A Complete Approximation Theory for Weighted Transition Systems

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Abstract

In this paper we explore a proof theoretic approach to approximate reasoning about weighted transition systems. We introduce generalized weighted transition systems (GTS) as an extension of the classical notion of a weighted transition system by replacing exact transition weights with intervals over non-negative real numbers. We define a modal logic over GTSs that can reason about said intervals by taking operators known from probabilistic logics and adjusting their interpretation for a weighted context. Semantically we can then describe whether a transition with at least some weight or at most some weight can be taken to a state satisfying some property. We show that our logic has the Hennesy-Milner property, i.e. it is semantically invariant under an appropriate bisimulation relation.

As our main contribution we provide a sound and weak-complete axiomatization of our logic. To achieve the completeness result we have used a common technique for modal and Markovian logics involving the construction of a canonical model.

1 Introduction

Transition systems are often used to model concurrent and distributed systems. In particular, weighted transition systems (WTSs) can be used to model the case where some resource is involved, such as time, money, or energy. In practice, however, there is often some uncertainty attached to the resource cost, whereas weights in a WTS are precise. Thus the model may be too restrictive and unable to capture the uncertainties inherent in the domain that is being modeled.

A trend with increasing interest and potential is the notion of Cyber-Physical Systems (CPS) which considers the integration of computation and the physical world. Sensor feedback affects computation, and through machinery, computation can affect physical processes in the world. The accuracy of measurements depends both on the quality of sensors, but also the environment in which it senses. Not only does CPS operate in an unpredictable setting, system inputs from sensor readings or human input are also inherently imprecise. This calls for models and logic that can capture and reason with this uncertainty.

In this paper we explore a formalism which can be used for applications such as CPS, by expanding the realm of weighted systems. The quantitative nature of weighted systems is well-suited for the quantifiable inputs and sensor measurements of CPS, but their rigidity makes them less well suited for the uncertainty inherent in CPSs. We propose an extension of weighted systems, by introducing intervals on transitions instead of exact weight specifications. This allow us to model the imprecision introduced when gathering data from the world by making the intervals wider for greater uncertainties and narrower for small uncertainties.

Specifically we propose a generalization of weighted transition systems termed *generalized weighted transitions systems* (GTSs) where the classical notion of a transition relation is replaced with a transition function, associating to each subset of the state space an interval which is to be interpreted as the lower and upper bounds of transitioning into said set. Not only do intervals allow more loosely specified models, it also allows GTSs to represent an abstraction of sets of models of similar structure but different weights. This is another interesting aspect, namely the opportunity for independent reasoning; if the general GTS satisfies a formula, then all WTS instances it represents also satisfies the formula. Reality often relies on equipment of varying quality: If a model is not satisfactory, a sensor with greater accuracy may be substituted. One could imagine a model with intervals for varying precision. If the model satisfies the required properties, we can choose the cheapest equipment with specifications with the interval ranges. If the model does not satisfy the property, we tweak the intervals, and thus limit the amount of potential WTS instantiations it constitutes, and so equipment usable.

We give a formal semantics for our logic and introduce a new kind of bisimulation, under which our semantics is proven to be invariant. We also give an axiomatization of our logic that we prove not only sound, but also weak-complete with relation to GTS semantics.

2 Related Work

Rooted in the need for reasoning about resource consuming or producing behaviour, various logics have been developed and researched. Notably Larsen et al. [9] considers an infinite state and infinite branching model with resources on states and transitions. They provide an axiomatization for a weighted logic with state and transition modalities, and show both a weak and strong completeness result of their axiomatization. To achieve their strong-completeness result, they assume an infinitary rule similar to what is named the countable additivity rule used by Goldblatt in [5].

Juhl et al. [7] consider what is known as weighted modal transition systems, a formalism that can express optional and required behaviour. They associate transitions with intervals of weight values allowed rather than a specific weight, also supporting the idea of a "loose" specification. A refinement process of finding a common model among a number of models then eliminates potential weights from intervals, resulting in a concrete set of allowed weights. Further they use a variant of the well-known Computation Tree Logic with a constraint function that specify restrictions on accumulated weights.

Looking to probabilistic systems, logics have been proposed with modalities with the operators having semantics as for instance "φ holds with at least probability b". The just stated example is expressed as the formula $w_i(\varphi) \geq b$ in [2]. Operators with similar semantics appear in [8], [10], [13], [6] among others, reasoning about transition probabilities e.g. $L_r\varphi$ says that we can "reach a state satisfying φ with a probability at *least* r". While we shall adopt a syntactically similar notation in this paper, the model they are interpreted over, and consequently the semantics, are different.

To prove completeness, we use a similar technique and construction as in [8], [9]. In line with other completeness results [5], [9], we rely on the assumption of Lindenbaum's lemma [1]. It is also worth noting the work of [8] where they find a countable axiomatization, and applied the

[1] Lindenbaum's lemma states that every consistent set of formulae can be extended to maximally-consistent set. [12, Thm. 12]

Rasiowa-Sikorski lemma [2], which directly imply the Lindenbaum property. This method is also the used in [9] to show weak-completeness; however for their strong completeness result, they have to assume the Lindenbaum property.

3 Preliminaries

In this section we introduce some notation and various concepts that will be used heavily throughout this paper. Firstly, we denote by $\mathbb{N}$ the set of natural numbers. We let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_{\geq 0}$ the set of non-negative reals. In a similar manner we let $\mathbb{Q}$ denote the set of all rational numbers and $\mathbb{Q}_{\geq 0}$ the set of non-negative rationals.

In the following section we define the notion of generalized weighted transition systems. In order to do so we need the notion of an interval over the non-negative real numbers. Intuitively an interval can be seen as defining the lower and upper bounds on the weights that can be used to transition into some set of states. We will now formally define an interval and related operations.

Definition 1. An *interval* $I \in \mathbb{R}^2_{\geq 0}$ over the non-negative real numbers is a tuple $\langle x, y\rangle$ where $x \leq y$. The set of all such intervals including the empty set is denoted by $\mathfrak{I}$, i.e.

$$\mathfrak{I} = \{\langle x, y\rangle \in \mathbb{R}^2_{\geq 0} \mid x \leq y\} \cup \{\emptyset\}.$$

▲

We now define special union and intersection operators that extend union and intersection to intervals.

Definition 2. Let $\uplus$ be an operation defined for any $\mathcal{I} \in 2^{\mathfrak{I}}$ where $\sup\{y \mid \langle x, y\rangle \in \mathcal{I}\} \in \mathbb{R}_{\geq 0}$ as

$$\biguplus_{I \in \mathcal{I}} I = \begin{cases} \emptyset & \text{if } \mathcal{I} = \emptyset \\ \langle \inf\{x \mid \langle x, y\rangle \in \mathcal{I}\}, \sup\{y \mid \langle x, y\rangle \in \mathcal{I}\}\rangle & \text{otherwise.} \end{cases}$$

▲

Definition 3. Let $\Cap$ be an operation defined for any $\mathcal{I} \in 2^{\mathfrak{I}}$ as

$$\bigcap\!\!\!\!+_{I \in \mathcal{I}} I = \begin{cases} \emptyset & \text{if } \begin{array}{l}\mathcal{I} = \emptyset \text{ or} \\ \sup\{x \mid \langle x, y\rangle \in \mathcal{I}\} > \inf\{y \mid \langle x, y\rangle \in \mathcal{I}\}\end{array} \\ \langle \sup\{x \mid \langle x, y\rangle \in \mathcal{I}\}, \inf\{y \mid \langle x, y\rangle \in \mathcal{I}\}\rangle & \text{otherwise.} \end{cases}$$

▲

If $\uplus$ (respectively $\Cap$) is used on just two intervals $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$, we will write this as $\langle x_1, y_1\rangle \uplus \langle x_2, y_2\rangle$ (respectively $\langle x_1, y_1\rangle \Cap \langle x_2, y_2\rangle$).

We now define an ordering relation as a special type of set inclusion over two intervals.

Definition 4. Let $\Subset$ be a relation $\Subset: \mathfrak{I} \times \mathfrak{I}$ which satisfies for all $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in \mathfrak{I}$,

$$\langle x_1, y_1\rangle \Subset \langle x_2, y_2\rangle \text{ iff } x_2 \leq x_1 \text{ and } y_1 \leq y_2,$$

and $\emptyset \Subset I$ for any $I \in \mathfrak{I}$. ▲

[2] For any multimodal logic where the provability relation admits an axiomatization with only countable many instances, the Rasiowa-Sikorski lemma states that any consistent formula is contained in a maximally-consistent set of formulae. [8]

Given two intervals I_1, I_2 we say that I_1 is *included* in I_2 iff $I_1 \Subset I_2$.

Proposition 5. $(\mathfrak{I}, \uplus, \Cap, \Subset)$ *forms a lattice over* $\mathfrak{I}$*, with join and meet defined as* $\uplus$ *and* $\Cap$ *respectively, ordered by* $\Subset$*.*

Proof. We first prove that $\Subset$ is reflexive, antisymmetric, and transitive.

Reflexivity: For any $\langle x, y\rangle \in \mathfrak{I}$, $x \leq x$ and $y \leq y$, so $\langle x, y\rangle \Subset \langle x, y\rangle$, and for $\emptyset \in \mathfrak{I}$, $\emptyset \Subset \emptyset$.

Antisymmetry: If $I_1 \Subset I_2$ and $I_2 \Subset I_1$, then $x_{I_1} = x_{I_2}$ and $y_{I_1} = y_{I_2}$, so $I_1 = I_2$. If any of I_1 and I_2 are $\emptyset$, then they must both be $\emptyset$, so $I_1 = I_2$.

Transitivity: If $I_1 \Subset I_2$ and $I_2 \Subset I_3$, then $x_{I_1} \leq x_{I_2} \leq x_{I_3}$ and $y_{I_1} \geq y_{I_2} \geq y_{I_3}$, so $I_1 \Subset I_3$. This also holds if any of the three are $\emptyset$.

Thus $\Subset$ is a partial order.

Next we prove that $I_1 \Cap I_2$ is the greatest lower bound of I_1 and I_2. If $I_1 \Cap I_2 = \emptyset$, then $\emptyset \Subset I_1$ and $\emptyset \Subset I_2$. If $I_1 \Cap I_2 = \langle x, y\rangle$, then $x \geq x_{I_1}$, $x \geq x_{I_2}$, $y \leq y_{I_1}$, and $y \leq y_{I_2}$, so $\langle x, y\rangle \Subset I_1$ and $\langle x, y\rangle \Subset I_2$, so $I_1 \Cap I_2$ is a lower bound.

