Weighted Branching Systems: Behavioural Equivalence, Metric Structure, and their Characterisations

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Abstract

In this report, we extend the notion of branching bisimulation to weighted systems, more specifically weighted Kripke structures. We show that a version weighted CTL that excludes the next-operator characterises this weighted branching bisimulation. Due to the restrictive nature of exact quantitative behavioural equivalences, we introduce a notion of expanding our bisimulation by allowing transitions to be matched by runs that accumulate a weight within an expanded interval. Intuitively, we allow a transition \( s \xrightarrow{w} s' \) to be matched by a run that accumulates a weight within the expanded interval \([we^{-1}, we]\) were \( \varepsilon \) is the factor of expansion. We extend this expansion to our logic and show that for a particular class of formulae our expanded formulae characterise our expanded bisimulations. From our notion of expanding our bisimulation, we derive a behavioural pseudometric and prove that states that are close together are guaranteed to satisfy similar formulae. Lastly, we show that specific formulae can be considered closed in the open-ball topology induced by our distance and define a distance between these closed formulae.
1 Introduction

For concurrent and interactive systems the notion of semantic equality has always held particular importance and in general forms the groundwork for most further reasoning about such systems. To capture this equality between systems, many behavioural preorders and equivalences have been considered, including the now classical notion of bisimulation introduced by Hennesy, Milner [1] and Park [2]. Alongside the development of behavioural equivalences, there has been an effort in describing systems with the use of various modal and temporal logics. In general, when one has a behavioural equivalence, we would like to produce a logic that corresponds with this equivalence, in the sense that two states are behaviourally equivalent if and only if they satisfy the same logical formulae, e.g. Hennesy Milner Logic and bisimulation [1].

In conjunction with this, there has also been an emphasis on discovering behavioural equivalences that allow us to abstract away from the internal behaviour of systems and only require equivalence at an external level. The original notion of observational equivalence by Milner [3] serves this purpose, as does the later notion of branching bisimulation introduce by Weijland and Glabbeck [4]. Branching bisimulation has the additional property of being completely characterised by several temporal logics [5], including Computation Tree Logic (CTL) without the next-operator.

Today, the most common way to model concurrent systems has been by the use of process algebras such as the Calculus of Communicating Systems (CCS) introduced by Milner [3] or by coalgebraic structures such as labelled transition systems (LTS). While models with only labels are sufficient for reasoning about the reactive and functional behaviour of systems, they cannot encode quantitative aspects that may be of importance to actual systems, e.g. resource consumption. This has motivated the introduction and study of weighted transition systems, for which transitions are labelled with quantities – e.g. real numbers – allowing for the modelling of consumption or production of resources. Analogously to LTS, weighted transition systems also have a well-developed notion of semantic equivalence [6], namely weighted bisimulation.

In this report, we revisit weighted transition systems with the intent of identifying behavioural equivalences similar to that of branching bisimulation; meaning we remain sensitive to quantitative behaviour, yet abstract away from internal activity. We develop a notion of weighted branching bisimulation, in which we require the transitions of behaviourally equivalent states to be matched by finite paths of equal accumulated weight.

Example 1. Consider the small weighted transition system shown below. Conceptually, we would like for t to be similar to s, as there exists a path from t that accumulates
a weight of 5, thereby matching the transition $s \xrightarrow{5} s'$.

$$s \xrightarrow{5} s'$$

$$t \xrightarrow{3} t' \xrightarrow{2} t''$$

As with [4], we aim to characterise our weighted branching bisimulation; as such we consider a weighted extension of CTL without the next operator, for which the until operator has been equipped with closed intervals of rational numbers. To this end, we develop a notion of branching-finiteness, which requires that the possible ways of accumulating a weight within a particular interval from a state be finite. We show that our weighted logic characterises our notion of weighted branching bisimulation on branching-finite weighted Kripke structures (WKS).

Due to the restrictive nature of exact quantitative behavioural relations – i.e. the fact that small deviations in weights will cause otherwise equivalent systems to be non-equivalent – we develop a notion of expanding our weighted branching bisimulation by some real-valued factor. This approach is based upon similar work done for probabilistic systems by Giacalone et al. [7] and Desharnais et al. [8]. The idea being, that if we expand our weighted branching bisimulation by $\varepsilon \in \mathbb{R}_{\geq 1}$, then any transition of weight $w$ from bisimilar states have to be matched by finite paths of accumulated weight within the interval $[we^{-1}, we]$. Parallel to this, we develop a corresponding notion of expanding our logic and show that states that are expanded-weighted branching bisimilar are characterised by the expansion of our logic.

From this notion of expanding our bisimulation, we derive a distance between states: the greatest lower bound of factors, such that the states in question are expanded-weighted branching bisimilar. We show that this distance behaves much akin to a pseudometric (and that the logarithm of the distance is a pseudometric). We show that states that are within a certain distance of each other are guaranteed to satisfy similar formulae.

Lastly, we show that a particular class of formulae, the formulae using only negation on atomic propositions, are closed in the open-ball topology induced by our distance, i.e. if a sequence of states that all satisfy such a formula converge to some state, then their limit also satisfy said formula. We then define a distance between these closed formulae, namely the greatest lower bound of factors such that the satisfaction of the expanded formula implies the satisfaction of the non-expanded other and vice versa.

1.1 Related Work

Model checking of a weighted extension of CTL on weighted systems is presented by Buchholz and Kemper in [9].
In [10], Hansen et al. introduce a generalised version of weighted bisimulation and show that it is characterised by a weighted modal logic inspired by Markovian logics for image-compact WKS. This concept of image-compactness was the primary inspiration to the concept of branching-finiteness presented in this paper.

Efficient algorithms are given for model-checking upper-bounded WCTL formulae and completely bounded formulae are shown to be NP-hard by Jensen et al. in [11]. The weighted branching logic of this report is based upon the version of weighted CTL presented in this paper.

The concept of relaxing quantitative behavioural equivalences is first presented by Giacalone et al. in [7]. Desharnais et al. later expands upon this work and develops the notion of a bisimulation metric, i.e. bisimilar states should be at distance 0 from each other and states relatively close together should be relatively bisimilar.

A deeper analysis of metrics for weighted systems is done by Fahrenberg, Thrane and Larsen in [12, 13, 14].

Larsen et al. show in [15] that for discrete and continuous Markov processes, certain classes of formulae can be considered closed, open, $G_\delta$ or $F_\sigma$ under an appropriate bisimulation metric, something they call a dynamically continuous bisimulation metric. We show that our distance has properties very much akin to such a metric.

The work of this report is based upon unpublished ideas developed by Foshammer, Larsen, Mardare, and Xue at the Department of Computer Science at Aalborg University.

2 Weighted Branching Bisimulation

In this section, we introduce the basis for the quantitative model that we will be working with in this report – the weighted Kripke structure (WKS) – which is just the straightforward extension of adding positively real numbered weights to the transitions of conventional Kripke structures. Kripke structures are well suited for reasoning about temporal properties of systems [16]. By extending Kripke structures with weights, we will be able to reason about quantitative aspects in a temporal setting as well.

We introduce a notion of observational behavioural equivalence that abstracts away from internal behaviour. To this end, we present the concepts of runs and prefixes of runs – i.e. respectively infinite and finite paths in a given WKS – which are used to define our behavioural equivalence. For any given state and closed interval of real numbers, we describe a base set of prefixes that we show later on entirely characterises specific properties for all runs starting in said state.

Lastly, we introduce a notion of finiteness, which similarly to the classical idea of image-finiteness, allow us to completely characterise our behavioural equivalence with a temporal logic.
Definition 2.1 (Weighted Kripke Structure)
A weighted Kripke structure is a tuple $(S, \rightarrow, AP, V)$ where

- $S$ is a set of states,
- $\rightarrow \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is the transition relation,
- $AP$ is a set of atomic propositions and
- $V : S \rightarrow 2^{AP}$ is the function assigning a set of atomic propositions to each state.

Whenever $(s, w, s') \in \rightarrow$ we use the shorthand $s \xrightarrow{w} s'$. For a given WKS $K = (S, \rightarrow, AP, V)$ we say that it is non-blocking if for all $s \in S$ there exists a $w \in \mathbb{R}_{\geq 0}$ and $s' \in S$ such that $s \xrightarrow{w} s'$. In this text we only consider non-blocking WKS, and as such for all future defined WKS it will be implicitly implied that they are non-blocking. This is done for purely notational reasons and all results could easily be extended to blocking WKS. Furthermore any blocking WKS can easily be made into a non-blocking version of itself, by adding zero-loops to any blocking state. See Figure 1 for a graphical representation of a WKS.

![Diagram](image.png)

Figure 1: A simple graphical representation of a finite non-blocking WKS $K$

As mentioned, in this report we would like to abstract away from single transitions and instead focus on sequences of transitions. We therefore now introduce the notion of runs and prefixes of runs.

Definition 2.2 (Runs)
Let $K = (S, \rightarrow, AP, V)$ be a WKS and $s_0 \in S$. A run $\rho$ starting in $s_0$ is a countably infinite sequence of transitions

$$\rho = (s_0, w_1, s_1), (s_1, w_2, s_2), \ldots, (s_i, w_{i+1}, s_{i+1}), \ldots$$

where for all $i \in \mathbb{N}$, $s_i \in S$ and $s_i \xrightarrow{w_{i+1}} s_{i+1}$. For $k \in \mathbb{N}$ the $k$-th transition of $\rho$ is denoted $\rho(k)$ and the $k$-th state of $\rho$ is defined as $\rho[k] = s_k$ where $\rho(k) = (s_k, w_{k+1}, s_{k+1})$. Furthermore, the accumulated weight of $\rho$ at position $k$ is defined as
\[ W(\rho)(k) = \begin{cases} 
0 & \text{if } k = 0 \\
\sum_{i=1}^{k} w_i, \text{ where } \rho(i-1) = (s_{i-1}, w_i, s_i) & \text{otherwise} 
\end{cases} \]

Given a WKS \( \mathcal{K} = (S, \rightarrow, \mathcal{A}, \mathcal{P}, \mathcal{V}) \) and \( s \in S \), let \( \text{Runs}(s) \) be the set of all runs starting in \( s \) and \( \text{Runs}(\mathcal{K}) = \bigcup_{s \in S} \text{Runs}(s) \) the set of all runs in \( \mathcal{K} \). Lastly, for \( k \in \mathbb{N} \), let

\[ \text{Runs}_k^+(s) = \{ \rho \in \text{Runs}(s) \mid \nexists m, n \in \mathbb{N} : m < n \leq k, \rho[m] = \rho[n] \text{ and } W(\rho)(m) = W(\rho)(n) \} \]

be the set of runs starting in \( s \) who before position \( k \) has no zero-cycles.

