a) Let I_1 be a valid investment for Player 1. We need to find a Player 1 strategy δ_1 which ensures that any play from q_{in} either reaches q_{out} or is loosing for Player 2. The Player 1 strategy δ_1 is; when the play reach a state in $Q_1 \setminus \{q_{end}\}$ then take the action in $\Sigma_2 \setminus I_2$ available from that state.

We now evaluate how any play can evolve starting from q_{in} when Player 1 play according to δ_1. As $q_{in} \in Q_1$ the play continues according to δ_1 sending the play *right*, one step closer to q_{out}, and the play continues from a state in Q_2.

For this state there are two cases, if $x'_1 \in I_1$ or if $x_1 \in I_1$. If $x'_1 \in I_1$ then Player 2 has a choice, either to take a x_1 transition *up* or to take a τ transition *right*. If Player 2 takes the x_1 transition *up* he loses, as the run generated would be maximal and valid. If Player 2 takes the τ transition *right* the play continues from a state in Q_1, and the play will be one step closer to q_{out}.

If $x_1 \in I_1$ then the state only has one successor and Player 2 must take the τ transition *right*, the play will continue from a state in Q_1, and it will be one step closer to q_{out}.

Now we know how any play from q_{in} where Player 1 plays according to δ_1 either takes two transitions *right*, two steps closer to q_{out}, or is loosing for Player 2.

There are $2n$ steps from q_{in} to q_{out} and if the argument is repeated to cover all

$2n$ steps, then it gives that any play from q_{in} where Player 1 plays according to δ_1 either reaches q_{out} or is loosing for Player 2.

(b) Let I_1 be a valid investment for Player 1. We need to find a Player 2 strategy δ_2 which ensures that any play from q_{in} either reaches q_{out} or is loosing for Player 1. The Player 2 strategy δ_2 is; take the $\tau, 0$ transition *right* when possible.

We now evaluate how a play can evolve starting from q_{in} when Player 2 plays according to δ_2.

From q_{in} Player 1 has a choice, either to take the $\tau, -1$ self-loop transition or move *right* along either a x_1 or x'_1 transition. Player 1 does eventually lose if enough $\tau, -1$ self-loop transition is taken, by breaking the lower bound of the interval.

If Player 1 takes a transition *right* the play continues from a state in Q_2, one step closer to q_{out}. From there Player 2 plays according to δ_2 and takes $\tau, 0$ transition *right* to a state in Q_1, again one step closer to q_{out}.

Now we know how any play from q_{in} where Player 2 plays according to δ_2, Player 1 loses or the play evolves *right*, two steps closer to q_{out}.

The argument can be repeated to cover all $2n$ steps through the game, and gives that any play from q_{in} where Player 2 plays according to δ_2 either reaches q_{out} or is loosing for Player 1.

(c) Let I_1 not be a valid investment for Player 1. This means that there is a smallest i such that $\{x_i, x'_i\} \in I_2$ or $\{x_i, x'_i\} \notin I_2$ and for $1 \leq j < i$ is either $x_j \in I_1$ or $x'_j \in I_1$, but not both. We need to find a Player 2 strategy which ensures that Player 2 wins any play starting from q_{in}.

The Player 2 strategy is; take $\tau, 0$ transition *right* when possible until the play reach q_i. By the similar argument as in *(b)* we know that any play from q_{in} reaches q_i or is winning for Player 2.

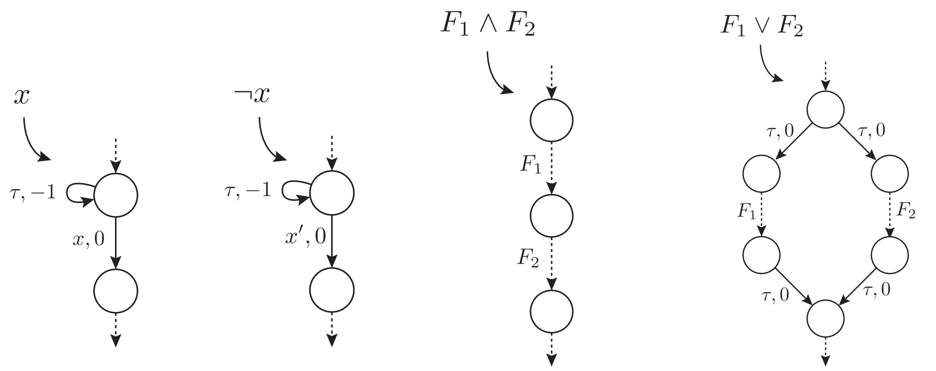
If the play reaches q_i is there two possibility, either $\{x_i, x'_i\} \in I_1$ or $\{x_i, x'_i\} \notin I_1$. We will now find a Player 2 strategy for each possibility. If $\{x_i, x'_i\} \in I_2$ wins Player 2 as Player 1 is forced to do a infinite amount of $q_i \xrightarrow{\tau,-1} q_i$, transitions

If $\{x_i, x'_i\} \notin I_2$ wins Player 2 by taking two transitions *up* to t_i as any play from t_i gives a infinite invalid run which is winning for Player 2. $\square$

5.1.3 Gadget $\mathcal{G}_\varphi(\varphi)$

The purpose of the gadget $\mathcal{G}_\varphi(\varphi)$ is to construct an AIG, $\mathcal{A}_\varphi$, from a boolean formula $\varphi(x_1 \dots x_n)$. We assume without loss of generality that all negations are pushed to the variables in φ.

