MODELING OF AN INTERFACE Crack with Bridging Effects Between Two Fibrous Composite Layers

MASTER OF SCIENCE THESIS Design of Mechanical Systems



Brian Lau Verndal Bak

Thomas Bro Henriksen

DEPARTMENT OF MECHANICAL AND Manufacturing Engineering Aalborg University Spring 2010



Study Board of Industry and Global Business Development *Design of Mechanical Systems*

Fibigerstræde 16 Telephone: 99 40 93 09 Email: swe@me.aau.dk http://industri.aau.dk/

Title: Modeling of an Interface Crack with Bridging Effects Between Two Fibrous Composite Layers
Theme: Design of Mechanical Systems
Project Period: DMS4, Spring 2010
Project Group: Group 48F

Supervisor:

Jens H. Andreasen

Authors:

Brian Lau Verndal Bak

Thomas Bro Henriksen

Copies: 4

Report: 118 pages

DVD: 1

Finished: June 2nd 2010

The content of this report is freely accessible, though publication (with reference) may only occur after permission from the authors.

Abstract

In this report a procedure for analyzing the fracture resistance of interface cracks between similar or dissimilar layers of fibrous composite materials is developed. This is intended for a detailed simulation of delamination defects using fracture mechanics, as they cannot be analyzed by use of point stress or strain based criteria.

In advanced composite structures a significant amount of resources are spent on inspection and repair of production errors like dry-spots, which are areas where fibers have not been wetted by the resin. The developed procedure is intended to be used to evaluate whether or not a dry-spot or another kind of detected delamination in a composite structure is critical for the structural integrity, and therefore has to be repaired.

The propagation of cracks in fibrous materials is considered to be dependent on the critical energy release rate $G_{c,0}$ at the crack tip, as well as the crack length and end-opening due to fiber bridging across the crack faces behind the crack tip. Fiber bridging increases the fracture resistance against crack propagation until reaching a steady state crack length for which it is assumed constant, and this is accounted for in the developed procedure.

The criteria for determining propagation implies that the energy release rate at the crack tip must reach a material dependent critical value $G_0 \ge G_{c,0}$. The energy release rate G_0 is calculated from the stress intensity factors, and these are estimated by use of the analytical expression for crack face displacements near the crack tip. The increase in fracture resistance from fiber bridging influences this criterion by causing tractions between the crack faces and thereby reduce the relative crack displacement. This means that an increased load can be applied before the critical values are reached.

An extensive study of fracture mechanical theory concerning interface cracks between two isotropic or anisotropic materials, with focus on obtaining expressions for the relative displacement of the crack faces, has been conducted and is described in this report.

A numerical method for simulating this criterion for a specific delamination crack based on solving the structural response using a finite element model is described. Results for the relative crack face displacement near the crack tip are exported to a parameter estimation used for determining the stress intensity factors related to the analytical expressions. This parameter estimation is formulated as an optimization problem minimizing the least squares difference between the numerical results and the analytical expressions by use of the conjugate gradient method with a golden section search.

In fracture mechanics the expressions for the relative crack face displacement are derived as infinite series of eigenfunctions, but in most literature they are presented by the first eigenfunction which only applies very close to the crack tip. As this calls for using small elements compared to structural dimensions in order to yield reliable results in the numerical procedure, analytical expressions for the relative displacement including the second and third eigenfunctions are derived.

These are implemented in the parameter estimation, and studies of the determination of the stress intensity factors show that it yields reliable estimation results for elements up to 10-20 times larger than when using the expressions with the first eigenfunction only. Similar studies have been conducted for orthotropic materials with the same conclusion.

A simulation of propagation of a crack in two similar orthotropic fiber materials with bridging is conducted for a plane DCB model in order to verify the modeling of fiber bridging based on assumed test results. The resulting R-curves are compared to these and show good correlation for the pure mode I and mode II, while for a mixed mode case the steady state fracture resistance is overestimated.

Preface

This Master of Science Thesis is the product of the DMS4 project period of Spring 2010 at Aalborg University, Department of Mechanical and Manufacturing Engineering. It has been published by group 48F. The purpose of this project is to simulate an interface crack with bridging effects between two fibrous composite layers.

The report is most of all directed to the supervisor and censor but also fellow students with an interest in this subject. The reader is assumed familiar with linear fracture mechanics of isotropic materials as well as fibrous composite materials in order to understand the contents of this report.

Structure of the Report

This report consists of a main report accompanied by an appendix in the back of the report. A detailed description of the approach to the project and the structure of the report is given in chapter 1. The appendix part consists of supplementary information and calculations to the parts of the report. There are references from the report to the appendix where appropriate.

Computational calculations are made in MATLAB, analytical expressions are derived using Maple and all finite element analysis are made in ANSYS. All programs and macros used to build and analyze the models studied in this project can be found on the appertaining DVD.

Instructions for Reading

It is expected that the chapters are read together and in their chronological order. The notation used for references in the report is as is in agreement with the Harvard citation style. The used sources are listed in the bibliography in the back of the main report. Figures and tables are numbered sequentially in each chapter and references are made to all figures and tables. The report contains a nomenclature located at on page ix.

Contents of the appertaining DVD

The contents of the enclosed DVD are:

- PDF files of this report
- All MATLAB and Maple programs used in the project
- All macros for generating the finite element models

Nomenclature

$(\delta_y)_i$	Relative displacement between nodepair <i>i</i>
α, β	Dundurs' parameters
δ	Crack mouth opening displacement, CMOD
δ_n, δ_t	Relative normal and tangential displacement of crack faces
$\delta_x, \delta_y, \delta_z$	Relative displacement in x -, y - and z -direction
Г	J-integral path
κ	Poisson parameter for plane stress or strain
λ	Singularity parameter
d (<i>x</i>)	Relative displacement vector
$\mathbf{f}(z)$	Complex potential vector
$\mathbf{h}(x)$	Vector relating the complex potential vectors for material 1 and 2
t (<i>x</i>)	Traction vector
w ₃	Eigenvector
w	Eigenvector
μ	Shear module
μ	Complex root of sixth order characteristic equation
ν	Poission's ratio
$\Phi(z)$	Complex stress potential defined as $\frac{d}{dz}\phi(z)$
$\phi(z)$	Complex stress potential
$\Psi(z)$	Complex stress potential defined as $\frac{d}{dz}\psi(z)$
$\psi(z)$	Complex stress potential
σ_{nn}, σ_{tt}	Stress components rotated according to the nt-coordinate system
σ_n, σ_t	Normal and tangential stresses on crack faces from fiber bridging
θ	Argument for complex number
θ	Rotation angle
ε	Bimaterial constant
Α	Matrix describing the mechanical properties of an anisotropic material
A_j, B_j, C_j, D_j	Complex material parameters
В	Matrix describing the mechanical properties of an anisotropic material
В	Width of the DCB test specimen
С	Constant dependent on material and geometry

Ε	Youngs Modulus for an isotropic material
G	Energy release rate
G_c	Critical energy release rate
Н	Height of each beam of the DCB test specimen
Н	Matrix describing the mechanical properties of an anisotropic bimaterial combination
Ι	Second area moment
j	Index for material 1 or material 2
J_R	Critical fracture resistance
Κ	Complex stress intensity factor $(K_I + iK_{II})$
k	Index for eigenvalue: 0 = main eigenvalue, 1 = second eigenvalue,
K_I	Stress intensity factor for mode I
K _{III}	Stress intensity factor for mode III
K_{II}	Stress intensity factor for mode II
L	Length
L	Matrix describing the mechanical properties of an anisotropic material
M_1	Moment on the upper beam
M_2	Moment on the lower beam
q_1 , q_2	Weighting factors
r _i	Distance from the crack tip to nodepair <i>i</i>
S_{ij}	Compliance matrix
t	Thickness
u, v, w	Displacement in x -, y - and z -direction
Z_i	Parameters used in the parameter estimation procedure
r	Distance from crack tip

<u>X</u>_____

xi

.

Contents

Ab	ostract	vii
Pro	eface	viii
No	omenclature	Х
1	Project Description 1.1 Project Introduction	1 1
	1.2Problem Statement1.3Project Approach	5 5
	 Project limitations	· · · · · · · 7 · · · · · · · 7
2	Crack in a Single Isotropic Material	9
	2.1 Stresses and Displacements Around a Crack Tip	
3	Interface Crack between Two Isotropic Materials	21
	3.1 Theory of Interfacial Crack Between Two Isotropic Materials	
	3.2 Simulation of a Crack Between Two Dissimilar Isotropic Materials .	
	3.3 Presentation of Simulation Results	40
	3.4 Evaluation of the Results of the Parameter Estimation	43
4	Interface Crack between Two Dissimilar Anisotropic Materials	45
	4.1 Theory of Interface Crack between Dissimilar Anisotropic Materials	45
	4.2 Fiber Bridging	53
	4.3 Finite Element Modeling of Bridging	57
	4.4 Description of Test Data	60
	4.5 Simulation of Crack between Two Orthotropic Materials	
	4.6 Analysis of DCB test specifien with a solid FE model	
5	Discussion	77
	5.1 Application of the Developed Procedure	77
	5.2 Further Development	80
Со	onclusion	88
Bil	bliography	89
<u>AP</u>	PPENDIX	93
Ap	pendix A Simulation Results for Isotropic Bimaterial Problem	93
•	A.1 Results Based on Complete 2 Termed Displacement Expansion	93

A.1Results Based on Complete 2 Termed Displacement Expansion93A.2Results Based on Simplified 2 Termed Displacement Expansion94

A.3	Results Based on Complete 3 Termed Displacement Expansion	98
A.4	Results Based on Simplified 3 Termed Displacement Expansion	100
Appendi	ix B Simulation Results for Anisotropic Bimaterial Problem	103

Chapter

1

Project Description

In this chapter the subject of the project is presented and the motivation for treating it is explained. Furthermore, the problem statement and the limitations to the project are clarified and a description of the approach adopted for conducting the project is given in order to give an overview of the contents of the project.

1.1 Project Introduction

In this report the development of a numerical procedure based on fracture mechanics to be used for evaluating stability of a crack in the interface between two fiber composite materials is described. A significant number of mechanical structures such as wind turbine blades, boats and aircrafts are designed with use of fiber composite materials in order to reduce weight, and this gives rise to a wish for improved methods for predicting design limits for these structures. At present, fiber composite structures are almost exclusively designed and analyzed on basis of layered shell element models with design limits being determined from in-plane stress or strain criteria. These criteria do not take out-of-plane effects into consideration which are one of the major causes to fiber layer delamination and related failure of the structure. It is the intension to be able to model this for details in the form of sub-models of a larger model/structure with the procedure developed in this project.

The intended use of the numerical procedure developed in this project is to be able to predict whether a delamination crack in a structure caused by for example a production flaw is critical to the integrity of the structure or not. In this way it can be assessed if the crack should be repaired or not. The goal is to be able to determine a stability criteria for a crack and the effect on this criteria caused by fibers across the two crack surfaces behind the crack tip, the so-called *bridging effect*.

Research in the field of modeling of crack propagation in composite structures is extensive and ongoing. In the wind turbine and aviation industry the long termed goals are to be able to model and analyze crack propagation in large structures such as wind turbine blades or aircraft wings, and in a lager perspective to be able to assess and predict the consequences of fatigue crack growth in composite structures.

The main alternative to the method used in the procedure treated in this project is cohesive modeling which uses elements formulated with a cohesive constitutive law that implies both

crack tip growth and bridging effects. Cohesive modeling has been a significant area of research at Aalborg University in recent years, see for instance (Hansen, 2009).

1.1.1 Concept of Crack Instability Criteria and Bridging

The concept of crack instability criteria with bridging is introduced here in order to clarify the purpose of the method used in the numerical procedure developed. A crack in any given structure of common materials have a fracture toughness against crack growth. The fracture toughness, which is material dependent, is quantified by the critical energy release rate $G_c \left[\frac{J}{m^2}\right]$.

When a crack propagates through an interface between two fibrous composites or in a single fibrous composite material along the fibers, a bridging zone develops behind the crack tip as illustrated in figure 1.1. The fibers across the crack opening create tractions between the crack surfaces which increase the overall fracture toughness of the crack. This is illustrated by so-called *R*-curves (resistance curves) that show the increase in the critical energy release rate with respect to crack opening.





Figure 1.1: (*a*) *Picture of a double cantilever beam (DCB) test specimen with clear visual bridging (Sørensen* et al., 2006). (*b*) *Sketch of a DCB test specimen with fiber bridging across the crack surfaces.*

In figure 1.2 an example of an *R*-curve based on test results from (Sørensen and Jacobsen, 2009) for a specimen similar to the one shown in figure 1.1bis shown. It is seen from the curves that the fracture toughness increases with the opening of the crack until it reaches some assumed steady state level.



Figure 1.2: Test results for fracture resistance $J_R = G_c$ as function of Normal End-Opening of the crack for a test specimen similar to the one seen in figure 1.1b. The material is uni-directional fiber composite, and the specimen is loaded with equal, opposite moments in the test. (Sørensen et al., 2009)

The implications of this R-curve behaviour is explained with the following example for the specimen shown in figure 1.3.



Figure 1.3: DCB test specimen of isotropic material loaded with forces.

The energy release rate of the specimen for this case is on the form (Andreasen, 2008):

$$G(\delta, P) = C \cdot P^{\frac{4}{3}} \cdot \delta^{\frac{2}{3}} \tag{1.1}$$

where:

C δ Constant dependent on material and geometry Crack mouth opening displacement, CMOD

Equation (1.1) is plotted together with an assumed *R*-curve in figure 1.4a. The plane surface represent the initial critical energy release rate, $G_{c,0}$. In this plot it is possible to follow the equilibrium path (in red) during crack propagation. When the driving parameter *P* is increased, $G(\delta, P)$ follows the blue surface up until the initial critical energy release is reached at the intersection point of the three surfaces. Then the crack starts to propagate and as the

crack propagates a bridging zone develops behind the crack tip which due to the *R*-curve effect increases the fracture toughness illustrated by the yellow surface. At some point the force *P* causes the energy release rate $G(\delta, P)$ to exceed the increase in fracture toughness. This point is marked by the red dot in figure 1.4b and is called the limit point of the equilibrium path.



Figure 1.4: (a) General view of fracture toughness without bridging $G_{c,0}$ (black surface) and with bridging G_c (yellow surface) as well as the energy release rate $G(\delta, P)$ for a DCB loaded by equal, opposite forces (blue surface). (b) Plane view of the equilibrium path and indication of the limit point for crack instability.

In figure 1.5a the equilibrium path is shown in the P- δ plane. It is seen that beyond the limit point the equilibrium path has a negative slope meaning that unstable crack propagation occurs beyond this point. The magnitude of P when the instability of the crack occurs is seen to be significantly higher than the force implying the initial crack propagation. In figure 1.5b The limit point is plotted in the G_c - δ plane and it is seen that the steady state of the R-curve has not been reached.



Figure 1.5: (a) The equilibrium path shown in the P- δ plane together with the limit point. (b) The R-curve shown together with the limit point in the G_c - δ plane.

Thus ignoring the bridging effect and only determining the fracture toughness by $G_{c,0}$ is conservative relative to what is actually the case. Considering the steady state fracture toughness $G_{c,SS}$ to be the limiting value on the other hand is rather optimistic in this case. By knowing the two limits $G_{c,0}$ and $G_{c,SS}$ and the shape of the curve G_c from the former to the latter, it is possible to predict the stability of a given crack exposed to a given loading more precisely from a curve as shown in figure 1.5b. In this context it is the objective to be able to predict the course of the fracture toughness curve from $G_{c,0}$ to $G_{c,ss}$ for a specific crack by the procedure developed in this project.

1.2 Problem Statement

The objective of this project is to develop a procedure for determining if a crack in an interface between two dissimilar materials is critical to the integrity of a particular structure. Furthermore, the objective is to gain understanding of theory of advanced fracture mechanics describing interface cracks and how this can be implemented in numerical modeling procedures.

In order to do this, theory describing the physical behavior and growth of a crack in interfaces must be studied. This concern among other things analytical descriptions of the stresses and crack face displacements near the crack tip including treatment of the coupled oscillating stress singularity, determination of the stress intensity factors and the effects of crack mode mixity. Furthermore, the increase in the fracture resistance, R-curve effects, caused by bridging fibers behind the crack tip is treated. A numerical procedure for application of the full formulation fracture mechanics near the crack tip, with a simulation of bridging effects in the crack opening between the crack surfaces, has to be developed.

Finally, an application of the procedure on a two dimensional case for orthotropic fiber composites has to be carried out in order to evaluate the simulation of the R-curve effects by comparing it to R-curves inspired from tests conducted in (Sørensen and Jacobsen, 2009).

1.3 Project Approach

The approach pursued in order to meet the objectives of the problem statement consists of a development procedure in five steps that are described below and are illustrated in figure 1.6:

- 1. The first step treats the theory of fracture mechanics on cracks in a homogeneous isotropic material and the development of a numerical procedure (1) for determining stress intensity factors for mode I and II for a double cantilever beam (DCB) specimen. The numerical results are compared to an analytical solution from (Andreasen, 2008, p. 10) based on beam theory.
- 2. The second step extends the theory and numerical procedure from step 1 to be applicable to a plane crack in the interface between two dissimilar isotropic materials. The numerical results from the extended numerical procedure (2) are compared to an analytical solution from (Suo and Hutchinson, 1990, p. 5) that is also based on beam assumptions.
- 3. The third step deals with a further extended theory and numerical procedure (3) which makes it possible to determine the stress intensity factors for mode I, II and III for a crack in an interface between two anisotropic materials. No analytical solution has been available for this simulation.
- 4. The fourth step concerns an implementation of bridging effects to the numerical procedure in order to model the increase in fracture resistance caused by this physical phenomenon.

5. In the fifth and final step the simulation of R-curve behavior by the numerical procedure developed is evaluated for pure mode I, pure mode II and a mixed mode example. The results are compared to R-curves inspired from tests conducted in (Sørensen and Jacobsen, 2009).



Figure 1.6: Flowchart illustrating the project approach. The analytical solutions for (1) and (2) are from (Andreasen, 2008, p. 10) and (Suo and Hutchinson, 1990, p. 5), respectively

The structure of the project approach is based on a wish to treat the theory and numerical simulation simultaneously and a stepwise verification of the developed procedure with relatively simple examples.

1.4 Project limitations

The development of a procedure for determining stability of an interface crack in this report is treated within the following limitations:

- The theory applied in the developed procedure is based on the assumptions of linear fracture mechanics.
- The main focus is limited to straight crack growth in plane problems.
- The procedure is limited to existing cracks only, and it does not treat simulation of crack initiation.
- The procedure is limited to simulating crack growth in a straight plane between two materials only. Thus changes of crack planes are not treated.

1.5 Structure of the report

This report is structured in the following four parts:

- **Introduction to the project** In this part the general problem of interface crack growth is described with an introduction to significant fracture mechanical concepts related to it. The motivation for working with this type of crack problem and the aim of the project is covered as well as a contextual connection to application areas is described.
- **Theory and simulations of a crack in isotropic materials** This part covers two chapters that describe the fracture mechanical theory used and the numerical procedures developed for determining the stress intensity factors for a crack in a single and in the interface between two dissimilar isotropic materials. Each chapter contains a description of the theory followed by the application in the numerical procedure for each of the two cases.
- **Theory and simulation for crack in anisotropic materials** This part contains a chapter containing a description of theory of an analytical description of crack face displacements for anisotropic materials, which is implemented in the simulation.

Furthermore the concept of fiber bridging is treated and a simulation method is described and implemented in the simulation procedure.

The test and results from (Sørensen and Jacobsen, 2009) is described, and assumptions made in this article are discussed based on a simulation of a three dimensional model of the test specimen.

Finally a two dimensional simulation of a double cantilever specimen of orthotropic fiber materials is described in order to evaluate the R-curve effects in comparison with R-curves inspired from the article.

Discussion of the procedure developed This covers a general discussion and evaluation of the simulation procedure developed, and puts it in into perspective with suggestions for prospective developments and possible applications of the procedure.

Chapter

2

Crack in a Single Isotropic Material

The purpose of this chapter is to briefly describe the stresses and displacement expressions for a crack in an isotropic material. This makes the base for a description of a numerical determination of stress intensity factors for this type of crack by fitting finite element result for crack face displacements to analytical expressions.

2.1 Stresses and Displacements Around a Crack Tip

In this section the expressions for stresses and displacements around the crack tip from the linear elastic fracture mechanics (LEFM) for a crack in a single isotropic material are briefly described and plotted.

The objective of this report is to develop a numerical method for determining stability of a crack between two similar or dissimilar materials. In order to do this the energy release rate for the crack has to be calculated for a specific load, and then compared to a critical value that is material dependent. It is intended to do this by calculating the stress intensity factor(s) based on finite element results of stresses or displacements. From the stress intensity factors the energy release rate can be calculated.

The stresses around the crack tip for the three fracture modes are (Kildegaard, 2002, p. 26, 32, 37-38):

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix}$$
(2.1)

Mode II

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix} -\sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \\ \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \end{bmatrix}$$
(2.2)

Mode III

$$\begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}} \begin{bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}$$
(2.3)

These stress expressions are dependent on the stress intensity factors K_I , K_{II} and K_{III} for the three modes respectively. This makes it possible to determine the stress intensity factors from these expressions, if the stresses in the vicinity of the crack tip are determined numerically. The stress expressions each contain a singularity at the crack tip for $r \rightarrow 0$ which is clearly seen in the 3D plots in figures 2.1a and 2.1b for pure mode I loading. This singularity makes it difficult to simulate the stresses in a finite element model, and this subject is treated in the following section.



Figure 2.1: (a) Stresses σ_{yy} for mode I with $K_I = 1.0$ (b) Stresses τ_{xy} for mode I with $K_I = 1.0$.