However, a lot of the time we are only interested in runs up to a certain position and therefore we introduce the notion of prefixes.

**Definition 2.3 (Prefixes)**

Let \( \mathcal{K} = (S, \rightarrow, \mathcal{A}, \mathcal{P}, \mathcal{V}) \) be a WKS and \( \rho \in \text{Runs}(\mathcal{K}) \). The **prefix** of \( \rho \) of length \( k \) is denoted \( \rho \uparrow^k \) and is the finite sequence of transitions

\[ \rho \uparrow k = \rho(0), \rho(1), ..., \rho(k-1) \]

Note that prefixes with equal lengths but of differing runs may be equal (see Example 2). Furthermore, when we write \( \rho \uparrow k \in X \) where \( X \) is a set of prefixes, it is implied that \( \rho \in \text{Runs}(\mathcal{K}) \) and \( k \in \mathbb{N} \).

**Example 2.** Consider the WKS \( \mathcal{K} \) shown in Figure 1. Some examples of runs in \( \mathcal{K} \) are

\[
\rho = (s_0, 2, s_1), (s_1, 0, s_1), (s_1, 0, s_1), ...
\]
\[
\pi = (s_0, 5, s_2), (s_2, 0, s_2), (s_2, 0, s_2), ...
\]
\[
\sigma_1 = (s_0, 3, s_0), (s_0, 5, s_2), (s_2, 0, s_2), (s_2, 0, s_2), ...
\]
\[
\sigma_2 = (s_0, 3, s_0), (s_0, 3, s_0), (s_0, 5, s_2), (s_2, 0, s_2), (s_2, 0, s_2), ...
\]

and some examples of prefixes of said runs are

\[
\rho \uparrow 2 = (s_0, 2, s_1), (s_1, 0, s_1)
\]
\[
\pi \uparrow 2 = (s_0, 5, s_2), (s_2, 0, s_2)
\]
\[
\sigma_1 \uparrow 1 = \sigma_2 \uparrow 1 = (s_0, 3, s_0)
\]

Note how the prefixes \( \sigma_1 \uparrow 1 \) and \( \sigma_2 \uparrow 1 \) are equivalent despite being described by different runs. In a sense, the set of all prefixes of length \( k \) can be seen as a partitioning of all runs, where runs who up till position \( k \) are equivalent are grouped together. *
We now present our notion of behavioural equivalence that allows us to abstract away from internal behaviour. This is done by allowing for a single transition to be matched by a prefix that preserves behaviour along the way, preserves the end behaviour, and accumulates that same weight as the transition. This equivalence is in many ways similar to the classical branching bisimulation with the exception that we also take the weight of transitions into account.

**Definition 2.4 (Weighted Branching Bisimulation)**

Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, a *weighted branching bisimulation* (WBB) is a relation $R \subseteq S \times S$ such that whenever $(s, t) \in R$ then

1. $\mathcal{V}(s) = \mathcal{V}(t)$
2. For all $s \xrightarrow{w} s'$ there exists a run $\rho \in \text{Runs}(s)$ and $k \in \mathbb{N}$ such that $\forall i < k : (s, \rho[i]) \in R$, $(s', \rho[k]) \in R$, and $W(\rho)(k) = w$
3. For all $t \xrightarrow{w} t'$ there exists a run $\rho \in \text{Runs}(s)$ and $k \in \mathbb{N}$ such that $\forall i < k : (t, \rho[i]) \in R$, $(t', \rho[k]) \in R$, and $W(\rho)(k) = w$

If there exists a weighted branching bisimulation relating $s$ and $t$, we say that $s$ and $t$ are bisimilar and denote it $s \sim t$. The relation $\sim$ is the largest weighted branching bisimulation and will be referred to as weighted branching bisimilarity.

**Example 3.** Consider the WKS illustrated in Figure 2, any transitions from $s_0$, $t_0$ or $p_0$ can be matched by a run from any of the others according to the three conditions given in Definition 2.4, e.g. $s_0 \xrightarrow{3} s_2$ can be matched by both $t_0 \xrightarrow{2} t_2 \xrightarrow{3} t_1$ and $p_0 \xrightarrow{2} p_0 \xrightarrow{3} p_1$.

![Figure 2: A simple WKS where $s_0 \sim t_0$, $t_0 \sim p_0$ and $p_0 \sim s_0$.](image)

We now introduce a concept of sets of prefixes that within given closed interval of real numbers characterise certain properties about all runs that accumulated a weight.
within that interval. We will use this concept of sets of characterising prefixes to define a finiteness property similar to that of image-finiteness, but for weighted branching systems.

**Definition 2.5 (Characterising Prefixes)**

Given a WKS $\mathcal{K} = (S, \rightarrow, AP, V)$, $s \in S$, and an arbitrary closed interval of positive real numbers $I$, let

$$\mathcal{P}(s)(I) = \{ \rho \uparrow k \mid \rho \in \text{Runs}_k^+(s), \ k \in \mathbb{N}, \ W(\rho)(k) \in I\}$$

Intuitively, $\mathcal{P}(s)(I)$ can be seen as the set of prefixes that start in $s$, has no zero-cycles, and accumulates a weight within the interval $I$.

As mentioned, we use these sets of prefixes to define a concept of finiteness for weighted systems similar to that of image-finiteness, namely branching-finiteness. This new concept will prove important in proving a lot of desired properties for WKS, including characterising our behavioural equivalence with a temporal logic and later on for defining a behavioural pseudometric.

**Definition 2.6 (Finiteness in WKS)**

Let $\mathcal{K} = (S, \rightarrow, AP, V)$ be a WKS. We say that $\mathcal{K}$ is

- **finite** if $S$ and $\rightarrow$ are finite,
- **image-finite** if for all $s \in S$ the set $\{ s' \in S \mid \exists w \in \mathbb{R}_{\geq 0}, s \xrightarrow{w} s' \}$ is finite,
- **branching-finite** if for all $s \in S$ and all closed intervals of positive real numbers $I$, the set $\mathcal{P}(s)(I)$ is finite.

**Example 4.** Consider the two WKS presented in Figure 3. Clearly $\mathcal{K}_t$ is image-finite, as any state has at most two successors. It is however not weighted branching-finite, as the set of prefixes starting in $t_0$ that accumulates a weight within $[1, 1]$ and excludes prefixes with zero-cycles – i.e. $\mathcal{P}(t)([1, 1])$ – is infinite. In other words, the number of non-redundant ways we can accumulate the weight 1 from $t_0$ is infinite.

As for $\mathcal{K}_s$, clearly it is not image-finite, as the state $s$ has an infinite number of successors. It is however weighted branching-finite, as for any closed positive real interval $I$, the set $\mathcal{P}(s)(I)$ is finite.

Lastly, we show that for branching-finite WKS, our sets of characterising prefixes can themselves be distinguished by using only rational numbers.

**Lemma 2.1.** Let $\mathcal{K} = (S, \rightarrow, AP, V)$ be a branching-finite WKS, and $s \in S$. For all closed intervals of real numbers $[v, u]$, there exists a pair $v_M, u_M \in \mathbb{Q}_{\geq 0}$ such that $[v, u] \subseteq [v_M, u_M]$ and $\mathcal{P}(s)([v, u]) = \mathcal{P}(s)([v_M, u_M])$. 

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Proof. First note that for any two closed intervals of real numbers \( I \) and \( J \) such that \( I \subseteq J \) we have that \( P(s)(I) \subseteq P(s)(J) \). Now, let \( (v_n)_{n \in \mathbb{N}} \) be an increasing sequence of rational numbers, such that \( \lim_{n \to \infty} v_n = v \). Additionally, let \( (u_n)_{n \in \mathbb{N}} \) be a decreasing sequence of rational numbers, such that \( \lim_{n \to \infty} u_n = u \). Clearly, for all \( n \in \mathbb{N} \) we have that \( [v_n, u_n] \supseteq [v_{n+1}, u_{n+1}] \), implying that for all \( n \in \mathbb{N} \) that \( P(s)([v_n, u_n]) \supseteq P(s)([v_{n+1}, u_{n+1}]) \).

Since \( K \) is branching-finite we have that \( (P(s)([v_n, u_n]))_{n \in \mathbb{N}} \) is a non-increasing sequence of finite sets of prefixes and that there must exists a \( M \in \mathbb{N} \) such that for all \( m > M \), \( P(s)([v_m, u_m]) = P(s)([v_M, u_M]) \). As \( [v, u] \subseteq [v_M, u_M] \) we have that \( P(s)([v, u]) \subseteq P(s)([v_M, u_M]) \).

Assume now towards a contradiction that \( P(s)([v_M, u_M]) \not\subseteq P(s)([v, u]) \), meaning there exists a prefix \( \rho \uparrow k \in P(s)([v_M, u_M]) \) such that \( \rho \uparrow k \not\in P(s)([v, u]) \). This implies that \( W(\rho)(k) \in ([v_M, u_M] \setminus [v, u]) \). We can now choose an \( m > M \) such that \( W(\rho)(k) \not\in [v_m, u_m] \) implying that \( \rho \uparrow k \not\in P(s)([v_m, u_m]) \), thereby contradicting that \( \forall m > M : P(s)([v_m, u_m]) = P(s)([v_M, u_M]) \). \qed

3 Logical Characterisation

In this section, we introduce the temporal logic that we will use to reason about WKS, namely Weighted Branching Logic (WBL). Weighted Branching Logic can be seen as a weighted extension of Computation Tree Logic without the next-operator. The
reason for the exclusion of the next-operator is that we do not wish to reason about single transitions, but instead only about sequences thereof. The weighted extension of CTL is the addition of rationally bounded closed intervals on until-formulae, signifying that the accumulated weight of any prefix satisfying an until-formulae must be within the given interval. We show that for branching-finite WKS, Weighted Branching Logic characterises our weighted branching bisimulation. Meaning that two states are behaviourally equivalent if and only if they satisfy the same WBL formulae.