The gadget is inductively construct using the four figures, in Figure 5.3.



(a) Positive literal (b) Negative literal (c) Conjunction (d) Disjunction

Figure 5.3: Building blocks for $\mathcal{G}_\varphi$

The representation of the gadget is shown in Figure 5.4.

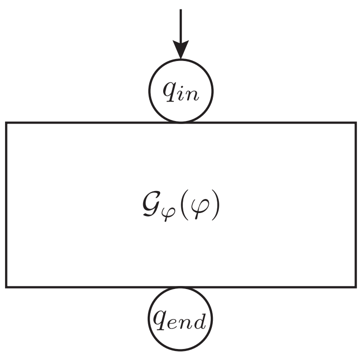


Figure 5.4: Representation of $\mathcal{G}_\varphi$

Recall that $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Lemma 15. *Let I_1, I_2 be a valid investments and let*

$$v(x) = \begin{cases} \mathit{true} & \text{if } x' \in I_1 \text{ or } x \in I_2 \\ \mathit{false} & \text{if } x \in I_1 \text{ or } x' \in I_2 \end{cases}$$

(a) If φ is true under v then Player 1 has a strategy to win any play starting from q_{in} in $\mathcal{A}_\varphi$.

(b) If φ is false under v then Player 2 is the winner of any play starting from q_{in} in $\mathcal{A}_\varphi$.

Proof. ***(a)*** Let φ be true under v and we prove by induction in the structure of $\mathcal{A}_\varphi$ that Player 1 has a strategy to win any play starting from q_{in} on $\mathcal{A}_\varphi$.

Basis: We have two base cases, $\varphi = x$ and $\varphi = \neg x$.

If $\varphi = x$, then $\mathcal{A}_\varphi$ is as in Figure 5.3a and the Player 1 strategy needs to cover the first state where a $x, 0$ and a $\tau, -1$ transition is possible. The Player 1 strategy is to take the $x, 0$ transition. The $x, 0$ action is possible as φ is true under v. Any play with this strategy is winning for Player 1 as the resulting run is valid and maximal.

If $\varphi = \neg x$ then $\mathcal{A}_\varphi$ is as in Figure 5.3b and the Player 1 strategy needs to cover the first state where a $x, 0$ and a $\tau, -1$ transition is possible. The Player 1 strategy is to take the $x', 0$ transition. The $x', 0$ action is possible as φ is true under v. Any play with this strategy is winning for Player 1 as the resulting run is valid and maximal.

Induction Step: Assume by induction hypothesis (IH) that for $\varphi = F$ Player 1 has a strategy to win any play on $\mathcal{A}\varphi$.

We now prove that Player 1 has a strategy to win any play where $\varphi = F_1 \wedge F_2$ and where $\varphi = F_1 \vee F_2$.

If $\varphi = F_1 \wedge F_2$, we know that φ is true under v, this implies that F_1 and F_2 is true, and we know by the IH that Player 1 has a strategy to win both F_1 and F_2, therefor he also has a strategy to win when $\varphi = F_1 \wedge F_2$.

If $F_1 \vee F_2$ we know that φ is true under v, this implies that either F_1 or F_2 is true. We know by the IH that if F_1 is true Player 1 has a winning strategy for F_1 or if F_2 is true Player 1 has a winning strategy for F_2. The Player 1 strategy is to take the transition *Left* $\tau, 0$ if F_1 is true and take *Right* $\tau, 0$ if F_2 is true. Player 1 now has a strategy to win if either F_1 or F_2 is true.

We now know by induction that Player 1 has a winning strategy to win any play on $\mathcal{A}\varphi$ if φ is true under v.

(b) Let φ be false under v and we prove by induction in the structure of $\mathcal{A}_\varphi$ that any play starting from q_{in} on $\mathcal{A}_\varphi$ is Player 2 winning.

Basis: We have two base cases, $\varphi = x$ and $\varphi = \neg x$.

If $\varphi = x$, then $\mathcal{A}_\varphi$ is as in Figure 5.3a and

If $\varphi = x$, then is $\mathcal{A}_\varphi$ as Figure 5.3a We know φ is $false$ under υ meaning that either $x \in I_1$ or $x' \in I_2$ implying that no x transition is present in the resulting WEG. Any play is therefore Player 2 winning it produces infinite and invalid run.

If $\varphi = \neg x$, then is $\mathcal{A}_\varphi$ as Figure 5.3b We know φ is $false$ under υ meaning that either $x' \in I_1$ or $x \in I_2$ implying that no x' transition is present in the resulting WEG. Any play is therefore Player 2 winning it produces infinite and invalid run.