In the figures 2.2a and 2.2b the stresses around the crack tip are plotted as contourplots in order to illustrate that the normal stresses σ_{yy} are symmetric and the shear stresses σ_{xy} are anti-symmetric about the crack line for a single isotropic material.



Figure 2.2: (a) Contourplot of stresses σ_{yy} for mode I with $K_I = 1.0$. This plot shows that these stresses are symmetric about the crack plane. (b) Contouplot of stresses τ_{xy} for mode I with $K_I = 1.0$. This plot shows that these stresses are anti-symmetric about the crack plane.

The displacements around the crack tip for the three fracture modes are (Kildegaard, 2002, p. 27, 32, 38):

Mode I

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{2(1+v)}{E} K_I \sqrt{\frac{r}{2\pi}} \begin{bmatrix} \cos\frac{\theta}{2} & \frac{\kappa-1}{2} + \sin^2\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \left[\frac{\kappa+1}{2} - \cos^2\frac{\theta}{2} \right] \end{bmatrix}$$
(2.4)

. .

Mode II

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{2(1+v)}{E} K_{II} \sqrt{\frac{r}{2\pi}} \begin{bmatrix} \sin\frac{\theta}{2} \left\lfloor \frac{\kappa+1}{2} - \cos^2\frac{\theta}{2} \right\rfloor \\ \cos\frac{\theta}{2} \left\lfloor \frac{\kappa-1}{2} + \sin^2\frac{\theta}{2} \right\rfloor \end{bmatrix}$$
(2.5)

Mode III

$$\left[w\right] = 4 \frac{1+v}{E} K_{III} \sqrt{\frac{r}{2\pi}} \left[\sin\frac{\theta}{2}\right]$$
(2.6)

Similarly to the stresses these expressions are also dependent on the stress intensity factors K_I , K_{II} and K_{III} for the three modes respectively, and this makes it possible to determine these from displacement results from a finite element analysis. The factor \sqrt{r} causes the shape of the displacements to be parabolic, and this is seen in figures 2.3a and 2.3b for pure mode I loading.



Figure 2.3: (a) Displacements v in y-direction for mode I for $K_I = 1.0$ (b) Displacements u in x-direction for mode I for $K_I = 1.0$.

The corresponding contourplots in figures 2.4a and 2.4b show that the displacements *v* are anti-symmetric while *u* are symmetric about the crack line.



Figure 2.4: (a) Contouplot of displacements v in y-direction for mode I for $K_I = 1.0$ (b) Contourplot of displacements u in x-direction for mode I for $K_I = 1.0$

Expressions for the crack face displacements are obtained by setting $\theta = +\pi$ for the upper face and $\theta = -\pi$ for the lower face, and the relative displacements of these are described for the three modes by the following expressions (Kildegaard, 2002, p. 28, 33, 38).

Mode I

$$\delta_y = \nu^+ + \nu^- = 2 \frac{(1+\nu)(\kappa+1)}{E} K_I \sqrt{\frac{r}{2\pi}}$$
(2.7)

Mode II

$$\delta_x = u^+ + u^- = 2 \frac{(1+\nu)(\kappa+1)}{E} K_{II} \sqrt{\frac{r}{2\pi}}$$
(2.8)

Mode III

$$\delta_z = w^+ + w^- = 8 \frac{1+v}{E} K_{III} \sqrt{\frac{r}{2\pi}}$$
(2.9)

The following section describes a numerical determination of the stress intensity factors by using the expressions described here. This is based on finite element analyses for a specific crack in an isotropic material.

2.2 Simulation of a Crack in a Isotropic Material.

In this section the numerical application of the fracture mechanical theory for cracks in a single isotropic material is treated. The focus is on a selection of a robust method for calculating the stress intensity factors.

A numerical calculation of the stress intensity factors for a crack involves a method for modeling and treating the stress singularity at the crack tip. The numerical model used here is a double cantilever beam (DCB) because the test data from (Sørensen and Jacobsen, 2009), which are modeled in chapter 4, are for this type of test specimen. Dimensions and crack length of the model are seen in figure 2.5a for a deformed plot of the DCB test specimen.



Figure 2.5: (a) Deformed plot of the double cantilever beam (DCB) specimen used in the finite element model. (b) The model geometry with subdivision into 8 areas and load configurations: 1) = Mode I force load 2) = Mode II force load 3 and 4) = Mode I or mode II moment load. (c) Meshed finite element model of the DCB specimen with mode I deflection.

The DCB is modeled in ANSYS as a 2D plane stress model using Plane183 elements with unit thickness, and load options for force loads in both coordinate directions and for moment loading. The DCB model is divided into four rectangular areas with the crack running between the two to the left as indicated in figure 2.5b. This division into four parts is mainly intended for the situation with two different materials and thus not directly useful here.

Four quarter circular areas are defined around the crack tip in order to control the mesh and mesh size in this area which is further commented below. At the left side end of the model a number of keypoints are defined which are used for applying the loads to the model. An example of the meshed model deformed by mode I loading is seen in figure 2.5c.

2.2.1 Elements and Mesh

A common method for modeling the stress singularity when approaching the crack tip is by using quarter point elements (Barsoum, 1977) instead of ordinary plane eight node rectangular elements. Plane quarter point elements are eight node rectangular elements for which three nodes at one side are collapsed to be coincident at the crack tip node, which turns it into a triangular element. Furthermore, the mid-side nodes of the two sides intersecting at the collapsed node are moved to one fourth of the side length as illustrated in figure 2.6. This alteration of the element gives stresses at the collapsed nodes with a $\frac{1}{\sqrt{r}}$ singularity which complies with the

theory of linear fracture mechanics. As will be shown later, it is not possible to obtain stresses on the exact same form as the analytical expression predict, but the quarter point elements represents the stress singularity far better than regular 8 node elements.



Figure 2.6: Transformation by collapse of three side nodes of a rectangular eight node element to a triangular six node quarter point element

Quarter point elements are available in ANSYS and the triangular shape of these is the reason for the circular area around the crack tip in the model. The mesh around the crack tip is seen in figure 2.7. Some of the rectangular elements away from the crack tip mesh are collapsed during the meshing procedure which often means that the stiffness of these elements is increased. This influences the stresses locally, but is ignored as the number of distorted elements is insignificant close to the crack tip where the stresses are sought.



Figure 2.7: The mesh of the model around the crack tip. The triangular quarter point elements are placed in a circular formation around the crack tip.

2.2.2 Methods for Determining the Stress Intensity Factors

The methods presented here for determining the stress intensity factors for a given crack is based on the stress and relative displacement expressions in equation (2.1) and (2.7). These give the stress and displacement at the crack tip as a function of the stress intensity factors. The approach for determining the stress intensity factors is to determine the stress or displacement at the crack tip using a FE model. These data is then used to determine what value of the stress intensity factors ensures the best agreement between the data from the FE model and the analytical expression for the stresses and displacements.

A study is conducted on whether it is best to use the stress or displacement expression for determining the stress intensity factors. This is done using the before mentioned FE model of a DCB specimen. The DCB specimen is given the material properties of aluminum and is loaded in mode I by pure moments, see figure 2.8.



A plot of the FE determined σ_{yy} values is seen in figure 2.9 along with the analytically determined stress values. Both graph has been multiplied by \sqrt{r} in order better to see how they diverge from each other. It is seen that the FE values are almost constant except for the two points closest to the crack tip which deviate significantly. The stress values close to the crack tip are important for the determination of the stress intensity factors because of the limitation of the functions to only apply in the close vicinity of the crack tip. Based on the large deviation of the FE values compared to the analytical values, it is concluded that the stress data from the FE model are not suited for determining the stress intensity factors.



Figure 2.9: Plot of analytical and finite element stresses σ_{yy} in front of the crack tip multiplied by \sqrt{r} . The deviation of the finite element results close to the crack tip should be noted

As will be clarified in the following, the displacements from the FE model are better suited for determining the stress intensity factors. Two methods based on the displacements are presented and evaluated against each other. These are fitting the FE data for the relative displacement of the crack faces to (1) the analytical expression for the relative displacement for mode I in equation (2.7) and to (2) an extension of the expression in equation (2.7). The first is denoted one termed displacement expression and the second is denoted two termed displacement expression, which will be elaborated on later. The fitting criterion used is the least squares method. The results from the two fits are compared to an analytical determined K_I value based on a beam solution obtained from (Andreasen, 2008):

$$K_I = \sqrt{-\frac{1}{t}\frac{M^2}{I}} \tag{2.10}$$

$$=\sqrt{-\frac{1}{1mm}\frac{(1000\text{Nmm})^2}{83.3\text{mm}^4}}$$
(2.11)

$$= 109.54 MPa\sqrt{mm} \tag{2.12}$$

where: I Second area moment t Thickness

2.2.3 Determining K_I with One Termed Displacement Expression

The relative *y*-displacement of the crack faces δ_y is given by.

$$\delta_y = \frac{2(1+\nu)(\kappa+1)}{\sqrt{2\pi}E} K_I \sqrt{r} \tag{2.13}$$

For use with the least squares method, the equation is rewritten into the following where the constant *C* is determined when fitting the expression.

$$\frac{\delta_y}{\sqrt{r}} = \frac{2(1+\nu)(\kappa+1)}{\sqrt{2\pi}E} K_I \equiv C$$
(2.14)

Then K_I is determined as:

$$K_{I} = \frac{\sqrt{2\pi}E}{2(1+\nu)(\kappa+1)}C$$
(2.15)

In the discretized case of the numerical model with values for the 1, ..., *n* FE nodes from the crack tip, *C* is calculated from the linear least square method as (Kreyszig, 2005, pp. 859-862):

$$C = \frac{\sum_{i=1}^{n} (\delta_{y})_{i}}{\sum_{i=1}^{n} \sqrt{r_{i}}}$$
 for i=1,...,n (2.16)

where: $(\delta_y)_i$ Relative displacement between nodepair i r_i Distance from the crack tip to nodepair i

2.2.4 Determining K_I with Two Termed Displacement Expression

The two termed curve fit is adapted from the ANSYS documentation and is based on a two termed series expansion for the displacements which is described in the next chapter. This means that the term $Br^{\frac{3}{2}}$ is added to the relative displacement expression in equation (2.13).

$$\delta_y = \frac{2(1+\nu)(\kappa+1)}{\sqrt{2\pi}E} K_I \sqrt{r} + Br^{\frac{3}{2}}$$
(2.17)

$$=Ar^{\frac{1}{2}} + Br^{\frac{3}{2}} \quad \Rightarrow \tag{2.18}$$

$$\frac{\delta_y}{\sqrt{r}} = A + Br \tag{2.19}$$

In the discretized case of the numerical model with values for the 1,..., *n* FE nodes from the crack tip, *A* and *B* are determined from the linear least square method which yields the equations in (2.20) (Kreyszig, 2005, pp. 859-862):

$$A \cdot n + B \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} \left(\frac{(\delta_y)_i}{\sqrt{r_i}} \right)$$

$$A \sum_{i=1}^{n} r_i + B \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} r_i \left(\frac{(\delta_y)_i}{\sqrt{r_i}} \right)$$
(2.20)

The coefficients *A* and *B* are the determined as:

$$B = \frac{n \sum_{i=1}^{n} r_i \left(\frac{(\delta_y)_i}{\sqrt{r_i}}\right) - \sum_{i=1}^{n} \left(\frac{(\delta_y)_i}{\sqrt{r_i}}\right) \sum_{i=1}^{n} r_i}{n \sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} \left(\frac{(\delta_y)_i}{\sqrt{r_i}}\right)}$$
(2.21)
$$A = \frac{\sum_{i=1}^{n} \left(\frac{(\delta_y)_i}{\sqrt{r_i}}\right) - B \sum_{i=1}^{n} r_i}{n}$$
(2.22)

When A and B are determined, then by letting $r \rightarrow 0$ the first order term vanishes and then K_I can be determined from A equivalently to the first method.

$$K_{I} = \frac{\sqrt{2\pi}E}{2(1+\nu)(\kappa+1)}A$$
(2.23)

2.2.5 Comparison of the One and Two Term Fit

The one term and two term fits for calculating the stress intensity factor are compared to each other in order to select the most robust one. Results for the two fits are plotted together with the analytical displacements of the crack surface in figure 2.10 and divided by \sqrt{r} in figure 2.11. These plots are for an element length around the crack tip of 0.15mm and show that the two term-fit is significantly closer to the analytical solution than the one term fit. This is expected to be due to the higher order term of r in the curve fit.

Furthermore, the displacements from an FE analysis with and one without quarter point elements around the crack tip are plotted in the two figures, which show that the displacements are lower without quarter point elements causing the calculated value of K_I to be significantly lower than the analytical value.

The sensitivity of the one term and two term fits to the number of nodes on the crack faces used in the curve fit when calculating K_I has been tested on the FE model, and the results



Figure 2.10: Plot of analytical solution and curve fit of the mode I crack faces displacement δ_y for the one term and two term fit. 7 node values from the FE analyses with and without QP elements (Element size = 0.15 mm) from which the two node pairs closest to the crack tip are used in the curve fit for both methods.



Figure 2.11: Plot of analytical solution and curve fit of the mode I crack surface displacement δ_y divided by \sqrt{r} for the one term and two term fit, and with 7 node values from the FE analyses with and without QP elements (Element size = 0.15 mm) from which the two node pairs closest to the crack tip are used in the curve fit for both methods.

are seen in figure 2.12. It shows that the determination of K_I becomes more precise when the number of nodes are decreasing.



Figure 2.12: Sensitivity to number of nodes used in the calculation of K_I by the one term and two term fit. The value from beam theory calculation is independent of node number.



Figure 2.13: Sensitivity to size of quarter point and rectangular elements around the crack tip of the one term- and two term-fit for element sizes of [0.05, 0.15, 0.25, 0.50[mm]].

Furthermore the sensitivity of the two fits to the element length around the crack tip is tested. This is done for element lengths of [0.05, 0.15, 0.25, 0.50[mm]] and the results are plotted in figure 2.13 together with the result for the calculation based on beam theory.

The plot clearly shows that the two term-fit is roughly independent of element length and that the one term-fit goes towards the analytical value for the element length going towards zero. This shows that in order to get adequately precise results, the one term-fit is only applicable to nodes in the close vicinity of the crack tip in accordance with the theory. On the contrary, due to the second term in the two term-fit, this fit can be used on nodes a larger distance away from the crack tip. This makes the two term-fit more robust and facilitates use of a more coarse mesh that still gives satisfactory results. Because of this, the two term-fit is considered most suitable for the calculation and is chosen as the method to be used further on in the report.

The two curve fit methods have been tested and compared for mode II with the same conclusions.

Chapter

3

Interface Crack between Two Isotropic Materials

The purpose of this chapter is to derive and explain the complete expressions for the stresses and displacements at the crack tip of an interface crack between two dissimilar isotropic materials. Furthermore, the energy release rate as a function of the complex stress intensity factor is given as a function of the stress intensity factors. The expressions for the relative crack surface displacements are applied in a simulation of a crack between to dissimilar isotropic materials where different studies are conducted on how to determine the stress intensity factors and the energy release rate.

In contrast to a crack in a single isotropic material each stress or displacement component in the case of a crack in the interface of two dissimilar isotropic materials do not depend uniquely on a single outer loading mode. Instead, an outer loading mode I on a two material specimen implies a local mode mixity at the crack tip of both mode I and II. See figure 3.1 for an illustration.



Figure 3.1: Illustration of an outer mode I loading and indication of the mode mixity at the crack tip.

The reason for this local mode mixity is that the two materials have different stiffness properties. This implies that the two materials strain differently, which causes shear stress along the interface and thus mode II deformation of the crack surface and around the crack tip.

The motivation for the derivation of the stress and displacement description in this section is to gain a deeper understanding of the background for the stress and displacement description for an interface crack between two dissimilar isotropic materials described in article (Suo and Hutchinson, 1990). In this article the stress and displacement are given for the crack interface ($\theta = 0$), only.

Although the stress or displacement description at the interface is sufficient for determining the stress intensity factors by numeric simulation, the full field solutions for stresses and displacements around the crack tip are considered relevant to obtain. This is seen in the perspective of achieving a wide comprehension of the stresses and displacements at an interfacial crack tip and mathematical method used for describing them.

Furthermore, it has only been possible to obtain the first eigenfunction expressions for the stresses and displacements for the interface between the two materials from the article (Suo and Hutchinson, 1990). This means that these expressions are only valid very close to the crack tip and not suited for use with the FE analysis for determination of the stress intensity factors that must be conducted because this calls for very small elements in the model. Hence, the rest of the terms in the series describing the stresses and displacements are derived here.

The approach when dealing with interface cracks between two dissimilar isotropic materials and the idea of terms like the bimaterial constant and oscillating singularities, which are treated later in this chapter, are equivalent to the ideas and terms figuring in the more general description of an interfacial crack between anisotropic materials which are treated in chapter 4. Therefore a thorough treatment of these terms and ideas are presented here both for use in the simulation of a crack between two dissimilar isotropic materials as well as a basis for the theory presented in chapter 4.

3.1 Theory of Interfacial Crack Between Two Isotropic Materials

The derivation of the stress and displacement description at the crack tip follows the approach of (Muskhelishvili, 1977) where the stress and displacement fields are expressed by complex stress potentials, which are to be determined. The software Maple 13 has been used as a tool for the derivation process and illustrations. All relevant Maple documents can be found on the appertaining DVD.



Figure 3.2: Definition of coordinate systems used for the derivation of the complex stress and displacement. The *r*-axis starts at the crack tip.

In the approach for determining stresses and displacements at the crack tip they are formulated on complex form as $(\sigma_{yy} + i\sigma_{xy})$ and $(u_x + iu_y)$ which is called complex stress and complex displacement, respectively. This involves describing the stress and displacement in terms of complex stress potentials which are functions of the complex variable *z*. For the crack
illustrated in figure 3.2 the following complex displacement is given (Rathkjen, 1984, p. 33).

$$u_x + i u_y = \frac{1}{2\mu_j} \left(\kappa_j \cdot \phi(z) - z \cdot \overline{\Phi(z)} - \overline{\psi(z)} \right)$$
(3.1)

where:	$\phi(z)$	Complex stress potential
	$\psi(z)$	Complex stress potential
	$\Phi(z)$	Complex stress potential defined as $\frac{d}{dz}\phi(z)$
	μ	Shear module
	κ	Poisson parameter for plane stress or strain
	j	Index for material 1 or material 2

In the same way the so called fundamental stress combinations expressed by stress potentials in cartesian coordinates are given as (Rathkjen, 1984, p. 27).

$$\sigma_{xx} + \sigma_{yy} = 2\left(\Phi(z) + \overline{\Phi(z)}\right) \tag{3.2}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left(\overline{z}\left(\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z)\right) + \Psi(z)\right)$$
(3.3)

where: $\Psi(z)$ Complex stress potential defined as $\frac{d}{dz}\psi(z)$

The sum of these fundamental stress combinations eliminates σ_{xx} and yields the following complex expression for the σ_{yy} and σ_{xy} which are of interest for the stress analysis at the crack tip, because these stresses are the only stresses that influence the propagation of a crack orientated as in figure 3.2.

$$\sigma_{yy} + i\sigma_{xy} = \Phi(z) + \overline{\Phi(z)} + \overline{z} \left(\frac{\mathrm{d}}{\mathrm{d}z} \Phi(z)\right) + \Psi(z) \tag{3.4}$$

In order to obtain the complex stress and complex displacement the task is to determine the stress potentials. To do this the complex stress and complex displacement are transformed to a *nt*-coordinate system that is rotated θ in relation to the *xy*-coordinate system, see figure 3.2. This is partly due to the way these types of problems has been solved in the past and because it makes it possible to solve for wedge formed crack geometries instead of closed crack geometries which in the end, though, are treated in the present derivation.

The complex stress and displacement transformed to the *nt*-system expressed by stress potentials are given as (Rathkjen, 1984, p.33):

$$\sigma_{tt} + i\sigma_{nt} = \Phi(z) + \overline{\Phi(z)} + \left(\overline{z} \left(\frac{\mathrm{d}}{\mathrm{d}z} \Phi(z)\right) + \Psi(z)\right) e^{2i\theta}$$
(3.5)

$$u_n + i u_t = \frac{1}{2\mu} \left(\kappa_j \cdot \phi(z) - z \cdot \overline{\Phi(z)} - \overline{\psi(z)} \right) e^{-i\theta}$$
(3.6)

The complex stress and displacement in the nt-coordinates (3.5) and (3.6) are used to determine the stress potentials figuring in the complex stress and displacement in equation (3.1) and (3.4). This is done by applying the boundary conditions for the stresses on the crack faces and behind the crack tip, and boundary conditions for the displacement across the interface. The boundary conditions are described in section 3.1.1 With these boundary conditions

around the crack tip in mind the stress potentials are guessed to be in the form showed in the following equations (Williams, 1959, p. 200):

$$\phi_j(z) = A_j z^{\lambda + i\varepsilon} + B_j z^{\lambda - i\varepsilon} \tag{3.7}$$

$$\psi_j(z) = C_j z^{\lambda + i\varepsilon} + D_j z^{\lambda - i\varepsilon}$$
(3.8)

$$\Phi_j(z) = \frac{\mathrm{d}}{\mathrm{d}z}\phi_j = \frac{A_j z^{\lambda+i\varepsilon} \left(\lambda+i\varepsilon\right)}{z} + \frac{B_j z^{\lambda-i\varepsilon} \left(\lambda-i\varepsilon\right)}{z}$$
(3.9)

$$\Psi_j(z) = \frac{\mathrm{d}}{\mathrm{d}z} \psi_j = \frac{C_j z^{\lambda+i\varepsilon} \left(\lambda+i\varepsilon\right)}{z} + \frac{D_j z^{\lambda-i\varepsilon} \left(\lambda-i\varepsilon\right)}{z}$$
(3.10)

where: A_j, B_j, C_j, D_j Complex material parameters λ Singularity parameter ε Bimaterial constant

The reason that the stress potentials are guessed to be in this form is that they have the ability to produce the stress singularity at the crack tip from the power of the real number λ as $z^{\lambda+i\varepsilon} = r^{\lambda+i\varepsilon}e^{i\theta(\lambda+i\varepsilon)}$ and the denominator, z in the Φ and Ψ stress potentials. In addition this form of the stress potentials shows a discontinuity across the crack surfaces as θ is uniquely defined in the interval $[-\pi, \pi]$ only. Thus when going from the upper crack surface to the lower, θ changes from $+\pi$ to $-\pi$ (see figure 3.3).