**Definition 3.1 (Syntax of Weighted Branching Logic)**
Let \( \mathcal{AP} \) be a set of atomic propositions, the state-formulae of weighted branching logic, denoted \( \mathcal{L} \), are induced by the following grammar

\[
\mathcal{L} : \quad \phi ::= a | \neg \phi | \phi \land \phi | E\psi | A\psi
\]

and the path-formulae of weighted branching logic by

\[
\psi ::= \phi U_I \phi
\]

where \( a \in \mathcal{AP} \) and \( I = [l, u], l, u \in \mathbb{Q}_{\geq 0}, l \leq u \).

**Definition 3.2 (Semantics of Weighted Branching Logic)**
The semantics of weighted branching logic are given by the satisfaction relation defined inductively for an arbitrary WKS \( \mathcal{K} = (S, \rightarrow, \mathcal{AP}, \mathcal{V}) \), \( s \in S \) and \( \rho \in \text{Runs}(\mathcal{K}) \) as follows

\[
s \models a \quad \text{iff} \quad a \in \mathcal{V}(s)
\]

\[
s \models \neg \phi \quad \text{iff} \quad \text{not} \ s \models \phi
\]

\[
s \models \phi_1 \land \phi_2 \quad \text{iff} \quad s \models \phi_1 \text{ and } s \models \phi_2
\]

\[
s \models E\psi \quad \text{iff} \quad \exists \rho \in \text{Runs}(s) : \rho \models \psi
\]

\[
s \models A\psi \quad \text{iff} \quad \forall \rho \in \text{Runs}(s) : \rho \models \psi
\]

\[
\rho \models \phi_1 U_I \phi_2 \quad \text{iff} \quad \exists k \in \mathbb{N} : \forall i \leq k : \rho[i] \models \phi_1, \rho[k] \models \phi_2, \text{ and } \mathcal{W}(\rho)(k) \in I
\]

where \( a \in \mathcal{AP} \) and \( I = [l, u], l, u \in \mathbb{Q}_{\geq 0}, l \leq u \). Furthermore the satisfaction of until-formulae can be extended to prefixes, for an arbitrary WKS \( \mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow) \), \( \rho \in \text{Runs}(\mathcal{K}) \), and \( k \in \mathbb{N} \) as follows

\[
\rho \uparrow k \models \phi_1 U_I \phi_2 \quad \text{iff} \quad \exists k' \leq k : \forall i \leq k' : \rho[i] \models \phi_1, \rho[k'] \models \phi_2, \text{ and } \mathcal{W}(\rho)(k') \in I
\]

When it is not the case that \( s \models \phi \) for some \( \phi \in \mathcal{L} \) we write \( s \not\models \phi \). Furthermore, for \( \mathcal{K} = (S, \mathcal{AP}, \mathcal{V} \rightarrow) \) and \( s \in S \) let \( \mathcal{L}(s) = \{ \phi \in \mathcal{L} \mid s \models \phi \} \) denote the set of state
formulae satisfied by \( s \) and lastly, for a formula \( \phi \in \mathcal{L} \) let \( [\phi] = \{ s \in S \mid s \models \phi \} \). denote the set of states that satisfy \( \phi \).

We now show that when trying to prove that all runs do not satisfy until-formulae, we can ignore, without a loss of generality, all prefixes that have zero-cycles. For branching finite WKS this implies that in order to prove that all runs starting in some state does not satisfy an until-formulae, we only have to check a finite number of prefixes.

**Lemma 3.1.** Let \( K = (S, \rightarrow, AP, V) \) be a WKS, \( s \in S \), and \( l, u \in \mathbb{Q}_{\geq 0} \), such that \( l < u \). For arbitrary \( \phi_1, \phi_2 \in \mathcal{L} \)

\[
\forall \rho \uparrow k \in \mathcal{P}(s)([l, u]) : \rho \uparrow k \not\models \phi_1 U_{[l,u]} \phi_2 \iff \forall \rho \in \text{Runs}(s) : \rho \not\models \phi_1 U_{[l,u]} \phi_2
\]

**Proof.** (\( \Rightarrow \)) Assume that \( \forall \rho \uparrow k \in \mathcal{P}(s)([l, u]) : \rho \uparrow k \not\models \phi_1 U_{[l,u]} \phi_2 \). Assume now towards a contradiction that \( \exists \rho \in \text{Runs}(s) : \rho \models \phi_1 U_{[l,u]} \phi_2 \). We know that \( \rho \models \phi_1 U_{[l,u]} \phi_2 \) if and only if \( \exists k \in \mathbb{N} \) such that \( \forall i < k : \rho[i] \models \phi_1, \rho[k] \models \phi_2 \), and \( \mathcal{W}(\rho)(k) \in [l, u] \). Consider the set of zero-cycles in \( \rho \) up to \( k \),

\[
\Omega_{\rho,k} = \{(m, n) \in \mathbb{N}^2 \mid m < n \leq k, \rho[m] = \rho[n], \mathcal{W}(\rho)(m) = \mathcal{W}(\rho)(n)\}
\]

Consider now the following function \( f_\rho : \mathbb{N} \to \text{Runs}(K) \) defined as follows

\[
f_\rho(m) = \begin{cases} f_\rho(n) & \text{if } \exists n \in \mathbb{N} : (m, n) \in \Omega_{\rho,k} \\ \rho(m)f_\rho(m+1) & \text{otherwise} \end{cases}
\]

where \( \text{run}(m)f_\rho(m+1) \) is the concatenation of two sequences of transitions. Intuitively, \( f_\rho(0) \) can be seen as \( \rho \) without any zero-cycles before \( k \). Clearly \( f_\rho(0) \in \text{Runs}(s) \). Furthermore as we only removed zero-cycles up to the position \( k \) of \( \rho \) we know that there exists a \( h \leq k \) such that \( f_\rho(0)[h] = \rho[k] \), hence \( f_\rho(0)[h] \models \phi_2 \). Secondly we know that \( \forall j < h : \exists i < k : f_\rho(0)[j] = \rho[i] \), hence \( \forall j < h : f_\rho(0)[j] \models \phi_1 \). Lastly as we only removed zero-cycles, we know that \( \mathcal{W}(f_\rho(0))(h) \in [l, u] \). We therefore get that \( f_\rho(0) \uparrow h \models \phi_1 U_{[l,u]} \phi_2 \) and that \( f(0) \uparrow h \in \mathcal{P}(s)([l, u]) \), as it starts in \( s \), contains no zero-cycles, and accumulates a weight within \([l, u]\); contradicting our assumption that \( \forall \rho \uparrow k \in \mathcal{P}(s)([l, u]) : \rho \uparrow k \not\models \phi_1 U_{[l,u]} \phi_2 \).

(\( \Leftarrow \)) By definition, the existence of a prefix \( \rho \uparrow k \models \phi_1 U_{[l,u]} \phi_2 \) implies that \( \rho \models \phi_1 U_{[l,u]} \phi_2 \). Therefore, \( \forall \rho \in \text{Runs}(s) : \rho \not\models \phi_1 U_{[l,u]} \phi_2 \) implies \( \forall \rho \uparrow k \in \mathcal{P}(s)([l, u]) : \rho \uparrow k \not\models \phi_1 U_{[l,u]} \phi_2 \).

Lastly, we conclude this section by showing that our behavioural equivalence, weighted branching bisimulation, induces the same relation as weighted branching
logic. In other words, we show that states that behave in the same way satisfy the same formulae.

**Theorem 3.2.** Let $\mathcal{K} = (S, \rightarrow, AP, V)$ be a weighted branching-finite WKS, then for all $s, t \in S$

$$s \sim t \iff \mathcal{L}(s) = \mathcal{L}(t)$$

**Proof.** $(\Rightarrow)$ We show that $s \sim t \implies \mathcal{L}(s) = \mathcal{L}(t)$. Assume $s \sim t$ and $s \models \phi$ for some $\phi \in \mathcal{L}$. We show by induction on the structure of $\phi$ that if $s \sim t$ then $s \models \phi \implies t \models \phi$. This is sufficient as $\sim$ is symmetric.

**Case $\phi = a$:**
We know that $s \models a$ if and only if $a \in \mathcal{V}(s)$ and since $s \sim t$ implies $\mathcal{V}(s) = \mathcal{V}(t)$ we know that $t \models a$.

**Case $\phi = \neg \phi_1$:**
We know that $s \models \neg \phi_1$ if and only if $s \not\models \phi_1$. Assume now that $t \models \phi_1$, since $s \sim t$ then by structural induction $s \models \phi_1$ which contradicts our assumption that $s \not\models \neg \phi_1$. Therefore $t \models \neg \phi_1$.

**Case $\phi = \phi_1 \land \phi_2$:**
We know that $s \models \phi_1 \land \phi_2$ if and only if $s \models \phi_1$ and $s \models \phi_2$. Since $s \sim t$ then by structural induction we have that $t \models \phi_1$ and $t \models \phi_2$. Therefore by definition $t \models \phi_1 \land \phi_2$.

**Case $\phi = E \phi_1 U_I \phi_2$:**
We know that $s \models E \phi_1 U_I \phi_2$ if and only if there exists a run $\rho \in \text{Runs}(s)$ and a $k \in \mathbb{N}$ such that $\forall i < k : \rho[i] \models \phi_1$, $\rho[k] \models \phi_2$, and $\mathcal{W}(\rho)(k) \in I$. Since $s \sim t$ we know that there exists a run $\pi \in \text{Runs}(t)$ matching $\rho$ whereby there exists a sequence $(h_0, \ldots, h_k) \in \mathbb{N}^{k+1}$ where $h_0 = 0$ and $\forall i, 0 < i \leq k : h_{i-1} \leq h_i$ such that $\forall i < k : \rho(i) = \rho[i] \xrightarrow{w_{i+1}} \rho[i + 1]$ we have that $\forall j, h_i \leq j < h_{i+1} : \rho[i] \sim \pi[j]$, $\rho[i + 1] \sim \pi[i + 1]$, and $\mathcal{W}(\pi)(h_{i+1}) - \mathcal{W}(\pi)(h_i) = w_{i+1}$. By structural induction we have that $\forall j < h_k : \pi[j] \models \phi_1$ and that $\pi[h_k] \models \phi_2$. Furthermore we have that $\mathcal{W}(\pi)(h_k) = \mathcal{W}(\rho)(k) \in I$ and therefore by definition $t \models E \phi_1 U_I \phi_2$.