Now assume there exists some $I \in \mathfrak{I}$ such that $I_1 \Cap I_2 \Subset I$, $I \Subset I_1$, and $I \Subset I_2$. If $I_1 \Cap I_2 = \emptyset$, then $\max\{x_{I_1}, x_{I_2}\} > \min\{y_{I_1}, y_{I_2}\}$, and since $I \Subset I_1$ and $I \Subset I_2$, we get $x_I \geq \max\{x_{I_1}, x_{I_2}\}$ and $\min\{y_{I_1}, y_{I_2}\} \geq y_I$, but this implies that $x_I > y_I$, so $I = \emptyset$.

If $I_1 \Cap I_2 = \langle x, y\rangle$, then $x_{I_2} \geq x_I \geq \max\{x_{I_1}, x_{I_2}\}$, which implies that either $x_I = x_{I_1} = x$ or $x_I = x_{I_2} = x$. Also $y_{I_2} \leq y_I \leq \min\{y_{I_1}, y_{I_2}\}$, which means that either $y_I = y_{I_1} = y$ or $y_I = y_{I_2} = y$. In any case, $I = I_1 \Cap I_2$, so $I_1 \Cap I_2$ is the greatest lower bound.

Now we prove that $I_1 \uplus I_2$ is the least upper bound of I_1 and I_2. If $I_1 \uplus I_2 = \emptyset$, then $I_1 = I_2 = \emptyset$, so $I_1 \Subset I_1 \uplus I_2$ and $I_2 \Subset I_1 \uplus I_2$. If $I_1 \uplus I_2 = \langle x, y\rangle$, then $x \leq x_{I_1}$, $x \leq x_{I_2}$, $y \geq y_{I_1}$, and $y \geq y_{I_2}$, so $I_1 \Subset \langle x, y\rangle$ and $I_2 \Subset \langle x, y\rangle$. Hence $I_1 \uplus I_2$ is an upper bound on I_1 and I_2.

Now assume there exists some $I \in \mathfrak{I}$ such that $I_2 \Subset I \Subset I_1 \uplus I_2$. If $I_1 \uplus I_2 = \emptyset$, then $I = I_1 = I_2 = \emptyset$. If $I_1 \uplus I_2 = \langle x, y\rangle$, then $\min\{x_{I_1}, x_{I_2}\} \geq x_I \geq x_{I_2}$ which implies that either $x = x_{I_1} = x_I$ or $x = x_{I_2} = x_I$, and we also know that $\max\{y_{I_1}, y_{I_2}\} \leq y_I \leq y_{I_2}$, which means either $y = y_{I_1} = y_I$ or $y = y_{I_2} = y_I$. In any case, $I = I_1 \uplus I_2$, so $I_1 \uplus I_2$ is the least upper bound.

We conclude that $(\mathfrak{I}, \uplus, \Cap, \Subset)$ is a lattice. ∎

We now proceed to prove that $\Cap$ is distributive over $\uplus$ and that $\uplus$ is distributive over $\Cap$.

Proposition 6 ($\Cap$ and $\uplus$ are distributive)**.** *Let* $B \subseteq 2^{\mathfrak{I}}$ *be a possibly infinite set of intervals and* $a \in \mathfrak{I}$ *an arbitrary interval such that* $a \neq \emptyset$ *and* $b \neq \emptyset$ *for every* $b \in B$*, then*

$$(i) \quad a \Cap \left(\biguplus_{b \in B} b\right) = \biguplus_{b \in B} \Big(b \Cap a\Big) \text{ and}$$

$$(ii) \quad a \uplus \left(\bigcap_{b \in B} b\right) = \bigcap_{b \in B} \Big(b \uplus a\Big).$$

Proof. Let $a = \langle x_a, y_a\rangle, B = \{b_0 = \langle x_{b_0}, y_{b_0}\rangle, \ldots, b_i = \langle x_{b_i}, y_{b_i}\rangle, \ldots\}$, notationally we denote arbitrary members of B as $b = \langle x_b, y_b\rangle$.

(i)

Using the definitions of $\uplus$ and $\Cap$, we can write out the left hand side of the equation, add superfluous infimum and supremum, move out infimum and supremum and finally get the right hand side.

$$
\begin{aligned}
a \mathbin{\cap\!\!\!\!\!{\scriptstyle +}} \left(\biguplus_{b\in B} b\right) &= \left\langle \max\{x_a, \inf_{b\in B}\{x_b\}\}, \min\{y_a, \sup_{b\in B}\{y_b\}\} \right\rangle \\
&= \left\langle \max\{\inf_{b\in B}\{x_a\}, \inf_{b\in B}\{x_b\}\}, \min\{\sup_{b\in B}\{y_a\}, \sup_{b\in B}\{y_b\}\} \right\rangle \\
&= \left\langle \inf_{b\in B}\{\max\{x_a, x_b\}\}, \sup_{b\in B}\{\min\{y_a, y_b\}\}, \right\rangle \\
&= \biguplus_{b\in B}\Big(a \mathbin{\cap\!\!\!\!\!{\scriptstyle +}} b\Big).
\end{aligned}
$$

(ii)

We show this in a similar fashion as for (i),

$$
\begin{aligned}
a \uplus \left(\bigcap\!\!\!\!\!\!{\scriptstyle +}_{b\in B} b\right) &= \left\langle \min\{x_a, \sup_{b\in B}\{x_b\}\}, \max\{y_a, \inf_{b\in B}\{y_b\}\} \right\rangle \\
&= \left\langle \sup_{b\in B}\{\min\{x_a, x_b\}\}, \inf_{b\in B}\{\max\{y_a, y_b\}\} \right\rangle \\
&= \bigcap\!\!\!\!\!\!{\scriptstyle +}_{b\in B}\Big(b \uplus a\Big).
\end{aligned}
$$

■

4 Model

In this section we introduce the models studied in this paper as well as the notion of a bisimulation relation, relating model states that that exhibit equivalent behavior. The models we address are generalizations of weighted transition systems, where the classical notion of a transition relation is replaced with a transition function, that assigns an interval to each subset of the state space. First we recap the definition of a weighted transition system. A *Weighted Transition System* (WTS) is an extension of regular transition systems, where transitions are labeled with real numbers. Each state in a WTS is labeled with a subset of *atomic propositions* from the fixed countable set $\mathcal{AP}$. A WTS is formally defined in the following manner:

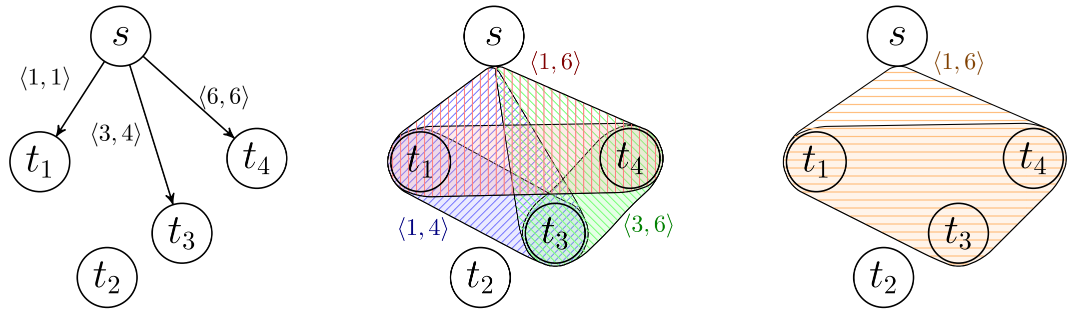
Definition 7. A *Weighted Transition System (WTS)* is a tuple $\mathcal{M} = (S, \rightarrow, \boldsymbol{\ell})$, where

- S is a non-empty set of *states*,
- $\rightarrow \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is the *transition relation*, and
- $\boldsymbol{\ell} : S \rightarrow 2^{\mathcal{AP}}$ is a *labeling function* mapping to each state a set of atomic propositions.

▲

We now define the notion of generalized weighted transition systems (GTSs), using intervals over real numbers as transition weights. The idea behind GTSs is that we can have transitions from a single state to a set of states, and the weight of such a transition is given by an interval rather than a single number. See Figure 1 for a visual representation of this. Intuitively, this interval gives a bound on the lowest and the highest weight to reach any state in the target set of states.

We can now define GTSs formally.



(a) Every transition to a state has an interval associated to it.

(b) Taking a transition to a set of two states changes the intervals.

(c) Taking a transition to a set of three states also changes the intervals.

Figure 1: Illustration of a GTS transition function, showing related states to s.

Definition 8. A *Generalized Weighted Transition System (GTS)* is a tuple $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$, where

(1) S is a non-empty set of *states*,

(2) $\theta : S \to (2^S \to \mathfrak{I})$ is a *transition function* satisfying the following conditions:

(I) $$\theta\,(s)\,(\emptyset) = \emptyset,$$

(II) $$\theta\,(s)\left(\bigcup_i S_i\right) = \biguplus_i \theta\,(s)\,(S_i)\,,\ \text{and}$$

(III) $$\theta\,(s)\left(\bigcap_i S_i\right) \neq \emptyset \implies \theta\,(s)\left(\bigcap_i S_i\right) = \bigcap\!\!\!\!+_i\ \theta\,(s)\,(S_i)\,.$$

(3) $\boldsymbol{\ell} : S \to 2^{\mathcal{AP}}$ is a *labeling function* mapping to each state a set of atomic propositions.

▲

We denote by $\mathfrak{G}$ the set of all GTSs, and if $\theta\,(s)\,(T) = \langle x, y\rangle$, then we also write $\theta^-\,(s)\,(T)$ to denote x and $\theta^+\,(s)\,(T)$ to denote y.

A consequence of the conditions imposed upon θ is that it is monotonic.

Lemma 9 (Monotonicity of θ)**.** *For arbitrary $\mathcal{G} = (S, \theta, \boldsymbol{\ell}) \in \mathfrak{G}$, $s \in S$ and $S', S'' \subseteq 2^S$ where $\theta\,(s)\,(S') \neq \emptyset$ it holds that*

$$S'' \subseteq S' \quad \textit{implies} \quad \theta\,(s)\,(S'') \Subset \theta\,(s)\,(S')$$

Proof. If $S'' = \emptyset$ then by condition **I** of θ we know that $\theta\,(s)\,(S'') = \emptyset$ and therefore $\theta\,(s)\,(S'') \Subset \theta\,(s)\,(S')$.