The following part of the derivation concerns a verification showing that stress potentials on this form are applicable and a determination of the constants figuring in the stress potentials as well as a solution for λ and ε .

By inserting the stress potentials Ψ_j and Φ_j into (3.5) and by converting the complex variable z to polar form as $z = re^{i\theta}$ and $z^{\lambda+i\varepsilon} = r^{\lambda+i\varepsilon}e^{i\theta(\lambda+i\varepsilon)}$ the complex stress in nt-coordinates is expressed as:

$$\sigma_{tt} + i\sigma_{nt} = \frac{1}{r^{1-\lambda}} \left(\left[(\lambda + i\varepsilon)^2 e^{i\theta(\lambda + i\varepsilon - 1)} A_j + (\lambda - i\varepsilon)^2 e^{i\theta(\lambda - i\varepsilon - 1)} B_j + (\lambda + i\varepsilon) e^{i\theta(\lambda + i\varepsilon)} e^{i\theta} C_j + (\lambda - i\varepsilon) e^{i\theta(\lambda - i\varepsilon)} e^{i\theta} D_j + (\lambda - i\varepsilon) e^{-i\theta(\lambda - i\varepsilon + 1)} \overline{A_j} + (\lambda + i\varepsilon) e^{-i\theta(\lambda + i\varepsilon + 1)} \overline{B_j} \right] \cos(\ln(r)\varepsilon) + i \left[(\lambda + i\varepsilon)^2 e^{i\theta(\lambda + i\varepsilon - 1)} A_j - (\lambda - i\varepsilon)^2 e^{i\theta(\lambda - i\varepsilon - 1)} B_j + (\lambda + i\varepsilon) e^{i\theta(\lambda + i\varepsilon)} e^{i\theta} C_j - (\lambda - i\varepsilon) e^{i\theta(\lambda - i\varepsilon)} e^{i\theta} D_j - (\lambda - i\varepsilon) e^{-i\theta(\lambda - i\varepsilon + 1)} \overline{A_j} + (\lambda + i\varepsilon) e^{-i\theta(\lambda + i\varepsilon + 1)} \overline{B_j} \right] \sin(\ln(r)\varepsilon) \right)$$
(3.11)

The cosine and sine terms in equation (3.11) emerges from $r^{i\varepsilon} = e^{i(\ln(r)\varepsilon)} = \cos(\ln(r)\varepsilon) + i\sin(\ln(r)\varepsilon)$ and cause a so-called oscillating singularity to the stress description for $r \to 0$. It should be noted in equation (3.11) that the polar representation uses the same angle θ as is used in the coordinate transformation from the *xy*-system to the *nt*-system. This implies that a chosen point $z = re^{i\theta}$ always lies on the *n*-axis.

In the same manner as for the complex stress the complex displacement from equation (3.6) is expressed as:

$$u_{n} + iu_{t} = \frac{r^{\lambda}}{e^{i\theta}} \cdot \frac{1}{2\mu_{j}} \cdot \left(\left[\kappa_{j} \left(A_{j} e^{i\theta(\lambda + i\varepsilon)} - B_{j} e^{i\theta(\lambda - i\varepsilon)} \right) + A_{j} e^{i2\theta} e^{-i\theta(\lambda - i\varepsilon)} \right] + B_{j} e^{i2\theta} e^{-i\theta(\lambda + i\varepsilon)} \right) + \frac{1}{A_{j}} e^{i2\theta} e^{-i\theta(\lambda - i\varepsilon)} - \frac{1}{D_{j}} e^{-i\theta(\lambda + i\varepsilon)} = i\sin(\ln(r)\varepsilon) + \left[\kappa_{j} \left(A_{j} e^{i\theta(\lambda + i\varepsilon)} + B_{j} e^{i\theta(\lambda - i\varepsilon)} \right) + \left[\kappa_{j} \left(A_{j} e^{i\theta(\lambda - i\varepsilon)} + B_{j} e^{i\theta(\lambda - i\varepsilon)} \right) + \frac{1}{A_{j}} e^{i2\theta} e^{-i\theta(\lambda - i\varepsilon)} \right] \cos(\ln(r)\varepsilon) + i\varepsilon \right) - \frac{1}{A_{j}} e^{i2\theta} e^{-i\theta(\lambda - i\varepsilon)} - \frac{1}{D_{j}} e^{-i\theta(\lambda + i\varepsilon)} = \cos(\ln(r)\varepsilon) \right]$$
(3.12)

It is seen that the expression for the complex displacement consists of an oscillating sine and cosine term similar to the complex stress.

3.1.1 Applying the Boundary Conditions

It is assumed that boundaries other than the crack faces and the unbroken interface between the two materials can be ignored because a solution is sought in the close vicinity of the crack tip only. In this way the outer geometry of a given structure does not influence the solution. As will be shown later the outer geometry and external load are accounted for in the complex stress intensity factor $K = K_I + iK_{II}$. The boundary conditions for the crack faces and the interface are illustrated in figure 3.3 and given as:

- 1. No stresses on the crack faces for material 1 and 2.
- 2. Continuity of σ_{xy} and σ_{yy} across the interface between material 1 and 2.
- 3. Continuity of u_x and u_y across the interface between material 1 and 2.

These boundary conditions are used in the following to determine the parameters $A_1, ..., D_2$. This results in an eigenvalue problem with a system of 8 equations.

Applying boundary condition 1 The C_1 and D_1 coefficients of the complex stress potentials for material 1 are obtained by applying the boundary conditions of item 1 in the list above. For the complex stress component on the crack faces to equal zero for material 1, that is $\sigma_{yy} + i\sigma_{xy} = 0$ at $\theta = \pi$, the coefficients of the cosine and sine terms in equations (3.11) must equal zero. From the cosine term $C_1(A_1, B_1)$ is obtained and from the sine term the expression for $D_1(A_1, B_1, C_1)$ is obtained. $C_2(A_2, B_2)$ and $D_2(A_2, B_2, C_2)$ are obtained similarly for $\sigma_{yy} + i\sigma_{xy} = 0$ at $\theta = -\pi$.

Applying boundary condition 2 The A_2 and B_2 coefficients are obtained by applying the boundary conditions of item 2. In order for the stresses to be continuous across the interface



Figure 3.3: Illustration of boundary conditions for the crack faces and the interface ahead of the crack tip. The situation with $\theta = \pm \pi$ is the limiting case of a wedge shaped crack and is the type concerned in this project.

they must be equal here. Hence, the difference between the stresses in material 1 and 2, respectively, must be zero. This difference can be rewritten into a form similar to equation (3.11) with a sine and cosine term, and the coefficients of the sine and cosine terms must both equal zero for the difference to be zero. From the coefficient of the sine term $A_2(A_1)$ is obtained and from the coefficient of the cosine term $B_2(A_1, B_1, A_2)$ is obtained.

Applying boundary condition 3 The B_1 coefficient is obtained by applying the boundary conditions of item 3. For the displacements across the interface to be continuous the difference between the displacement of material 1 and 2 at the interface must equal zero, which is similar to the application of item 2 of the boundary conditions. Again the difference is rewritten to contain a sine term and a cosine term, only. By setting the coefficient of the sine term equal to zero $B_1(A_1)$ is obtained.

If the found expression for $A_1(B_1)$ is inserted into the coefficient of the cosine term the following equation must be fulfilled in order to obtain nontrivial solution where $A_1 \neq 0$.

$$0 = (1+\beta) e^{i2\pi(\lambda+i\varepsilon)} - \beta + 1$$
(3.13)

Where β is one of Dundurs' parameters that are defined as (Suo and Hutchinson, 1990, p.3):

$$\beta = \frac{\frac{\mu_1}{\mu_2} (\kappa_2 - 1) - \kappa_1 + 1}{\frac{\mu_1}{\mu_2} (\kappa_2 + 1) + \kappa_1 + 1}$$
(3.14)

Hence, the determination of the coefficients figuring in the complex stress potentials is the same as solving for the eigenvalues of the system of equations defining the coefficients. All eigenvalues for the system can be found as roots of equation (3.13). An eigenvalue for this system is given in equation (3.15) which is referred to as the main solution and is the one the formulation of the one termed expressions for the stress and displacement are based on.

$$\lambda + i\varepsilon = \frac{1}{2} - i\frac{1}{2\pi}\ln\left(\frac{1-\beta}{1+\beta}\right)$$
(3.15)

All other eigenvalues for this problem are given as:

$$\lambda + i\varepsilon = \frac{(1+2k)}{2} - i\frac{1}{2\pi}\ln\left(\frac{1-\beta}{1+\beta}\right) \quad \text{where} \quad k \in \mathbb{Z}$$
(3.16)

The solutions of the eigenvalue problem is illustrated in figure 3.4 where it is seen that ε remains constant while λ has infinitely many solutions.



Figure 3.4: Illustration of equation (3.13) for a given value of β

All negative λ eigenvalues implies so-called super singularities in the complex stress expression defined as $\frac{1}{(\sqrt{r})^n}$ where n is an integer larger than 1, see table 3.1. The negative eigenvalues that implies super singularities are not used in the following treatment of deriving the complex stress and complex displacement expressions because it can be shown that they cause nonphysical material behavior which involves infinite strain energy at the crack tip which is treated in (Hutchinson *et al.*, 1990). All positive λ_k eigenvalues imply finite strain energy. From the positive λ_k eigenvalues the main solution $(\lambda_1 = \frac{1}{2})$ is the most dominating factor for $r \rightarrow 0$. Hence, the other positive λ_k eigenvalues are ignored as only the area in the close vicinity of the crack tip is of interest. When *r* approaches 1.0 and above the dominating factors for higher positive values for λ_k become significant and cannot be neglected.



Table 3.1: Table showing the dominating factor when r approaches zero in the complex stress expression (3.11) for λ_k eigenvalues close to zero.

As discovered in the simulation of a crack in a single isotropic material in chapter 2 there is in some cases reason for including the next eigenvalue in the complex stress or displacement description in order to utilize the use of larger elements at the crack tip while still being able to obtain good estimates of the stress intensity factors. Hence, the displacement expression to be used in the curve fit for determining the stress intensity factor for an interface crack must also contain terms for the first and the second positive eigenvalue.

In order to fit the numerical results to the analytical solution containing these terms it is in principle necessary just to know the dominating factor for the second eigenvalue term and not the coefficient. For the first eigenvalue term though, the dominating factor and coefficient of the main eigenvalue must be known. This is because the stress intensity factors can be calculated from the coefficient of the main dominating factor alone.

By inserting the eigenvalues λ_k and ε into the seven coefficients A_2 , B_1 , B_2 , C_1 , C_2 , D_1 and D_2 determined from solving the equations obtained from the boundary conditions they reduce significantly which simplifies the expression for the complex stress and displacement. The reduced coefficients are:

$$A_2 = \frac{1-\beta}{1+\beta}A_1 \tag{3.17}$$

$$B_1 = 0$$
 (3.18)

$$B_2 = 0$$
 (3.19)

$$C_1 = -\left(\lambda_k + i\varepsilon\right)A_1\tag{3.20}$$

$$C_2 = -\frac{1-\beta}{1+\beta} \left(\lambda_k + i\varepsilon\right) A_1 \tag{3.21}$$

$$D_1 = \frac{1-\beta}{1+\beta}\overline{A_1} \tag{3.22}$$

$$D_2 = \overline{A_1} \tag{3.23}$$

From the complex stress equation (3.4) the following eigenfunctions describing the complex stress $\sigma(\lambda_k)$ is obtained by inserting the above coefficients.

$$\sigma(\lambda_k) = \frac{1}{r^{\lambda_k}} \left(\frac{r^{i\varepsilon} \cdot \overline{K}}{\sqrt{2\pi}} c^{(1)} + \frac{r^{-i\varepsilon} \cdot K}{\sqrt{2\pi}} c^{(2)} \right)$$
(3.24)

where:

Complex stress intensity factor $(K_I + iK_{II})$

Index for eigenvalue: 0 = main eigenvalue, 1 = second eigenvalue,...

The coefficients $c^{(1)}$ and $c^{(2)}$ are given by:

Κ

k

$$c^{(1)} = -\frac{1}{2}(1+\beta)(\lambda_k - i\varepsilon - 1)e^{i\theta(\lambda_k + i\varepsilon)} \cdot \left(e^{-3i\theta} - e^{-i\theta}\right)$$
(3.25)

$$c^{(2)} = \frac{1}{2} \left((1+\beta) \cdot e^{i\theta(-\lambda_k + i\varepsilon + 1)} + (1-\beta) \cdot e^{-i\theta(-\lambda_k + i\varepsilon + 1)} \right)$$
(3.26)

The eigenfunction series describing the complex stress is then given as a linear combination of the eigenfunctions.

$$\sigma_{yy} + i\sigma_{xy} = 1.0 \cdot \sigma \left(\lambda_1 = \frac{1}{2}\right) + q_1 \cdot \sigma \left(\lambda_2 = \frac{3}{2}\right) + q_2 \cdot \sigma \left(\lambda_3 = \frac{5}{2}\right) + \dots$$
(3.27)

where: q_1, q_2 Weighting factors

The reason that the weight in front of the first eigenfunction is 1.0 is because consistency is sought between the way the stress intensity factors are defined for a single isotropic material. In this way the expression in equation (3.49) reduce to the stress description for a single isotropic material if material properties are the same for material 1 and 2.

From comparing equation (3.24) for any specific value of λ_k and equation (3.4) with the determined stress potentials inserted the complex stress intensity factor is determined to be:

$$A_{1,k} = \frac{\overline{K}(1+\beta)}{2(\lambda_k - i\varepsilon)\sqrt{2\pi}}$$
(3.28)

The stress components σ_{yy} and σ_{xy} are plotted in 3D in figure 3.5 as the real and imaginary part of equation (3.49) for arbitrary chosen K = 83.3 + i76.3 and the material properties given in table 3.2 on page 34. The same stress components are plotted as contours in figure 3.6 as this emphasizes the asymmetry about the crack plane due to the difference of the materials.



Figure 3.5: (a) Plot showing σ_{xy} for a single termed series expansion. (b) Plot showing σ_{yy} for a single termed series expansion. Both plots are based on a mode I loading and are for the material properties given in table 3.2.

The corresponding eigenfunctions for the complex displacement $u(\lambda_k)$ is obtained by inserting the parameters $A_1, ..., D_2$ into equation (3.12):

$$u_j(\lambda_k) = \frac{r^{\lambda_k}}{\sqrt{2\pi}} \left(\overline{K} r^{i\varepsilon} c_j^{(3)} + K r^{-i\varepsilon} c_j^{(4)} \right)$$
(3.29)



Figure 3.6: (a) Contourplot of σ_{xy} for a single termed series expansion. (b) Contourplot of σ_{yy} for a single termed series expansion. Both plots are based on a mode I loading and are for the material properties given in table 3.2, and it is seen that the compared with the analogous plot for the single isotropic material case, σ_{xy} is not anti-symmetric and σ_{yy} is not symmetric about the crack plane.

with the coefficients $c_{i}^{(3)}$ and $c_{i}^{(4)}$ being:

$$c_1^{(3)} = \frac{(1+\beta) \cdot e^{i\theta\lambda_k} \cdot e^{-\theta\cdot\varepsilon}\kappa_1 - (1-\beta)e^{-i\theta\lambda} \cdot e^{\theta\cdot\varepsilon}}{4(\lambda_k + i\varepsilon)\mu_1}$$
(3.30)

$$c_{2}^{(3)} = \frac{\left(1-\beta\right) \cdot e^{i\theta\lambda_{k}} \cdot e^{-\theta\cdot\varepsilon}\kappa_{2} - (1+\beta)e^{-i\theta\lambda} \cdot e^{\theta\cdot\varepsilon}}{4(\lambda_{k}+i\varepsilon)\mu_{2}}$$
(3.31)

$$c_1^{(4)} = \frac{1}{4\mu_1} \left(1 + \beta \right) \cdot e^{-i\theta\lambda_k} \cdot e^{-\theta\cdot\varepsilon} (1 - e^{2i\theta})$$
(3.32)

$$c_2^{(4)} = \frac{1}{4\mu_2} \left(1 - \beta \right) \cdot e^{-i\theta\lambda_k} \cdot e^{-\theta\cdot\varepsilon} \left(1 - e^{2i\theta} \right)$$
(3.33)

The eigenfunction series describing the complex displacement is then given as

$$u_x + i u_y = 1 \cdot u \left(\lambda_1 = \frac{1}{2}\right) + q_1 \cdot u \left(\lambda_2 = \frac{3}{2}\right) + q_2 \cdot u \left(\lambda_3 = \frac{5}{2}\right) + \dots$$
(3.34)

The displacement components u_x and u_y are plotted in figure 3.7 as the real and imaginary part of equation (3.34) for K = 83.3 + i76.3 and the material properties given in table 3.2, and as contourplots in figure 3.8. These contourplots are composed by a separate plot for the upper and lower material, hence the continuity of the displacements in front of the crack tip (to the right) is not clear, but the asymmetry is clearly illustrated.



Figure 3.7: (a) Plot showing u_x for a single termed series expansion. (b) Plot showing u_y for a single termed series expansion. Both plots are based on a mode I loading and are for the material properties given in table 3.2.



Figure 3.8: (a) Plot showing u_x for a single termed series expansion. (b) Plot showing u_y for a single termed series expansion. Both plots are based on a mode I loading and are for the material properties given in table 3.2, and it is seen that compared with the analogous plot for the single isotropic material case, u_x is not anti-symmetric and u_y is not symmetric about the crack plane.

3.1.2 Stresses at $\theta = 0$

The complex stress is found for $\theta = 0$, by inserting this into equation (3.49). This corresponds to the stresses at the interface between the two materials in front of the crack tip.

$$(\sigma_{xy} + i\sigma_{yy})_{(\theta=0)} = \frac{1}{\sqrt{2\pi r}} \Big[\cos(\ln(r)\varepsilon)K_I + \sin(\ln(r)\varepsilon)K_{II} \Big] + i\frac{1}{\sqrt{2\pi r}} \Big[\cos(\ln(r)\varepsilon)K_{II} - \sin(\ln(r)\varepsilon)K_I \Big]$$
(3.35)

3.1.3 Displacements at $\theta = \pm \pi$

The complex displacement is found for $\theta = \pi$ and $\theta = -\pi$, which corresponds to the displacement of the crack face for the two materials. For $\theta = \pi$ the eigenfunctions of the displacement of material 1 is given as:

$$u(\lambda_k)_{j=1} = \frac{\overline{K}r^{\lambda_k}}{\sqrt{2\pi}} \cdot r^{i\varepsilon} \cdot \frac{(1+\beta)(i\sin(\pi\varepsilon))e^{-\pi\varepsilon}\kappa_1 + (1-\beta)(i\sin(\pi\lambda_k))e^{\pi\varepsilon}}{4(\lambda_k + i\varepsilon)\mu_1}$$
(3.36)

For $\theta = -\pi$ the eigenfunctions for the displacement of material 2 are expressed as:

$$u(\lambda_k)_{j=2} = \frac{\overline{K}r^{\lambda_k}}{\sqrt{2\pi}} \cdot r^{i\varepsilon} \cdot \frac{(1-\beta)(i\sin(-\pi\varepsilon))e^{\pi\varepsilon}\kappa_2 + (1+\beta)(i\sin(-\pi\lambda_k))e^{-\pi\varepsilon}}{4(\lambda_k + i\varepsilon)\mu_2}$$
(3.37)

In practice it only makes sense to measure the relative displacement between the crack faces in the numeric simulation. Hence, the relative displacement is derived for the relative displacement in the *x*-direction and *y*-direction, δ_x and δ_y , respectively and are given as:

$$\begin{cases} \delta_x = u_{x,j=1} - u_{x,j=2} \\ \delta_y = u_{y,j=1} - u_{y,j=2} \end{cases}$$
 \Leftrightarrow (3.38)

$$\delta_x + i \cdot \delta_y = (u_x + iu_y)_{j=1} - (u_x + iu_y)_{j=2}$$
(3.39)

In order to obtain the relative complex displacement without *K* being conjugated as in equation (3.36) and (3.37) the relative complex displacement is written as:

$$\delta_y + i \cdot \delta_x = i(\overline{\delta_x + i\delta_y}) \tag{3.40}$$

$$= i(\overline{(u_x + iu_y)_{j=1} - (u_x + iu_y)_{j=2}})$$
(3.41)

(3.42)

This implies that the eigenfunctions describing the relative complex displacement $\Delta(\lambda_k)$ are given as:

$$\Delta(\lambda_k) = \frac{Kr^{\lambda_k}r^{-i\varepsilon}}{\sqrt{2\pi}} \left(-i(-1)^{\lambda_k}\right) \frac{\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_2+1}{\mu_2}}{4(\lambda_k - i\varepsilon)\cosh(\pi\varepsilon)}$$
(3.43)

The factor $(-i(-1)^{\lambda_k})$ implies that the sign of the eigenfunctions change for every λ_k in the following way:

$$\frac{\lambda_k \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{5}{2} \quad \frac{7}{2} \quad \dots}{-i(-1)^{\lambda_k} \quad 1 \quad -1 \quad 1 \quad -1 \quad \dots}$$

In figure 3.9 the first 11 eigenfunctions for the relative complex displacement are shown for r = 0 - 1.0mm. It clearly illustrates that the first eigenfunction is dominating close to the crack tip, and that the influence from the higher order eigenfunctions increases away from the crack tip.