**Case $\phi = A \phi_1 U_I \phi_2$:**
We know that $s \models A \phi_1 U_I \phi_2$ if and only for all runs $\rho \in \text{Runs}(s)$ there exists some $k \in \mathbb{N}$ such that $\forall i < k : \rho[i] \models \phi_1$, $\rho[k] \models \phi_2$, and $\mathcal{W}(\rho)(k) \in I$. Assume now towards a contradiction that $t \not\models A \phi_1 U_I \phi_2$ meaning that there exists some run $\pi \in \text{Runs}(t)$ such that $\forall k \in \mathbb{N}$ either (a) $\exists i < k : \pi[i] \not\models \phi_1$, (b) $\pi[k] \not\models \phi_2$, or (c) $\mathcal{W}(\pi)(k) \not\in I$. 
Hence by definition \( \rho \in R \) is a weighted branching bisimulation. We first show that \( \exists s \) such that \( (s, t) \in R \). We now show that \( \exists s \) such that \( (s, t) \in R \). Now assume towards a contradiction that \( \forall \exists s \). Clearly \( \exists s \) is finite. We therefore know that the set \( \mathcal{P}(t)([w, w]) \) is finite.

(a) For each \( \rho \uparrow k \in \mathcal{P}(t)([w, w]) \) such that \( \exists \exists s \). Let \( \Phi_1 \) be the set of all such formulae. Clearly \( \Phi_1 \) is finite as \( \mathcal{P}(t)([w, w]) \) is finite.

(b) For each \( \rho \uparrow k \in \mathcal{P}(t)([w, w]) \) such that \( (s', \rho[k]) \notin R \) there exists a formula \( \phi_2 \) such that \( s' \models \phi_2 \) and \( \rho[k] \notin \phi_2 \). Let \( \Phi_2 \) be the set of all such formulae. Clearly \( \Phi_2 \) is finite as \( \mathcal{P}(t)([w, w]) \) is finite.

By Lemma 2.1 we have that there exists a pair \( l, u \in \mathbb{Q}_{\geq 0} \) such that \([w, w] \subseteq [l, u]\) and \( \mathcal{P}(t)([w, w]) = \mathcal{P}(t)([l, u]) \). We can now create an until-formula

\[
\psi = \bigwedge_{\phi_1 \in \Phi_1} \phi_1 \cup \bigwedge_{\phi_2 \in \Phi_2} \phi_2
\]

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for which \( \forall \rho \uparrow k \in \mathcal{P}(t)([w, w]) : \rho \uparrow k \not\models \psi \). By Lemma 3.1 we get that \( \forall \rho \in \text{Runs}(t) : \rho \not\models \psi \) which is equivalent with \( t \not\models E \psi \). We now have that \( s \models E \psi \) and \( t \not\models E \psi \), contradicting \((s, t) \in \mathcal{R}\). Therefore \( \mathcal{R} \) satisfies condition II (and condition III by symmetry).

\[ \square \]

4 A Pseudometric for Weighted Branching Systems

The notion of an exact bisimulation can often be too restrictive due to the quantitative nature of weighted systems and the requirement that the accumulated weights of runs be matched exactly by bisimilar states. Often the measurements of the real world upon which we build our models are given with some margin of error. We, therefore, introduce a class of relaxations of our bisimulation relation, expanded bisimulations, allowing the accumulated weight of runs to be matched within some factor of error by bisimilar states. E.g. if \( s \) can do a transition of weight \( w \) to \( s' \), then \( t \) can match with a run which at some position \( k \) accumulates a weight within \([w \varepsilon^{-1}, w \varepsilon] \), where \( \varepsilon \) is the factor for the allowed margin of error. The reason for multiplying and dividing by some factor instead of just adding and subtracting by some constant is that we would like to model transitions of zero-weight as special (as transitioning with zero weight is similar to not doing anything in order to transition). Therefore, as anything multiplied by zero is zero, we have that no factor of expansion can make a non-zero transition into a zero transition or vice versa. Giacalone et al. first introduced the notion of relaxing bisimulation relations in \([7]\), by allowing probabilistic processes with almost identical behaviour (i.e. diverting within the margin of some constant \( \varepsilon \)) to be bisimilar. Desharnais et al. later developed upon this notion in \([8]\) by providing a logical characterisation of a metric derived from these relaxed bisimulations.

Hereafter, we define a notion of distance between states by taking the infimum of factors that allow for two states to be expanded bisimilar. We show that this distance forms a pseudometric when functionally composed with logarithm, and we introduce a notion of expanding our logical formulae such that they behave in accordance with our expanded bisimulations. We show that if two states are within a certain distance from each other, then they are guaranteed to satisfy similar formulae.

Lastly, we show that certain sets are closed in the topology induced by our behavioural pseudometric, meaning that if a sequence of states converges and all satisfy a closed formula, then the limit also satisfy said formula.

**Definition 4.1** (Expanded Weighted Branching Bisimulation)
Given a WKS \( \mathcal{K} = (S, \mathcal{AP}, V, \rightarrow) \) and \( \varepsilon \in \mathbb{R}_{\geq 1} \), an \( \varepsilon \)-expanded weighted branching bisimulation (\( \varepsilon \text{WBB} \)) is a relation \( \mathcal{R} \subseteq S \times S \) such that whenever \((s, t) \in \mathcal{R}\) then

1. \( V(s) = V(t) \)
for all $s \xrightarrow{w} s'$ there exists a run $\rho \in \text{Runs}(t)$ and $k \in \mathbb{N}$ such that $\forall i < k: (s, \rho[i]) \in \mathcal{R}$, $(s', \rho[k]) \in \mathcal{R}$, and $\mathcal{W}(\rho)(k) \in [w\varepsilon^{-1}, w\varepsilon]$

III for all $t \xrightarrow{w} t'$ there exists a run $\rho \in \text{Runs}(s)$ and $k \in \mathbb{N}$ such that $\forall i < k: (t, \rho[i]) \in \mathcal{R}$, $(t', \rho[k]) \in \mathcal{R}$, and $\mathcal{W}(\rho)(k) \in [w\varepsilon^{-1}, w\varepsilon]$

If there exists an $\varepsilon$-expanded weighted branching bisimulation relating $s$ and $t$, we say that $s$ and $t$ are $\varepsilon$-expanded bisimilar and denote it $s \overset{\varepsilon}{\sim} t$. The relation $\overset{\varepsilon}{\sim}$ is the largest $\varepsilon$-expanded weighted branching bisimulation and will be referred to as $\varepsilon$-expanded weighted branching bisimilarity.

**Example 5.** Consider the WKS shown in Figure 4. The states $s_0$ and $t_0$ are similar to a degree, with the exception that the self loop on $s_0$ is represented as a cycle between $t_0$ and $t_1$. Clearly $s_0 \not\sim t_0$ as the transition $t_0 \xrightarrow{1.5} t_1$ cannot be matched by any prefix from $s_0$. If we however expand by a factor of 2 then $t_0 \xrightarrow{1.5} t_1$ can be matched by $s_0 \xrightarrow{3} s_0$. This in turn requires that $s_0 \overset{2}{\sim} t_1$. This is also the case as $t_1 \xrightarrow{1.5} t_0$ again can be matched by $s_0 \xrightarrow{3} s_0$ and vice versa.

Lastly, as $s_0 \xrightarrow{2} s_1$ and $s_0 \xrightarrow{5} s_2$ can be matched by $t_1 \xrightarrow{1.5} t_0 \xrightarrow{2} t_2$ and $t_1 \xrightarrow{1.5} t_0 \xrightarrow{5} t_3$ respectively within an expansion of 2, we have that $s \overset{2}{\sim} t$.

Unlike weighted branching bisimulation, the class of expanded bisimulations does not form equivalence relations (except for $1 \sim$), this is due to them not satisfying the transitive property. Furthermore, clearly if $s$ and $t$ are is $\varepsilon$-expanded bisimilar, then for any $\gamma \geq \varepsilon$, $s$ and $t$ would also be $\gamma$-expanded bisimilar.

**Lemma 4.1.** Let $K = (S, \rightarrow, \mathcal{AP}, \mathcal{V})$ be a WKS, $s, t \in S$, and $\varepsilon \in \mathbb{R}_{\geq 1}$.

If $s \overset{\varepsilon}{\sim} t$ then $\forall \gamma \geq \varepsilon : s \overset{\gamma}{\sim} t$

**Proof.** Assume $s \overset{\varepsilon}{\sim} t$, we now show that $s \overset{\gamma}{\sim} t$, for some $\gamma \geq \epsilon$. Condition I is implied by the fact $s \overset{\varepsilon}{\sim} t$ if and only if $\mathcal{V}(s) = \mathcal{V}(t)$. For condition II (and III by
symmetry), clearly if all transitions from $s$ of weight $w$ can be matched by a run from $t$ within an expanded interval of $[w\varepsilon^{-1}, w\varepsilon]$ then $t$ can also match that transition within any interval $I$ such that $[w\varepsilon^{-1}, w\varepsilon] \subseteq I$. And since $\varepsilon \leq \gamma$ we have that $[w\varepsilon^{-1}, w\varepsilon] \subseteq [w\gamma^{-1}, w\gamma]$.

We now introduce a way of expanding WBL state formulae in an intuitive way that corresponds with the idea of expanding our behavioural equivalence relation. The expansion of a formula leaves most things intact, except that the intervals given on until formulae are enlarged by a given factor in both directions.

**Definition 4.2 (Expansion of Formulae)**
For a set of atomic propositions, $AP$, the recursive expansion of formulae is given for an arbitrary $\varepsilon \in \mathbb{R}_{\geq 1}$ on the structure of $\phi \in \mathcal{L}$ by the following

- If $\phi = a$ then $\phi^{\varepsilon} = a$,
- If $\phi = \neg \phi_1$ then $\phi^{\varepsilon} = \neg \phi_1^{\varepsilon}$,
- If $\phi = \phi_1 \land \phi_2$ then $\phi^{\varepsilon} = \phi_1^{\varepsilon} \land \phi_2^{\varepsilon}$,
- If $\phi = E \phi_1 U_I \phi_2$ then $\phi^{\varepsilon} = E \phi_1^{\varepsilon} U_I^{\varepsilon} \phi_2^{\varepsilon}$,
- If $\phi = A \phi_1 U_I \phi_2$ then $\phi^{\varepsilon} = A \phi_1^{\varepsilon} U_I^{\varepsilon} \phi_2^{\varepsilon}$

where $a \in AP$, $\phi_1, \phi_2 \in \mathcal{L}$ and $I^{\varepsilon} = [l, u]^{\varepsilon} = [l\varepsilon^{-1}, u\varepsilon]$. Furthermore, expansion of formulae can be extended to path formulae as: if $\psi = \phi_1 U_I \phi_2$ then $\psi^{\varepsilon} = \phi_1^{\varepsilon} U_I^{\varepsilon} \phi_2^{\varepsilon}$

We also introduce two subsets of our WBL formulae, a positive and a negative one.