Suppose therefore that $S'' \neq \emptyset$. Since $S'' \subseteq S'$ it must be the case that $S'' = S' \cap S''$ and therefore, by condition **III** of θ that $\theta\,(s)\,(S'') = \theta\,(s)\,(S') \Cap \theta\,(s)\,(S'')$. From Definition 3 we know that

$$\theta\,(s)\,(S'') \Cap \theta\,(s)\,(S') = \left\langle \max\left\{\theta^-\,(s)\,(S')\,, \theta^-\,(s)\,(S'')\right\}, \min\left\{\theta^+\,(s)\,(S')\,, \theta^+\,(s)\,(S'')\right\}\right\rangle$$

Since

$$\theta^-\,(s)\,(S') \leq \max\left\{\theta^-\,(s)\,(S')\,, \theta^-\,(s)\,(S'')\right\}$$

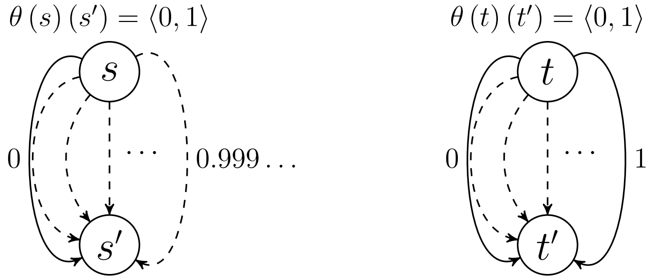


Figure 2: $s \sim t$, but are not bisimilar in the classical sense.

and

$$\min\left\{\theta^+(s)(S'), \theta^+(s)(S'')\right\} \leq \theta^+(s)(S'),$$

we can therefore conclude $\theta(s)(S'') \Subset \theta(s)(S')$. ■

As usual we would like some way of relating model states with equivalent behavior. To this end we define the notion of a bisimulation relation. The notion of a bisimulation relation presented here resembles probabilistic bisimulation [11] for Markov processes more than it does the usual notion of a bisimulation relation for weighted systems. As our models do not have exact weights on the transitions, we do not impose the classical Zig/Zag conditions[1] of a bisimulation relation, but instead require that intervals be matched for any bisimulation class.

Definition 10. Given a GTS $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$, an equivalence relation $\mathcal{R}$ on S is called a *bisimulation relation* iff $s\mathcal{R}t$ implies

- $\boldsymbol{\ell}(s) = \boldsymbol{\ell}(t)$ and
- $\theta(s)(S') = \theta(t)(S')$ for all equivalence classes $S' \in S/\mathcal{R}$.

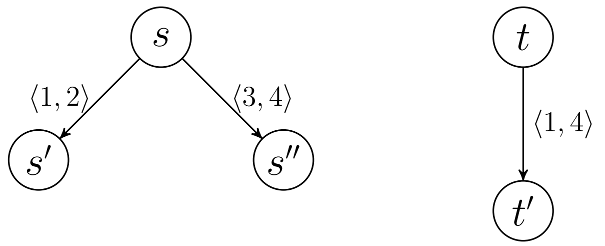
▲

Given a GTS $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$ and two states $s, t \in S$ we say that s and t are bisimilar, written $s \sim t$, iff there exists a bisimulation relation $\mathcal{R}$ such that $s\mathcal{R}t$. Bisimilarity, $\sim$, is the largest bisimulation relation.

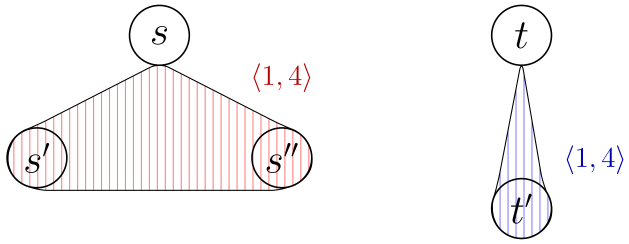
In Example 1 and Example 2 we show some of the ways in which our notion of bisimilarity differs from the standard notion of bisimilarity.

Example 1. Consider the states s and t of the GTS depicted in Figure 2. Both s and t have exactly one successor and the possible transition weights are bounded by the same interval $\langle 0, 1\rangle$. However, s can only take transitions with weights arbitrarily close to but not including 1 whereas t can take a transition with the weight exactly 1. Thus s and t are clearly **not** bisimilar in the classical sense where transitions weights are required to be matched exactly. As we, in this paper, are dealing with approximations of weighted systems we are only interested in the intervals bounding the possible transition weights being matched, and as such s and t are bisimilar in the sense defined in this paper. ◆

Example 2. Consider the GTS depicted in Figure 3a. We see that s can take transitions with weights in the interval $\langle 1, 2\rangle$ to s' and to s'' in the interval $\langle 3, 4\rangle$ where t can take transitions with weights in the interval $\langle 1, 4\rangle$ to t'. If we consider the set $S' = \{s', s'', t'\}$ a bisimulation class, clearly s has no transitions strictly within the interval $\langle 2, 3\rangle$ going to S' whereas t' can



(a) $\theta(s)(s') = \langle 1,2\rangle$, $\theta(s)(s'') = \langle 3,4\rangle$ and $\theta(t)(t') = \langle 1,4\rangle$



(b) $\theta(s)(\{s', s'', t'\}) = \langle 1,4\rangle$ and $\theta(t)(\{s', s'', t'\}) = \langle 1,4\rangle$

Figure 3: $\theta(s)(\{s', s'', t'\}) = \theta(t)(\{s', s'', t'\})$

take transitions with any weights in the interval $\langle 1,4\rangle$ to S'. Again, the models are **not** bisimilar in the classical sense for weighted systems, but as depicted in Figure 3b the lower and upper bounds on transitions going into S' coincide and they are therefore bisimilar in the sense defined in this paper. ♦

5 Logic

In this section we introduce a modal logic with semantics based on GTSs. Our aim is that our logic should be able to capture the notion of bisimilar states as presented in the previous section, and as such it must be able reason about the lower and upper bounds on transition weights. The syntax of our logic and the intuitive meaning of the formulae is inspired by the work on Markov processes in [10], but although our syntax is the same we present entirely different semantics.

Definition 11. The formulae of the logic $\mathcal{L}$ are induced by the abstract syntax

$$\mathcal{L}: \quad \varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid L_r\varphi \mid M_r\varphi$$

where $r \in \mathbb{Q}_{\geq 0}$ and $p \in \mathcal{AP}$. ▲

$\neg$ and $\wedge$ are the usual negation and conjunction operators, whereas L_r and M_r are modal operators. Intuitively, $L_r\varphi$ means that with weight *at least* r we can take a transition to where φ holds, and $M_r\varphi$ means that with weight *at most* r we can take a transition to where φ holds. We now give the precise semantics using GTSs.

Definition 12. Given a GTS $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$, a state $s \in S$ and a formula $\varphi \in \mathcal{L}$, the satisfiability

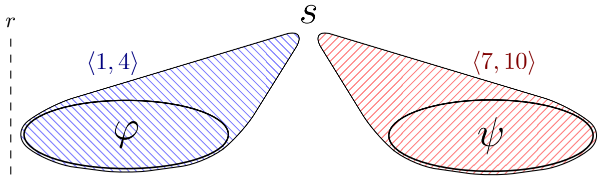


Figure 4: For any $r < 1$, $L_r\varphi$ and $L_r\psi$ does not imply $L_r(\varphi \wedge \psi)$, as there may be no transition to $\varphi \wedge \psi$. The φ and ψ ellipses illustrate sets of states satisfying φ and ψ.

relation $\models$ is defined inductively by

$$
\begin{array}{lll}
\mathcal{G}, s \models p & \text{iff} & p \in \boldsymbol{\ell}(s), \\
\mathcal{G}, s \models \neg\varphi & \text{iff} & \text{it is not the case that } \mathcal{G}, s \models \varphi, \\
\mathcal{G}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{G}, s \models \varphi \text{ and } \mathcal{G}, s \models \psi, \\
\mathcal{G}, s \models L_r\varphi & \text{iff} & \theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset \text{ and } \theta^-(s)(\llbracket\varphi\rrbracket) \geq r, \\
\mathcal{G}, s \models M_r\varphi & \text{iff} & \theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset \text{ and } \theta^+(s)(\llbracket\varphi\rrbracket) \leq r.
\end{array}
$$

where $\llbracket\varphi\rrbracket$ denotes the set of all GTS states that have the property φ, i.e. $\llbracket\varphi\rrbracket = \{s \mid \exists \mathcal{G} = (S, \theta, \boldsymbol{\ell}) \in \mathfrak{G} \text{ s.t. } s \in S \text{ and } \mathcal{G}, s \models \varphi\}$. ▲

If it is not the case that $\mathcal{G}, s \models \varphi$, we write this as $\mathcal{G}, s \not\models \varphi$. If $\mathcal{G}, s \models \varphi$ we say that s is a model of φ. A formula is said to be *satisfiable* if it has at least one model, i.e. a formula $\varphi \in \mathcal{L}$ is satisfiable iff $\llbracket\varphi\rrbracket \neq \emptyset$. We say that φ is a *validity* and write $\models \varphi$ if $\neg\varphi$ is not satisfiable, i.e. φ is valid iff $\llbracket\neg\varphi\rrbracket = \emptyset$. In addition to the operators defined by the syntax of $\mathcal{L}$, we also have the following derived operators:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\bot & \stackrel{\text{def}}{=} & \varphi \wedge \neg\varphi & \top & \stackrel{\text{def}}{=} & \neg\bot \\
\varphi \vee \psi & \stackrel{\text{def}}{=} & \neg(\neg\varphi \wedge \neg\psi) & \varphi \to \psi & \stackrel{\text{def}}{=} & \neg\varphi \vee \psi
\end{array}
$$

Note that $\llbracket\bot\rrbracket = \emptyset$ because it can not both be the case that $\mathcal{G}, s \models \varphi$ and $\mathcal{G}, s \not\models \varphi$ at the same time.