Induction Step: Assume by induction hypothesis (IH) that for $\varphi = F$ is any play on $\mathcal{A}_\varphi$ Player 2 winning.

We now prove that any play on $\mathcal{A}_\varphi$ where $\varphi = F_1 \wedge F_2$ and where $\varphi = F_1 \vee F_2$ is Player 2 winning.

If $F_1 \wedge F_2$, we know that φ is $false$ if either F_1 or F_2 is $false$ under υ. By the IH do and the fact that F_1 and F_2 are in sequence wins Player 2 any play as any produces run is infinite and invalid.

If $F_1 \vee F_2$, we know that φ is $false$ if both F_1 and F_2 is $false$ under υ. By the IH do and the fact that F_1 and F_2 are in parallel wins Player 2 any play as any produces run is infinite and invalid. $\square$

5.1.4 Linking gadgets

Gadgets can be linked together in sequence and thereby create a combined AIG. This is done by adding a $\tau, 0$ transition from q_{out} in one gadget to the state q_{in} in another gadget. If it is necessary then rename states and updating successor relation, union action sets, union state sets, union successor relation, sum budgets.

The short hand notation of linking gadgets is an arrow. The representation of three linked gadgets $\mathcal{G}_\exists(\vec{x}) \rightarrow \mathcal{G}_\forall(\vec{y}) \rightarrow \mathcal{G}_\varphi(\varphi)$ is shown in Figure 5.5.

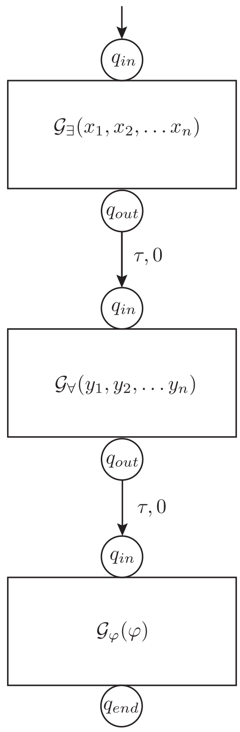


Figure 5.5: Linking of gadget $\mathcal{G}_\exists(\vec{x})$ with $\mathcal{G}_\forall(\vec{y})$ and $\mathcal{G}_\varphi(\varphi)$

The construction of linking gadgets is useful for reducing (Q)SAT problems to AIG problems. A construction of a reduction from a given QSAT problem to a AIG problem is shown in Example 16.

Example 16. *Reduction from the QSAT problem* $\exists x_1, x_2 \forall y_1, y_2 (x_1 \vee (\neg y_2 \wedge y_1) \wedge (\neg y_1 \vee y_2 \vee \neg x_2)) = 1$ *to an AIG problem. The construction is three gadgets linked together,* $\mathcal{G}_\exists(\vec{x}) \rightarrow \mathcal{G}_\forall(\vec{y}) \rightarrow \mathcal{G}_\varphi(\varphi)$*, the representation of this shown in Figure 5.5.*

The full drawn out AIG is shown in Figure 5.6.

The intuitions is that if there exists a x_1, x_2 *for which the QSAT problem is true, then Player 1 is the winner of the AIG. Similar if there does not exists a* x_1, x_2 *for which the QSAT problem is true then Player 2 is the winner of the AIG.*