**Definition 4.3 (Positive and Negative subset of WBL)**
Given a set of atomic propositions, $AP$, we define the following two subsets of WBL

- $\mathcal{L}^+ : \phi ::= a | \neg a | \phi \land \phi | \phi \lor \phi | E \phi U_I \phi | A \phi U_I \phi$
- $\mathcal{L}^- = \{ \neg \phi \mid \phi \in \mathcal{L}^+ \}$

where $a \in AP$.

The satisfiability of expanding positive WBL formulae (positive formulae being ones containing only negation on propositions) behave in accordance with the notion of expanding our bisimulation. Furthermore, the satisfiability of expanding negative WBL formulae (negative formulae being the negations of positive formulae) behaves inversely compared to positive formulae. This is due the fact that intervals of real numbers increase in size when expanded and that the complement of such intervals decrease in size.

As an auxiliary result, we get that we can characterise bisimulation using only positive or negative formulae.
Theorem 4.2. Let $\mathcal{K} = (S, AP, \mathcal{V}, \rightarrow)$ be a branching-finite WKS, $s, t \in S$, and $\varepsilon \in \mathbb{R}_{\geq 1}$.

$$s \sim t \text{ iff } \forall \phi \in \mathcal{L}^+: [s \models \phi \implies t \models \phi^\varepsilon \text{ and } t \models \phi \implies s \models \phi^\varepsilon]$$

Proof. ($\Rightarrow$) Assume that $s \sim t$ and that $s \models \phi$ for some $\phi \in \mathcal{L}^+$. We show by structural induction on $\phi$ that $s \models \phi \implies t \models \phi^\varepsilon$. This is sufficient as $\sim$ is symmetric.

Case $\phi = a$:
We know that $s \models a$ if and only if $a \in \mathcal{V}(s)$ and since $s \sim t$ implies $\mathcal{V}(s) = \mathcal{V}(t)$ we know that $a \in \mathcal{V}(t)$ and therefore $t \models a^\varepsilon = a$.

Case $\phi = \neg a$:
We know that $s \models \neg a$ if and only if $a \notin \mathcal{V}(s)$ and since $s \sim t$ implies $\mathcal{V}(s) = \mathcal{V}(t)$ we know that $a \notin \mathcal{V}(t)$ and therefore $t \models \neg a^\varepsilon = \neg a$.

Case $\phi = \phi_1 \land \phi_2$:
By definition $s \models \phi_1 \land \phi_2$ if and only if $s \models \phi_1$ and $s \models \phi_2$, by structural induction we have that $t \models \phi_1^\varepsilon$ and $t \models \phi_2^\varepsilon$ and therefore $t \models \phi_1^\varepsilon \land \phi_2^\varepsilon$.

Case $\phi = \phi_1 \lor \phi_2$:
By definition $s \models \phi_1 \lor \phi_2$ if and only if $s \models \phi_1$ or $s \models \phi_2$, by structural induction we have that $t \models \phi_1^\varepsilon$ or $t \models \phi_2^\varepsilon$ and therefore $t \models \phi_1^\varepsilon \lor \phi_2^\varepsilon$.

Case $\phi = E \phi_1 U_I \phi_2$:
We know that $s \models E \phi_1 U_I \phi_2$ if and only if there exists a run $\rho \in \text{Runs}(s)$ and a $k \in \mathbb{N}$ such that $\forall i < k : \rho[i] \models \phi_1$, $\rho[k] \models \phi_2$, and $\mathcal{W}(\rho)(k) \in I$. Since $s \sim t$ we know that there exists a run $\pi \in \text{Runs}(t)$ matching $\rho$ whereby there exists a sequence $(h_0, ..., h_k) \in \mathbb{N}^{k+1}$ where $h_0 = 0$ and $\forall i, 0 < i \leq k : h_i \leq h_{i-1}$ such that $\forall i < k : \rho(i) = \rho[i] \xrightarrow{u_{i+1}} \rho[i + 1]$ we have that $\forall j, h_i \leq j < h_{i+1} : \rho[i] \sim \pi[j]$, $\rho[i + 1] \sim \pi[i + 1]$, and $\mathcal{W}(\pi)(h_{i+1}) - \mathcal{W}(\pi)(h_i) \in [w_{i+1}\varepsilon^{-1}, w_{i+1}\varepsilon]$. By structural induction we have that $\forall j < h_k : \pi[j] \models \phi_1^\varepsilon$ and that $\pi[h_k] \models \phi_2^\varepsilon$. Furthermore we have that $\mathcal{W}(\pi)(h_k) \in [\mathcal{W}(\rho)(k)\varepsilon^{-1}, \mathcal{W}(\rho)(k)\varepsilon] \subseteq I^\varepsilon$ and therefore by definition $t \models E \phi_1^\varepsilon U_I^\varepsilon \phi_2^\varepsilon$.

Case $\phi = A \phi_1 U_I \phi_2$:
We know that $s \models A \phi_1 U_I \phi_2$ if and only for all runs $\rho \in \text{Runs}(s)$ there exists some $k \in \mathbb{N}$ such that $\forall i < k : \rho[i] \models \phi_1$, $\rho[k] \models \phi_2$, and $\mathcal{W}(\rho)(k) \in I$. Assume now towards a contradiction that $t \not\models A \phi_1^\varepsilon U_I^\varepsilon \phi_2^\varepsilon$ meaning that there exists some run $\pi \in \text{Runs}(t)$ such that $\forall k \in \mathbb{N}$ either (a) $\exists i < k : \pi[i] \not\models \phi_1^\varepsilon$, (b) $\pi[k] \not\models \phi_2^\varepsilon$, or (c) $\mathcal{W}(\pi)(k) \notin I^\varepsilon$.  

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Since $s \sim t$ we know that there exists a run $\rho' \in \text{Runs}(s)$ matching $\pi$ whereby there exists a sequence $(h_0, \ldots, h_k) \in \mathbb{N}^{k+1}$ where $h_0 = 0$ and $\forall i, 0 < i \leq k : h_{i-1} \leq h_i$ such that $\forall i < k : \rho(i) = \rho[i] \xrightarrow{w_{i+1}} \rho[i + 1]$ we have that $\forall j, h_i \leq j < h_{i+1} : \pi[i] \sim \rho'[j]$, $\pi[i + 1] \sim \rho'[h_{i+1}]$, and $W(\rho')(h_{i+1}) - W(\rho')(h_i) = v_{i+1}$. We therefore have that for all $k \in \mathbb{N}$ either

(a) $\exists i < k : \pi[i] \not\models \phi'_i$ then by structural induction $\forall j, h_i \leq j < h_{i+1} : \rho'[j] \not\models \phi_1$,

(b) $\pi[k] \not\models \phi'_2$ then by structural induction $\rho'[h_k] \not\models \phi_2$, or

(c) As $W(\pi)(k) \notin I^-$ we have that $[W(\pi)(k)\varepsilon^{-1}, W(\pi)(k)\varepsilon] \cap I = \emptyset$. From $s \sim t$ we have that $W(\rho')(h_k) \in [W(\pi)(k)\varepsilon^{-1}, W(\pi)(k)\varepsilon]$ which implies $W(\rho')(h_k) \notin I$.

Hence by definition $\rho' \not\models \phi_1 U_I \phi_2$ which contradicts our assumption that $s \models A \phi_1 U_I \phi_2$ since $\rho' \in \text{Runs}(s)$. Therefore $t \models A \phi'_1 U_I \phi'_2$.

$(\Leftarrow)$ We show that $s \sim t \iff \forall \phi \in \mathcal{L}^+ : [s \models \phi \implies t \models \phi^\varepsilon$ and $t \models \phi \implies s \models \phi^\varepsilon$.

It is sufficient to show that the relation

$$\mathcal{R} = \{(s, t) \in S \times S | \forall \phi \in \mathcal{L}^+ : [s \models \phi \implies t \models \phi^\varepsilon$ and $t \models \phi \implies s \models \phi^\varepsilon \}$$

is a $\varepsilon$-expanded weighted branching bisimulation. We first show that $\mathcal{R}$ satisfies condition I for $\varepsilon$WBB. Assume that $(s, t) \in \mathcal{R}$. Now assume towards a contradiction that $V(s) \neq V(t)$. This means that there exists an element $a \in V(s) \cup V(t)$ such that either $s \models a$ and $t \not\models a^\varepsilon$ or $t \models a$ and $s \not\models a^\varepsilon$, as $a = a^\varepsilon$, contradicting that $(s, t) \in \mathcal{R}$. Therefore $V(s) = V(t)$.

We now show that $\mathcal{R}$ also satisfies condition II of $\varepsilon$WBB (and by the symmetry of $\mathcal{R}$ condition III). Assume that $(s, t) \in \mathcal{R}$ and that $s \xrightarrow{w} s'$ for some $w \in \mathbb{R}_{\geq 0}$ and $s' \in S$. Assume now towards a contradiction that no run $\rho \in \text{Runs}(t)$ can match $s \xrightarrow{w} s'$ within an expansion of $\varepsilon$, meaning that $\forall \rho \in \text{Runs}(t)$ and $\forall k \in \mathbb{N}$ either (a) $\exists i < k : (s, \rho[i]) \notin \mathcal{R}$, (b) $(s', \rho[k]) \notin \mathcal{R}$, or (c) $W(\rho)(k) \notin [w\varepsilon^{-1}, w\varepsilon]$.

As $K$ is branching-finite we know that the set $P(t)([w\varepsilon^{-1}, w\varepsilon])$ is finite.