The formula $L_0\varphi$ has special significance in our logic, as this formula means that it is possible to take some transition to where φ holds. In fact, it follows in a straightforward manner from the semantics that

$$\mathcal{G}, s \models L_0\varphi \quad \text{iff} \quad \theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset.$$

Notice also that in general, the following schemes **do not hold**.

$$
\begin{gathered}
L_r\varphi \wedge L_r\psi \to L_r(\varphi \wedge \psi) \\
M_r\varphi \wedge M_r\psi \to M_r(\varphi \wedge \psi)
\end{gathered}
$$

The reason that they do not hold in general is that there may be no transition to where $\varphi \wedge \psi$ holds, i.e. $\neg L_0(\varphi \wedge \psi)$. This is illustrated in Figure 4. If we assume $L_0(\varphi \wedge \psi)$, then both the schemes hold.

Next we wish to prove that our logic is invariant under bisimulation, which is also sometimes known as the Hennessy-Milner property. This property is captured by the following theorem.

Theorem 13 (Bisimulation Invariance)**.** *Given a GTS $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$ and two states $s, t \in S$*

$$s \sim t \quad \textit{iff} \quad [\forall \varphi \in \mathcal{L} : \ \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi].$$

Proof. Let $\mathcal{G} = (S, \theta, \boldsymbol{\ell})$ be a GTS and $s, t \in S$ states. Suppose that $s \sim t$. We will now show that

$$\forall \varphi \in \mathcal{L} : \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi$$

by structural induction on φ. In each case we suppose that $\mathcal{G}, s \models \varphi$ and show that it must also be the case that $\mathcal{G}, t \models \varphi$. In each case symmetrical arguments can be made for the reverse direction of the biconditional. Since $s \sim t$ we know that there exists a bisimulation relation $\mathcal{R} \subseteq S \times S$ such that $s\mathcal{R}t$.

Case: $\varphi = p$

Since $s\mathcal{R}t$ we must have that $\boldsymbol{\ell}(s) = \boldsymbol{\ell}(t)$ and thus $\mathcal{G}, t \models p$.

Case: $\varphi = \neg\psi$

By the semantics of $\models$ it cannot be the case that $\mathcal{G}, s \models \psi$. Suppose towards a contradiction that $\mathcal{G}, t \models \psi$. Since $s\mathcal{R}t$ we have by symmetry of $\mathcal{R}$ that $t\mathcal{R}s$ and thus, by structural induction, that $\mathcal{G}, s \models \psi$. This contradicts our assumption that $\mathcal{G}, s \models \neg\psi$ and therefore it cannot be the case that $\mathcal{G}, t \models \psi$.

Case: $\varphi = \psi \wedge \psi'$

By the semantics of $\models$ we must have that $\mathcal{G}, s \models \psi$ and $\mathcal{G}, s \models \psi'$ which, by structural induction, implies $\mathcal{G}, t \models \psi$ and $\mathcal{G}, t \models \psi'$ and thus $\mathcal{G}, t \models \psi \wedge \psi'$.

Case: $\varphi = L_r\psi$

$\mathcal{G}, s \models L_r\psi$ implies $\theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$ and $\theta^-(s)(\llbracket\psi\rrbracket) \geq r$ implying the existence of a state $s' \in \llbracket\psi\rrbracket$ such that $\theta^-(s)(\{s'\}) \geq \theta^-(s)(\llbracket\psi\rrbracket) \geq r$. Since $\mathcal{R}$ is an equivalence relation we must have $s'\mathcal{R}s'$ which implies the existence of an equivalence class $S' \subseteq \llbracket\psi\rrbracket$, $S' \in S/\mathcal{R}$ such that $s' \in S'$. Because $s\mathcal{R}t$ we must have that $\theta(s)(S') = \theta(t)(S')$ and therefore $\theta(t)(S') \neq \emptyset$ implying $\theta(t)(\llbracket\psi\rrbracket) \neq \emptyset$.

Suppose towards a contradiction that $\theta^-(t)(\llbracket\psi\rrbracket) < r$ implying the existence of a state $t' \in \llbracket\psi\rrbracket$ such that $\theta^-(t)(\llbracket\psi\rrbracket) \leq \theta(t)(\{t'\}) < r$. Because $\mathcal{R}$ is an equivalence relation we must have $t'\mathcal{R}t'$ implying the existence of an equivalence class $S'' \subseteq \llbracket\psi\rrbracket, S'' \in S/\mathcal{R}$ such that $t' \in S''$. Since $t' \in S''$ we must have that $\theta^-(t)(S'') \leq \theta^-(t)(\{t'\}) < r$ and because $s\mathcal{R}t$ we have that $\theta(s)(S'') = \theta(t)(S'')$ implying $\theta^-(s)(\llbracket\psi\rrbracket) < r$ which is a contradiction.

Case: $\varphi = M_r\psi$

$\mathcal{G}, s \models M_r\psi$ implies $\theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$ and $\theta^+(s)(\llbracket\psi\rrbracket) \leq r$ implying the existence of a state $s' \in \llbracket\psi\rrbracket$ such that $\theta^+(s)(\{s'\}) \leq \theta^+(s)(\llbracket\psi\rrbracket) \leq r$. Since $\mathcal{R}$ is an equivalence relation we must have $s'\mathcal{R}s'$ which implies the existence of an equivalence class $S' \subseteq \llbracket\psi\rrbracket$, $S' \in S/\mathcal{R}$ such that $s' \in S'$. Because $s\mathcal{R}t$ we must have that $\theta(s)(S') = \theta(t)(S')$ and therefore $\theta(t)(S') \neq \emptyset$ implying $\theta(t)(\llbracket\psi\rrbracket) \neq \emptyset$.

Suppose towards a contradiction that $\theta^+(t)(\llbracket\psi\rrbracket) > r$ implying the existence of a state $t' \in \llbracket\psi\rrbracket$ such that $\theta^+(t)(\llbracket\psi\rrbracket) \geq \theta(t)(\{t'\}) > r$. Because $\mathcal{R}$ is an equivalence relation we must have $t'\mathcal{R}t'$ implying the existence of an equivalence class $S'' \subseteq \llbracket\psi\rrbracket, S'' \in S/\mathcal{R}$ such that $t' \in S''$. Since $t' \in S''$ we must have that $\theta^+(t)(S'') \geq \theta^+(t)(\{t'\}) > r$ and because $s\mathcal{R}t$ we have that $\theta(s)(S'') = \theta(t)(S'')$ implying $\theta^+(s)(\llbracket\psi\rrbracket) > r$ which is a contradiction.

We have thus shown that

$$s \sim t \quad \text{implies} \quad [\forall \varphi \in \mathcal{L} : \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi].$$

Now, suppose that

$$\forall \varphi \in \mathcal{L} : \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi.$$

We will now show that $s \sim t$. In order to do so, we define a relation on S and show that it is a bisimulation relation. Consider the relation

$$\mathcal{R} = \{(s', t') \mid \forall \varphi \in \mathcal{L} : \ \mathcal{G}, s' \models \varphi \iff \mathcal{G}, t' \models \varphi\}.$$

It is clear that $\mathcal{R}$ is an equivalence relation and $(s, t) \in \mathcal{R}$. We will first show that $\boldsymbol{\ell}(s) = \boldsymbol{\ell}(t)$. Suppose towards a contradiction that $\boldsymbol{\ell}(s) \neq \boldsymbol{\ell}(t)$, then there must exist $p \in \mathcal{AP}$ such that either $[\ \mathcal{G}, s \models p$ and $\mathcal{G}, t \not\models p\ ]$ or $[\ \mathcal{G}, s \not\models p$ and $\mathcal{G}, t \models p\ ]$, which contradicts our assumption that

$$\forall \varphi \in \mathcal{L} : \ \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi.$$

What remains to show is that $\theta\,(s)\,(S'') = \theta\,(t)\,(S'')$ for each equivalence class $S'' \in S/\mathcal{R}$.

We will first show that $\theta\,(s)\,(S') \neq \emptyset$ implies $\theta\,(t)\,(S') \neq \emptyset$ for all equivalence classes $S' \in S/\mathcal{R}$. Suppose $\theta\,(s)\,(S') \neq \emptyset$ for some equivalence class $S' \in S/\mathcal{R}$.
Let $(\!|S'|\!) = \{\varphi \in \mathcal{L} \mid \exists s' \in S' : \mathcal{G}, s \models \varphi\}$, then $S' = \bigcap_{\varphi \in (\!|S'|\!)} [\![\varphi]\!]$. Since $\mathcal{L}$ is countable we can enumerate the formulae in $(\!|S'|\!)$ such that $(\!|S'|\!) = \{\varphi_1, \ldots, \varphi_k, \ldots\}$. For $i \in \mathbb{N}$ we define $\psi_1 = \varphi_1$ and $\psi_i = \psi_{i-1} \wedge \varphi$. We then have a decreasing sequence such that $[\![\psi_i]\!] \supseteq [\![\psi_{i+1}]\!]$ for any $i \in \mathbb{N}$ and $S' = \sup_i [\![\psi_i]\!]$. Now, either $\theta\,(t)\,([\![\psi_i]\!]) \neq \emptyset$ for any $i \in \mathbb{N}$ implying $\theta\,(t)\,(S') \neq \emptyset$ or there exists some $i \in \mathbb{N}$ such that $\theta\,(t)\,([\![\psi_i]\!]) = \emptyset$ implying $\mathcal{G}, s \models L_0\psi_i$ and $\mathcal{G}, t \not\models L_0\psi_i$ contradicting our assumption that

$$\forall \varphi \in \mathcal{L} : \ \mathcal{G}, s \models \varphi \iff \mathcal{G}, t \models \varphi.$$

We thus conclude that $\theta\,(s)\,(S') \neq \emptyset$ implies $\theta\,(t)\,(S') \neq \emptyset$ for any equivalence class $S' \in S/\mathcal{R}$. Symmetrical arguments show that also $\theta\,(t)\,(S') \neq \emptyset$ implies $\theta\,(s)\,(S') \neq \emptyset$. Suppose towards a contradiction that there exists an equivalence class $S' \in S/\mathcal{R}$ such that $\theta\,(s)\,(S') \neq \theta\,(t)\,(S')$. We have four cases to consider, namely $\theta^-\,(s)\,(S') < \theta^-\,(t)\,(S')$, $\theta^-\,(s)\,(S') > \theta^-\,(t)\,(S')$, $\theta^+\,(s)\,(S') < \theta^+\,(t)\,(S')$ and $\theta^+\,(s)\,(S') > \theta^+\,(t)\,(S')$.