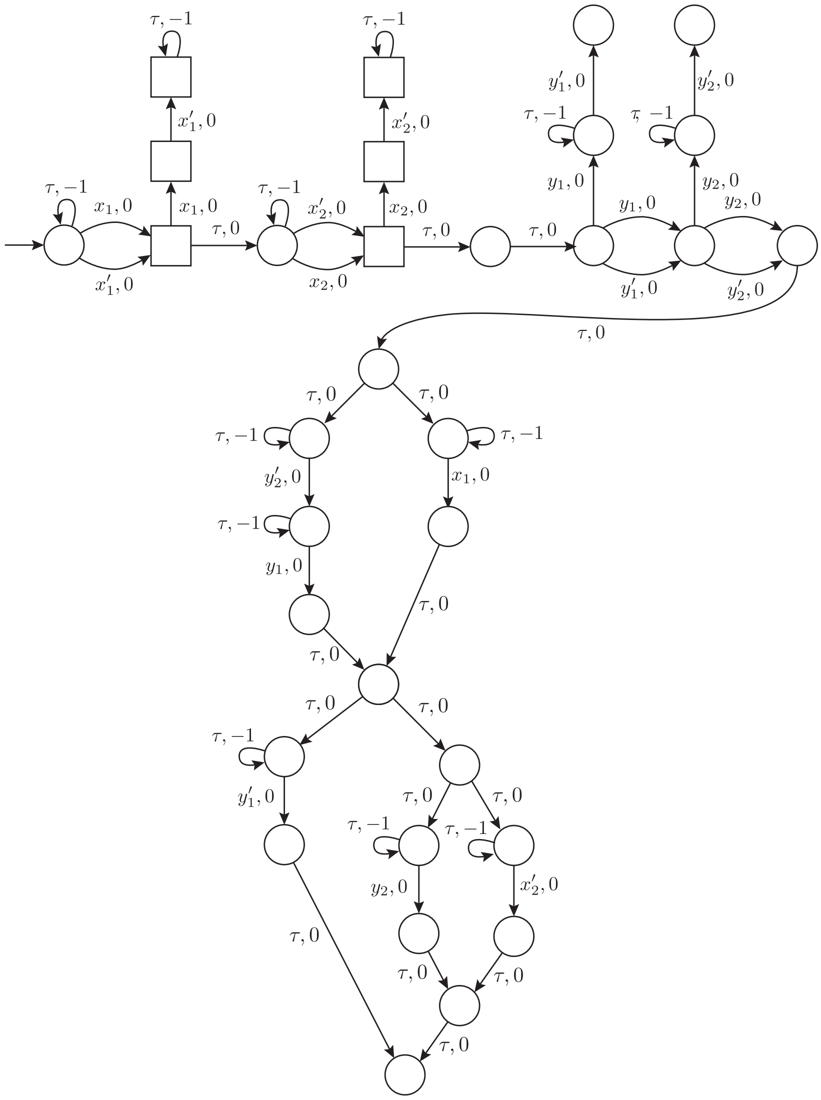


Figure 5.6: Translation

We now argue that this transformation can be done in polynomial time. For a problem with n literals is at most $5n$ states introduced in the game by the first two gadgets, and for the last gadget with the worst possible φ, all n literals disjoint introduce $4n+2$ states. Therefor the AIG constructed is at most $20n+2$ larger than the QSAT problem, which is definitely polynomial in the size of the QSAT problem.

5.2 Complexity results

This section covers all complexity results.

Budget restrictions	Interval	Type	
		Existential $Q_2 = \emptyset$	Game
$B_1 = 0$, $B_2 = 0$	$[a, \infty]$	$\in$ P, [5]	$\in$ UP $\cap$ coUP, [5]
	$[a, b]$	NP-Hard, $\in$ PSPACE, [5]	EXPTIME-complete, [5]
$B_2 = 0$	$[a, \infty]$	$\in$ P, Lem. 18	NP-Complete, Lem.24
	$[a, b]$	NP-Hard, $\in$ PSPACE, Lem. 19	EXPTIME-complete, Lem. 27
$B_1 = 0$	$[a, \infty]$	Π_1^P-complete, Lem. 20	Π_1^P-Complete, Lem. 25
	$[a, b]$	Π_2^P-hard, Claim 21, $\in$ PSPACE, Lem. 22	EXPTIME-complete, Lem. 27
–	$[a, \infty]$	Π_1^P-complete, Lem. 23	Σ_2^P-complete Lem. 26
	$[a, b]$	Π_2^P-hard, Claim 21, $\in$ PSPACE, Lem. 22	EXPTIME-complete, Lem. 27

Table 5.1: Complexity for AIG problems for an interval

Proposition 17. *Let $\mathcal{A}_\mathcal{G}$ be an AIG where $Q_2 = \emptyset$. The complexity of solving the AIG problem for $\mathcal{A}_\mathcal{G}$ in an interval with no restrictions on B_1 and is equivalent to the complexity of solving the AIG for problem for an $\mathcal{A}_\mathcal{G}$ interval where $B_1 = 0$.*

Proof. Let $\mathcal{A}_\mathcal{G}$ be an AIG where $Q_1 = \emptyset$ and with no restrictions on B_1. Now we solve the AIG problem for an $\mathcal{A}_\mathcal{G}$ interval. Player 1 choses an investment $I_1 \subseteq \Sigma_1$ where $actCost(I_1) \leq B_1$. But all states are in Q_1, meaning that Player 1 controls any play on the resulting WEG, hence Player 1 does not benefit from making an investment different from $I_1 = \emptyset$ as this could remove options for him in the resulting WEG.

Therefore the budget B_1 can be ignored and restricted to 0, as Player 1 in any AIG where $Q_2 = \emptyset$ for any budget makes the investment $I_1 = \emptyset$. This makes the two problems equivalent, and the budget B_1 can be set to 0 when $Q_2 = \emptyset$. □

Lemma 18. *The AIG problem where $Q_2 = \emptyset$ and $B_2 = 0$, for the interval $[a, \infty]$ is in P.*

Proof. This follows from Proposition 17 and the fact that the AIG problem where $Q_2 = \emptyset$, $B_2 = 0$ and $B_1 = 0$ for the interval $[a, \infty]$ is in P. □

Lemma 19. *The AIG problem where $Q_2 = \emptyset$, and $B_2 = 0$, for the interval $[a, b]$ is NP-Hard and in PSPACE.*

Proof. This follows from Proposition 17 and the fact that the AIG problem where $Q_2 = \emptyset$, $B_2 = 0$ and $B_1 = 0$ for the interval $[a, \infty]$ is NP-Hard and in PSPACE. $\square$

Lemma 20. *The AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$, for the interval $[a, \infty]$ is Π_1^P-complete.*

Proof. Lower bound: The AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$ for the interval $[a, \infty]$ is Π_1^P-hard by reduction from Π_1^P-SAT.