(a) For each $\rho \uparrow k \in P(t)([w\varepsilon^{-1}, w\varepsilon])$ such that $\exists i < k : (s, \rho[i]) \notin \mathcal{R}$, without a loss of generality we can assume there exists a formula $\phi_1 \in \mathcal{L}^+$ such that $s \models \phi_1$ and $\rho[i] \not\models \phi'_1$. Let $\Phi_1$ be the set of all such formulae. Clearly $\Phi_1$ is finite as $P(t)([w\varepsilon^{-1}, w\varepsilon])$ is finite.

(b) For each $\rho \uparrow k \in P(t)([w\varepsilon^{-1}, w\varepsilon])$ such that $(s', \rho[k]) \notin \mathcal{R}$, without a loss of generality we can assume there exists a formula $\phi_2 \in \mathcal{L}^+$ such that $s' \models \phi_2$.
and $\rho[k] \not\models \phi_2$. Let $\Phi_2$ be the set of all such formulae. Clearly $\Phi_2$ is finite as $\mathcal{P}(t)([w^{-1}, w])$ is finite.

By Lemma 2.1 we have that there exists a $l, u \in \mathbb{Q}_{\geq 0}$ such that $[w, w] \subseteq [l, u]$ and $\mathcal{P}(t)([w, w]) = \mathcal{P}(t)([l, u])$. We can now create an until-formula

$$\psi = ( \bigwedge_{\phi_1 \in \Phi_1} U_{[l, u]} ( \bigwedge_{\phi_2 \in \Phi_2} \phi_2)$$

for which $\forall \rho \uparrow k \in \mathcal{P}(t)([w^{-1}, w]) : \rho \uparrow k \not\models \psi^\varepsilon$. By Lemma 3.1 we get that $\forall \rho \in \text{Runs}(t) : \rho \not\models \psi^\varepsilon$ which is equivalent with $t \not\models E \psi^\varepsilon$. We now have that $s \models E \psi$ and $t \not\models E \psi^\varepsilon$, contradicting $(s, t) \in \mathcal{R}$. Therefore $\mathcal{R}$ satisfies condition $\text{II}$ (and condition $\text{III}$ by symmetry).

From the contrapositive of the right side of Theorem 4.2 we get that negated formulae behave inversely to positive formulae, i.e. if a state does not satisfy the expanded version of a negated formula, then it does not satisfy the non-expanded version.

**Corollary 4.2.1.** Let $\mathcal{K} = (S, AP, V, \rightarrow)$ be a branching-finite WKS, $s, t \in S$, and $\varepsilon \in \mathbb{R}_{\geq 1}$.

$$s \sim t \iff \forall \phi \in \mathcal{L}^- : [s \models \phi^\varepsilon \implies t \models \phi \text{ and } t \models \phi^\varepsilon \implies s \models \phi]$$

**Proof.** The right side of the bi-implication is the contrapositive of the respective term in Theorem 4.2. □

We also quickly extend these results to that of runs. We show that if a run satisfies some positive until-formulae, then it also satisfies any expanded version of that until-formulae.

**Lemma 4.3.** Let $\mathcal{K} = (S, AP, V, \rightarrow)$ be a branching-finite WKS, $\rho \in \text{Runs}(\mathcal{K})$, and $\varepsilon \in \mathbb{R}_{\geq 1}$. Furthermore, let $\psi = \phi_1 U_I \phi_2$ where $I$ is a closed interval of reals and $\phi_1, \phi_2 \in \mathcal{L}^+$.

$$\rho \models \psi \implies \rho \models \psi^\varepsilon$$

**Proof.** Assume that $\rho \models \psi$, meaning that there exists a $k \in \mathbb{N}$ whereby $\forall i < k : \rho[i] \models \phi_1$, $\rho[k] \models \phi_2$, and $\mathcal{W}(\rho)(k) \in I$. By Lemma 4.1 we have that $\forall i < k : \rho[i] \sim \rho[i]$ and that $\rho[k] \sim \rho[k]$; hence, by Theorem 4.2 we have that $\forall i < k : \rho[i] \models \phi_1^\varepsilon$ and $\rho[k] \models \phi_2^\varepsilon$. Lastly, as $I \subseteq I^\varepsilon$, implying $\mathcal{W}(\rho)(k) \in I^\varepsilon$, we have by definition that $\rho \models \psi^\varepsilon$. □

Furthermore, due to the WKS in question being branching-finite we can argue that if a state satisfies an expanded positive formula to an arbitrary degree of closeness, then it must also satisfy the expanded formula itself.
Theorem 4.4. Let $\mathcal{K} = (S, \rightarrow, AP, \mathcal{V})$ be a branching-finite WKS, $s \in S$, $\phi \in \mathcal{L}^+$, and $\varepsilon \in \mathbb{R}_{\geq 1}$. Lastly, let $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 1}$ be a non-increasing sequence of real numbers greater than or equal to 1.

If $\forall n \in \mathbb{N} : s \models \phi_{\varepsilon_n}$ and $\lim_{n \to \infty} \varepsilon_n = \varepsilon$ then $s \models \phi^\varepsilon$.

Proof. Assume that $\forall n \in \mathbb{N} : s \models \phi_{\varepsilon_n}$ and $\lim_{n \to \infty} \varepsilon_n = \varepsilon$. We show by induction on the structure of $\phi \in \mathcal{L}^+$ that $s \models \phi^\varepsilon$.

Case $\phi = a$:
We know that $\forall n \in \mathbb{N} : a_{\varepsilon_n} = a^\varepsilon = a$ and since $\forall n \in \mathbb{N} : s \models a_{\varepsilon_n}$ therefore $s \models a^\varepsilon$.

Case $\phi = \neg a$:
We know that $\forall n \in \mathbb{N} : \neg a_{\varepsilon_n} = \neg a^\varepsilon = \neg a$ and since $\forall n \in \mathbb{N} : s \models \neg a_{\varepsilon_n}$ therefore $s \models \neg a^\varepsilon$.

Case $\phi = \phi_1 \land \phi_2$:
We know that $\forall n \in \mathbb{N} : s \models \phi_{1\varepsilon_n} \land \phi_{2\varepsilon_n}$ if and only if $\forall n \in \mathbb{N} : s \models \phi_{1\varepsilon_n}$ and $s \models \phi_{2\varepsilon_n}$. By structural induction we have that $s \models \phi_{1\varepsilon}$ and $s \models \phi_{2\varepsilon}$. Therefore, by definition $s \models \phi_{1\varepsilon} \land \phi_{2\varepsilon}$.

Case $\phi = \phi_1 \lor \phi_2$:
We know that $\forall n \in \mathbb{N} : s \models \phi_{1\varepsilon_n} \lor \phi_{2\varepsilon_n}$ if and only if $\forall n \in \mathbb{N} : s \models \phi_{1\varepsilon_n}$ or $s \models \phi_{2\varepsilon_n}$. By structural induction we have that either $s \models \phi_1^\varepsilon$ or $s \models \phi_2^\varepsilon$. Therefore, by definition $s \models \phi_1^\varepsilon \lor \phi_2^\varepsilon$.

Case $\phi = E \phi_1 U \phi_2$:
We know that $\forall n \in \mathbb{N} : s \models E \phi_{1\varepsilon_n} U_{I_{\varepsilon_n}} \phi_{2\varepsilon_n}$ if and only if $\forall n \in \mathbb{N} : \exists \rho \in \text{Runs}(s)$ and $\exists k \in \mathbb{N}$ such that $\forall i < k : \rho[i] = \phi_{1\varepsilon_n}$, $\rho[k] = \phi_{2\varepsilon_n}$, and $W(\rho)(k) \in I_{\varepsilon_n}$.

Since $\mathcal{K}$ is branching-finite we have that for all $n \in \mathbb{N}$, $\mathcal{P}(s)(I_{\varepsilon_n})$ is finite. Additionally, as $(\varepsilon_n)_{n \in \mathbb{N}}$ is a non-increasing sequence we have that $(\mathcal{P}(s)(I_{\varepsilon_n}))_{n \in \mathbb{N}}$ is a non-increasing sequence of finite sets of prefixes where $\exists M \in \mathbb{N} : \forall m > M : \mathcal{P}(s)(I_{\varepsilon_M}) = \mathcal{P}(s)(I_{\varepsilon_m})$. By Lemma 4.3 we have that if $\rho \models \phi_{1\varepsilon_M} U_{I_{\varepsilon_M}} \phi_{2\varepsilon_M}$ then $\forall n < M : \rho \models \phi_{1\varepsilon_n} U_{I_{\varepsilon_n}} \phi_{2\varepsilon_n}$.

This implies that there must exists a $\rho \uparrow k \in \mathcal{P}(s)(I_{\varepsilon_M})$ whereby $\forall n \in \mathbb{N} : [\forall i < k : \rho[i] = \phi_{1\varepsilon_n}, \rho[k] = \phi_{2\varepsilon_n}$, and $W(\rho)(k) \in I_{\varepsilon_n}]$. By structural induction we have that $\forall i < k : \rho[i] = \phi_1^\varepsilon$ and $\rho[k] = \phi_2^\varepsilon$. Furthermore, we have that $\bigcap_{n \in \mathbb{N}} I_{\varepsilon_n} = I^\varepsilon$ and since $\forall n \in \mathbb{N} : W(\rho)(k) \in I_{\varepsilon_n}$ we have that $W(\rho)(k) \in I^\varepsilon$. Therefore, by definition $s \models E \phi_1^\varepsilon U_{I^\varepsilon} \phi_2^\varepsilon$. 

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Case $\phi = A \phi_1 U_f \phi_2$:

We know that $\forall n \in \mathbb{N} : s \models A \phi_1^{\varepsilon_n} U_f \phi_2^{\varepsilon_n}$ if and only if $\forall n \in \mathbb{N} : \forall \rho \in \text{Runs}(s)$ there exists a $k \in \mathbb{N}$ such that $\forall i < k : \rho[i] = \phi_1^{\varepsilon_n}, \rho[k] = \phi_2^{\varepsilon_n}$, and $W(\rho)(k) \in I_\varepsilon$.