Case: $\theta^-\,(s)\,(S') < \theta^-\,(t)\,(S')$

$\theta^-\,(s)\,(S') < \theta^-\,(t)\,(S')$ implies the existence of a formula $\varphi \in (\!|S'|\!)$ and a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $\theta^-\,(s)\,(S') < q < \theta^-\,(t)\,([\![\varphi]\!]) \leq \theta^-\,(t)\,(S')$ implying $\mathcal{G}, s \not\models L_q\varphi$ and $\mathcal{G}, t \models L_q\varphi$ which is a contradiction.

Case: $\theta^-\,(s)\,(S') < \theta^-\,(t)\,(S')$

$\theta^-\,(s)\,(S') > \theta^-\,(t)\,(S')$ implies the existence of a formula $\varphi \in (\!|S'|\!)$ and a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $\theta^-\,(s)\,(S') \geq \theta^-\,(s)\,([\![\varphi]\!]) > q > \theta^-\,(t)\,(S')$ implying $\mathcal{G}, s \not\models L_q\varphi$ and $\mathcal{G}, t \models L_q\varphi$ which is a contradiction.

Case: $\theta^+\,(s)\,(S') < \theta^+\,(t)\,(S')$

$\theta^+\,(s)\,(S') < \theta^+\,(t)\,(S')$ implies the existence of a formula $\varphi \in (\!|S'|\!)$ and a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $\theta^+\,(s)\,(S') < q < \theta^+\,(t)\,([\![\varphi]\!]) \leq \theta^-\,(t)\,(S')$ implying $\mathcal{G}, s \models M_q\varphi$ and $\mathcal{G}, t \not\models M_q\varphi$ which is a contradiction.

Case: $\theta^+\,(s)\,(S') > \theta^+\,(t)\,(S')$

$\theta^+\,(s)\,(S') > \theta^+\,(t)\,(S')$ implies the existence of a formula $\varphi \in (\!|S'|\!)$ and a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $\theta^+\,(s)\,(S') \geq \theta^+\,(s)\,([\![\varphi]\!]) > q > \theta^-\,(t)\,(S')$ implying $\mathcal{G}, s \not\models M_q\varphi$ and $\mathcal{G}, t \models M_q\varphi$ which is a contradiction.

■

6 Metatheory

In this section we propose an axiomatization of our logic that we aim to prove not only sound, but also weak-complete with relation to the GTS semantics.

We first define the notion of filters on a lattice, which will be used in our axiomatization. By ordering the formulae of our logic such that $\varphi \sqsubseteq \psi$ iff $\vdash \varphi \to \psi$, the partially ordered set $(\mathcal{L}, \sqsubseteq)$ becomes a lattice.

Definition 14. A non-empty subset F of $\mathcal{L}$ is called a *filter* on $\mathcal{L}$ iff

- $\perp \notin F$,
- $\varphi \in F$ and $\vdash \varphi \to \psi$ implies $\psi \in F$, and
- $\varphi \in F$ and $\psi \in F$ implies $\varphi \wedge \psi \in F$.

▲

We denote by $\mathcal{F}$ the set of all filters on $\mathcal{L}$. The smallest filter containing a given formula $\varphi \in \mathcal{L}$ is a *principal filter*. The principal filter for φ is given by the set $\{\psi \in \mathcal{L} \mid \vdash \varphi \to \psi\}$ and is denoted by $\uparrow \varphi$. Given any filter F we write $\mathcal{G}, s \models F$ to mean that $\mathcal{G}, s \models \psi$ for all $\psi \in F$.

Definition 15. A filter $F \in \mathcal{F}$ is called an *ultrafilter* iff for all formulae $\varphi \in \mathcal{L}$ either $\varphi \in F$ or $\neg\varphi \in F$. ▲

The ultrafilters on $\mathcal{L}$ correspond to the maximal consistent sets of $\mathcal{L}$. We let $\mathcal{U}$ denote the set of all ultrafilters.

6.1 Axiomatic System

Now let $r, s \in \mathbb{Q}_{\geq 0}$ and F be a filter. Then the deducibility relation $\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is a classical conjunctive deducibility relation, and is defined as the smallest relation which satisfies the axioms of propositional logic in addition to the axioms given in Table 1 as well as the axioms D1-D5 from [4]. We will write $\vdash \varphi$ to mean $\emptyset \vdash \varphi$, and we say that a formula or a set of formulae is *consistent* if it can not derive $\perp$.

For the remainder of this text, for arbitrary sets of formulae $\Phi \in 2^{\mathcal{L}}$, we write $\varphi \vdash \Phi$ to mean that $\varphi \vdash \psi$ for all $\psi \in \Phi$, and we use $\lceil\!\lceil \Phi \rceil\!\rceil$ to denote the set of formulae *derivable* from Φ, and $\lfloor\!\lfloor \Phi \rfloor\!\rfloor$ to denote the set of formulae *forming* Φ, i.e.

$$\lceil\!\lceil \Phi \rceil\!\rceil = \begin{cases} \{\perp\} & \text{if } \Phi = \emptyset \\ \{\varphi \in \mathcal{L} \mid \Phi \vdash \varphi\} & \text{otherwise,} \end{cases} \qquad \lfloor\!\lfloor \Phi \rfloor\!\rfloor = \begin{cases} \{\perp\} & \text{if } \Phi = \emptyset \\ \{\varphi \in \mathcal{L} \mid \varphi \vdash \Phi\} & \text{otherwise.} \end{cases}$$

Axiom **A1** captures the notion that since $\perp$ is never satisfied, we can never take a transition to where $\perp$ holds. Axioms **A2-A7′** describe the relationship between conjunction and disjunction, whereas axioms **A8** and **A9** describe the relationship between L_r and M_r. Notice also that **A9** and **A2** together gives us that $\neg L_0\varphi$ implies $\neg L_r\varphi$ and $\neg M_r\varphi$ for any $r \in \mathbb{Q}_{\geq 0}$.

The axioms **R1** and **R1′** establish the Archimedean property for rational numbers, and axioms **R2** and **R2′** give a sort of monotonicity for L_r and M_r. Axiom **R3** says that if ψ follows from φ, then if it is possible to take a transition to where φ holds, it is also possible to take a transition to where ψ holds. The axiom **R4** ensures that we can not have transition intervals that are infinite.

The axioms **R5** and **R5′** state that for any consistent formula φ, the bounds on the of set of formulae provable from φ must coincide with the bounds on the formulae that prove φ. Axioms **R6** and **R6’** lift this property to filters.

We suspect that some of our axioms are not independent, such as **R5** and **R6**, as well as some of the axioms from **A3-A7′**, but since they are all sound, as we will prove next, this does not pose a problem.

We now proceed to prove the soundness of each of the axioms in Table 1.

(A1):	$\vdash \neg L_0 \bot$
(A2):	$\vdash L_{r+s}\varphi \to L_r\varphi,\ s > 0$
(A2′):	$\vdash M_r\varphi \to M_{r+s}\varphi,\ s > 0$
(A3):	$\vdash L_r\varphi \wedge L_s\psi \to L_{\min\{r,s\}}(\varphi \vee \psi)$
(A3′):	$\vdash M_r\varphi \wedge M_s\psi \to M_{\max\{r,s\}}(\varphi \vee \psi)$
(A4):	$\vdash ((L_r\varphi) \wedge (L_s\psi)) \to \big(L_0(\varphi \wedge \psi) \to L_{\max\{r,s\}}(\varphi \wedge \psi)\big)$
(A4′):	$\vdash ((M_r\varphi) \wedge (M_s\psi)) \to \big(L_0(\varphi \wedge \psi) \to M_{\min\{r,s\}}(\varphi \wedge \psi)\big)$
(A5):	$\vdash ((L_0\varphi) \wedge (\neg L_r\varphi) \wedge (L_0\psi) \wedge (\neg L_s\psi)) \to \neg L_{\max\{r,s\}}(\varphi \wedge \psi)$
(A5′):	$\vdash ((L_0\varphi) \wedge (\neg M_r\varphi) \wedge (L_0\psi) \wedge (\neg M_s\psi)) \to \neg M_{\min\{r,s\}}(\varphi \wedge \psi)$
(A6):	$\vdash L_r(\varphi \vee \psi) \to L_r\varphi \vee L_r\psi$
(A6′):	$\vdash M_r(\varphi \vee \psi) \to M_r\varphi \vee M_r\psi$
(A7):	$\vdash \neg L_0\psi \to (L_r\varphi \to L_r(\varphi \vee \psi))$
(A7′):	$\vdash \neg L_0\psi \to (M_r\varphi \to M_r(\varphi \vee \psi))$
(A8):	$\vdash L_{r+s}\varphi \to \neg M_r\varphi,\ s > 0$
(A9):	$\vdash M_r\varphi \to L_0\varphi$
(R1):	$\{L_s\varphi \mid s < r\} \vdash L_r\varphi$
(R1′):	$\{M_s\varphi \mid s > r\} \vdash M_r\varphi$
(R2):	$\vdash \varphi \to \psi \implies \vdash ((L_r\psi) \wedge (L_0\varphi)) \to L_r\varphi$
(R2′):	$\vdash \varphi \to \psi \implies \vdash ((M_s\psi) \wedge (L_0\varphi)) \to M_s\varphi$
(R3):	$\vdash \varphi \to \psi \implies \vdash L_0\varphi \to L_0\psi$
(R4):	$\{\neg M_r\varphi \mid r \in \mathbb{Q}_{\geq 0}\} \vdash \neg L_0\varphi$
(R5):	$\dfrac{\{\varphi_i \mid i \in \mathbb{N}\} \vdash \varphi \quad \vdash \varphi_{i+1} \to \varphi_i \quad \vdash \varphi \to \varphi_i}{\therefore \quad \{\neg L_r\varphi_i \mid i \in \mathbb{N}\} \vdash \neg L_{r+s}\varphi}, \quad s > 0$
(R5′):	$\dfrac{\{\varphi_i \mid i \in \mathbb{N}\} \vdash \varphi \quad \vdash \varphi_{i+1} \to \varphi_i \quad \vdash \varphi \to \varphi_i}{\therefore \quad \{\neg M_{r+s}\varphi_i \mid i \in \mathbb{N}\} \vdash \neg M_r\varphi}, \quad s > 0$
(R6):	$\{L_{r+s}\varphi \mid \varphi \vdash F\} \cup \{\neg L_r\psi \mid F \vdash \psi\} \vdash \bot$
(R6′):	$\{M_{r+s}\varphi \mid \varphi \vdash F\} \cup \{\neg M_r\psi \mid F \vdash \psi\} \vdash \bot$

Table 1: The axioms for our axiomatic system, where F is a filter and $r, s \in \mathbb{Q}_{\geq 0}$.