The relative complex displacement can be described by a series of linear combinations of the above eigenfunctions in the form of:

$$\delta_y + i \cdot \delta_x = 1 \cdot \Delta \left(\lambda_1 = \frac{1}{2} \right) + q_1 \cdot \Delta \left(\lambda_2 = \frac{3}{2} \right) + q_2 \cdot \Delta \left(\lambda_3 = \frac{5}{2} \right) + \dots$$
(3.44)



Figure 3.9: Plot of the first 11 eigenfunctions of the total relative displacement between the crack faces.

3.1.4 Relation between Energy Release Rate and K

When the stress intensity factors K_I and K_{II} are determined, the energy release rate is given as (Suo and Hutchinson, 1990, p. 4).

$$G = \frac{\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}}{16\cosh^2(\pi\varepsilon)} |K_I + iK_{II}|^2$$
(3.45)

3.2 Simulation of a Crack Between Two Dissimilar Isotropic Materials

The purpose and goals of the simulation described in this section, where a crack in the interface between two dissimilar isotropic materials is simulated, are given in the following list:

- To test methods for determining K_I and K_{II} from displacement data.
- To conduct a sensitivity analysis of the influence of the number of data points (nodes) used.
- To conduct a sensitivity analysis of influence of the element size.
- To test the influence of the choice of data used to determine K_I and K_{II} : δ_x , δ_y or $\delta_x + \delta_y$ data.

The first item concerns determining K_I and K_{II} by fitting displacement data to the analytical expression consisting of one, two and three terms of the series expansion in both full form, and a simplified form for term 2 and three, which is described in subsection 3.2.3 on page 35.

The sensitivity analyses of the influence of the number of data points and the element size are conducted in order to evaluate the robustness of the method for determining the stress intensity factors, and what level of details of the model is required in order to obtain sufficiently precise results compared to an analytical solution from (Suo and Hutchinson, 1990, p. 5).

In contrast to the determination of the stress intensity factors for at single isotropic material, the coupling between the stress intensity factors for the bimaterial case implies that it is not obvious which displacement data gives the best estimation of the stress intensity factors. Hence, a study of this is conducted. Different estimates based on the x-displacement, y-displacement and the sum of the x and y displacement are compared.

3.2.1 Model used in the Simulation

The model used in the simulation of a crack between two dissimilar isotropic materials is the same DCB model with the same dimensions as the one used in the simulation of a single isotropic material in section 2.2, and it is illustrated in figure 3.10. The material properties used in the model are given in table 3.2.



Figure 3.10: Double cantilever beam model of two dissimilar isotropic materials used in the simulation.

Material	Young's module	Poisson's ratio	
1	70 GPa	0.35	
2	8 GPa	0.35	

 Table 3.2: Material properties used in the simulation and examples throughout this chapter.

The FE-model is programmed to contain the loading scenarios listed below and illustrated in figure 3.11. The loads in the figure are categorized with numbers in the way that:

- 1. Outer mode 1 loading by forces
- 2. Outer mode 2 loading by forces
- 3. Outer mode 1 loading by pure moments
- 4. Outer mode 2 loading by pure moments



Figure 3.11: The model shown as lines and keypoints with the different loading modes.

The outer mode I loading by pure moments is used in all the studies conducted in this chapter, as performing the test for all four loading scenarios has been considered too far-reaching in this project.

3.2.2 Approach for Determining the Stress Intensity Factors

... . . .

An estimation of the stress intensity factors K_I and K_{II} is conducted using a least squares formulation for fitting the FE displacement data to the analytical expression analogous to method used in chapter 2. This estimation is nonlinear because the eigenfunctions (equation (3.43)) in the expression for the relative displacement (equation (3.44)) contain the factor

$$r^{-i\varepsilon} = e^{-i(ln(r)\varepsilon)} = \cos(ln(r)\varepsilon) - i\sin(ln(r)\varepsilon)$$
(3.46)

where the cosine and sine terms cause the oscillating singularity with a nonlinear dependence on r. This mean that the linear method used on the simulation for a single isotropic material cannot be used here. Instead, a numeric search routine is applied to solve an optimization problem in which values for K_I and K_{II} that minimizes the least square difference between the FE and analytical data for the relative displacements are estimated. The routine used for estimating the stress intensity factors which fit the displacement data closest, is the *conjugated gradient method* with a *golden section line search method*. This routine has been programmed in MATLAB according to an approach described in (Arora, 2004).

From an FE analysis of the DCB model, relative displacements in the *x*- and *y*-directions between the two crack faces are extracted node-wise. The parameter estimation of the stress intensity factors is conducted for three displacement data configurations, that is *x*-displacement data, *y*-displacement data and the sum of *x*- and *y*-displacement data, respectively. The reason for using these three configurations is that they are not expected to yield the same results because of deviations of the FE displacement data from to the analytical solution. The reason for fitting the parameters to the sum of the *x*- and *y*-displacement data is to obtain stress intensity factors that fit both the *x*-displacement data and *y*-displacement data in the best way.

3.2.3 Estimation of the Stress Intensity Factors

The numeric search routine used shows to some degree to be sensitive to the initial guess of the values for the stress intensity factors with regards to calculation speed and reliability of the solution. Due to this a simplified form of the relative displacement expressed with the first eigenfunction is used to give an initial guess in the numeric search routine. This ensures a more robust parameter estimation of the stress intensity factors with two or more eigenfunctions in the expression for the relative displacement, equation (3.44). With the first eigenfunction only $(\lambda_k = \frac{1}{2})$, this expression simplifies to the one in equation (3.47), and the relative displacement components δ_x and δ_y are defined as the real and complex parts of it. These components are

given in equation (3.48) and (3.49).

$$\delta_{y} + i\delta_{x} = \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot (1-2i\varepsilon) \cdot \cosh(-\pi\varepsilon)} K\sqrt{r}r^{-i\varepsilon}$$

$$= \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot (1-2i\varepsilon) \cdot \cosh(-\pi\varepsilon)} (K_{I} + iK_{II}) \sqrt{r} \left(\cos(\ln(r)\varepsilon) - i\sin(\ln(r)\varepsilon)\right)$$

$$= \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \sqrt{r} \left[\frac{(K_{II} + 2\varepsilon K_{I})\sin(\ln(r)\varepsilon) + (K_{I} - 2\varepsilon K_{II})\cos(\ln(r)\varepsilon)}{1 + 4\varepsilon^{2}} + i \left(\frac{(2\varepsilon K_{II} - K_{I})\sin(\ln(r)\varepsilon) + (K_{II} + 2\varepsilon K_{I})\cos(\ln(r)\varepsilon)}{1 + 4\varepsilon^{2}} \right) \right]$$
(3.47)

$$\delta_{x} = \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \sqrt{r} \left[\frac{(2\varepsilon K_{II} - K_{I})\sin(\ln(r)\varepsilon) + (K_{II} + 2\varepsilon K_{I})\cos(\ln(r)\varepsilon)}{1 + 4\varepsilon^{2}} \right]$$
(3.48)

$$\delta_{y} = \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \sqrt{r} \left[\frac{(K_{II} + 2\varepsilon K_{I})\sin(\ln(r)\varepsilon) + (K_{I} - 2\varepsilon K_{II})\cos(\ln(r)\varepsilon)}{1 + 4\varepsilon^{2}} \right]$$
(3.49)

Linearized Expression for the Relative Displacements

In the expression for δ_x and δ_y the factors $\sin(\ln(r)\varepsilon)$ and $\cos(\ln(r)\varepsilon)$ cause the oscillating singularity and thus the nonlinearity with respect to r mentioned above. The variation of these two terms with r is investigated in order to determine if it is reasonable to circumvent this non-linearity linearizing the expression in equation (3.48) and (3.49). The cosine and sine terms are plotted in figure 3.12 respectively for $\varepsilon = 0.084$ corresponding to the material properties given in table 3.2.



Figure 3.12: (*a*) Plot of the consine term for $\varepsilon = 0.084$. (*b*) Plot of the sine term for $\varepsilon = 0.084$.

It is seen that the cosine and sine terms are close to 1.0 and 0.0, respectively, for $r \in [0.0, 5.0]$. Due to this, these constant values are inserted in (3.48) and (3.49) which gives the following linearized approximations for δ_x and δ_y with respect to r.

$$\delta_{x} = \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \sqrt{r} \left[\frac{(K_{II} + 2\varepsilon K_{I})}{1 + 4\varepsilon^{2}} \right]$$
(3.50)

$$\delta_{y} = \frac{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \sqrt{r} \left[\frac{(K_{I} - 2\varepsilon K_{II})}{1 + 4\varepsilon^{2}} \right]$$
(3.51)

The linearization makes the estimation of the initial guess simpler and faster because it is possible to use the linear least squares approach from the simulation in chapter 2 to solve for K_I and K_{II} . This implies the following form of the equations (3.50) and (3.51) where A_x and A_y are determined with the linear least squares method in the same way as in equations (2.13) - (2.15) in subsection 2.2.3 on page 16.

$$\frac{\delta_x}{\sqrt{r}} = \frac{\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \left[\frac{(K_{II} + 2\varepsilon K_I)}{1 + 4\varepsilon^2} \right] = A_x \tag{3.52}$$

$$\frac{\delta_y}{\sqrt{r}} = \frac{\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}}{2\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \left[\frac{(K_I - 2\varepsilon K_{II})}{1 + 4\varepsilon^2} \right] = A_y \tag{3.53}$$

The stress intensity factors as functions of the fit parameters A_x and A_y are then given as:

$$K_{I} = \frac{2\sqrt{2}\cosh(\pi\varepsilon)(2\varepsilon A_{x} + A_{y})}{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}$$
$$K_{II} = \frac{2\sqrt{2}\cosh(\pi\varepsilon)(2\varepsilon A_{y} - A_{x})}{\frac{\kappa_{1}+1}{\mu_{1}} + \frac{\kappa_{2}+1}{\mu_{2}}}$$
(3.54)

For a arbitrary set of stress intensity factors $K_I = 83.3[MPa\sqrt{mm}]$ and $K_{II} = 76.3[MPa\sqrt{mm}]$ and the material properties in table 3.2, the linearized approximation of δ_x in equation (3.50) and the first term expansion of the relative *x*-displacement in equation (3.48), are plotted in figure 3.13. The same expressions divided by \sqrt{r} are plotted in figure 3.14. In figure 3.13 it is seen that the approximations are fairly close to each other.



Figure 3.13: Plot of the relative displacement δ_x of the crack surfaces for one eigenvalue term (equation (3.48)), and linearized for $\cos(\ln(r)\varepsilon) = 1.0$ and $\sin(\ln(r)\varepsilon) = 0.0$ (equation (3.50)).



Figure 3.14: Plot of the relative displacement δ_x of the crack surfaces from equations (3.48) and (3.50) divided by \sqrt{r} , respectively. Illustration of a set of numerical displacement data marked by \times and a random curve fit to these data points on the form $\delta_x = A_x \sqrt{r}$

In figure 3.14 the red parabolic curve shows δ_x from equation (3.48) described by the full first eigenfunction. This expression is divided by \sqrt{r} and plotted for $K_I = 83.3[MPa\sqrt{mm}]$ and $K_{II} = 76.3[MPa\sqrt{mm}]$. An example of data points from an FE analysis are indicated, and the yellow line shows an expected linearized fit to these data points. The green line shows the linearized δ_x from equation (3.50) divided by \sqrt{r} and for the same values for K_I and K_{II} . The point illustrated by this is, that if the linearized δ_x is fit to the data points it would lie on the yellow line and the values for K_I and K_{II} would be determined to be higher than the "true" values that give the green line. Hence, the approximation of K_I and K_{II} from the linearized expressions overestimates the values compared to the full expression, but is considered a reasonable initial guess.

3.2.4 Analytic Beam Solution for Reference

In the article (Suo and Hutchinson, 1990) an analytical method for determining the stress intensity factors for DCB geometries is described, and this method has been used for calculating two loading scenarios which are used for reference in the simulation of for two isotropic materials. The calculations are enclosed on the appertaining DVD and the results are presented in figure 3.15.



Figure 3.15: *Results of the analytic solution for the stress intensity factors following the approach described in (Suo and Hutchinson, 1990). The material properties used are given in table 3.2.*

3.2.5 Preparation of Expressions for Parameter Estimation

In the parameter estimation of K_I and K_{II} , a curve fit of the two and three termed series expansion of the complex relative displacement is used on the displacement data from the FE simulation. The three termed expansion is given as:

$$\delta_y + i \cdot \delta_x = 1 \cdot \Delta \left(\lambda_1 = \frac{1}{2} \right) + q_1 \cdot \Delta \left(\lambda_2 = \frac{3}{2} \right) + q_2 \cdot \Delta \left(\lambda_3 = \frac{5}{2} \right)$$
(3.55)

Here Δ are eigenfunctions for the relative displacement defined in equation (3.43). The two termed expansion is given by the same equation by setting $q_2 = 0$.

Because of the arbitrary constants q_1 and q_2 in front of the second and third term there may be no need to operate with a full expression of the second and third term. It is investigated if it is possible to substitute the second and third term with $r^{\frac{3}{2}}$ and $r^{\frac{5}{2}}$ multiplied by an arbitrary constant, respectively. Obviously this introduces an error because the constant is in reality a function of r figuring in $r^{-i\varepsilon} = \cos(\ln(r)\varepsilon) - i\sin(\ln(r)\varepsilon)$, cf. equation (3.43). By using one of these simplified two and three term expressions a sufficiently accurate estimation of K_I , K_{II} obtained. This is favorable as it makes the parameter estimation less cumbersome.

The optimization problem of identifying the stress intensity factors K_I and K_{II} is ill conditioned due to the large difference between the value of K_I , K_{II} , q_1 and q_2 . In order to circumvent this, the values of these parameters must all be within the same decade. To ensure this the following normalization of these entities is substituted into the expression (3.55):

$$\begin{bmatrix} K_I \\ K_{II} \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ 10^{-2} \cdot Z_3 \\ 10^{-3} \cdot Z_4 \end{bmatrix}$$
(3.56)

The magnitudes of the constant factors in front of Z_3 and Z_4 have been obtained by trying different values until reaching a configuration that seems to ensure a fast and robust solution of the minimization problem. This substitution yields the following normalized expressions

for the relative displacements.

$$\delta_x = r^{\frac{1}{2}} R_x \left(\lambda_1 = \frac{1}{2} \right) + \frac{Z_3}{10^2} r^{\frac{3}{2}} R_x \left(\lambda_2 = \frac{3}{2} \right) + \frac{Z_4}{10^3} r^{\frac{5}{2}} R_x \left(\lambda_3 = \frac{5}{2} \right)$$
(3.57)

$$\delta_{y} = r^{\frac{1}{2}} R_{y} \left(\lambda_{1} = \frac{1}{2} \right) + \frac{Z_{3}}{10^{2}} r^{\frac{3}{2}} R_{y} \left(\lambda_{2} = \frac{3}{2} \right) + \frac{Z_{4}}{10^{3}} r^{\frac{5}{2}} R_{y} \left(\lambda_{3} = \frac{5}{2} \right)$$
(3.58)

Where:

$$R_{x}(\lambda_{k}) = \frac{c_{1} + c_{2}}{\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \cdot \left[\frac{(\varepsilon Z_{2} - \lambda_{k} Z_{1}) \sin(\ln(r)\varepsilon) + (\lambda_{k} Z_{2} + \varepsilon Z_{1}) \cos(\ln(r)\varepsilon)}{4(\lambda_{k})^{2} + 4(\varepsilon)^{2}} \right]$$
(3.59)

$$R_{y}(\lambda_{k}) = \frac{c_{1} + c_{2}}{\sqrt{2\pi} \cdot \cosh(-\pi\varepsilon)} \cdot \left[\frac{(\lambda_{k}Z_{2} + \varepsilon Z_{1})\sin(\ln(r)\varepsilon) + (\lambda_{k}Z_{1} - \varepsilon Z_{2})\cos(\ln(r)\varepsilon)}{4(\lambda_{k})^{2} + 4(\varepsilon)^{2}} \right]$$
(3.60)

The weight factor q_1 in equation (3.55) has shown to have the magnitude of approximately 0.0026 and q_2 is even smaller for mode I opening. Because these factors are so small it is likely that the dependency of r in the $sin(ln(r)\varepsilon)$ and $cos(ln(r)\varepsilon)$ factors of the second and third eigenfunction of the relative displacement are insignificant in the determination of the stress intensity factors. It is therefore investigated if it is possible to obtain satisfactory estimations of the stress intensity factors if the dependency of r in the $sin(ln(r)\varepsilon)$ and $cos(ln(r)\varepsilon)$ factors is ignored. The simplified normalized expressions for the relative displacements are given as:

$$\delta_x = R_x \left(\lambda_1 = \frac{1}{2} \right) r^{\frac{1}{2}} + \frac{Z_3}{10^4} r^{\frac{3}{2}} + \frac{Z_4}{10^6} r^{\frac{5}{2}}$$
(3.61)

$$\delta_{y} = R_{y} \left(\lambda_{1} = \frac{1}{2} \right) r^{\frac{1}{2}} + \frac{Z_{3}}{10^{4}} r^{\frac{3}{2}} + \frac{Z_{4}}{10^{6}} r^{\frac{5}{2}}$$
(3.62)

The simplified expression is normalized in the following way:

$$\begin{bmatrix} K_I \\ K_{II} \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ 10^{-4} \cdot Z_3 \\ 10^{-6} \cdot Z_4 \end{bmatrix}$$
(3.63)

3.3 Presentation of Simulation Results

In the following the obtained results for the strength intensity factors are given for different versions of the relative displacement expressions. The approach of determining the stress intensity factors is sketched in figure 3.16. First the FE model is solved for loading mode I with different size of elements at the crack tip. From this analysis results for the relative displacements are obtained as well as an initial guess for the stress intensity factors. These data are exported to a parameter estimation routine in MATLAB. In this the different forms of the relative displacement expressions described in section 3.2.5 are fitted to the FE displacement data using the numerical search method *conjugated gradient method* with a *golden section line search*. The four forms of the relative displacement expressions are:

• Complete two termed relative displacement expressions, (3.57) and (3.58) with $Z_4 = 0$

- Simplified two termed relative displacement expressions, (3.61) and (3.62) with $Z_4 = 0$
- Complete three termed relative displacement expressions, (3.57) and (3.58)
- Simplified three termed relative displacement expressions, (3.61) and (3.62)

Each of the four expressions are evaluated separately with regards to calculation time and sensitivity to elements size and number of nodes used in the parameter estimation of K_I and K_{II} . It is done for 3 to 12 nodes pairs behind the crack tip for element sizes [0.15, 0.30, 0.60, 1.20]. Based upon these evaluations, the expression that is considered to determine the stress intensity factors most precisely is selected.



Figure 3.16: Flowchart of the determination of stress intensity factors in the simulation of an interface crack between two isotropic materials.

A summary of the results is presented in the following. For a more detailed presentation of the results see appendix A.

For all four forms of the displacement expressions the determination of the stress intensity factors is better using the *y*-displacement instead of the *x*-displacement. The reason is that the *x*-displacement in the transition from the quarter point element to the adjacent eight node element is not as smooth as for the *y*-displacement.

This means that whenever it is possible the *y*-displacement should be used to determine the stress intensity factors. This is always possible when material 1 and 2 have significantly different stiffness properties meaning that the *y* and *x*-displacement have a strong dependency on both K_I and K_{II} . For materials combinations where the properties are identical or almost identical the *y*-displacement are only or mainly a function of K_I . In this case K_{II} should be obtained from the *x*-displacement.

The best results obtained from the four displacement expressions are obtained with the simplified 3 termed displacement expression. These results showed the best correspondence with the analytical solution, cf. figure 3.15. Furthermore, the results are almost independent of the element size and the number of nodes used to determine the stress intensity factors.

In figure 3.17 and 3.18 the curve fits and the obtained stress intensity factors based on the simplified 3 termed displacement expansion is plotted for element size 0.15mm, 0.30mm, 0.60mm, 1.20mm on the crack faces, respectively.



Figure 3.17: Plot of the best fit for 3 to 12 node pairs from the crack tip to the simplified three termed series expansion in equation (3.62). The plots are based on the y-displacement data and is shown for element sizes 0.15mm, 0.30mm, 0.60mm and 1.20mm.



Figure 3.18: Plot of the stress intensity factors corresponding to the curve fits shown in figure 3.17. The green and red line indicates the analytically determined stress intensity factors.

3.4 Evaluation of the Results of the Parameter Estimation

Based on the results from the parameter estimations of the stress intensity factors for the four different expressions for the relative displacements, the one using the simplified 3 termed relative *y*-displacement to fit to the *y*-displacement data obtained from the FE model yields values closest to the analytical ones. This is because this method shows no tendency to converge towards local minima or sensitivity to the different element sizes.

All parameter estimations have a relatively long computation time (30 sec. - 5 min.) This becomes critical as the method is intended to be implemented in a progressive crack propagation simulation, in which the parameter estimation is carried out for each increment in crack length. It is expected that it is possible to improve the numerical search method significantly, for instance by conditioning the expression even better, but this lies outside the scope of this project and is not treated further. Hence, a discussion of the calculation speed of each of the parameter estimations for the four different relative displacement expressions treated is excluded.

Due to the results obtained for the simulation of a crack between two dissimilar isotropic materials in this section, the parameter estimation for determining the stress intensity factors for cracks in anisotropic materials in chapter 4 is based on the simplified three termed relative *y*-displacement whenever possible, and for the *x*-displacement if necessary.