Let $\forall \vec{x} \varphi(\vec{x}) = 1$ be a Π_1^P-SAT problem. We construct the AIG $\mathcal{A}_{\mathcal{G}}$ given by $\mathcal{G}_{\forall}(\vec{x}) \to \mathcal{G}_{\varphi}(\varphi)$ and define B_2 to be any sufficient budget.

We want to show:

(a) If $\forall \vec{x} \varphi(\vec{x}) = 1$ is $true$, Player 1 wins $\mathcal{A}_{\mathcal{G}}$.

(b) If $\forall \vec{x} \varphi(\vec{x}) = 1$ is $false$, Player 2 wins $\mathcal{A}_{\mathcal{G}}$.

(a) Suppose that $\forall \vec{x} \varphi(\vec{x}) = 1$ is $true$ under the assignment of υ. We now want to show that Player 1 wins the AIG $\mathcal{A}_{\mathcal{G}}$.

Player 2 can pick either an invalid or valid investment I_2. If I_2 is an invalid investment we know from Lemma 13 (c) that Player 1 has a strategy to win any play starting from q_{in} in the AIG $\mathcal{A}_{\forall}$ constructed by $\mathcal{G}_{\forall}(\vec{x})$. If I_2 is a valid investment we know from Lemma 13 (b) that Player 1 has a strategy such that any play from q_{in} reaches q_{out} in $\mathcal{A}_{\forall}$.

If the play reaches q_{out} in $\mathcal{A}_{\forall}$ then by the construction of linking the play continues in the AIG $\mathcal{A}_{\varphi}$ constructed from $\mathcal{G}_{\varphi}(\varphi)$.

We know that $\forall \vec{x} \varphi(\vec{x}) = 1$ is $true$. Hence for every υ this imply by Lemma 15 (a) that Player 1 win any play starting from q_{in} in $\mathcal{A}_{\varphi}$.

Therefore do Player 1 have a strategy to win no what investment I_2 Player 2 does.

(b) Suppose that $\forall \vec{x} \varphi(\vec{x}) = 1$ is $false$. There there is an assignment υ such that $\forall \vec{x} \varphi(\vec{x}) = 1$ is $false$. We now want to show that Player 2 wins the AIG $\mathcal{A}_{\mathcal{G}}$.

Player 2 chooses the valid investment $I_2 = \{x \mid \upsilon(x) = true\} \cup \{x' \mid \upsilon(x) = false\}$.

By Lemma 13 (a) we know that any play starting in q_{in} gets to q_{out} in the AIG $\mathcal{A}_{\forall}$ constructed by $\mathcal{G}_{\exists}(\vec{x})$, or Player 2 wins.

If the play reaches q_{out} in $\mathcal{A}_{\forall}$ then by the construction of linking the play continues in the AIG $\mathcal{A}_{\varphi}$ constructed from $\mathcal{G}_{\varphi}(\varphi)$.

We know that $\forall \vec{x} \varphi(\vec{x}) = 1$ is $false$ under the assignment of υ and therefore by Lemma 15 (b) we know that Player 2 is winning any play starting from q_{in} in $\mathcal{A}_{\varphi}$.

We have now shown that Player 2 wins any on play on $\mathcal{A}_{\mathcal{G}}$.

This construction shows that a solution to the AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$ for the interval $[a, \infty]$ can solve any Π_1^P-SAT problem, hence the AIG problem is Π_1^P-hard.

Upper bound:
The AIG problem where $Q_2 = \emptyset$ and $B_1 = 0$ for the interval $[a, \infty]$ is in Π_1^P by the following algorithm.

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve interval bound problem for the resulting WEG ($Q_2 = \emptyset$) in the interval $[a, \infty]$ if all results are positive return yes, else return no.

The algorithm uses one universal quantifier over a polynomial time problem, therefore in Π_1^P. □

Claim 21. *The AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is Π_2^P-hard.*

Lemma 22. *The AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is in PSPACE.*

Proof. We want to prove that the AIG problem where $Q_2 = \emptyset$, for the interval $[a, b]$ is in PSPACE, this is done by finding an algorithm in PSPACE which solves the problem. The algorithm is

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve interval bound problem for the resulting WEG ($Q_2 = \emptyset$) in the interval $[a, b]$ if all results are positive return yes, else return no.

Step 1 is a universal quantifier over step 2. Step 2 is a problem in PSPACE, hence the algorithm is in PSPACE. □

Lemma 23. *The AIG problem where $Q_2 = \emptyset$, for the interval $[a, \infty]$ is Π_1^P-complete.*

Proof. This follows from Proposition 17 and Lemma 20. □

Lemma 24. *The AIG problem where $B_2 = 0$ for an interval $[a, \infty]$ is NP-complete.*

Proof. Lower bound:
The AIG problem where $B_2 = 0$ for the interval $[a, \infty]$ is NP-hard by reduction from SAT.