Lemma 16 (Soundness)**.**

$$\vdash \varphi \quad \textit{implies} \quad \models \varphi.$$

Proof. We now prove the soundness of each axiom in turn.

A1

In order for $\mathcal{G}, s \models \neg L_0 \bot$ to hold, it must be the case that $\theta(s)(\llbracket \bot \rrbracket) = \emptyset$ or $\theta^-(s)(\llbracket \bot \rrbracket) < 0$. Since $\llbracket \bot \rrbracket = \emptyset$, we immediately get $\theta(s)(\llbracket \bot \rrbracket) = \emptyset$, so $\mathcal{G}, s \models \neg L_0 \bot$.

A2

Assume $\mathcal{G}, s \models L_{r+x}\varphi$ where $x > 0$. Then $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \varphi \rrbracket) \geq r + x > r$, so $\theta(s)(\llbracket \varphi \rrbracket) \geq r$ and hence $\mathcal{G}, s \models L_r\varphi$.

A2′

Assume $\mathcal{G}, s \models M_r\varphi$. Then $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \varphi \rrbracket) \leq r < r + x$, where $x > 0$. This means that $\theta(s)(\llbracket \varphi \rrbracket) \leq r + x$, so $\mathcal{G}, s \models M_{r+x}\varphi$.

A3

Suppose $\mathcal{G}, s \models L_x\varphi \wedge L_y\psi$. Since $\mathcal{G}, s \models L_x\varphi$ and $\mathcal{G}, s \models L_y\psi$, we know that $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$, which implies that $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \uplus \theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$. We also know that $\theta^-(s)(\llbracket \varphi \rrbracket) \geq x$ and $\theta^-(s)(\llbracket \psi \rrbracket) \geq y$, so $\theta^-(s)(\llbracket \varphi \vee \psi \rrbracket) = \min\{\theta^-(s)(\varphi), \theta^-(s)(\psi)\} \geq \min\{x, y\}$, and hence $\mathcal{G}, s \models L_{\min\{x,y\}}(\varphi \vee \psi)$.

A3′

Suppose $\mathcal{G}, s \models M_x\varphi \wedge M_y\psi$. Since $\mathcal{G}, s \models M_x\varphi$ and $\mathcal{G}, s \models M_y\psi$, we know that $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$, which implies that $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \uplus \theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$. We also know that $\theta^+(s)(\llbracket \varphi \rrbracket) \leq x$ and $\theta^+(s)(\llbracket \psi \rrbracket) \leq y$, so $\theta^-(s)(\llbracket \varphi \vee \psi \rrbracket) = \max\{\theta^-(s)(\varphi), \theta^-(s)(\psi)\} \leq \max\{x, y\}$, and hence $\mathcal{G}, s \models L_{\max\{x,y\}}(\varphi \vee \psi)$.

A4

Suppose $\mathcal{G}, s \models (L_q\varphi) \wedge (L_r\psi)$ implying $\theta^-(s)(\llbracket \varphi \rrbracket) \geq q$ and $\theta^-(s)(\llbracket \psi \rrbracket) \geq r$. We want to show that $\mathcal{G}, s \models L_0(\varphi \wedge \psi) \to L_{\max\{q,r\}}(\varphi \wedge \psi)$. Suppose $\mathcal{G}, s \models L_0(\varphi \wedge \psi)$ implying that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$. Since $\llbracket \varphi \wedge \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ and $\llbracket \varphi \wedge \psi \rrbracket \subseteq \llbracket \psi \rrbracket$ Lemma 9 yields that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \Subset \theta(s)(\llbracket \varphi \rrbracket)$ and $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \Subset \theta(s)(\llbracket \psi \rrbracket)$ and therefore $\theta^-(s)(\llbracket \varphi \wedge \psi \rrbracket) \geq \max\{\theta^-(s)(\llbracket \varphi \rrbracket), \theta^-(s)(\llbracket \psi \rrbracket)\} \geq \max\{q, r\}$ implying that $\mathcal{G}, s \models L_{\max\{q,r\}}(\varphi \wedge \psi)$.

A4′

Suppose $\mathcal{G}, s \models (M_q\varphi) \wedge (M_r\psi)$ implying $\theta^+(s)(\llbracket \varphi \rrbracket) \leq q$ and $\theta^+(s)(\llbracket \psi \rrbracket) \leq r$. We want to show that $\mathcal{G}, s \models L_0(\varphi \wedge \psi) \to M_{\min\{q,r\}}(\varphi \wedge \psi)$. Suppose $\mathcal{G}, s \models L_0(\varphi \wedge \psi)$ implying that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$. Since $\llbracket \varphi \wedge \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ and $\llbracket \varphi \wedge \psi \rrbracket \subseteq \llbracket \psi \rrbracket$ Lemma 9 yields that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \Subset \theta(s)(\llbracket \varphi \rrbracket)$ and $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \Subset \theta(s)(\llbracket \psi \rrbracket)$ and therefore $\theta^+(s)(\llbracket \varphi \wedge \psi \rrbracket) \leq \min\{\theta^+(s)(\llbracket \varphi \rrbracket), \theta^+(s)(\llbracket \psi \rrbracket)\} \leq \min\{q, r\}$ implying that $\mathcal{G}, s \models M_{\min\{q,r\}}(\varphi \wedge \psi)$.

A5

Suppose $\mathcal{G}, s \models (L_0\varphi) \wedge (\neg L_q\varphi) \wedge (L_0\psi) \wedge (\neg L_r\psi)$ implying $\mathcal{G}, s \models L_0\varphi$, $\mathcal{G}, s \models \neg L_q\varphi$, $\mathcal{G}, s \models L_0\psi$ and $\mathcal{G}, s \models \neg L_r\psi$. From $\mathcal{G}, s \models L_0\varphi$ we know that $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$, from $\mathcal{G}, s \models \neg L_q\varphi$ that $\theta^-(s)(\llbracket \varphi \rrbracket) < q$, from $\mathcal{G}, s \models L_0\psi$ that $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$ and from $\mathcal{G}, s \models \neg L_r\psi$ that $\theta^-(s)(\llbracket \psi \rrbracket) < r$. It is trivial that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) = \emptyset$ implies

$\mathcal{G}, s \models \neg L_{\max\{q,r\}}(\varphi \wedge \psi)$. Suppose $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$ and towards a contradiction that $\mathcal{G}, s \models L_{\max\{q,r\}}(\varphi \wedge \psi)$ implying $\theta^-(s)(\llbracket \varphi \wedge \psi \rrbracket) \geq \max\{q,r\}$. Since $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ and $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$ we have by condition **III** of the transition function that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \Cap \theta(s)(\llbracket \psi \rrbracket) = \langle \max\{\theta^-(s)(\llbracket \varphi \rrbracket), \theta^-(s)(\llbracket \psi \rrbracket)\}, \min\{\theta^+(s)(\llbracket \varphi \rrbracket), \theta^+(s)(\llbracket \psi \rrbracket)\}\rangle$. $\mathcal{G}, s \models L_{\max\{q,r\}}(\varphi \wedge \psi)$ therefore implies $\max\{\theta^-(s)(\llbracket \varphi \rrbracket), \theta^-(s)(\llbracket \psi \rrbracket)\} \geq \max\{q,r\}$ which is a contradiction and thus $\mathcal{G}, s \models \neg L_{\max\{q,r\}}(\varphi \wedge \psi)$.

A5′

Suppose $\mathcal{G}, s \models (L_0\varphi) \wedge (\neg M_q\varphi) \wedge (L_0\psi) \wedge (\neg M_r\psi)$ implying $\mathcal{G}, s \models L_0\varphi$, $\mathcal{G}, s \models \neg M_q\varphi$, $\mathcal{G}, s \models L_0\psi$ and $\mathcal{G}, s \models \neg M_r\psi$. From $\mathcal{G}, s \models L_0\varphi$ we know that $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$, from $\mathcal{G}, s \models \neg L_q\varphi$ that $\theta^-(s)(\llbracket \varphi \rrbracket) > q$, from $\mathcal{G}, s \models L_0\psi$ that $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$ and from $\mathcal{G}, s \models \neg L_r\psi$ that $\theta^-(s)(\llbracket \psi \rrbracket) > r$. It is trivial that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) = \emptyset$ implies $\mathcal{G}, s \models \neg L_{\max\{q,r\}}(\varphi \wedge \psi)$. Suppose $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$ and towards a contradiction that $\mathcal{G}, s \models M_{\min\{q,r\}}(\varphi \wedge \psi)$ implying $\theta^+(s)(\llbracket \varphi \wedge \psi \rrbracket) \leq \min\{q,r\}$. Since $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ and $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) \neq \emptyset$ we have by condition **III** of the transition function that $\theta(s)(\llbracket \varphi \wedge \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \Cap \theta(s)(\llbracket \psi \rrbracket) = \langle \max\{\theta^-(s)(\llbracket \varphi \rrbracket), \theta^-(s)(\llbracket \psi \rrbracket)\}, \min\{\theta^+(s)(\llbracket \varphi \rrbracket), \theta^+(s)(\llbracket \psi \rrbracket)\}\rangle$. $\mathcal{G}, s \models M_{\min\{q,r\}}(\varphi \wedge \psi)$ therefore implies $\min\{\theta^+(s)(\llbracket \varphi \rrbracket), \theta^+(s)(\llbracket \psi \rrbracket)\} \leq \min\{q,r\}$ which is a contradiction and thus $\mathcal{G}, s \models \neg M_{\min\{q,r\}}(\varphi \wedge \psi)$

A6

Assume $\mathcal{G}, s \models L_r(\varphi \vee \psi)$. Then $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) \neq \emptyset$ and $\theta^-(s)(\llbracket \varphi \vee \psi \rrbracket) \geq r$. From the first we get that $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \uplus \theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$, which means that at least one of the two intervals are not the empty set. If only $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$, then $\theta^-(s)(\llbracket \varphi \rrbracket) \geq r$, so $\mathcal{G}, s \models L_r\varphi$, and if only $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$, then $\theta^-(s)(\llbracket \psi \rrbracket) \geq r$, so $\mathcal{G}, s \models L_r\psi$. If both $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$, then $\theta^-(s)(\llbracket \varphi \vee \psi \rrbracket) = \min\{\theta^-(s)(\llbracket \varphi \rrbracket), \theta^-(s)(\llbracket \psi \rrbracket)\} \geq r$, so we have $\mathcal{G}, s \models L_r\varphi$ and $\mathcal{G}, s \models L_r\psi$.