Summary of Simulation of a Crack Between Two Dissimilar Isotropic Materials

In this chapter the complete expressions for the stress and displacement field at the crack tip between two dissimilar isotropic materials is derived. From this the relative displacements between the two crack faces have been derived. A simulation of a crack in the interface between the two materials has been conducted in ANSYS with the relative displacements between the crack faces as output.

Different forms of the equation for the relative displacement have been reformulated into a minimization problem by use of the least squares method, where the purpose is to determine the stress intensity factors based on the displacement data from the simulation. Different formulations of the relative displacement together with different element sizes at the crack tip have been studied. The conclusion is that by using the simplified displacement formulation for two and three terms, respectively, the problem of determining the stress intensity factors seems to be convex and satisfactory results are obtained. In addition the simplified three termed displacement formulation makes it possible to use larger element sizes without compromising the results.

On the other hand if the minimization problem is based on the complete displacement formulation with two and three terms, respectively, several local minima seem to exist close to the true values for the stress intensity factors, and this makes it difficult to obtain satisfactory results.

Chapter



Interface Crack between Two Dissimilar Anisotropic Materials

In this chapter the theory describing the stress and displacement fields at a crack tip between two dissimilar anisotropic materials is presented. The procedure for determining the stress intensity factors explained in the previous chapter is extended to the case of a crack between two anisotropic materials. Furthermore, the R-curve behavior from fiber bridging in fibrous composites is treated and the concept and modeling of fiber bridging is presented and implemented in a FE model. A test of a double cantilever beam specimen made of orthotropic fiber materials presented in (Sørensen and Jacobsen, 2009) is briefly described and this specimen is modeled as a solid FE model. The elastic response of the test specimen is studied and evaluated in comparison to the assumptions in the theory of a crack between dissimilar anisotropic materials and the assumptions of the article. Finally a simulation of the test specimen with application of the parameter estimation for determining the energy release rate at the crack tip and bridging across the crack surfaces is carried out for pure mode I and II, and for mixed mode loading in order to obtain R-curves for these three scenarios.

4.1 Theory of Interface Crack between Dissimilar Anisotropic Materials

In continuation of the development of the method used to determine the stress intensity factors in the simulation of two dissimilar isotropic materials, a theory describing the relative crack face displacements for a crack in the interface between two dissimilar anisotropic materials is described in this section. It is presented for anisotropic materials in general, but is in this chapter only applied to a simulation of two orthotropic fiber composite materials. Both of these have unidirectional fibers orientated in the *x*-direction (see figure 4.1) as this is the case for the test specimen in (Sørensen and Jacobsen, 2009).

The change from considering a crack between two isotropic materials to two anisotropic materials involves a number of additional challenges. The description of the stress and displacement field around an interface crack between two anisotropic materials involves constitutive laws for the materials which compared to the isotropic material problem are extended from 2 to 21 independent material parameters. I the general case this implies coupling between all three crack opening modes for a given external load in opposition to the isotropic bimaterial problem, where only coupling between modes I and II is present. The is because



Figure 4.1: DCB model used for two dissimilar orthotropic materials and indication of fiber direction. Indices 1 and 2 used in this chapter refers to the respective materials indicated in this figure.

each of the relative displacements of the crack faces δ_x , δ_y and δ_z are functions of all three stress intensity factors, K_I , K_{II} , K_{III} . Furthermore, the parameters figuring in the expression for the relative displacement are functions of parameters which cannot be determined analytically, and are thus determined by insertion of numeric values and using numerical methods. This means that numerous calculations must be done for every material combination in order to obtain the expression for the relative displacement. This is not the case for the isotropic bimaterial problem where the relative displacement expression is explicitly dependent of the material parameters, only.

The theory and method used for describing the displacements of the crack faces for a crack in the interface between two dissimilar anisotropic materials is proposed in (Suo, 1990). It covers a description of the near-tip displacements that contains a two-dimensional coupled oscillating singularity for modes I and II, and a one-dimensional square root singularity for mode III. Remember that in the case of two isotropic materials there is only a single coupled singularity for modes I and II. The theory presented in this chapter is obtained from (Suo, 1990), (Lekhnitskii, 1981) and (Lekhnitskii, 1968). The theory is presented with focus on application. Hence, only steps relevant for obtaining the relative displacement of the crack faces in the x, yand z direction is given here.

The general constitutive law for an anisotropic material can be expressed by 21 independent elastic constants in the compliance matrix S_{ij} as:

$$\varepsilon_i = \sum_{j=1}^6 S_{ij} \sigma_j \tag{4.1}$$

where

$$\varepsilon_i = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{23} & 2\varepsilon_{31} & 2\varepsilon_{12} \end{bmatrix}^T$$
(4.2)

$$\sigma_i = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{31} & \sigma_{12} \end{bmatrix}^T$$
(4.3)

The approach for determining the stress and displacement fields at the crack tip between two dissimilar anisotropic materials is based on a solution of the elastic field proposed by Lekhnitskii (1981). The solution is derived on the assumptions of plane stress deformation or plane strain deformation. Plane stress deformation which is also referred to as *generalized plane stress* refers to a stress state that is only dependent on two coordinates. This means that all 6 stress components are present in general, but that they do not change in the direction parallel to the crack front (*z*-direction). Plane stress deformation is in many cases assumed for two dimensional problems with respect to geometry and external load in the sense that they are invariant in the direction normal to the *xy*-plane.

As in the case of the isotropic bimaterial problem the elastic field around the crack can be described by stress functions that are functions of complex potentials. In this case it is the three complex potentials $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$ where $z_j = x + \mu_j y$, j = 1,2,3 with μ_j being three distinct complex numbers with positive imaginary part. These are function of the mechanical material properties and are determined in the derivation of the displacement expression. The not yet determined complex potentials are presented as a function vector

$$\mathbf{f}(z) = \begin{bmatrix} f_1(z) & f_2(z) & f_3(z) \end{bmatrix}^T$$
(4.4)

where $z = x + \mu y$ for which μ has not been determined. With these complex potentials Lekhnitskii has shown that the displacement vector for the crack face (y=0) and the traction vector at the interface can be described by the complex potentials in the following way.

$$\mathbf{u}(x) = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A\mathbf{f}(x) + \overline{A\mathbf{f}(x)}$$
(4.5)

$$\mathbf{t}(x) = \begin{bmatrix} \tau_{yx} \\ \sigma_{yy} \\ \tau_{yz} \end{bmatrix} = L\mathbf{f}'(x) + \overline{L\mathbf{f}'(x)}$$
(4.6)

These equations corresponds to the complex stress and complex displacement equations for the isotropic bimaterial problem in equation (3.4) and (3.1), respectively.

The stresses σ_x , σ_y , σ_z , τ_{xy} , τ_{xz} and τ_{yz} can in the case of a two dimensional problem (as defined on page 46) be expressed by two stress functions which are functions of the complex potentials in equation (4.4) (Lekhnitskii, 1981, pp. 99-114). This corresponds to the way the stress components σ_x , σ_y and τ_{xy} can be expressed by a single Airy stress function for plane stress or strain problems in the case for isotropic materials. The two stress functions are shown by Leknitskii to fulfil two homogenous equations corresponding to the biharmonic equation in Airy's work. By rewriting these two homogenous equations, the following sixth-order equation of μ is obtained for plane stress deformation:

$$l_2(\mu) l_4(\mu) - \left(l_3(\mu)\right)^2 = 0 \tag{4.7}$$

where

$$l_2(\mu) = S_{55}\mu^2 - 2S_{45}\mu + S_{44}$$
(4.8)

$$l_4(\mu) = S_{11}\mu^4 - 2S_{16}\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22}$$
(4.9)

$$l_3(\mu) = S_{15}\mu^3 - (S_{14} + S_{56})\mu^2 + (S_{25} + S_{26})\mu - S_{24}$$
(4.10)

The six roots μ in equation (4.10) form three complex conjugate pairs under the requirement that the compliance matrices of the two materials are positive definite, which is the case for orthotropic materials rotated in the *xz*-plane. By selecting the three complex roots with positive imaginary part as μ_1 , μ_2 and μ_3 , two matrices *A* and *L* are defined for each material based on these roots and the elastic constants in the compliance matrix S_{ij} . The *L*-matrix is defined as:

$$L = \begin{bmatrix} -\mu_1 & \mu_2 & -\mu_3\eta_3\\ 1 & 1 & \eta_3\\ -\eta_1 & -\eta_2 & -1 \end{bmatrix}$$
(4.11)

Where

$$\eta_{\alpha} = -l_3(\mu_{\alpha}) \cdot (l_2(\mu_{\alpha}))^{-1}$$
, $\alpha = 1, 2$ (4.12)

$$\eta_3 = -l_3(\mu_3) \cdot (l_4(\mu_3))^{-1} \tag{4.13}$$

For $\alpha = 1,2$ the *A*-matrix for plane stress deformation is given as:

$$A_{1\alpha} = S_{11}\mu_{\alpha}^{2} + S_{12} - S_{16}\mu_{\alpha} + \eta_{\alpha}(S_{15}\mu_{\alpha} - S_{14})$$

$$A_{2\alpha} = S_{21}\mu_{\alpha} + S_{22}\mu_{\alpha}^{-1} - S_{26} + \eta_{\alpha}(S_{25} - S_{24}\mu_{\alpha}^{-1})$$

$$A_{3\alpha} = S_{41}\mu_{\alpha} + S_{42}\mu_{\alpha}^{-1} - S_{46} + \eta_{\alpha}(S_{45} - S_{44}\mu_{\alpha}^{-1})$$

$$A_{13} = \eta_{3}(S_{11}\mu_{3}^{2} + S_{12} - S_{16}\mu_{3}) + S_{15}\mu_{3} - S_{14}$$

$$A_{23} = \eta_{3}(S_{21}\mu_{3} + S_{22}\mu_{3}^{-1} - S_{26}) + S_{25} - S_{24}\mu_{3}^{-1}$$

$$A_{33} = \eta_{3}(S_{41}\mu_{3} + S_{42}\mu_{3}^{-1} - S_{46}) + S_{45} - S_{44}\mu_{3}^{-1}$$
(4.14)

The *L*- and *A*-matrix can be obtained for plane strain displacement as well by using a compliance matrix *S'* where $S'_{ij} = S_{ij} - S_{i3}S_{j3}(S_{33})^{-1}$. Similar to the case of plane stress deformation, this means that the strain state is independent of *z*.

The A and L matrices are used to construct a B-matrix

$$B = iAL^{-1} \tag{4.15}$$

which is a positive definite matrix. A *B*-matrix is determined for material 1 and 2, and these are referred two as B_1 and B_2 , respectively. From B_1 and B_2 a bimaterial matrix *H* is defined for convenience.

$$H = B_1 + \overline{B_2} \tag{4.16}$$

The *H*-matrix is, as will be shown later, the only information of the stiffness properties of the two materials that figures in the expression for the relative displacement for the anisotropic bimaterial problem.

In the work with implementing the calculations of the *L*, *A*, *B* and *H* matrices in Maple it was discovered that the roots μ_1 , μ_2 and μ_3 have to be sorted in a certain way in order for the B-matrix to be hermitian, which is a requirement for the theory to work. This has been presented to the author of the article (Suo, 1990) and it seems that there is no specific definition on how they should be sorted. In the calculations of the *B*-matrix for the materials used in the simulations conducted in this master thesis, it seems that if the choice of $\mu_i = \begin{bmatrix} a & b & c \end{bmatrix}$ makes the B-matrix hermitian then $\mu_i = \begin{bmatrix} b & a & c \end{bmatrix}$ also makes the B-matrix hermitian.

Now that the matrices describing the material properties of each material and the bimaterial properties are given, the procedure for obtaining the expression for the relative displacement is analogous to the one used for deriving the solution to the isotropic bimaterial problem in the previous chapter. The vector containing the relative displacement between the crack faces in the *x*, *y* and *z* direction is defined from $\mathbf{u}(x)$ in equation (4.5) as

$$\mathbf{d}(x) = \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = \mathbf{u}(x, 0^+) - \mathbf{u}(x, 0^-)$$
(4.17)

where the superscript + and – refer to the displacement of the upper and lower crack face respectively. The relative displacement $\mathbf{d}(x)$ and the stresses on the interface between material 1 and 2 $\mathbf{t}(x)$ are given by (Suo, 1990, p. 341) as:

$$i\mathbf{d}'(x) = HL_{1}\mathbf{f}'_{1}(x) - HL_{2}\mathbf{f}'_{2}(x)$$

$$\mathbf{t}(x) = L_{1}\mathbf{f}'_{1}(x) + L_{2}\mathbf{f}'_{2}(x)$$
(4.18)

By use of analytic continuation techniques it can be shown that the stress and displacement field can be described by use of either the expressions describing the stress or displacement for material 1 and 2, respectively. In other words the stress and displacement fields in both materials are functions of a single function $\mathbf{h}(z)$ given as:

$$\mathbf{h}(z) = L_1 \mathbf{f}_1(z) = H^{-1} \overline{H} L_2 \mathbf{f}_2(z)$$
(4.19)

By inserting $\mathbf{h}(z)$ into equation (4.18) the following is obtained.

$$i\mathbf{d}'(x) = H[\mathbf{h}^+(x) - \mathbf{h}^-(x)]$$
 (4.20)

$$\mathbf{t}(x) = \mathbf{h}^{+}(x) + \overline{H}^{-1}H\mathbf{h}^{-}(x)$$
(4.21)

This means that instead of determining a complex potential for each material, a single function $\mathbf{h}(z)$ is determined instead. By applying the boundary conditions for stresses on the crack faces (all stresses are zero) the following equation is obtained.

$$\mathbf{h}^{+}(x) + \overline{H}^{-1}H\mathbf{h}^{-}(x) = 0, \qquad z \in \mathbb{C}$$
(4.22)

As in the case of the isotropic bimaterial problem, a guess on the complex stress potentials is made followed by a test of whether or not the guessed form fulfills the boundary conditions. h(z) is guessed to be on the following form (Suo, 1990, p. 345).

$$\mathbf{h}(z) = \mathbf{w} z^{-\lambda + i\varepsilon} \tag{4.23}$$

By inserting $\mathbf{h}(z)$ into (4.22), the following eigenvalue problem is obtained for which the task is to determine the eigenvectors w and eigenvalues $-\lambda + i\varepsilon$.

$$\left(\overline{H} - He^{i\pi\lambda}e^{2\pi\varepsilon}\right)\mathbf{w} = 0 \tag{4.24}$$

It is seen that the eigenvalues for λ is $\lambda = \frac{1+k}{2}, k \in \mathbb{Z}$.

In the end it is only the first eigenfunction that is sought because it is the simplified three termed expression of the relative displacement (see subsection 3.2.5 in chapter 3) that is to be used to determine the stress intensity factors. Hence the eigenvalue for $\lambda = 1/2$ is inserted.

$$\left(\overline{H} - He^{2\pi\varepsilon}\right)\mathbf{w} = 0 \tag{4.25}$$

The solution to the eigenvalue problem in equation (4.24) yields the following three distinct eigenpairs

$$(\varepsilon, \mathbf{w}), \quad (-\varepsilon, \overline{\mathbf{w}}), \quad (0, \mathbf{w}_3)$$
 (4.26)

and the vector $\mathbf{h}(z)$ is expressed as a linear combination of the three eigenvectors for $\lambda = \frac{1}{2}$ as

$$\mathbf{h}(z) = z^{-\frac{1}{2}} [a\mathbf{w}z^{i\varepsilon} + b\overline{\mathbf{w}}z^{-i\varepsilon} + c\mathbf{w}_3] \quad \text{where} \quad a, b, c \in C$$
(4.27)

From $\mathbf{h}(z)$ and equation (4.20) the displacements of the crack surfaces as a function of *r* is derived as (Suo, 1990, p. 347):

$$\mathbf{d}(r) = \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = (H + \overline{H}) \sqrt{\frac{r}{2\pi}} \left[\frac{(K_I + iK_{II})r^{i\varepsilon}\mathbf{w}}{(1 + 2i\varepsilon)\cosh(\pi\varepsilon)} + \frac{(K_I - iK_{II})r^{-i\varepsilon}\overline{\mathbf{w}}}{(1 - 2i\varepsilon)\cosh(\pi\varepsilon)} + K_{III}\mathbf{w}_3 \right]$$
(4.28)

The displacement vector **d**(r) contains only the first eigenvalue term of the series expansion, that is for $\lambda = \frac{1}{2}$, and is thus restricted to describing the near crack tip field.

In order to define the stress intensity factors in the same way as done for the single isotropic material as well as the isotropic bimaterial problem the following normalization of the eigenvectors **w** proposed in (Suo, 1990, p. 347)

$$\mathbf{w} = [-\frac{1}{2}i, *, *], \quad \overline{\mathbf{w}} = [\frac{1}{2}i, *, *], \quad \mathbf{w}_3 = [*, *, 1]$$
(4.29)

is made where * indicates some value. By use of this normalization the eigenvectors recovers the stress intensity factor definition for isotropic bimaterials, cf. equation (3.44). Equation (4.28) is used for determining the stress intensity factors K_I , K_{II} and K_{III} by use of a similar curve fitting method as the one used in the simulation of the isotropic bimaterial problem in the previous chapter.

4.1.1 Approach for Determining the Relative Displacement

- - -

The theory describing the relative displacement of the crack faces for the anisotropic bimaterial problem presented in the previous section is implemented in a number of Maple routines which are enclosed on the appertaining DVD. In figure 4.2 the overall procedure for obtaining



Figure 4.2: Flow chart showing the procedure for obtaining the relative displacement vector for a given selection of material 1 and 2. The relative displacement vector is showed as a simplified three term expression with arbitrary constants q_1 and q_2 equivalent to the those described in subsection 3.2.5 on page 39 in chapter 3. All steps shown in the figure are programmed in Maple.

the simplified 3 termed expression for the relative crack face displacements is given.

4.1.2 Agreement with Formulas for Isotropic Bimaterials

In the following the agreement between the theory presented for isotropic bimaterials in section 3.1 and the theory for anisotropic bimaterials is investigated. It is shown that the predicted displacement for anisotropic bimaterials approaches the displacement predicted for isotropic bimaterials when the anisotropic materials approaches the isotropic materials. The material properties used in the isotropic bimaterial displacement are given in table 4.1.

Material	E [GPa]	ν	(<i>G</i> [GPa])	
1	70	0.3	(26.9)	
2	150	0.3	(57.7)	

Table 4.1: Isotropic material properties used to calculate the displacement for an isotropic bimaterial problem which is used for reference.

The anisotropic materials used are given in table 4.2. Material 1 is near isotropic and almost equal to material 1 in table 4.1. Material 2 is approaching material 2 in table 4.1 in four steps from 2a to 2d.

Material no.	E_1 [GPa]	E_2 [GPa]	E_3 [GPa]	G ₁₂ [GPa]	G ₁₃ [GPa]	G ₂₃ [GPa]
1	70.0	69.0	69.0	26.9	26.9	24.2
2a	150.0	10.0	10.0	5.0	5.0	2.5
2b	150.0	50.0	50.0	25.0	25.0	15.0
2c	150.0	100.0	100.0	50.0	5.0	40.0
2d	150.0	149.0	149.0	57.7	57.7	57.7

Table 4.2: Material data used for the plots in figure 4.3. The values are fictive and used for illustrating that the procedure used approaches the isotropic bimaterial solution when the orthotropic materials approach isotropy. Poisson's constants are $v_{12} = v_{13} = v_{23} = 0.30$ for all materials.

Displacements δ_x and δ_y for each of the four material sets are plotted in figures 4.3 and 4.4 in blue, and the isotropic bimaterial results from using the materials in table 4.1 are shown in black. All displacement curves are plotted for the arbitrarily chosen complex stress intensity factor $K = K_I + i \cdot K_{II} = 86[MPa\sqrt{mm}] + i \cdot 56[MPa\sqrt{mm}]$. It is seen that when the two orthotropic materials approach the two isotropic materials, the displacements for the orthotropic materials approach those for the isotropic case. It is therefore concluded that there is a good agreement between the theory given for the displacement in the isotropic bimaterial problem and the anisotropic bimaterial problem.



Figure 4.3: Plot of the displacement $\delta_y = d_2(r)$ from (4.28) in the y-direction for material 1 with material data held constant as 1 in table 4.2. Material 2 is varied from 2a that is a characteristic orthotropic fiber material to 2d that is an almost isotropic material. The black curve is displacements in the y-direction for the isotropic bimaterial solution with the data from table 4.1.



Figure 4.4: Plot of the displacement $\delta_x = d_1(r)$ from (4.28) in the x-direction for material 1 with material data held constant as 1 in table 4.2. Material 2 is varied from 2a that is a characteristic orthotropic fiber material to 2d that is an almost isotropic material. The black curve is displacements in the x-direction for the isotropic bimaterial solution with the data from table 4.1.

4.2 Fiber Bridging

When considering cracks in fibrous composite materials there is often a tendency towards that the resistance against propagation of the crack increases as it develops - also called *R*-curve effect, see figure 4.5. This is caused by fibers connecting the crack faces and thereby exerting a closing effect on the crack and this phenomenon is called fiber bridging. This effect is advantageous from a design view of perspective and therefore fiber bridging is implemented in the simulation of orthotropic fiber material. In the following fiber bridging and mechanisms causing it are presented, and the type of fiber bridging considered is so-called large scale bridging with involves bridging lengths that are comparable or exceed a specimen dimension (Sørensen and Jacobsen, 1998, p. 1443).



Figure 4.5: *Experimental R-curve data from a mode I loading of a glass-polyester DCB test specimen Sørensen and Jacobsen (2009).*

An example of fiber bridging is seen in figure 4.6 which shows propagation of a crack with a significant amount of fibers crossing between the crack surfaces.



Figure 4.6: *Photo of a distinct bridging zone for a double cantilever beam test specimen (Sørensen* et al., 2006).