Since $K$ is branching-finite we have that for all $n \in \mathbb{N}$, $\mathcal{P}(s)(I_\varepsilon^n)$ is finite. Additionally, as $(\varepsilon_n)_{n \in \mathbb{N}}$ is a non-increasing sequence we have that $(\mathcal{P}(s)(I_\varepsilon^n))_{n \in \mathbb{N}}$ is a non-increasing sequence where $\exists M \in \mathbb{N} : \forall m > M : \mathcal{P}(s)(I_\varepsilon^M) = \mathcal{P}(s)(I_\varepsilon^m)$. By Lemma 4.3 we have that if $\rho = \phi_1^{\varepsilon_M} U_f \phi_2^{\varepsilon_M}$ then $\forall n < M : \rho = \phi_1^{\varepsilon_n} U_f \phi_2^{\varepsilon_n}$.

This implies that for each $\rho \in \text{Runs}(s)$ there must exist a $k \in \mathbb{N}$ such that $\rho \uparrow k \in \mathcal{P}(s)(I_\varepsilon^M)$ and $\forall n \in \mathbb{N} : \forall i < k : \rho[i] = \phi_1^{\varepsilon_n}, \rho[k] = \phi_2^{\varepsilon_n}$, and $W(\rho)(k) \in I_\varepsilon$. By structural induction we have that $\forall i < k : \rho[i] = \phi_1^{\varepsilon_n}$ and $\rho[k] = \phi_2^{\varepsilon_n}$. Furthermore, we have that $\bigcap_{n \in \mathbb{N}} I_\varepsilon^n = I_\varepsilon$ and since $\forall n \in \mathbb{N} : W(\rho)(k) \in I_\varepsilon$ we have that $W(\rho)(k) \in I_\varepsilon$. Therefore, by definition $s = A \phi_1 U_f \phi_2$.

From this expanded behavioural relation we can now derive a distance function between states: the greatest lower bound of possible factors such that the states in question are expanded bisimilar.

**Definition 4.4** (Weighted Branching Bisimulation Distance)

Given a WKS $K = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, the weighted branching bisimulation distance between two states, $s, t \in S$, is given by the function $d : (S \times S) \rightarrow \mathbb{R}_{\geq 1} \cup \{\infty\}$ such that

$$d(s, t) = \inf\{\varepsilon \in \mathbb{R}_{\geq 1} \mid s \sim^\varepsilon t\}$$

where $\inf \emptyset = \infty$.

Clearly, due to atomic proposition being unchanged by the expansion of bisimulation, we have that if two states do not share the same propositions, they are at distance infinity.

**Lemma 4.5.** Let $K = (S, \rightarrow, \mathcal{AP}, \mathcal{V})$ be a WKS and $s, t \in S$.

If $\mathcal{V}(s) \neq \mathcal{V}(t)$ then $d(s, t) = \infty$

*Proof.* Assume that $\mathcal{V}(s) \neq \mathcal{V}(t)$. Assume now towards a contradiction that $d(s, t) = \varepsilon$ for some $\varepsilon \in \mathbb{R}_{\geq 1}$. This implies that there exists some $\delta > \varepsilon$ such that $s \sim^\delta t$, as $d(s, t) = \inf\{\varepsilon \in \mathbb{R}_{\geq 1} \mid s \sim^\varepsilon t\}$. From Definition 4.1 condition $I$, we have that $s \sim^\delta t$ if and only if $\mathcal{V}(s) = \mathcal{V}(t)$, contradicting our assumption that $\mathcal{V}(s) \neq \mathcal{V}(t)$. □

Note that unlike the bisimulation distances given in [8] where a distance of 0 between states would imply that they are bisimilar and vice versa, our distance function assigns bisimilar states the distance of 1. Furthermore, not all states that are at a distance of 1 from each other are necessarily bisimilar (see Example 6). This is
because our distance function is the greatest lower bound of factors of expansions, and as such states that are behavioural equivalent to an arbitrary degree of closeness take the distance of multiplicative identity. We correct this inconsistency by requiring the systems from which we derive our distance function to be branching-finite and by composing it with a logarithmic function.

**Proposition 4.6.** Let $K = (S, AP, V, \rightarrow)$ be a WKS. For the distance function $d$ and arbitrary $x, y, z \in S$ we have that

1. $d(x, y) = 1 \iff x \sim y$ (Multiplicative Identity)
2. $d(x, y) = d(y, x)$ (Symmetry)
3. $d(x, z) \leq d(x, y) \cdot d(y, z)$ (Submultiplicity)

**Proof.**

1. Consequence of the fact that $[w1^{-1}, w1] = w$
2. Implied by the symmetry of $\varepsilon$WBB.
3. Assume that $x \overset{\alpha}{\sim} y$ and $y \overset{\beta}{\sim} z$ for some $\alpha, \beta \in \mathbb{R}_{\geq 1}$, we show that this implies $x \overset{\alpha\beta}{\sim} z$. Since $x$ can be matched by $y$ within an expansion of $\alpha$ and $y$ can be matched by $z$ within an expansion of $\beta$, we therefore get that $x$ can be matched by $z$ within an expansion of $\alpha\beta$.

$\square$

And as said, we can turn our distance into an extended pseudometric by taking the logarithm of our distance.

**Corollary 4.6.1.** Let $K = (S, AP, V, \rightarrow)$ be a WKS. $\log d$ is an extended pseudometric, i.e. for and arbitrary $x, y, z \in S$

1. $\log d(x, y) = 0 \iff x \sim y$ (Identity)
2. $\log d(x, y) = \log d(y, x)$ (Symmetry)
3. $\log d(x, z) \leq \log d(x, y) + \log d(y, z)$ (Triangular inequality)

where $\log \infty = \infty$.

**Proof.** Consequence of the fact that $\log : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 0}$ is an isomorphism on the abelian monoids $(\mathbb{R}_{\geq 1}, \ast, 1)$ and $(\mathbb{R}_{\geq 0}, +, 0)$.

$\square$

However, as mentioned our distance does not have the desired property of in turn implying that states at distance $\varepsilon$ are also $\varepsilon$-expanded bisimilar on arbitrary WKS. We illustrate this problem in Example 6. It is however of little consequence, as we will show that for branching-finite WKS this property is indeed satisfied.
Example 6. Consider the WKS shown in Figure 5. Clearly \( s \not\sim t \) as the transition \( s \xrightarrow{a} s' \) cannot be matched by \( t \). It can however be matched to an arbitrary degree of closeness and as such \( d(s, t) = \inf\{\varepsilon \in \mathbb{R}_{\geq 1} \mid s \sim_{\varepsilon} t\} = 1 \).

![Figure 5: A WKS where \( d(s, t) = 1 \) even though \( s \not\sim t \).](image)

We now show that for branching-finite WKS, that our distance implies expanded bisimulation.

**Proposition 4.7.** Let \( \mathcal{K} = (S, \rightarrow, AP, V) \) be a branching-finite WKS, \( s, t \in S \), and \( \varepsilon \in \mathbb{R}_{\geq 1} \).

\[
d(s, t) = \varepsilon \implies s \sim_{\varepsilon} t
\]

**Proof.** As \( d(s, t) = \inf\{\varepsilon \in \mathbb{R}_{\geq 1} \mid s \sim_{\varepsilon} t\} = \varepsilon \) we have that there exists a non-increasing sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 1} \) such that \( s \sim_{\varepsilon_n} t \) and \( \lim_{n \to \infty} \varepsilon_n = \varepsilon \). By Theorem 4.2 (\( \Rightarrow \)), we have that

\[
\forall \phi \in \mathcal{L}^+ : [s \models \phi] \implies \forall n \in \mathbb{N} : t \models \phi_{\varepsilon_n} \text{ and } t \models \phi \implies \forall n \in \mathbb{N} : s \models \phi_{\varepsilon_n}
\]

As \( \mathcal{K} \) is branching-finite, we have by Theorem 4.4 that

\[
\forall \phi \in \mathcal{L}^+ : [s \models \phi] \implies t \models \phi \text{ and } t \models \phi \implies s \models \phi
\]

Therefore, by Theorem 4.2 (\( \Leftarrow \)), we get that \( s \sim_{\varepsilon} t \).}

Lastly, we show that the distance between states correlates with the satisfiability of expanding formulae for branching-finite WKS. Intuitively this can be stated as: similar states also satisfy similar formulae.
Theorem 4.8. Let $\mathcal{K} = (S, \rightarrow, \mathcal{AP}, \mathcal{V})$ be a branching-finite WKS, $s, t \in S$, $\varepsilon \in \mathbb{R}_{\geq 1}$, and $\phi \in \mathcal{L}^+$. If $d(s, t) \leq \varepsilon$ then $s \models \phi \implies t \models \phi^\varepsilon$.

Proof. Assume $d(s, t) \leq \gamma$ where $1 \leq \gamma \leq \varepsilon$. Since $\mathcal{K}$ is branching-finite we have by Proposition 4.7 that $s \sim t$. Since $\varepsilon \geq \gamma$, we have by Lemma 4.1 that $s \sim t$. Therefore, by Theorem 4.2 we have that $s \models \phi \implies t \models \phi^\varepsilon$. \hfill \qed

4.1 Topology of Weighted Branching Systems

We now induce a topological space from our distance between states, with the purpose of showing how converging sequences of states behave as they approach a limit. More precisely, we show that for positive formulae, if a sequence of states, all satisfying a positive formula $\phi$, converges, then the limit of this sequence also satisfies $\phi$. Similar results have been shown for Markov processes and Markov logic by Larsen et al. in [15].

Example 7. Consider the WKS shown in Figure 6 where we have a sequence of states $(s_n)_{n \in \mathbb{N}}$ which converge to the state $s$.

![Figure 6: A simple WKS where the sequence of states $(s_n)_{n \in \mathbb{N}}$ converges to $s$.](image)

Clearly for all $n \in \mathbb{N}$ we have that $s_n \models E \top U_{[1,2]} \top$ and $s_n \models \neg E \top U_{[0,1]} \top$. However, at the limit $s$ we only have that $s \models E \top U_{[0,1]} \top$, as due to the transition $s \frac{1}{2} \rightarrow s$ we get that $s \not\models \neg E \top U_{[1,1]} \top$.

We now define the topological space on which we prove that our positive formulae are closed. Note, that unlike usual distance functions, ours is not a pseudometric, but rather an exponent thereof. As a consequence, for converging sequences the distance to the limit approaches 1, whereas for classical distance functions this would converge to 0.

Definition 4.5 (Topology of Weighted Kripke Structures) Given a branching-finite WKS $\mathcal{K} = (S, \rightarrow, \mathcal{AP}, \mathcal{V})$, let $\mathcal{T}_d$ be the open-ball topology induced on the distance $d$.