Let $\exists \vec{x} \varphi(\vec{x}) = 1$ be a SAT problem. We construct the AIG $\mathcal{A}_{\mathcal{G}}$ given by $\mathcal{G}_{\exists}(\vec{x}) \rightarrow \mathcal{G}_{\varphi}(\varphi)$ and define B_1 to be any sufficient budget.

We want to show:

(a) If $\exists \vec{x} \varphi(\vec{x}) = 1$ is *true*, Player 1 wins $\mathcal{A}_{\mathcal{G}}$.

(b) If $\exists \vec{x} \varphi(\vec{x}) = 1$ is $false$, Player 2 wins $\mathcal{A}_{\mathcal{G}}$.

(a) Suppose that $\exists \vec{x} \varphi(\vec{x}) = 1$ is $true$ under some assignment υ. We now want to show that Player 1 wins the AIG $\mathcal{A}_{\mathcal{G}}$. Player 1 chooses the valid investment $I_1 = \{x' \mid \upsilon(x) = true\} \cup \{x \mid \upsilon(x) = false\}$.

Player 1 has by Lemma 14 (a) a strategy such that any play starting in q_{in} gets to q_{out} in the AIG $\mathcal{A}_{\exists}$ constructed by $\mathcal{G}_{\exists}(\vec{x})$, or Player 1 wins.

If the play reaches q_{out} in $\mathcal{A}_{\exists}$ then by the construction of linking the play continues in the AIG $\mathcal{A}_{\varphi}$ constructed from $\mathcal{G}_{\varphi}(\varphi)$.

We know that $\exists \vec{x} \varphi(\vec{x}) = 1$ is $true$ under the assignment of υ and therefore Player 1 in $\mathcal{A}_{\varphi}$ by Lemma 15 (a) has a strategy such that any play starting in q_{in} is winning for Player 1.

We have now shown that Player 1 has an investment I_1 and a strategy to win such that he wins $\mathcal{A}_{\mathcal{G}}$.

(b) Suppose that $\exists \vec{x} \varphi(\vec{x}) = 1$ is $false$ under any assignment υ. We now want to show that Player 2 wins the AIG $\mathcal{A}_{\mathcal{G}}$ no matter what Player 1 does.

Player 1 can pick either an invalid or valid investment I_1. If I_1 is an invalid investment we know from Lemma 14 (c) that Player 2 has a strategy to win any play starting from q_{in} in the AIG $\mathcal{A}_{\exists}$ constructed by $\mathcal{G}_{\exists}(\vec{x})$.

If I_1 is a valid investment we know from Lemma 14 (b) that Player 2 has a strategy such that any play from q_{in} either reaches q_{out} in $\mathcal{A}_{\exists}$ or is winning for Player 2.

If the play reaches q_{out} in $\mathcal{A}_{\exists}$ then by the construction of linking the play continues in the AIG $\mathcal{A}_{\varphi}$ constructed from $\mathcal{G}_{\varphi}(\varphi)$.

We know that $\exists \vec{x} \varphi(\vec{x}) = 1$ is $false$ under the assignment of υ this implies by Lemma 15 (b) that Player 2 wins any play starting from q_{in} in $\mathcal{A}_{\varphi}$.

Therefore Player 2 has a strategy to win no matter what Player 1 does.

This construction shows that a solution to the AIG problem where $B_2 = 0$ for the interval $[a, \infty]$ can solve any SAT problem, hence the AIG problem is NP-hard.

Upper bound:
The AIG problem where $B_2 = 0$ for an interval $[a, \infty]$ is in NP by the fallowing algorithm.

1. Guess $I_1 \subseteq \Sigma_1$ where $actCost(I_1) \leq B_1$, let $I_2 = \emptyset$ and construct the resulting WEG.
2. Solve the interval bound problem for the resulting WEG in the interval $[a, \infty]$, if the result is positive return yes, else return no.

Step 1 is a guess of polynomial size, this is done in polynomial time. Step 2 is problem in UP $\cup$ coUP, which is a subclass of NP. Therefore the algorithm is in

NP. □

Lemma 25. *The AIG problem where $B_1 = 0$ for an interval $[a, \infty]$ is Π_1^P-complete.*

Proof. Lower bound: The AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ is Π_1^P-hard by reduction from Π_1^P-SAT.

Let $\forall \vec{x} \varphi(\vec{x}) = 1$ be a Π_1^P-SAT problem. We construct the AIG $\mathcal{A}_\mathcal{G}$ given by $\mathcal{G}_\forall(\vec{x}) \to \mathcal{G}_\varphi(\varphi)$ and define B_2 to be any sufficient budget.

The proof for the two following properties of the construction is similar to the proof in Lemma 20.

(a) If $\forall \vec{x} \varphi(\vec{x}) = 1$ is $true$, Player 1 wins $\mathcal{A}_\mathcal{G}$.

(b) If $\forall \vec{x} \varphi(\vec{x}) = 1$ is $false$, Player 2 wins $\mathcal{A}_\mathcal{G}$.

This construction shows that a solution to the AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ can solve any Π_1^P-SAT problem, hence the AIG problem is Π_1^P-hard.