Figure 4.7a illustrate the concept of fiber cross over bridging for crack opening. When the crack is loaded, the fibers are stretched and act like springs that exert reaction forces on the

crack faces attempting to prevent crack opening. These forces create a traction field on the crack faces as illustrated in figure 4.7b. These tractions are related to the crack end-opening by a so-called bridging law that is described in the way shown in figure 4.7c. The tractions increase rapidly to reach a peak σ_0 and then they decrease more steadily and diminish at a crack end-opening of δ_0 .



Figure 4.7: (*a*) Illustration of cross over bridging fibers for a crack loaded by pure moments. (b) Illustration of tractions on the crack faces caused by bridging fibers. (c) Illustrative example of a bridging law.

Several mechanisms cause fiber bridging. In figure 4.8 three mechanisms that are considered dominant in the case of fibrous, (Spearing and Evans, 1992), (Bao and Suo, 1992), are illustrated. The first mechanism develops from intralaminar micro cracks evolving in the area in front of the crack tip (figure 4.8a). When these micro cracks grow, a number of differently sized cracks propagate simultaneously. Eventually the micro cracks can turn into the main crack. This causes bundles or bands of fibers of various sizes to be left across the crack surfaces behind the primary crack tip (figure 4.8b). These fiber bundles vary from single fibers to bundles or laminae bands. These fiber bands have to be torn apart before the crack can propagate and this implies higher fracture resistance.

The second mechanism is fiber pull-out of crossing fibers that are broken within the material (figure 4.8c) and are pulled out during increasing crack opening (figure 4.8d). The friction between fiber and matrix material of this pull-out process contributes to the fracture resistance of the crack.

The third mechanism become effective if the fiber bands in the first fiber bridging mechanism is not torn apart. This implies the propagation of cracks between the crack face and the fiber bands which increases the toughness, see figure 4.8e.

In order to implement these mechanisms in the numerical modeling of bridging, a bridging law on the general form shown in figure 4.9 is used. This form of the bridging law is adapted from (Sørensen and Jacobsen, 2009, p. 452). The bridging law is not to be understood as a cohesive law as it describes the bridging zone only, and does not take the propagation of the crack tip into consideration. The bridging law is interpreted in the way that the first part (1) models



Figure 4.8: Three dominant fiber bridging mechanisms causing increasing fracture resistance. (a) Intralaminar micro cracks initiating in front of the crack tip and (b) bridging fiber bands or bundles caused by development of the micro cracks. (c) Crossing fiber that is broken within the material due to micro cracks in front of the crack tip (d) and causes increased fracture resistance by fiber pull-out resistance as friction between fiber and matrix material. (e) Steady degrading of fracture toughness from propagation of parallel cracks between fiber bundle and laminate until breakage.

the elastic response from the bridging fibers close to the crack tip as the crack opens until the tractions reaches a peak level σ_0 . From this point the tractions degrade (2) with increase of the crack end-opening because the bridging fibers begin to break. At a certain crack end-opening, the slope of the tractions levels out (3) until reaching a crack end-opening δ_0 for which the tractions vanishes and the bridging zone diminishes at this point.

The tractions in the bridging law are for general loading expressed on the form

$$\sigma_n = \sigma_n(\delta_n, \delta_t)$$
 and $\sigma_t = \sigma_t(\delta_n, \delta_t)$ (4.30)

where σ_n and σ_t are normal and shear tractions between the crack surfaces, and δ_n and δ_t are the normal and tangential opening of the crack. The J-integral around the crack tip and bridging zone for crack propagation can be expressed as

$$J_R = \int_0^{\delta_t^*} \sigma_t(\delta_n, \delta_t) d\delta_t + \int_0^{\delta_n^*} \sigma_n(\delta_n, \delta_t) d\delta_n + J_0(\Gamma_0)$$
(4.31)

and takes the tractions on the crack faces into consideration. In figure 4.10 an illustration of equation (4.31) is shown. $J_0(\Gamma_0)$ is the J-integral around the crack tip calculated for the contour Γ_0 and corresponds to the critical energy release rate for crack tip propagation. The sum of the two integrals for the bridging tractions are calculated as the difference $J_1(\Gamma_1) - J_0(\Gamma_0)$ between



Figure 4.9: Illustrative example of trilinear bridging law $\sigma_n(\delta_n)$ for normal end-opening with indication of which of the three bridging phenomena is dominant for each linear part.

 $J_1(\Gamma_1)$ calculated for the contour Γ_1 encircling the bridging zone and crack tip, and J_0 at the crack tip. The path independence of the J-integral (Rice, 1968) implies that $J_2(\Gamma_2) = J_1(\Gamma_1)$ and this is utilized for the J-integral in (Sørensen and Jacobsen, 2009), where it is calculated for the external boundaries of the test specimen as this simplifies the calculation.

 J_R corresponds to the critical energy release rate for the crack tip and bridging zone in $\frac{J}{m^2}$ and is the total fracture resistance of the crack.



Figure 4.10: Illustration of the J-integral in equation (4.31).

If J_0 is assumed constant in equation (4.31), the normal and tangential traction components are the derivative of the fracture resistance J_R with respect to the normal and tangential crack opening respectively, that is:

$$\sigma_n(\delta_n^*, \delta_t^*) = \frac{\partial J_R(\delta_n^*, \delta_t^*)}{\partial \delta_n^*}, \qquad \sigma_t(\delta_n^*, \delta_t^*) = \frac{\partial J_R(\delta_n^*, \delta_t^*)}{\partial \delta_t^*}$$
(4.32)

Thus for a specific R-curve for fracture resistance the bridging law can be calculated by the derivative. It should be noted that it is assumed that the tractions in the bridging law depends on the crack opening displacement but not on the opening history, and the same traction-opening relationship is assumed for every point along the bridging zone (Sørensen and Jacobsen, 2009, p. 447).

4.3 Finite Element Modeling of Bridging

The bridging effect described must be implemented in the finite element model in order to be able to predict a more precise fracture resistance for a particular crack. The tractions on the crack faces are implemented in the FE model by defining nonlinear springs between coincident nodes on the two crack surfaces within a specified bridging zone.

The springs used are modeled as *Combin39* elements in ANSYS for which it is possible to define a force-displacement law by a maximum of 20 points with varying stiffness for each step of the curve. Furthermore, these spring elements have the option of applying the stiffness in a specific direction only. For each node pair on the crack faces two spring elements are defined with one having a force-displacement law $F_n(\delta_n)$ for y-direction, opening normal to the crack faces, and one having a different force-displacement law $F_t(\delta_t)$ for the x-direction, opening tangential to the crack faces. This is illustrated in figure 4.11.



Figure 4.11: Illustration of the implementation of nonlinear springs in the finite element model with two springs defined for each coincident node pair on the crack surfaces. One spring exerts bridging tractions in the y-direction for normal end-opening of the crack, and the other exerts bridging tractions in the x-direction for tangential end-opening.

Owing to the nonlinearity of these spring elements, the simulation with bridging is solved by use of the nonlinear arc-length method solver in ANSYS.

In (Sørensen and Jacobsen, 2009, p. 452) a bridging law for normal end-opening for the DCB test calculated based on the measurements and is plotted as seen in figure 4.12. It is seen that this is of the form illustrated in figure 4.9 with rapidly increasing tractions reaching a critical value from which they decrease in a trilinear manner. This type of bridging law is applied to the spring elements for the normal opening and another one is applied for the springs in the tangential direction. These bridging laws are defined as the derivative of a R-curve as described in the previous section.

An example of a R-curve inspired from (Sørensen and Jacobsen, 2009) is seen in figure 4.13a for pure mode I loading. A corresponding bridging law for this mode is then determined by equation (4.32). This is plotted in figure 4.13b. It is seen that the shape of this bridging law resembles the one in figure 4.9.

The abrupt change in stiffness at the point of maximum stress may cause problems when solving the nonlinear problem. Hence, a number of additional points are defined at this point in order to smooth these changes by creating a circular shape as an attempt to ensure that the model converges. This is illustrated in the zoomed view of figure 4.13b.



Figure 4.12: Bridging law for normal tractions σ_n as function of the normal end-opening δ_n^* from tests in (Sørensen and Jacobsen, 2009, p. 452).



Figure 4.13: (a) *R*-curve with data inspired from the test, but assumed to be calculated based on the assumptions made in this report and not the ones from the article. This *R*-curve is used as base for modeling a corresponding material behavior. (b) Bridging law used for the nonlinear spring elements in the y-direction calculated as the derivative of the *R*-curve according to (4.32). The zoomed plot shows a rounding of the peak of the curve in order to facilitate the nonlinear solution procedure.
4.3.1 Distribution of Spring Elements

Based on a solid FE model of a DCB specimen it is illustrated how the spring elements modeling the bridging effects are distributed between the crack faces. The 3D FE model uses 20 node solid elements with mid-side nodes. In order to minimize the solving time of the nonlinear analysis, spring elements are defined for corner nodes only. This is illustrated in figure 4.14 showing a cut-out of the nodes on the crack faces with the red box marking the size of the bridging zone. The *Combin39* elements (in blue) are plotted as deformed from a mode I loading in order to visualize them. Regarding the 2D model, the spring elements are defined in the same way as for 3D with the exception that they are defined for the mid-side nodes as well. This is because this model is less extensive and faster to solve than the 3D model.

The bridging law in figure 4.13b must be defined for each of the spring elements by calculating resulting forces from the stresses. It is assumed that each spring element (roughly) represents the resulting force from stresses on an area corresponding to one element as illustrated in figure 4.14. Thus the stress values in the bridging law are multiplied by this area.



Figure 4.14: Illustration of the nodes and combine39 element in the bridging zone of a solid FE model of a DCB specimen. The red area indicates the bridging zone. The black dots indicate the nodes and the blue lines indicate the deformed combine39 elements.

When simulating the bridging effect, the traction level of resistance against crack opening in each spring lies somewhere on the bridging law curve. Figure 4.15a shows a crack with bridging zone in mode I loading. The nonlinear springs are exerting tractions (forces) on the crack faces. Selected springs are marked by a number and the corresponding traction level is indicated in figure 4.15b.



Figure 4.15: (a) Simulation of bridging zone modeled by nonlinear springs for pure mode I loading. (b) Bridging law for mode I opening with indication of traction and displacement for selected springs in the simulation.

It is seen that spring number 1 simulates undamaged bridging fibers that exert a rapidly increasing force. Spring number 2 simulates damaging of crossing fibers with degrading stiffness, and the springs 3 to 8 simulates bridging tractions dominated by fiber breakage and fiber pull-out with slowly degrading stiffness. Spring number 8 is just about to exceed the critical displacement for which it is considered completely broken meaning that it exerts no tractions on the crack surfaces, and the bridging zone has reached steady state.

The method for modeling bridging by nonlinear springs described it is possible to apply this together with the parameter estimation for determining the stress intensity factors. This makes simulation of the R-curve effects possible as higher loads can be applied before a critical opening displacement is reached.

4.4 Description of Test Data

In order to evaluate the method for predicting crack growth presented in the report so far, it is compared to experimental data. In this section the experiment used to evaluate the methods is summarized.

A test of the fracture resistance with respect to crack opening for at DCB test specimen made of unidirectional orthotropic fiber materials is described in (Sørensen and Jacobsen, 2009). The dimensions of the test specimen used in the experiment are illustrated in figure 4.16a, and the material used is unidirectional glass-epoxy composite material that in the article is assumed isotropic with E = 36 GPa and v = 0.3. An initial 60 mm crack is induced in the specimen. The tested specimen is illustrated in figure 4.16b loaded in mode mixity in a test setup described further in (Sørensen and Jacobsen, 2009).

The DCB specimen is tested for a range of different moment ratios $\frac{M_1}{M_2}$ related to particular mode mixities with $\frac{M_1}{M_2} = -1.0$ being for pure mode I to $\frac{M_1}{M_2} = 0.985$ for approximately pure mode II. During the testing procedure, the applied moments are measured along with the normal and tangential crack end-opening δ_n and δ_t at the initiation of the crack (a = 60mm), see figure 4.17.



Figure 4.16: (*a*) Double cantilever beam specimen used in the test. The specimen is made of 20 unidirectional E-glass fiber fabrics in polyester resin. (Sørensen and Jacobsen, 2009) (b) Photo of the test specimen during testing. (Sørensen and Jacobsen, 2009)



Figure 4.17: Illustration of bridging zone with indication of the magnitude of (δ^*) , normal (δ_n^*) and tangential (δ_t^*) crack opnening. (Sørensen and Jacobsen, 2009)

The J-integral is used to calculate the fracture toughness J_R from the applied moments (Sørensen and Jacobsen, 2009, p. 447):

$$J_R = \left(1 - \nu^2\right) \frac{21\left(M_1^2 + M_2^2\right) - 6M_1M_2}{4B^2H^3E}$$
(4.33)

where:	M_1	Moment on the upper beam		
	M_2	Moment on the lower beam		
	В	Width of the DCB test specimen		
	H	Height of each beam of the DCB test specimen		
	Ε	Youngs Modulus for an isotropic material		

Equation 4.33 is for a plane problem based on the assumption that the stresses and displacements are invariant in the z-direction.

The results of using this formula are illustrated in figure 4.18. In figure 4.18a the fracture toughness for different mode mixities are plotted, and it is seen that it increases for the mode mixity approaching mode II meaning that the shear toughness of the material is roughly three to four times the tensional toughness for pure mode I. The fracture toughness for pure mode I is plotted in figure 4.18b as a function of the normal end-opening δ_n^* . The results from tests of several specimens show to correlate and the general curve profile shows a resistance curve

reaching a steady state level at approximately four times the initial fracture toughness. The fracture toughness for approximately pure mode II Is plotted in figure 4.18c as function of the magnitude of the crack end-opening δ^* , and it shows that the fracture toughness increases with crack end-opening in a roughly linear manner. The mode II *R*-curve does not show a plateau within the end opening spectra tested. It is however expected that this plateau exits.



Figure 4.18: (a) Fracture resistance as a function of the magnitude of the end-opening δ^* for various moment ratios from the test. (b) Fracture resistance as function of normal end-opening δ^*_n for pure mode I loading. (c) Fracture resistance as function of the magnitude of end-opening δ^*_n for approximately pure mode II loading. All three figures are from (Sørensen and Jacobsen, 2009).

The moments applied to the test specimens and the corresponding end opening are not given directly in the article. It is necessary to know these when running the simulation, and they are calculated by use of equation (4.33) for the J-integral.

4.5 Simulation of Crack between Two Orthotropic Materials

The remaining sections of this chapter cover a simulation of a crack in a DCB specimen of orthotropic fiber materials similar to the one presented above from the test. Firstly, a plane FE model with bridging is analyzed with the purpose of investigating to what degree it is possible to model the R-curve effect for mode I, mode II and mixed mode with the spring elements described in the previous section.

Secondly, a solid FE model of the specimen is analyzed in order to identify a well suited element size and number of nodes for determining the stress intensity factors from the crack surface displacement for this model. The aim for identifying a well suited element size is to reduce model complexity in order to minimize calculation time without losing necessary precision.

The definition of necessary precision is difficult to quantify without knowing what implications a given error has on the stress intensity factors. Hence, this term needs further discussion, perhaps involving a business partner in order to connect the term to a real structure. This is considered beyond the limitations of this project. Hence, a guess on the necessary precision is that results must lie within \pm 5 % of the results of an analysis in which the elements near the crack tip are small relative to geometrical dimensions of the structure. This is because the results for the stress intensity factors are expected to converge to the "true" values when the element size approaches zero. The results of this analysis can be seen in appendix B. The conclusion is that an element size of up to 2 [mm] can be used to determine the stress intensity factors.

Furthermore, a simulation of the test specimen is analyzed using a solid FE model with bridging. The stress state in the solid double cantilever beam (DCB) specimen is evaluated as to determine if it should be assumed plane stress deformation or plane strain deformation. During this evaluation it was discovered that some of the assumptions used on the test specimen in article (Sørensen and Jacobsen, 2009) seem to be violated, and these have therefore been investigated further.

4.5.1 Plane Example of Simulation of Bridging

The expression for the relative crack face displacement for orthotropic material has been implemented in the parameter estimation of stress intensity factors, and the bridging modeling by use of nonlinear spring elements has been implemented in a plane and a solid FE model of the DCB specimen. In order to evaluate the modeling of bridging, the plane model has been analyzed for pure mode I, pure mode II and for a mixed mode example.

A displacement plot of the analyzed plane model loaded by opposite moments of different magnitude is seen in figure 4.19 with Combin39 elements bridging the crack faces. Plane strain is assumed for the plain eight node elements in this model because this has been assumed for the J-integral calculations in (Sørensen and Jacobsen, 2009) it is possible to compare the simulated results with these. As a thickness definition is unavailable for the plane strain configuration of the 8-node Plane183 element in ANSYS, unit thickness is used and the moment loads are scaled by division with the width of the specimen, ie. 30mm.



Figure 4.19: Displacement plot of the plane FE model loaded by pure uneven moments. The criterion $G \approx G_{c,0}$ for crack propagation indicating the local fracture toughness is determined from displacements around the crack tip, and the active bridging increases the overall fracture toughness.

The evaluation of the three cases, mode I, mode II and mixed mode has been done by the cyclic procedure shown in figure 4.20. A R-curve for pure mode I and one for pure mode II, determined by the J-integral from a set of test results are assumed available. By calculating the respective derivatives of these R-curves numerically, bridging laws are obtained for each of the two modes. These bridging laws are applied in the FE model for orthotropic materials and solved for a range of length of the bridging zone. For each bridging zone lengths the criterion that the energy release rate at the crack tip equals the critical value from R-curve must be fulfilled by alternating the moment loads on the model. The moment loads and crack opening for which the criterion is met are logged and a corresponding R-curve is calculated by use of the J-integral. This R-curve is then compared to the assumed R-curve from the test results and evaluated. The results of the procedure are described below.



Figure 4.20: Flowchart of the procedure for evaluating the bridging modeling for mode I, mode II and mixed mode.



Figure 4.21: (a) Initial R-curve for pure mode I used for determining a bridging law. (b) Bridging law for pure mode I determined by calculating the derivative of the R-curve. For a normal opening $\delta_n^* \approx 4$ the tractions from the springs are assumed to vanish.

For the analyses for mode I and II the R-curves are seen in figure 4.21a and figure 4.22a, respectively. These curves are assumed available from a test of the material combination to be

simulated. In order to use realistic R-curves, the shape and values of the R-curves used here are adapted from those in figure 4.18b, but should not be regarded identical as it is the intention to demonstrate and evaluate the simulation procedure rather than replicating the test results from (Sørensen and Jacobsen, 2009). It should be noted that the R-curve for pure mode II does not show a steady state opening, but is idealized as a straight line with constant slope. A steady state level is expected to be reached at some time though, but this is not described in (Sørensen and Jacobsen, 2009).



Figure 4.22: (a) Initial R-curve for pure mode II used for determining a bridging law. (b) Bridging law pure mode II determined by calculating the derivative of the R-curve. Due to the monotonous course of the R-curve, the bridging law has a constant value.

The R-curves are defined by a number of points that are interconnected by straight lines. The bridging law is then defined by the slopes of these lines, and these are shown in figures 4.21b and 4.22b, respectively. The bridging law for mode I has a course showing degrading stiffness until a steady state level is reached. The one for mode II on the other hand, is a constant value due to the shape of the R-curve. Thus the resistance force of the spring elements in the *x*-direction is constant for any opening displacement.

The results of the evaluation are plotted in three figures. Figure 4.23 shows the fracture resistance in terms of J_R as a function of the normal crack opening for mode I. It is seen from these results that the simulated curve resembles the course of the initial curve, but with lower values within a maximum deviation of 7-8 %. This is considered satisfactory seen in the light of the big difference between the measured R-curves in (Sørensen and Jacobsen, 2009), cf. figure 4.18b. The R-curve shows a steady state level at crack opening of approximately 3.5mm.

The curve for the bridging length shows the length L_{bz} of the bridging zone with respect to the crack opening. For mode I it increases with a shape similar to J_R until a length of $L_{bz} =$ 70 - 80mm for which a steady state value of J_R is reached. Regarding the steady state length of the bridging zone, the resulting forces for the nonlinear springs in the *y*-direction for pure mode I is plotted on the bridging law in figure 4.24 for $L_{bz} = 70mm$ and $L_{bz} = 80mm$. It is seen that for $L_{bz} = 70mm$ the outermost spring is deflected to $\delta_n \approx 3.2mm$ which is the close to $\delta_{n,0} = 3.5$ for which the tractions from the springs are considered to diminish. For $L_{bz} = 80mm$ the outermost spring has exceeded $\delta_{n,0}$ and thus the steady state length of the bridging zone has been reached. Thus the steady state length is regarded as $L_{ss} = 80mm$.

For mode II the initial and calculated R-curves have a similar course with the calculated lying lower which is illustrated in figure 4.25. The bridging zone length corresponding to the



Figure 4.23: Plot of the initial and calculated values for J_R and the bridging zone length L_{bz} with respect to the normal crack opening δ_n^* for mode I.



Figure 4.24: Bridging law for mode I with indication of the resistance forces in the bridging springs in the y-direction for bridging lengths of $L_{bz} = 70 mm$ and $L_{bz} = 80 mm$.

tangential opening for this mode are up to approximately three times the steady state length for mode I for magnitudes of tangential crack opening equal to normal opening. This mode shows no steady state level, or if it does it is beyond $L_{bz} > 300 mm$, and this has not been covered in (Sørensen and Jacobsen, 2009) or in this report.

One example has been calculated for mixed mode loading for which $\frac{M_1}{M_2} = 0.25$ meaning that the moments have the same direction but with M_2 being four times M_1 . This should introduce both mode I and II deformations around the crack tip and thus $K_I \neq 0$ and $K_{II} \neq 0$. The analysis has been made with application of the bridging laws obtained for pure mode I and mode II respectively in order to evaluate if, or to what degree, this is possible.