Equipped with this topology, we now show that on branching-finite WKS, the sets of states that satisfy a positive formulae are closed in $\mathcal{T}_d$. 24
Theorem 4.9. Let $\mathcal{K} = (S,\to,\mathcal{AP},\mathcal{V})$ be a branching-finite WKS.

If $\phi \in \mathcal{L}^+$ then $[[\phi]]$ is closed in $\mathcal{T}_d$

Proof. Induction on the structure of $\phi \in \mathcal{L}^+$.

Case $\phi = a$:
Assume that $(s_n)_{n \in \mathbb{N}} \subseteq S, \forall n \in \mathbb{N} : s_n \models a$ and that $\lim_{n \to \infty} s_n = s$ (in $\mathcal{T}_d$) for some $s \in S$. We know that $\forall n \in \mathbb{N} : s_n \models a$ if and only if $a \in \mathcal{V}(s_n)$. Assume towards a contradiction that $s \not\models a$, meaning that $a \not\in \mathcal{V}(s)$. By Lemma 4.5 we have that for all $n \in \mathbb{N}$, $d(s_n,s) = \infty$. This implies that $\lim_{n \to \infty} s_n \neq s$ contradicting our assumption that $\lim_{n \to \infty} s_n = s$. Therefore $s \models a$ and $[a]$ is closed in $\mathcal{T}_d$.

Case $\phi = \neg a$:
Assume that $(s_n)_{n \in \mathbb{N}} \subseteq S, \forall n \in \mathbb{N} : s_n \models \neg a$ and that $\lim_{n \to \infty} s_n = s$ (in $\mathcal{T}_d$) for some $s \in S$. We know that $\forall n \in \mathbb{N} : s_n \models \neg a$ if and only if $a \notin \mathcal{V}(s_n)$. Assume towards a contradiction that $s \models a$, meaning that $a \in \mathcal{V}(s)$. By Lemma 4.5 we have that for all $n \in \mathbb{N}$, $d(s_n,s) = \infty$. This implies that $\lim_{n \to \infty} s_n \neq s$ contradicting our assumption that $\lim_{n \to \infty} s_n = s$. Therefore $s \models \neg a$ and $[\neg a]$ is closed in $\mathcal{T}_d$.

Case $\phi = \phi_1 \land \phi_2$:
By structural induction we have that $[[\phi_1]]$ and $[[\phi_2]]$ are closed in $\mathcal{T}_d$. We know that $[[\phi_1 \land \phi_2]] = [[\phi_1]] \cap [[\phi_2]]$ and as the intersection of two closed sets is still closed we have that $[[\phi_1 \land \phi_2]]$ is closed in $\mathcal{T}_d$.

Case $\phi = \phi_1 \lor \phi_2$:
By structural induction we have that $[[\phi_1]]$ and $[[\phi_2]]$ are closed in $\mathcal{T}_d$. We know that $[[\phi_1 \lor \phi_2]] = [[\phi_1]] \cup [[\phi_2]]$ and as the union of two closed sets is still closed we have that $[[\phi_1 \lor \phi_2]]$ is closed in $\mathcal{T}_d$.

Case $\phi = E \phi_1 U_I \phi_2$:
Assume that $(s_n)_{n \in \mathbb{N}} \subseteq S, \forall n \in \mathbb{N} : s_n \models E \phi_1 U_I \phi_2$ and that $\lim_{n \to \infty} s_n = s$ (in $\mathcal{T}_d$) for some $s \in S$. Since $\lim_{n \to \infty} s_n = s$ we have that there exists a non-increasing sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 1}$ such that $\forall n \in \mathbb{N} : d(s_n,s) = \varepsilon_n$ and $\lim_{n \to \infty} \varepsilon_n = 1$. By Theorem 4.8 we have that $\forall n \in \mathbb{N} : s \models E \phi_1^{\varepsilon_n} U_I s_n \phi_2^{\varepsilon_n}$ and by Theorem 4.4 that $s \models E \phi_1 U_I \phi_2$.

Case $\phi = A \phi_1 U_I \phi_2$:
Assume that $(s_n)_{n \in \mathbb{N}} \subseteq S, \forall n \in \mathbb{N} : s_n \models A \phi_1 U_I \phi_2$ and that $\lim_{n \to \infty} s_n = s$ (in $\mathcal{T}_d$) for some $s \in S$. Since $\lim_{n \to \infty} s_n = s$ we have that there exists a non-increasing sequence
(e_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 1} \text{ such that } \forall n \in \mathbb{N} : d(s_n, s) = e_n \text{ and } \lim_{n \to \infty} e_n = 1. \text{ By Theorem 4.8 we that } \forall n \in \mathbb{N} : s \models A \phi_1^n U I s_n \phi_2^n \text{ and by Theorem 4.4 that } s \models A \phi_1 U \phi_2. \qed

As a consequence of our positive formulae being closed, we get that our negative formulae are open in $T_d$.

**Corollary 4.9.1.** Let $\mathcal{K} = (S, \to, \mathcal{AP}, \nu)$ be a branching-finite WKS.

If $\phi \in \mathcal{L}^-$ then $[\phi]$ is open in $T_d$.

**Proof.** Direct consequence of Theorem 4.9 as the complement of a closed set is an open set. \qed

Lastly, since our atomic propositions and their negations are both considered to be positive and negative formulae, i.e. they are both in $\mathcal{L}^+$ and $\mathcal{L}^-$, we get that these formulae are clopen, i.e. both closed and open.

**Corollary 4.9.2.** Let $\mathcal{K} = (S, \to, \mathcal{AP}, \nu)$ be a branching-finite WKS.

\[ \forall a \in \mathcal{AP} : [a] \text{ and } [-a] \text{ is clopen in } T_d \]

**Proof.** Since $a \in \mathcal{L}^+$ and $a \in \mathcal{L}^-$ we have by Theorem 4.9 and Theorem 4.9 that $[a]$ is closed and open respectively. Similarly, for $-a$. \qed

As conclusion to this section, we define a distance between positive formulae and show a similar robustness result as the one given for our distance between states (Theorem 4.8). Our distance between formulae can intuitively be seen as the greatest lower bound of factors required so that the expanded formula implies the non-expanded other and vice versa.

**Definition 4.6 (Distance Between Formulae)**

Let $\mathcal{K} = (S, \to, \mathcal{AP}, \nu)$ WKS, $\phi, \psi \in \mathcal{L}^+$. The distance between positive formulae is given by the function $\delta : \mathcal{L}^+ \times \mathcal{L}^+ \to \mathbb{R}_{\geq 1} \cup \{\infty\}$ as follows

\[ \delta(\phi, \psi) = \inf \{\varepsilon \in \mathbb{R}_{\geq 1} \mid [\phi] \subseteq [\psi] \text{ and } [\psi] \subseteq [\phi] \} \]

where $\inf \emptyset = \infty$.

As mentioned, we show that if a state satisfies a positive formulae, it also satisfies the $\varepsilon$-expanded formulae of any positive formulae within a distance of $\varepsilon$.

**Theorem 4.10.** Let $\mathcal{K} = (S, \to, \mathcal{AP}, \nu)$ be a branching-finite WKS, $s, t \in S$, $\phi, \psi \in \mathcal{L}^+$.

If $s \models \phi$ and $\delta(\phi, \psi) \leq \varepsilon$ then $s \models \psi^\varepsilon$. \[26\]
Proof. Assume that \( s \models \phi \) and that \( \delta(\phi, \psi) \leq \varepsilon \). Since \( \delta(\phi, \psi) = \inf \{ \varepsilon \in \mathbb{R}_{\geq 1} \mid [\phi] \subseteq [\psi^\varepsilon] \text{ and } [\psi] \subseteq [\phi^\varepsilon] \} \leq \varepsilon \) we have that there exists a non-increasing sequence \((\varepsilon_n)_{n\in\mathbb{N}}\) such that \( \lim_{n \to \infty} \varepsilon_n = \varepsilon \) and \( \forall n \in \mathbb{N} : [\phi] \subseteq [\psi^{\varepsilon_n}] \). In other words, for all \( n \in \mathbb{N} \) we have that \( s \in [\phi] \subseteq [\psi^{\varepsilon_n}] \) thereby implying that \( \forall n \in \mathbb{N} : s \models \psi^{\varepsilon_n} \). Hence, by Theorem 4.4 we have that \( s \models \psi^\varepsilon \).

\[\square\]

5 Conclusion and Future Work

In this report, we extended the classical notion of branching bisimulation to that of weighted systems. We developed a notion of observable behavioural equivalence, namely our Weighted Branching Bisimulation. This allows us to class together otherwise non-bisimilar systems in the classical sense that observably the same.

We develop a new concept of branching-finite, which is an analogue to image-finite but on weighted branching systems. We show that for branching-finite WKS, a weighted extension of CTL without the next-operator characterises our bisimulation.

We relax our bisimulation by expanding the interval by some factor wherein individual transitions can be matched by runs, thus giving us a more robust concept of bisimulation. We also introduce a way of expanding our formulae and show that for a positive subset our expansion of our bisimulation is characterised by the expansion of our formulae.

From the notion of expanding our bisimulation, we derive a distance between states: the greatest lower bound of factors such that two states are expanded bisimilar. We show that the logarithm of this distance behaves like a behavioural pseudometric on branching-finite WKS, and we show that states close to each other satisfy similar formulae.

Lastly, we showed that for branching-finite WKS, our positive formulae are closed in the open-ball topology induced by our distance and we defined a distance on said formulae.

For future work, it would be interesting to look into the complexity aspects of computing our bisimulation, distance, and so on. We would expect the complexity of deciding whether two states are weighted branching bisimilar to be NP-hard, as checking if there exists a path of exact accumulated weight is in NP-complete [17].

Furthermore, perhaps one could develop a weaker notion of weighted branching bisimulation, where we only required transitions to be matched by runs that accumulate an equal or lesser than weight, as checking if there exists a path with an accumulated weight below some upper-bound is in P.

Lastly, we would find it interesting to delve deeper into the topological properties of our distance function and weighted systems in general. Perhaps one could
define a Hausdorff distance between the sets of states satisfying a formula and show that restricted to our positive formulae that it is equal to our distance between said formulae.

References