A6′

Assume $\mathcal{G}, s \models M_r(\varphi \vee \psi)$. Then $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) \neq \emptyset$ and $\theta^+(s)(\llbracket \varphi \vee \psi \rrbracket) \leq r$, which implies that $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket) \uplus \theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$, so at least one of $\theta(s)(\llbracket \varphi \rrbracket)$ and $\theta(s)(\llbracket \psi \rrbracket)$ must be different from the empty set. If only one of them is different from the empty set, then either $\mathcal{G}, s \models M_r\varphi$ or $\mathcal{G}, s \models M_r\psi$. If they are both different from the empty set, then $\theta^+(s)(\llbracket \varphi \vee \psi \rrbracket) = \max\{\theta^+(s)(\llbracket \varphi \rrbracket), \theta^+(s)(\llbracket \psi \rrbracket)\} \leq r$, so we have both $\mathcal{G}, s \models M_r\varphi$ and $G, s \models M_r\psi$.

A7

Assume $\mathcal{G}, s \models \neg L_0\psi$. This means that $\theta(s)(\llbracket \psi \rrbracket) = \emptyset$. If $\mathcal{G}, s \models L_r\varphi$, then $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta^-(s)(\llbracket \varphi \rrbracket) \geq r$. Since $\theta(s)(\llbracket \psi \rrbracket) = \emptyset$, $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket)$, so $\mathcal{G}, s \models L_r(\varphi \vee \psi)$.

A7′

Assume $G, s \models \neg L_0\psi$, which means that $\theta(s)(\llbracket \psi \rrbracket) = \emptyset$. If $G, s \models M_r\varphi$, then $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta^+(s)(\llbracket \varphi \rrbracket) \leq r$, and since $\theta(s)(\llbracket \varphi \vee \psi \rrbracket) = \theta(s)(\llbracket \varphi \rrbracket)$, this means that $\mathcal{G}, s \models M_r(\varphi \vee \psi)$.

A8

Suppose $\mathcal{G}, s \models L_{r+s}\varphi$ where $s > 0$. This implies that $\theta^-(s)(\llbracket \varphi \rrbracket) \geq r + s$ and $\theta^-(s)(\llbracket \varphi \rrbracket) \geq r + s > r$ as $s > 0$. This means that $r < \theta^-(s)(\llbracket \varphi \rrbracket) \leq \theta^+(s)(\llbracket \varphi \rrbracket)$, so $\theta^+(s)(\llbracket \varphi \rrbracket) \not\leq r$, and hence $\mathcal{G}, s \not\models M_r\varphi$.

A9

Suppose $\mathcal{G}, s \models M_r\varphi$. This means that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$, and since by definition $\theta^-(s)(\llbracket\varphi\rrbracket) \not< 0$, then $\theta^-(s)(\llbracket\varphi\rrbracket) \geq 0$, implying that $\mathcal{G}, s \models L_0\varphi$.

R1

Suppose $\mathcal{G}, s \models L_s\varphi$ for any $s < r$ and towards a contradiction that $\mathcal{G}, s \not\models L_r\varphi$. Because $\mathcal{G}, s \models L_s\varphi$ we know that $\theta(s)(\varphi) \neq \emptyset$ and therefore $\mathcal{G}, s \not\models L_r\varphi$ implies $\theta^+(s)(\varphi) < r$. There must exist a rational number $t \in \mathbb{Q}$ such that $\theta^+(s)(\varphi) < t < r$. However, we know that $\mathcal{G}, s \models L_s\varphi$ for any $s < r$ implying $\mathcal{G}, s \models L_t\varphi$ contradicting $\theta^+(s)(\varphi) < t$ and therefore $\mathcal{G}, s \models L_r\varphi$.

R1′

Suppose $\mathcal{G}, s \models M_s\varphi$ for any $s > r$ and towards a contradiction that $\mathcal{G}, s \not\models M_r\varphi$. Because $\mathcal{G}, s \models M_s\varphi$ we know that $\theta(s)(\varphi) \neq \emptyset$ and therefore $\mathcal{G}, s \not\models M_r\varphi$ implies $\theta^+(s)(\varphi) > r$. There must exist a rational number $t \in \mathbb{Q}$ such that $\theta^+(s)(\varphi) > t > r$. However, we know that $\mathcal{G}, s \models M_s\varphi$ for any $s > r$ implying $\mathcal{G}, s \models M_t\varphi$ contradicting $\theta^+(s)(\varphi) > t$ and therefore $\mathcal{G}, s \models M_r\varphi$.

R2

Suppose $\vdash \varphi \to \psi$ and $\mathcal{G}, s \models L_r\psi \wedge L_0\varphi$. From $\mathcal{G}, s \models L_0\varphi$ we get $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$. Since $\vdash \varphi \to \psi$, we get that $\llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket$, which implies that $\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket = \llbracket\varphi\rrbracket$. Hence,
$\theta^-(s)(\llbracket\varphi\rrbracket) = \theta^-(s)(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket) = \max\{\theta^-(s)(\llbracket\varphi\rrbracket), \theta^-(s)(\llbracket\psi\rrbracket)\}$.
So if $\theta^-(s)(\llbracket\varphi\rrbracket) \geq \theta^-(s)(\llbracket\psi\rrbracket)$, then $\theta^-(s)(\llbracket\varphi\rrbracket) \geq \theta^-(s)(\llbracket\psi\rrbracket) \geq r$, and
if $\theta^-(s)(\llbracket\varphi\rrbracket) \leq \theta^-(s)(\llbracket\psi\rrbracket)$, then $\theta^-(s)(\llbracket\varphi\rrbracket) = \theta^-(s)(\llbracket\psi\rrbracket) \geq r$, so in any case
$\theta^-(s)(\llbracket\varphi\rrbracket) \geq r$. We conclude that $\mathcal{G}, s \models L_r\varphi$.

R2′

Suppose $\vdash \varphi \to \psi$ and $\mathcal{G}, s \models M_r\psi \wedge L_0\varphi$. From $\mathcal{G}, s \models L_0\varphi$ we get $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$. Since $\vdash \varphi \to \psi$, we get that $\llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket$, which implies that $\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket = \llbracket\varphi\rrbracket$. Hence,
$\theta^+(s)(\llbracket\varphi\rrbracket) = \theta^+(s)(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket) = \min\{\theta^+(s)(\llbracket\varphi\rrbracket), \theta^+(s)(\llbracket\psi\rrbracket)\}$.
So if $\theta^+(s)(\llbracket\varphi\rrbracket) \leq \theta^+(s)(\llbracket\psi\rrbracket)$, then $\theta^+(s)(\llbracket\varphi\rrbracket) \leq \theta^+(s)(\llbracket\psi\rrbracket) \leq r$, and
if $\theta^-(s)(\llbracket\varphi\rrbracket) \leq \theta^-(s)(\llbracket\psi\rrbracket)$, then $\theta^-(s)(\llbracket\varphi\rrbracket) = \theta^-(s)(\llbracket\psi\rrbracket) \leq r$, so in any case
$\theta^-(s)(\llbracket\varphi\rrbracket) \leq r$. We conclude that $\mathcal{G}, s \models M_r\varphi$.

R3

Suppose $\vdash \varphi \to \psi$ and $\mathcal{G}, s \models L_0\varphi$. $\mathcal{G}, s \models \varphi \to \psi$ implies $\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket = \llbracket\varphi\rrbracket$, and $\mathcal{G}, s \models L_0\varphi$ implies that $\theta(s)(\llbracket\varphi\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket) \Cap \theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$, so $\theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$. It can never be the case that $\theta(s)(\llbracket\psi\rrbracket) < 0$, so $\theta(s)(\llbracket\psi\rrbracket) \geq 0$, and hence $\mathcal{G}, s \models L_0\psi$.

R4

Suppose $\mathcal{G}, s \models \neg M_r\varphi$ for all $r \in \mathbb{Q}_{\geq 0}$. Then $\theta(s)(\llbracket\varphi\rrbracket) = \emptyset$ or $\theta^+(s)(\llbracket\varphi\rrbracket) > r$ for all $r \in \mathbb{Q}_{\geq 0}$. If $\theta(s)(\llbracket\varphi\rrbracket) = \emptyset$, then also $\mathcal{G}, s \models \neg L_0\varphi$. Otherwise, we get that $\theta^+(s)(\llbracket\varphi\rrbracket) > r$ for any $r \in \mathbb{Q}_{\geq 0}$, so $\theta^+(s)(\llbracket\varphi\rrbracket)$ can not be a real number, and since intervals are a subset of $\mathbb{R}^2_{\geq 0}$, it follows that $\theta(s)(\llbracket\varphi\rrbracket)$ can not be an interval, so $\theta(s)(\llbracket\varphi\rrbracket) = \emptyset$, and hence $\mathcal{G}, s \models \neg L_0\varphi$.

R5

Suppose arbitrary $\varphi \in \mathcal{L}$ and let $\Phi = \{\varphi_0, \varphi_1, \dots\}$, s.t. $\forall i, \vdash \varphi_{i+1} \to \varphi_i$ and $\forall i, \vdash \varphi \to \varphi_i$ and suppose that if $\forall \varphi_i \in \Phi, \mathcal{G}, s \models \varphi_i$ then $\mathcal{G}, s \models \varphi$. Suppose that $\mathcal{G}, s \models \neg L_r\varphi_i$, for all $\varphi_i \in \Phi$, implying that either $\theta(s)(\bigcap_i \llbracket\varphi_i\rrbracket) = \emptyset$ or $\theta^-(s)(\bigcap_i \llbracket\varphi_i\rrbracket) < r$.
Case $\theta(s)(\bigcap_i \llbracket\varphi_i\rrbracket) = \emptyset$

Suppose towards a contradiction that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$. Since by assumption $\vdash \varphi \to \varphi_i$ for any $i \in \mathbb{N}$, we get that $\theta(s)(\llbracket\bigcap_i \varphi_i\rrbracket) \neq \emptyset$, contradiction.