Regarding the determination of crack propagation by calculating G_0 at the crack tip this example simplifies in the way that the stress intensity factors K_I and K_{II} decouples because



Figure 4.25: Plot of the initial and calculated values for J_R and the bridging zone length L_{bz} with respect to the tangential crack opening δ_t^* for mode II.

the material of the upper and lower beam are approximately similar. This implies that

$$\delta_{x}(K_{I}, K_{II}) \Rightarrow \delta_{x}(K_{II}) \qquad \delta_{y}(K_{I}, K_{II}) \Rightarrow \delta_{x}(K_{I}) \tag{4.34}$$

which means that K_I must be determined by a parameter estimation of the FE displacements to δ_y with $K_{II} = 0$, and analogously for K_{II} . The values for the stress intensity factors at the crack tip with respect to L_{bz} are plotted in figure 4.26, and it shows that with no bridging K_I is approximately 75% of K_{II} but decreases rapidly to reach 0 at $L_{bz} = 90.0mm$ whereas K_{II} increases to reach a steady state level of approximately $101MPa\sqrt{mm}$ at the same bridging length. This shows that the mode II bridging is dominating and that for $L_{bz} \ge 90.0mm$ the bridging effect overcomes the mode I opening completely making the crack faces at the crack tip sliding relative to each other.



Figure 4.26: Plot of the stress intensity factors for mode I and mode II with respect to length L_{bz} of the bridging zone for mixed mode with $\frac{M_1}{M_2} = 0.25$.

The results for the fracture resistance J_R and L_{bz} with respect to the magnitude of the crack opening of the analysis are shown in figure 4.27. The black R-curve from assumed test data is idealized from a mixed mode curve in (Sørensen and Jacobsen, 2009, p. 449). The calculated curve initiates the crack at roughly the same G_0 as the test curve and shows a curved shape at the first part. As the mode I displacement succumb to the bridging effect, the calculated curve has a course of an increasing straight line which is due to the constant value of the mode II bridging law. It lies lower than the test curve until a crack opening of approximately 2.0*mm* where the test curve shows a steady state level but the calculated curve continues to increase. There are a number of aspects concerning this:

- Mode II bridging is too dominating and suppresses the mode I opening of the crack. This is because bridging law for mode II has no steady state level and this is replicated in the mixed mode curve. This entails the question if the mode II bridging law should have a decreasing shape at some crack opening instead of being constant.
- In (Sørensen and Jacobsen, 2009) the bridging law is proposed to be dependent on both δ_n^* and δ_t^* , are therefore a specific law corresponds to a specific loading mode. This has not been implemented in the model in this project but may cause a better resemblance between the test and calculated R-curves in figure 4.27.



Figure 4.27: Plot of the initial and calculated values for J_R and the bridging zone length L_{bz} with respect to the tangential crack opening δ_t^* for mixed mode with $\frac{M_1}{M_2} = 0.25$.

The results of the modeling of bridging demonstrate that it makes it possible to model pure mode I and pure mode II crack propagation with bridging within a satisfactory margin of deviations with the simulated values being lower than the tested ones, and thus are conservative.

For mixed mode a single example has demonstrated significant deviation between the tested and calculated results and that mode II dominates the fracture toughness. This is by using the mode I bridging law in the *y*-direction and mode I bridging law in the *x*-direction for the nonlinear springs. In order to improve this, the bridging laws should be alternated or a specific bridging law should be applied for a specific mixed mode case which is described in the article (Sørensen and Jacobsen, 2009).

Based on the curves for J_R in figure 4.27 it is considered that for this mixed mode case it is reasonable to apply this simulation as long as $\delta^* \leq 2.0 mm$, where the simulated results underestimates the tested ones and are conservative but roughly resembles the course of the tested curve. For $\delta^* \leq 2.0mm$ The simulated results over estimates the fracture resistance and it is not advisable to use the model in this case.

4.6 Analysis of DCB Test Specimen with a Solid FE Model

The simulation of crack propagation and bridging has been demonstrated for the plane case where the DCB model is assumed to be invariant through the width. In this section a simulation of the specific test specimen from (Sørensen and Jacobsen, 2009) with a solid FE model is treated. A solid model is used in order to study if this shows a plane response as assumed in the article. The solid model is illustrated in figure 4.28.



Figure 4.28: DCB model used for two dissimilar orthotropic materials. The model is shown with 2mm element size at the crack tip and external mode I loading by pure moments.

The mesh of the model has been changed compared to the model used for the isotropic single and bimaterial cases in the way that the circle around the crack tip has been removed. This is done because it added unnecessary many elements to the model when using elements at the crack tip that are of approximately the same size as the rest of the elements in the model. The mesh of the model is first created in the *xy*-plane and then extruded in the *z*-direction with 16 element divisions. In this way all elements have the same dimensions in the *z*-direction.

The test specimens in (Sørensen and Jacobsen, 2009) are made of a single composite material of glass-polyester with fibers oriented in the *x*-direction. The theory presented in section 4.1 is only applicable for dissimilar anisotropic materials. So in order to apply the theory, the material of the test specimen is modeled as two different materials with almost identical orthotropic properties. This is not expected to cause a significant deviation of the results compared to the case of identical materials. The compliance matrix for orthotropic materials is given as.

$$S_{ij} = \begin{bmatrix} \frac{1}{E_1} & -\frac{V_{21}}{E_2} & -\frac{V_{31}}{E_3} & 0 & 0 & 0\\ -\frac{V_{12}}{E_1} & \frac{1}{E_2} & -\frac{V_{32}}{E_3} & 0 & 0 & 0\\ -\frac{V_{13}}{E_1} & \frac{V_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$
(4.35)

Material	E_1 [GPa]	E_2 [GPa]	E_3 [GPa]	G_{12} [GPa]	G ₂₃ [GPa]	G ₁₃ [GPa]
1	36	11	11	4	4	4
2	36	10	10	4	4	4

The material properties of the materials used in the model is given in table 4.3.

Table 4.3: Material data for material 1 and 2 used in simulation 3. The Poisson ratios v_{12} , v_{23} and v_{13} are all 0.3.

In (Sørensen and Jacobsen, 2009) the material properties for the glass-polyester composite material are assumed isotropic, but the FE model of the test specimen is modeled with orthotropic material properties. No specific data has been retrieved for this, and therefore the values in table 4.3 are estimated based on other articles by Bent Sørensen concerning the same topic as (Sørensen and Jacobsen, 2009) and a general search for glass-polyester material properties.

4.6.1 Evaluation of Assumptions Made on the Test Specimen

The stress and strain state in the region around the crack tip for moment loading is investigated based on an analysis of the FE model. This is done in order to ensure that the assumptions for the theory of a crack between dissimilar anisotropic materials in (Suo, 1990) are valid, so that it is reasonable to simulate crack growth for the specimen using this theory. In the theory either plane stress or plane strain deformation is assumed. And furthermore this investigation is done to test the assumptions made for the test specimen in (Sørensen and Jacobsen, 2009). Based on the investigation an idealized stress or strain state assumption is selected.

As mentioned the material of the test specimens are assumed isotropic where $E_{isotropic} = E_{1,orthotropic}$ meaning that E = 36MPa. The stress state in the test specimens are considered as plane strain. This assumption implies that the out-of-plane strain components, ε_z , ε_{xz} and ε_{yz} are zero and that the relative displacement of the crack faces is constant in the *z*-direction meaning that the structural response is assumed two dimensional. A study of the stress state in the FE model shows that especially two parameters compromise this assumption. These are the width of the test specimen and Poison effects related to the bending of the beams of the DCB test specimen.

The Poison effects are illustrated in figure 4.29 by a scaled deformation plot of the specimen close to the crack for mode I loading. Due to the bending stresses in the beams the crack faces are contracted and the outer faces are widened. In the lower right figure an exaggerated sketch of the beam cross section during loading is shown. These bending effects increase with the width of the test specimen (when all other dimensions are kept constant).



Figure 4.29: View of the FE model where the most of the elements behind the crack tip are excluded. The crack faces are marked by red lines in order illustrate the curved shape caused by the Poison effects. The black lined figure in the lower right corner illustrates the cross sectional shape of the beam during bending.

This effect implies that the displacements are dependent on the *z*-coordinate which means that for the DCB specimen loaded with external mode I loading by pure moments, the displacement of the crack faces is smallest at the edges and largest in the middle. This is illustrated in figure 4.30. It is seen that the relative difference between the two curves is largest close to the crack tip.



Figure 4.30: (a) The relative displacement of the crack faces on the edge (z = 0mm) and in the middle (z = 0.15mm) of the DCB specimen. (b) Close up of the relative crack face displacement.

It is the relative displacements of the first 1-2 nodes behind the crack tip that are governing for the magnitude of the stress intensity factors. Hence, the selected *z*-coordinate has a signifi-

cant influence on the stress intensity factors. For an external mode I loading by pure moments, K_I at the center is approximately 80% larger than K_I at the edge. If for example the width is 10mm instead of 30mm this difference is only approximately 20%. It has been observed that the bending effect becomes more substantial the larger the transverse stiffness E_2 is. This is properly because the structure in front of the crack tip is able to suppress the bending effect if the transverse stiffness E_2 is very small.

This difference in K_I indicates that the assumption to treat the test specimen as a two dimensional problem is not valid. The difference of K_I must imply a curved crack front because it propagates first in the middle, and then gradually expands to the edges.

In figure 4.31 a sectional view of the strain components ε_y , ε_{xz} , ε_{yz} and ε_z are shown from the edge and into to the middle of the FE model of the test specimen. In figure 4.32 similar sectional views are shown for the stress components σ_y , σ_{xz} , σ_{yz} and σ_z . Comments on the stress state are given in the following two items:

- If the magnitude of the ε_y and ε_z are compared it is not clear that plane strain is present as assumed in (Sørensen and Jacobsen, 2009). The out-of-plane strain component ε_z should be significantly smaller than the in-plane strain component ε_y to assume plane strain. The test specimen is not constrained from contraction in the *z*-direction. Hence, a plane stress state on the edges of the specimen is expected. But as long as the edge effects are small compared to the specimen dimensions it is still fair to assume plane strain. However, it is assessed from the figure that the out of plane strain components ε_{xz} , ε_{yz} and ε_z can only be regarded as zero/insignificant for $z \in [6.4, 15]$ or maybe even only $z \in [8.6, 15]$. This corresponds to approximately 50-60 percent of the specimen width.
- The presented theory for a crack between two dissimilar anisotropic materials assumes a plane strain or stress deformation. This covers a state with stresses or strains out-ofplane but being invariant through the width. The change in the DCB test specimen between a dominating plane stress state at the edge and a plane strain state in the middle means that there is a change in the stresses and strains in the z-direction that cannot be neglected. Hence, it appears that neither plane stress deformation nor plane strain deformation can be assumed.



Figure 4.31: Sectional view of the tensor strain components ε_y , ε_{xz} , ε_{yz} and ε_z . The strain components are presented as nodal values. All *z* values are given in [mm]. The value *z* = 0mm indicates the edge of the specimen and *z* = 15mm indicates the middle of the test specimen. The colorbar is the same for ε_{xz} and ε_{yz} .



Figure 4.32: Sectional view of the stress components σ_y , σ_{xz} , σ_{yz} and σ_z . The stresses are presented as nodal values. All *z* values are given in [mm] and the stresses are given in [MPa]. The value z = 0mm indicates the edge of the specimen and z = 15mm indicates the middle of the test specimen. The colorbar is the same for σ_{xz} and σ_{yz} .

The assumptions made in (Sørensen and Jacobsen, 2009) seem not to be fulfilled based on the evaluation of the FE model of the test specimen. A summary of these assumptions and why they are violated is seen in the lest below.

Two dimensional structural response

From the displacement curves in figure 4.30 it is clear that due to the Poisson effects from bending, the structural response can not be regarded as two dimensional.

• Plane strain

The strain and stress plots of the FE model of the test specimen in figure 4.31 and 4.32 show that it is not reasonable to assume plane strain nor plane stress through the width of the specimen.

• Isotropic material

It seems from the orthotropic FE analysis that the elastic modulus E_2 (and maybe also E_3) have a large influence on to what degree the relative crack face displacement is influenced by the bending effects.

The fact that these assumptions seem not to be valid means that there are some significant uncertainties related to the R-curves presented in Sørensen and Jacobsen (2009). This implies that it is not reasonable to simulate R-curves for the solid FE model and compare these directly to the R-curves in the article because a single *R*-curve cannot be obtained for three dimensional cases with the method used in this project.

Thus in order to use the method on the solid model, it has to be further developed to make it possible to vary the propagation of the crack front with respect to the *z*-coordinate. This has not been done in this project, but is discussed in chapter 5.

4.6.2 Difference between R-curves for Solid FE model

A study of the difference between two R-curves determined for plane strain for relative displacements on the edge and in the middle of the solid model for pure mode I loading has been conducted. These curves are then compared to the R-curve from the test, in order to determine which one is closest to the test results.

A R-curve similar to the one for mode I from the plane simulation is used for determining the bridging law for the solid model. As this R-curve is used for the edge and the middle of the specimen, this will lead to apparently different fracture resistances for the two cases.

Because the calculation of the J-integral is not valid for the three dimensional case, the Rcurves plotted for moment load instead of fracture resistance with respect to crack opening. The loads from the test is obtained by solving for the moments, M_1 and M_2 in equation (4.33). In the case of pure mode I $M_1 = M_2$.

In figure 4.33 the moment measured in the test is shown together with the moments predicted from the crack face displacement of the edge and middle, respectively. It is seen that there is a good correspondence between the moment predicted for the crack face displacements in the middle and the measured moment. On the contrary the moment determined from the crack face displacement at the edge is far from the measured moment.

When the R-curves reach steady state there is approximately 30% difference between M_{measured} and M_{edge} . The difference between M_{measured} and M_{middle} is approximately -5%. From this



Figure 4.33: *R*-curves expressed as moments with respect to crack opening for pure mode I from the test, and from the relative displacement in the middle and at the edge for simulation of the solid model of the test specimen.

brief analysis it is shown that the crack face displacement in the middle of the test specimen is dominating for the fracture resistance of the test specimen.

Summary of Interface Crack between Two Anisotropic Materials

The purpose of this chapter has been to develop and implement a method for modeling and analyzing a crack between to fibrous composite materials. Experimental data from the article (Sørensen and Jacobsen, 2009) is simulated with these methods.

A theory regarding the determination of the stress intensity factors for a crack between two dissimilar anisotropic materials has been presented. The displacement expressions from this theory has been implemented in Maple and MATLAB routines which, based on the relative nodal displacements at the crack front of a plane or solid FE model, can determine the stress intensity factors.

Bridging effects for cracks in fibrous composite materials has been described an explanation of the dominating bridging mechanisms. The bridging effects have been modeled by nonlinear springs with two independent traction laws, $\sigma_n(\delta_n)$ and $\sigma_t(\delta_t)$. This has been implemented to a plane and solid FE analysis of the experimental data. For the plane FE model the bridging effects demonstrates good agreement for pure mode I or mode II compared to the experimental data. For mixed mode the results are less convincing.

An investigation of the assumptions made for the experiment based on analyses of the solid FE model has shown that assumptions concerning the test specimen as a 2D problem are not valid.

For the solid FE model the toughness of the test specimen is dependent on the *z*-coordinate for which the analysis is conducted and showed a difference approximately 30 percent between edge and middle of the crack.

Chapter

5

Discussion

The purpose of this chapter is to present and discuss suggested applications of the procedure developed in this project. Furthermore, suggestions for further development of the mesh strategy, bridging model and extension of the developed procedure for curved crack fronts, is presented.

5.1 Application of the Developed Procedure

In this section some applications of the developed procedure in the stage it has at the end of the project period are described. These concern:

- Progressive crack modeling
- Modeling of test data
- Sub-modeling
- Plane crack simulation

5.1.1 Simulation of Progressive Crack Propagation

The main steps in the simulation procedure developed in this project for simulating progressive crack propagation is illustrated in figure 5.1 and described below.

The simulation begins with a definition of the finite element model with a specification of the geometry, the two materials on either side of the crack, loads on the structure and the length and location of the initial crack without bridging between the crack faces. The expressions for the relative displacements of the crack faces are calculated for either isotropic or anisotropic materials, dependent on what type of materials are to be simulated.

When the definition of the model and the calculation of the displacement expressions are conducted, the progressive simulation is initiated in a loop as indicated in figure 5.1. The finite element model is analyzed and the relative nodal crack face displacements and the analytical expression are exported to the parameter estimation in MATLAB. The stress intensity factors K_I and K_{II} are estimated, and by use of these the energy release rate $G(K_I, K_{II})$ is calculated in order to compare this to the critical fracture toughness $G_{c,0}$ for the crack tip.

If $G(K_I, K_{II}) < G_{c,0}$ the crack is stable and the load can be increased, and the FE model is solved for the increased load and the loop is repeated. If $G(K_I, K_{II}) > G_{c,0}$ the crack propagates and must be increased a distance that is to be specified in the next run of the loop. A bridging zone develops behind the crack tip, and this must be added into the FE model. Then the FE



Figure 5.1: Flow chart of the procedure for progressive simulation of a crack propagation.

model is analyzed for the new crack length and the loop is repeated. The procedure within the loop is repeated until one of the following two conditions is reached:

- 1. A specified maximum loading for the structure has been applied and this has shown to lie below the load carrying ability of the structure with the propagated crack with bridging.
- 2. The steady state length of the bridging zone has been reached and the applied load causes that $G(K_I, K_{II}) > G_{c,0}$, which means that the limit of the fracture resistance is exceeded and the crack propagates unstably. This implies that the structure has collapsed or that the response of the structure is likely to become extensively nonlinear and a change of the loads on it must be assessed.

Thus, the developed procedure can be used for plane modeling of progressive propagation of a crack for generally all types of materials including bridging effects, and for an static evaluation of the fracture resistance of a specific crack in order to determine if it propagates for a specified loading.

5.1.2 Test Data for the Simulation Procedure

In the developed simulation procedure, reliable test data are necessary in order to determine the initial critical fracture toughness $G_{c,0}$ and R-curves for pure mode I, mode II and mixed mode combinations for the specific material combination to be simulated. From the R-curves, bridging laws can be determined and then applied in the simulation procedure. Thus, the reliability of the procedure depends on the reliability of these tests, and it is considered important to use a test specimen and configuration for which the assumption of plane stress or strain (deformation) state is indisputable.

In comparison to the test in Sørensen and Jacobsen (2009), it is expected that the stresses in a test specimen of smaller width, fx. 5mm instead of 30mm, can be regarded as plane stress, and that the bending effects treated in section 4.6.1 on page 70 are expected to be insignificant. Then the crack face displacement, and more importantly J_R , do not depend on the *z*coordinate which makes a comparison between the test and simulation more reasonable.

5.1.3 Primary Application on Sub-models

As the element mesh around the crack tip is dense with element sizes that are small compared to geometric sizes of the overall structure, the obvious application of the procedure is modeling of details on a large structure. This could be done by use of sub-modeling of the crack where the boundary conditions on the sub-model is determined by solving a coarse model of the overall structure. Eventually, it may be possible to implement the detailed models directly into full scale models of the overall structure, but this is so far limited by the computing resources available.

5.1.4 Application Perspectives of Plane Crack Simulation Procedure

An example on an application of the procedure developed for plane problems could be an evaluation of a delamination between a skin and stringer in a stiffened skin panel used in aircraft structures made of carbon fiber reinforced composite materials. A cut-out of a wing cover with a T-shaped stringer is seen in figure 5.2. The skin and stringer are loaded by the fuel pressure in the fuel tanks within the wing. As the fuel pressure can be assumed constant over the wing cover, it is reasonable to assume that the stress and strain state is invariant in the out-of plane direction along the stringer length, and thus the evaluation of a delamination is a plane problem.



Figure 5.2: Sketch of a stiffened skin panel.

A critical point for crack initiation and development is the interface between the stringer flange and skin as illustrated in the figure. It is possible to apply the developed procedure for evaluating such a delamination crack and the effect of varying degrees of fiber bridging. The procedure cannot predict crack initiation, but can be used to estimate damage tolerance if a delamination or production flaw is induced in the interface between the skin and stringer.

5.2 Further Development

During the project period ideas for further development and refinement of the procedure developed have arisen. Some of these are presented here and cover:

- Mesh procedure
- Modeling of the bridging effect
- · Automatization of the progressive crack analysis
- Extension of the procedure to curved crack fronts

5.2.1 Proposed Mesh Procedure for Progressive Simulation

In the procedure used in this report, the entire model build from the bottom and meshed for each increase of the crack length. This means that many operations is repeated for each crack increment without adding new information to the model, because the only location where there is a change in the model for each crack increment is at the crack front. For the DCB models used in this project the generation of the model and mesh consumes a relatively large amount of resources compared to the total solution time. Hence, by improving this the computational time can be reduced significantly.

In order to reduce the time used for re-meshing in progressive simulation an alternative meshing method is proposed. The main concepts in this are illustrated in figure 5.3. The main goal is to re-mesh the least number of elements possible for each increment of the crack, and to make the crack definition independent of the geometry of the simulated structure. The four steps of the procedure from the figures are as follows:

- 1. The entire geometry of the structure is meshed with specification of the element size in the area where the crack is to be defined. Figure 5.3a shows a illustrative cut-out of an ideal mesh for definition of the crack, with the left side being the outer boundary of the structure. By controlling the mesh it is ensured that no elements cross the line of propagation of the crack.
- 2. After the structure has been meshed, the crack tip is defined and the mesh around this is refined by adding quarter point elements at the crack tip. This is illustrated in figure 5.3b and is indicated by a red color.
- 3. Then the crack faces are defined on the blue line as shown in figure 5.3c by duplicating the nodes lying on this line. Each duplicated node one is ascribed to the element of the upper crack surface and the other one is ascribed to the element of the lower crack face. When this is done, there is no connection between the elements across the crack and the model is ready for analysis.
- 4. When the model has been analyzed and it has been determined that the crack tip propagates, a new crack length must be defined in the model as illustrated in 5.3d. This is



Figure 5.3: Illustration of the suggested mesh procedure for progressive crack simulation.

done by defining a new crack tip node and then refining the mesh around this node by relocating the quarter point elements and duplicating the nodes on the new crack faces in the same way as described in the previous step.