Upper bound:
The AIG problem where $B_1 = 0$ for the interval $[a, \infty]$ is in Π_1^P by the following algorithm.

1. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, let $I_1 = \emptyset$ and construct the resulting WEG.
2. Solve the interval bound problem for the resulting WEG in the interval $[a, \infty]$, if all results are positive return yes, else return no.

Step 1 is a universal guess over Step 2. Step 2 is problem in UP $\cup$ coUP, which is a subclass of coNP $= \Pi_1^P$. Since the algorithm only uses universal quantifiers, it is in Π_1^P. □

Lemma 26. *The AIG problem for an interval $[a, \infty]$ is Σ_2^P-complete.*

Proof. Lower bound: The AIG problem for the interval $[a, \infty]$ is Σ_2^P-hard by reduction from Σ_2^P-SAT.

Let $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ be a Σ_2^P-SAT problem. We construct the AIG $\mathcal{A}_\mathcal{G}$ given by $\mathcal{G}_\exists(\vec{x})\mathcal{G}_\forall(\vec{y}) \to \mathcal{G}_\varphi(\varphi)$ and define B_1 and B_2 to be any sufficient budget.

We want to show:

(a) If $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ is $true$, Player 1 wins $\mathcal{A}_\mathcal{G}$.

(b) If $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$ is $false$, Player 2 wins $\mathcal{A}_\mathcal{G}$.

(a) Suppose that $\exists \vec{x} \forall \vec{y} \varphi(\vec{x}, \vec{y}) = 1$, is $true$. Then υ is an assignment of the booleans in $\vec{x}$ such that for any assignments of $\vec{y}$, φ is true. We now want to show that Player 1 wins the AIG $\mathcal{A}_\mathcal{G}$.

Player 1 chooses the valid investment $I_1 = \{x' \mid \upsilon(x) = true\} \cup \{x \mid \upsilon(x) = false\}$.

Player 1 has by Lemma 14 (a) a strategy such that any play starting in q_{in} gets to q_{out} in the AIG $\mathcal{A}_\exists$ constructed by $\mathcal{G}_\exists(\vec{x})$, or Player 1 wins.

If the play reaches q_{out} in $\mathcal{A}_\exists$ then by the construction of linking the play continues in the AIG $\mathcal{A}_\forall$ constructed from $\mathcal{G}_\forall(\vec{y})$, where the play starts in q_{in}.

Player 2 can pick either an invalid or valid investment I_2. If I_2 is an invalid investment we know from Lemma 13 (c) that Player 1 has a strategy to win any play starting from q_{in} in the AIG $\mathcal{A}_\forall$.

If I_2 is a valid investment we know from Lemma 13 (b) that Player 1 has a strategy such that any play from q_{in} reaches q_{out} in $\mathcal{A}_\forall$.

If the play reaches q_{out} in $\mathcal{A}_\forall$ then by the construction of linking the play continues in the AIG $\mathcal{A}_\varphi$ constructed from $\mathcal{G}_\varphi(\varphi)$.

We know that $\exists\vec{x}\forall\vec{y}\varphi(\vec{x},\vec{y}) = 1$ is *true*, under the assignment υ this implies by Lemma 15 (a) that Player 1 wins any play starting from q_{in} in $\mathcal{A}_\varphi$.

Therefore Player 1 has an investment I_1 and a strategy to win no matter what investment I_2 and strategy Player 2 uses.

(b) Suppose that $\exists\vec{x}\forall\vec{y}\varphi(\vec{x},\vec{y}) = 1$ is $false$. Then υ is the assignment which, for any assignment of $\vec{x}$, assigns the booleans in $\vec{y}$ such that φ is false. We now want to show that Player 2 wins the AIG $\mathcal{A}_\mathcal{G}$ no matter what Player 1 does.

Player 1 can pick either an invalid or valid investment I_1. If I_1 is an invalid investment we know from Lemma 14 (c) that Player 2 has a strategy to win any play starting from q_{in} in the AIG $\mathcal{A}_\exists$ constructed by $\mathcal{G}_\exists(\vec{x})$.

If I_1 is a valid investment we know from Lemma 14 (b) that Player 2 has a strategy such that any play from q_{in} either reaches q_{out} in $\mathcal{A}_\exists$ or is winning for Player 2.

If the play reaches q_{out} in $\mathcal{A}_\exists$ then by the construction of linking the play continues in the AIG $\mathcal{A}_\forall$ constructed from $\mathcal{G}_\forall(\vec{y})$.

Player 2 choose the valid investment $I_2 = \{x \mid \upsilon(x) = true\} \cup \{x' \mid \upsilon(x) = false\}$.

By Lemma 13 (a) we know that any play starting in q_{in} gets to q_{out} in $\mathcal{A}_\forall$, or Player 2 wins.

If the play reaches q_{out} in $\mathcal{A}_\forall$ then by the construction of linking the play continues in the AIG $\mathcal{A}_\varphi$ constructed from $\mathcal{G}_\varphi(\varphi)$.