The exact procedure for implementing this re-meshing method has not been treated, and has to be created as part of a further development of the simulation procedure when extended to progressive modeling.

5.2.2 Improvement of the Parameter Estimation

During the development of the parameter estimation procedure for determining the stress intensity factors, a number of areas for improvement of it has been encountered, and are commented on here. The minimization problem for determining the stress intensity factors and the arbitrary constants in front of the second and third term eigenfunctions in the expression of the relative displacement is ill-conditioned. This is dealt with manually in the simulations conducted in this project, but will have to be automated in order to apply the procedure on different problems for arbitrary mode mixities.

Furthermore it is expected that the computing time for the parameter estimation, which for the DCB specimen in the simulations conducted in this report is about 2-3 min for five nodes on the crack faces included, could be improved by using improved alternative optimization algorithms customized to the specific problem.

5.2.3 Further Development of Bridging Model

The analysis of the plane model of the DCB specimen showed that the traction normal and tangential to the crack faces give the most correct prediction of the R-curve when the crack opening is pure mode I or mode II. When mode mixity is simulated, the prediction of the first part of the R-curve is well described, but the fracture resistance is overestimated for magnitudes of the end opening δ^* larger than the steady state opening. As previously mentioned this indicates that both the normal and tangential traction on the crack faces are dependent on both the tangential and normal end opening, δ_t and δ_n . In another way there is a coupling between tangential and normal traction.

In order to improve the R-curve simulation for mode mixity with the current method, it calls for a bridging law, which is dependent on the mode mixity at the crack tip and based on experimental data which cover a wide range of mode mixity ratios as is presented in the article (Sørensen and Jacobsen, 2009). But in this way the mode mixity has to be evaluated for every step in the analysis in order to use the correct bridging law, which is expected to add several iterations to the solving of the crack propagation.

It is under suspicion that the bridging model yields questionable results for mode mixity because the assumption that the traction law is independent of the crack propagation history is wrong when dealing with mode mixity. The reason for this suspicion is that the mode mixity close to the crack tip clearly changes as the crack propagates for a fixed external load, cf. figure 4.26 on page 67.

Another way of facing the problem of modeling mode mixity could be to rethink the way of applying the traction on the crack faces in the bridging model. In order to obtain a better understanding of the bridging effects it could be a good idea to analyze the micro mechanical mechanisms governing the bridging effects. In section 4.2 the bridging mechanisms have been described from a phenomenological viewpoint. In order to obtain a better model of the bridging law it would be an idea to model the bridging according to this behavior instead of breaking down the traction on the crack faces into tangential and normal traction which are only dependent on the corresponding crack opening by using the derivatives of the fracture resistance curves with respect to the end openings δ_n and δ_t .

As mentioned in this report there are a number of different mechanisms causing the bridging effect, but which one is dominating is not known. If the bridging is perceived alone as traction from fibers inclined an angle φ to the crack face for a given opening of the crack as in figure 5.4, the force exerted by the fiber F_f is also inclined an angle φ to the the crack face. In



Figure 5.4: Illustration of fiber bridging.

this way the normal an tangential traction force components F_n and F_t are a function of the orientation of the fiber. Therefore there may be a simple trigonometric relation between the normal and tangential traction which can be utilized to obtain a better results for mode mixity.

The prospects of modeling the traction, which are in better correspondence with the mechanisms of the bridging effect, would properly mean that the traction laws observed between two equally orientated materials can more easily be extended to modeling bridging between two materials with differently orientated fibers. Furthermore, it is expected that the more well described the mechanisms of the bridging effect is in the bridging model, the less experimental data is needed to model the traction law between the crack faces.

5.2.4 Automatization of Progressive Modeling

The implementation of the procedure in ANSYS and MATLAB has been developed to a stage where evaluation of the energy release rate and increase of crack length must be conducted manually. For use of the procedure in practice this evaluation should be automated.

The procedure is intended to be used in two different scenarios. In the first, a given load (determined from what load carrying ability is required of the structure) is ramped up while keeping track of the crack growth. In the second, the external load is simply ramped up until the crack has propagated up to a certain length in order to see how large a load, the structure can carry with specified restrictions on the propagation of the crack.

In both scenarios, a connection between MATLAB and ANSYS must be established which is invoked every time the displacement are used to determine the stress intensity factors. Alternatively, the numeric parameter identification could be programmed in ANSYS or as a Fortran code input for ANSYS in order to obtain a fast solution and a more integrated stress intensity factor determination.

5.2.5 Application to Curved Crack Fronts

In the following a suggestion on how the procedure developed for plane problems can be extended to three dimensional problems with straight cracks having curved crack fronts. The procedure is explained on the basis of a circular or oval shaped crack front in a laminate between two layers. In figure 5.5 a sketch of a solid submodel from a larger laminated structure is shown with a circular shaped crack. The shape of the crack is suggested to be modeled with splines trough a number of shape defining points. The circular crack is shown without elements in the middle to indicate that the there is no connection between the elements on each side of the crack.



Figure 5.5: Illustration of a submodel of a circular crack between two layers in a laminate. Planes at the shape defining points indicate where the crack face displacements are measured.

The elastic response of the sub-model is determined in a FE analysis. At every shape defining point the displacement expression for plane strain is used with the nodal displacements at planes normal to the crack front in order to obtain the energy release rate *G* at each point. If *G* at one of the shape defining points is larger than the critical energy release rate G_c this means that the crack will propagate at this point. This is modeled by moving the respective shape defining point outwards normal to the original crack front, see figure 5.6.



Figure 5.6: Figure showing the shape defining points and the splines going through them before and after crack propagation. For illustration it is assumed that the critical energy release rate at the two leftmost points which imply that these points are moved outwards as indicated in the figure to the right.

The reason for using the displacement expressions for plane strain deformation for determining the stress intensity factors and eventually the energy release rate is that these are conservative compared to the expressions for plane stress deformation. In fact the fracture toughness derived from using the displacement expressions for plane strain represents the lower bound of the fracture toughness of the material. When considering three dimensional fracture mechanical problems in the way described above, different areas of the modeling procedure for plane problems needs to be generalized to three dimensions. This involves for example:

- Traction laws that take the fiber orientation with respect to the crack front into consideration. This could strengthen the argument for choosing a more physically correct bridging model as mention in subsection 5.2.3.
- The compliance matrices used in the displacement expressions for anisotropic materials are needed in the coordinate system following the normal to the crack front. This means that the compliance matrix used in the displacement expressions in general is unique for each location on the crack front due to the coordinate transformation. This implies that numeric calculations of the displacement expressions for each location of the shape defining points has to be performed.

Conclusion

In this project a numerical procedure for determining if an interface crack between two dissimilar materials propagates for a specific outer load has been developed. This has been done in three main steps concerning a crack in a simple isotropic material, an interface crack in an isotropic bimaterial and an anisotropic bimaterial. This procedure applies a full fracture mechanical formulation for relative displacements of the crack surfaces near the crack tip to estimate the stress intensity factors. These are used to calculate the energy release rate and this is compared to a material dependent critical value in order to determine if the crack tip propagates.

An effort has been made for gaining a considerable understanding of the theory of fracture mechanics for describing stresses and displacements around the crack tip interface cracks in dissimilar materials. Particular focus has been on the derivation of complex stress and displacement for the isotropic bimaterial interface crack, involving a determination of complex potentials from crack boundary conditions. A theory on a description of the full field solution for an interface crack in an isotropic bimaterial has been treated and interpreted with application of it kept in mind.

The fracture mechanical expressions for the crack face displacements have been implemented in a numerical curve fit method which is used to determine the stress intensity factors from the nodal crack face displacements from a FE model. It has been made possible to use nodal displacement values a distance away from the crack tip that is 10-20 times larger than if only the first eigenfunction solution is used. This is done by applying two and three eigenfunctions in a simplified form in the expression for the crack face displacements.

The curve fit of the analytical expression to finite element results has been programmed as a least squares fit for the two bimaterial cases using the conjugate gradient method to estimate the stress intensity factors and a weighting factor for each of the simplified eigenfunctions.

The theory for anisotropic materials is used for simulating an interface crack between two orthotropic unidirectional fiber materials, and for this type of materials large scale bridging of fibers between the crack faces tend to cause a significant increase of the overall fracture resistance as the crack propagates. Dominating mechanisms causing fiber bridging has been described and a modeling of this by using nonlinear springs with specified traction-displacement descriptions has been implemented in a plane and a solid finite element model.

The capability for estimating propagation of a crack with bridging fibers of the developed procedure has been demonstrated for a plane model of two similar orthotropic materials. This has been done by applying bridging laws obtained from an R-curve based on test data, and comparing simulated R-curves with R-curves from the test. This has shown that for pure mode I and II it is possible to simulate the tested increase in the fracture resistance to a satisfactory degree (figures 4.23 and 4.25 on page 66-67), whereas for a mixed mode case the results are less convincing and has to be developed further (figure 4.27 on page 68).

The stress and strain state through the width of a solid model of a double cantilever beam specimen for two similar orthotropic materials used in the test in (Sørensen and Jacobsen, 2009) has been analyzed. It is shown that the assumptions regarding the specimen as a plane problem made in the article seem not to be valid. This is because the stress and strain state in

the specimen varies significantly through the width and that the crack face displacement are 80% larger in the middle than at the edges for the specific specimen. A R-curve from the test for pure mode I has been compared to R-curves simulated for the middle and the edge of the specimen, and this shows that it corresponds to the one in the middle.

Bibliography

Andreasen, J. H. (2008). Brudmekanik. Department of Mechanical Engineering.

Arora, Jasbir (2004). Introduction to Optimum Design. 2nd ed.. Academic Press.

- Bao, G. and Z. Suo (1992). Remarks on crack-bridging concepts. Applied Mechanics Reviews 45(8), 355-366.
- Barsoum, Roshdy S. (1977). Triangular quarter-point elements as elastic and perfectly-plastic crack tip elements. *International Journal for Numerical Methods in Engineering* **11**(1), 85–98.

Hansen, A. L. (2009). Simulation of crack growth and r-curve behavior using cohesive zone modeling.

Hutchinson, John W., Viggo Tvergaard and Fariborz Ghahremani (1990). Three-dimensional effects in microcrack nucleation in brittle polycrystals. *Journal of the American Ceramic Society* **73**(6), 1548–1554.

Kildegaard, Arne (2002). Kompendium i Brudmekanik. Department of Mechanical Engineering.

- Kreyszig, Erwin (2005). Advanced Engineering Mathematics. Wiley.
- Lekhnitskii, S. G. (1981). Theory of Elasticity of an Anisotropic Body. Mir Publishers.

Lekhnitskii, Sergei G. (1968). Anisotropic Plates. second ed.

Muskhelishvili, N. I. (1977). Some Basic Problems of the Mathematical Theory of Elasticity. 1 ed.. Springer.

- Rathkjen, A. (1984). Plan elasticitetsteori og komplekse variable. second ed.. Aalborg University.
- Rice, J.R. (1968). A path independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics* **35**, 379–386.
- Sørensen, Bent F. and Torben K. Jacobsen (2009). Characterizing delamination of fibre composites by mixed mode cohesive laws. *Composites Science and Technology*.
- Sørensen, Bent, Kenneth Jørgensen, Torben Jacobsen and Rasmus Østergaard (2006). Dcb-specimen loaded with uneven bending moments. *International Journal of Fracture* 141(1), 163–176.
- Spearing, S.M. and A.G. Evans (1992). The role of fiber bridging in the delamination resistance of fiber-reinforced composites. *Acta Metallurgica et Materialia* **40**(9), 2191 2199.
- Sørensen, Bent F. and Torben K. Jacobsen (1998). Large-scale bridging in composites: R-curves and bridging laws. *Composites Part A: Applied Science and Manufacturing* **29**(11), 1443 1451.
- Sørensen, Bent F., Stergios Goutianos and Torben K. Jacobsen (2009). Strength scaling of adhesive joints in polymermatrix composites. *International Journal of Solids and Structures* **46**(3-4), 741 – 761.
- Suo, Zhigang (1990). Singularities, interfaces and cracks in dissimilar anisotropic media. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **427**(1873), 331–358.
- Suo, Zhigang and John W. Hutchinson (1990). Interface crack between two elastic layers. *International Journal of Fracture* **43**(1), 1–18.
- Williams, M. L. (1959). The stresses around a fault or crack in dissimilar media. *Bulletin of the Seismological Society* of America **49**(2), 199–204.

APPENDIX

Appendix



Simulation Results for Isotropic Bimaterial Problem

A.1 Results Based on Complete 2 Termed Displacement Expansion

It shows to be difficult to obtain a reasonable curve fit using the complete two termed relative x or y-displacement expansion in equation (3.57). Either the solution diverged significantly from the analytical results or reached a local minima which was not a global minima being sought. An example hereof is shown in figure A.1 for the y-displacement with an element size of 1.20mm.



Figure A.1: Stress intensity factors based on the simplified 2 termed relative x-displacement expression. The element size at the crack tip is 1.20 mm

Because of this the complete 2 termed expressions are not treated further in the context of determining the stress intensity factors.

A.2 Results Based on Simplified 2 Termed Displacement Expansion

A.2.1 Simulation Results Based on *x*-displacement

In figure A.2 the node-wise relative displacement data are shown together with the curvefits of equation (3.61). The element size at the crack tip is 0.15mm. The more nodes that are used for the data fit, the higher the curve.



Figure A.2: Plot of the best fit for 3 to 12 node pairs from the crack tip to the two termed series expansion in equation (3.61). The more nodes are used for the data fit the higher the curve.

In figure A.3 the same data divided by the dominating factor, \sqrt{r} are shown. The division by \sqrt{r} emphasizes how much the data points from the simulation diverge from the analytical expressions. In the figure it is seen that the first two data points, corresponding to the nodal displacement of the quarter point element, diverge from the tendency of the other data points. These two points are considered to be the reason for the relatively large divergency between the curve fits.


Figure A.3: Plot of the best fit for 3 to 12 node pairs from the crack tip to the two termed series expansion divided by \sqrt{r} in equation (3.61). The smaller the number of nodes are used for the data fit the lower the curve.

In figure A.4 the determined stress intensity factors are shown together with the analytically determined stress intensity factors, cf. figure 3.15. From figure A.2, A.3 and A.4 it is seen that the curve fit is influenced by the number of data points used to fit the analytical expression. This means that the determination of the stress intensity factors based on the *x*-displacement is not robust to a satisfactory degree. The largest divergency between the values of the analytical determination of the stress intensity factors is seen when using the first 4 or 5 node pair data. This is assumed to be due to the large uncertainties of the *x*-displacement of the nodes of the quater point element. The more data points are used the less influence the 2 node pairs of the quarter point element have.



Figure A.4: Stress intensity factors based on the simplified 2 termed relative x-displacement expression. The element size at the crack tip is 0.15mm

The values for the *x*-displacements of the quarter point element diverge significantly from what is expected. It is therefore concluded that significant uncertainties are related to these *x*-displacements, and therefore they cannot be used for determining the stress intensity factors. Hence, only the obtained results for the stress intensity factors based on *y*-displacements is presented in the rest of this chapter.

A.2.2 Simulation Results Based on y-displacement

In figure A.5 the nodewise relative *y*-displacements from the FE solution are shown together with the curvefits of equation (3.61) for 3 to 12 nodes pairs from the crack tip. All curve fits except the curve fit of the 3 first node pairs are almost coincident which indicates that the determination of the stress intensity factors are more robust when based on the *y*-displacements.



Figure A.5: Plot of the best fit for 3 to 12 node pairs from the crack tip to the two termed series expansion in equation (3.61). The more nodes are used for the data fit the higher the curve.

In figure A.6 the same data divided by the dominating factor, \sqrt{r} is shown. In the figure it is seen that all points follows the overall tendency.



Figure A.6: Plot of the best fit for 3 to 12 node pairs from the crack tip to the two termed series expansion divided by \sqrt{r} in equation (3.61). The smaller the number of nodes are used for the data fit the lower the curve.

In figure A.7 the determined stress intensity factors are shown together with the analytically determined stress intensity factors. It is seen that the number of nodes used has no significant influence on the obtained values for the stress intensity factors.



Figure A.7: Stress intensity factors based on the simplified 2 termed relative y-displacement expression. *The element size at the crack tip is 0.15mm*

The figures A.8 and A.9 show plots of the curve fits and the obtained stress intensity factors for element sizes 0.15mm, 0.30mm, 0.60mm, 1.20mm on the crack surfaces, respectively.



Figure A.8: Plot of the best fit for 3 to 12 node pairs from the crack tip to the simplified two termed series expansion in equation (3.62). The plots are based on the y-displacement data and is shown for element sizes 0.15mm, 0.30mm, 0.60mm and 1.20mm.

It is seen that the robustness of the determination of the stress intensity factors reduces when the element size increases. The stress intensity factors are fairly constant and close to the analytical reference values for the element sizes 0.15mm and 0.30mm. Up to about 6 node pairs from the crack tip the model with the element sizes 0.60mm also gives reasonably values for the stress intensity factors. With 7 node pairs and above the values of the determined stress intensity factors diverge significantly from the expected analytical results. For the element sizes 1.20mm, subfigure A.9d, the parameter fitting routine finds a local minimum which is not the global one sought. This implies that the value of the stress intensity factors diverge considerably from the analytical values. It might be expected to be possible to find the global

minimum if an improved initial guess is used, but the initial guess in the particular case lies within $\pm 10.0 MPa\sqrt{mm}$ of the sought analytical values and it is therefore not considered practically possible to refine this method in this way.



Figure A.9: Plot of the stress intensity factors corresponding to the curve fits shown in figure A.8.

A.3 Results Based on Complete 3 Termed Displacement Expansion

In figure A.10 and A.11 the curve fits and the obtained stress intensity factors based on the complete 3 termed displacement expression for element sizes 0.15mm, 0.30mm, 0.60mm, 1.20mm on the crack surfaces, respectively, are plotted.

It is seen that generally the results obtained for all element sizes and number of nodes lie within the range of approximately $\pm 6.0 MPa\sqrt{mm}$ from the analytical values. The only values of the stress intensity factors that diverge significantly from this range is for element size 1.20mm based on 4 node pairs from the crack tip, which seems to have reached another minimum.



Figure A.10: Plot of the best fit for 3 to 12 node pairs from the crack tip to the complete 3 termed series expansion in equation (3.61). The plots are based on the y-displacement data and is shown for element sizes 0.15mm, 0.30mm, 0.60mm and 1.20mm.



Figure A.11: Plot of the stress intensity factors corresponding to the curve fits shown in figure A.8.

A.4 Results Based on Simplified 3 Termed Displacement Expansion

In figure A.12 and A.13 the curve fits and the obtained stress intensity factors based on the simplified 3 termed displacement expansion for element size 0.15mm, 0.30mm, 0.60mm, 1.20mm on the crack faces, respectively, are plotted.



Figure A.12: Plot of the best fit for 3 to 12 node pairs from the crack tip to the simplified three termed series expansion in equation (3.62). The plots are based on the y-displacement data and is shown for element sizes 0.15mm, 0.30mm, 0.60mm and 1.20mm.

It is seen in the figures that the obtained results are better than the results obtained with the complete 3 termed displacement expression. This indicates that the simplified 3 term displacement expression yields values for K_I and K_{II} that, in comparison to the other three expressions, are closest to the analytical values for all elements sizes and numbers of included node pairs tested. This indicates that using this expression in the parameter estimation gives a robust way of determining the stress intensity factors.



Figure A.13: Plot of the stress intensity factors corresponding to the curve fits shown in figure A.8.

Appendix

B

Simulation Results for Anisotropic Bimaterial Problem

Study of Best Suited Element Size and Number of Nodes for Solid Model Simulation

It has been studied what size of elements and number of nodes are best suited for determining the stress intensity factors when using results from an analysis of a solid model with orthotropic material properties.

An element size and number of nodes for estimation of the stress intensity factors have been selected based on analyses for five different element sizes 0.5, 1.0, 1.5, 2.0 and 2.3 [mm] and for 1 to 10 node pairs of the crack surfaces. Estimated values for the stress intensity factors for mode I loading for the five element sizes are illustrated in figures B.1a to B.1e. With regards to the element size, it is seen that K_I is roughly the same for all plots, and this is illustrated more distinctly in the box plot in figure B.1f where the difference in the mean value for K_I is roughly 0.1 $MPa\sqrt{mm}$. The value of 2.3mm is limited by the particular model used, and hence it may be possible to increase it even further, but for the specific DCB specimen this size must be compared to a beam height of 9.0mm and is therefore relatively large.

An element size of 2[mm] is used in the plane and solid models used to model the *R*-curve behavior as this yields good results and ensures a fast analysis. Regarding the number of nodepairs to use, the evaluation for the particular case has shown that including five nodepairs in the parameter estimation induce a fast and precise solution, and is selected for use in the following simulations, as a further investigation of this area of the numerical procedure is outside the scope of this project.



Figure B.1: (*a*) - (*e*) Shows the determined stress intensity factors for element sizes 0.5, 1.0, 1.5, 2.0, 2.3 mm. The lines in the plots represents the mean value. (f) A box plot of the K_I values. The red line indicates the mean value, the blue box indicates the 50th percentile, the whiskers indicates the 25th and 75th percentiles and the red cross indicates falling out of the other values.