We know that $\exists\vec{x}\forall\vec{y}\varphi(\vec{x},\vec{y}) = 1$ is $false$ under any assignment υ and therefore by Lemma 15 (b) we know that Player 2 is winning any play starting from q_{in} in $\mathcal{A}_\varphi$.

We have now shown that Player 2 wins any on play on $\mathcal{A}_\mathcal{G}$.

Upper bound:
The AIG problem where for the interval $[a, \infty]$ is in Σ_2^P by the following algorithm.

1. Guess $I_1 \subseteq \Sigma_1$ where $actCost(I_1) \leq B_1$.
2. For all $I_2 \subseteq \Sigma_2$ where $actCost(I_2) \leq B_2$, and construct the resulting WEG.
3. Solve the interval bound problem for the resulting WEG in the interval $[a, \infty]$ if all results are positive return yes, else return no.

Step 1 is a guess of polynomial size. Step 2 is a universal guess over Step 3. Step 3 is problem in UP $\cup$ coUP, which is a subclass of coNP $= \Pi_1^P$.

The algorithm uses one existential followed by two universal quantifiers, and is therefore in Σ_2^P. □

Lemma 27. *The AIG problem for an interval* $[a, b]$ *is EXPTIME-complete.*

Proof. Lower bound
The EXPTIME-hard lower bound is given by the AIG problem for an interval $[a, \infty]$ where $B_1 = 0$ and $B_2 = 0$.

Upper bound:
The EXPTIME upper bound is given by the following argument. There can be exponential many different combinations of investments I_1, I_2, this depends on the budget for each player, and for each combination we need to solve an EXPTIME problem, this gives an algorithm in EXPTIME. □

Chapter 6

Conclusion

We have introduced the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets. We have given a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We have given a comprehensive analysis of the complexity of the decision problem which is the foundation for reasoning about these trade-offs. The complexity problem is studied in different cases. In the existential case, where all states are controlled by Player 1, under the restriction that the budget for Player 2 is 0 then the complexity bounds follow those of energy games. In all other cases increases the complexity by at most two levels in the polynomial hierarchy.

It is expected that the complexity increases by two levels in the polynomial hierarchy as the problem introduces a existential choice and a universal quantifier before the energy game.

The future work is to write full proof of the claims, that the complexity for the existential action investment games problem for a closed interval is $\Pi_2^P - hard$. It is also of high priority to find a case study where the investment of Player 2 is used to model a more controllable phenomena than the weather.

Resume

We define the formalism for action investment games which can be used to reason about the trade-off between a constrained resource of energy and two budgets.

The formalism is an extended of energy games with budgets and prices on actions. An energy game are played by two players on a finite graph, where Player 1 wins if he has a strategy such that the accumulated energy is constrained within an given interval, and Player 2 wins if Player 1 loses.

We use the formalism in a motivating example of a network relay station where we study the trade-offs between battery size and the budget for each player.

We given a comprehensive analysis of the complexity of the decision problem which is the foundation for reasoning about these trade-offs. The complexity problem is studied in different cases. In the existential case, where all states are controlled by Player 1, under the restriction that the budget for Player 2 is 0 then the complexity bounds follow those of energy games.

In the worst cases increases the complexity by two levels in the polynomial hierarchy compared to the complexity of the corresponding energy game.

Bibliography

[1] Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach.* Cambridge University Press, New York, NY, USA, 1st edition, 2009.

[2] Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking (Representation and Mind Series).* The MIT Press, 2008.

[3] Nikola Benes, Jan Kretínský, Kim Guldstrand Larsen, Mikael H. Møller, and Jirí Srba. Dual-priced modal transition systems with time durations. In Nikolaj Bjørner and Andrei Voronkov, editors, *LPAR*, volume 7180 of *Lecture Notes in Computer Science*, pages 122–137. Springer, 2012.

[4] Antonia Bertolino. Software testing research: Achievements, challenges, dreams. In *2007 Future of Software Engineering*, FOSE '07, pages 85–103, Washington, DC, USA, 2007. IEEE Computer Society.

[5] Patricia Bouyer, Ulrich Fahrenberg, Kim Guldstrand Larsen, Nicolas Markey, and Jirí Srba. Infinite runs in weighted timed automata with energy constraints. In Franck Cassez and Claude Jard, editors, *FORMATS*, volume 5215 of *Lecture Notes in Computer Science*, pages 33–47. Springer, 2008.

[6] Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, and Jean-François Raskin. Generalized mean-payoff and energy games. In Kamal Lodaya and Meena Mahajan, editors, *FSTTCS*, volume 8 of *LIPIcs*, pages 505–516. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010.

[7] Uli Fahrenberg, Line Juhl, Kim G. Larsen, and Jirí Srba. Energy games in multiweighted automata. In Antonio Cerone and Pekka Pihlajasaari, editors, *ICTAC*, volume 6916 of *Lecture Notes in Computer Science*, pages 95–115. Springer, 2011.

[8] Christos H. Papadimitriou. *Computational complexity.* Addison-Wesley, 1994.