Case $\theta^-(s)(\bigcap_i \llbracket\varphi_i\rrbracket) < r$

We now show that $\mathcal{G}, s \models \neg L_{r+s}\varphi$. To this end, suppose towards a contradiction that $s \models L_{r+s}\varphi$. $s \models L_{r+s}\varphi$ implies that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$ and $\theta^-(s)(\llbracket\varphi\rrbracket) \geq r+s$.

$$\theta^-(s)\left(\bigcap_i \llbracket\varphi_i\rrbracket\right) < r < r+s \leq \theta^-(s)(\llbracket\varphi\rrbracket).$$

Suppose arbitrary $t \in \bigcap_i \llbracket\varphi_i\rrbracket$, then by the premise that $\mathcal{G}, s \models \varphi_i$ for all $\varphi_i \in \Phi$ implies $\mathcal{G}, s \models \varphi$, we also have that $t \in \llbracket\varphi\rrbracket$, thus $\llbracket\varphi\rrbracket = \bigcap_i \llbracket\varphi_i\rrbracket$. We then get the contradiction by the following inequalities,

$$\theta^-(s)(\llbracket\varphi\rrbracket) < r < r+s \leq \theta^-(s)(\llbracket\varphi\rrbracket).$$

R5′

Suppose arbitrary $\varphi \in \mathcal{L}$ and let $\Phi = \{\varphi_0, \varphi_1, \dots\}$, s.t. $\forall i, \vdash \varphi_{i+1} \to \varphi_i$ and $\forall i, \vdash \varphi \to \varphi_i$ and assume that if $\forall \varphi_i \in \Phi, \mathcal{G}, s \models \varphi_i$ then $\mathcal{G}, s \models \varphi$. Suppose that $\mathcal{G}, s \models \neg M_{r+s}\varphi_i$, for all $\varphi_i \in \Phi$, implying that either $\theta(s)(\bigcap_i \llbracket\varphi_i\rrbracket) = \emptyset$ or $\theta^+(s)(\bigcap_i \llbracket\varphi_i\rrbracket) > r+s$.

Case $\theta(s)(\bigcap_i \llbracket\varphi_i\rrbracket) = \emptyset$

Suppose towards a contradiction that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$. Since by assumption $\vdash \varphi \to \varphi_i$ for any $i \in \mathbb{N}$, we get that $\theta(s)(\llbracket\bigcap_i \varphi_i\rrbracket) \neq \emptyset$, contradiction.

Case $\theta^-(s)(\bigcap_i \llbracket\varphi_i\rrbracket) < r$

We now show that $\mathcal{G}, s \models \neg M_r\varphi$. To this end, suppose towards a contradiction that $s \models M_r\varphi$. $s \models M_r\varphi$ implies that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$ and $\theta^+(s)(\llbracket\varphi\rrbracket) \leq r$.

$$\theta^+(s)(\llbracket\varphi\rrbracket) \leq r < r+s < \theta^+(s)\left(\bigcap_i \llbracket\varphi_i\rrbracket\right).$$

Suppose arbitrary $t \in \bigcap_i \llbracket\varphi_i\rrbracket$, then by the premise that if $\mathcal{G}, s \models \varphi_i$ for all $\forall \varphi_i \in \Phi$ then $\mathcal{G}, s \models \varphi$, we also have that $t \in \llbracket\varphi\rrbracket$, thus $\llbracket\varphi\rrbracket = \bigcap_i \llbracket\varphi_i\rrbracket$. We then get the contradiction by the following inequalities,

$$\theta^+(s)(\llbracket\varphi\rrbracket) \leq r < r+s < \theta^+(s)(\llbracket\varphi_i\rrbracket).$$

R6

Assume $\mathcal{G}, s \models L_{r+q}\varphi$, $q > 0$, for all $\llbracket\varphi\rrbracket \subseteq \llbracket F\rrbracket$ and $\mathcal{G}, s \models \neg L_r\psi$ for all $\llbracket\psi\rrbracket \subseteq \llbracket F\rrbracket$. This means that $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$ and $\theta^-(s)(\llbracket\varphi\rrbracket) \geq r+q$, and that $\theta(s)(\llbracket\psi\rrbracket) = \emptyset$ or $\theta^-(s)(\llbracket\psi\rrbracket) < r$. Observe that we can write the states satisfied by F as

$$\llbracket F\rrbracket = \bigcup_{\llbracket\varphi\rrbracket \subseteq \llbracket F\rrbracket} \llbracket\varphi\rrbracket = \bigcap_{\llbracket\psi\rrbracket \supseteq \llbracket F\rrbracket} \llbracket\psi\rrbracket,$$

and that $\llbracket\varphi\rrbracket \subseteq \llbracket F\rrbracket \subseteq \llbracket\psi\rrbracket$.

Since $\llbracket\varphi\rrbracket \subseteq \llbracket F\rrbracket$, we have by condition **III** of θ that

$$\theta(s)(\llbracket\varphi\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket \cap \llbracket F\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket) \Cap \theta(s)(\llbracket F\rrbracket) \neq \emptyset,$$

which implies that $\theta(s)(\llbracket F\rrbracket) \neq \emptyset$ and also that $\theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$ for all $\llbracket\psi\rrbracket \supseteq \llbracket F\rrbracket$.

Now assume towards a contradiction that $\theta^{-}(s)(\llbracket F \rrbracket) < r+q$. Then

$$\theta^{-}(s)(\llbracket F \rrbracket) = \theta^{-}(s)\left(\bigcup_{\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket} \llbracket \varphi \rrbracket\right) = \inf_{\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket} \theta^{-}(s)(\llbracket \varphi \rrbracket) < r+q,$$

which means that there must exist some $\llbracket \varphi' \rrbracket \subseteq \llbracket F \rrbracket$ such that $\theta^{-}(s)(\llbracket \varphi' \rrbracket) < r+q$, but this contradicts the fact that $\theta^{-}(s)(\varphi) \geq r+q$ for all $\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket$.

Next assume towards a contradiction that $\theta^{-}(s)(\llbracket F \rrbracket) \geq r+q$. This means that

$$\theta^{-}(s)(\llbracket F \rrbracket) = \theta^{-}(s)\left(\bigcap_{\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket} \llbracket \psi \rrbracket\right) = \sup_{\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket} \theta^{-}(s)(\llbracket \psi \rrbracket) \geq r+q.$$

This implies that there must exist some $\llbracket \psi' \rrbracket \supseteq \llbracket F \rrbracket$ such that $\theta^{-}(s)(\llbracket \psi' \rrbracket) \geq r$, which contradicts that $\theta^{-}(s)(\llbracket \psi \rrbracket) < r$ for all $\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket$.

Either way we get a contradiction, so we conclude $\mathcal{G}, s \models \bot$.

R6′

Assume $\mathcal{G}, s \models M_{r+q}\varphi$ where $q > 0$ for all $\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket$ and $\mathcal{G}, s \models M_r \psi$ for all $\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket$. This means that $\theta(s)(\llbracket \varphi \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \varphi \rrbracket) \leq r+q$ and that $\theta(s)(\llbracket \psi \rrbracket) = \emptyset$ or $\theta(s)(\llbracket \psi \rrbracket) > r$.

Reasoning as in **R6**, we get that $\theta(s)(\llbracket F \rrbracket) \neq \emptyset$ and $\theta(s)(\llbracket \psi \rrbracket) \neq \emptyset$ for all $\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket$.

Now assume towards a contradiction that $\theta^{+}(s)(\llbracket F \rrbracket) > r+q$. This means that

$$\theta^{+}(s)(\llbracket F \rrbracket) = \theta^{+}(s)\left(\bigcup_{\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket} \llbracket \varphi \rrbracket\right) = \sup_{\llbracket \varphi \rrbracket \subseteq \llbracket F \rrbracket} \theta^{+}(s)(\llbracket \varphi \rrbracket) > r+q.$$

Then there must exist some $\llbracket \varphi' \rrbracket \subseteq \llbracket F \rrbracket$ such that $\theta^{+}(s)(\llbracket \varphi' \rrbracket) > r+q$, but this is a contradiction.

Assume towards a contradiction that $\theta^{+}(s)(\llbracket F \rrbracket) \leq r+q$. Then

$$\theta^{+}(s)(\llbracket F \rrbracket) = \theta^{+}(s)\left(\bigcap_{\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket}\right) = \inf_{\llbracket \psi \rrbracket \supseteq \llbracket F \rrbracket} \theta^{+}(s)(\llbracket \psi \rrbracket) \leq r+q,$$

which implies that there exists some $\llbracket \psi' \rrbracket \supseteq \llbracket F \rrbracket$ such that $\theta^{+}(s)(\llbracket \psi' \rrbracket) \leq r$, but this is a contradiction.

We therefore conclude that $\mathcal{G}, s \models \bot$.

■

Note also that we can now easily see that our logic is non-compact, i.e. there exists an infinite set $\Phi \subseteq \mathcal{L}$ where each finite subset of Φ admits a model, but Φ does not. To see this, consider the set

$$\Phi = \{\neg M_r \varphi \mid r \in \mathbb{Q}_{\geq 0}\} \cup \{L_0 \varphi\}.$$

Every finite subset of Φ admits a model, but Φ does not, since by axiom **R4** we can derive $\neg L_0 \varphi$ and thus prove $\bot$.

6.2 Canonical Model Construction

With our axioms proven sound, we are now ready to show weak-completeness for our axiomatization. To this end, we use a canonical model construction based on maximal consistent sets of formulae (ultrafilters). This approach has previously been used to show completeness of Markovian logics [8] and for showing completeness of weighted modal logic with relation to WTS semantics [9]. In the following section we present our canonical model construction together with various lemmas showing that our construction abides the definition of a GTS.

We wish to construct a GTS with ultrafilters as states. This means that we need to construct a transition function between ultrafilters, and we need to construct a labeling function for ultrafilters. The construction proceeds as follows. We first construct a transition function between ultrafilters and formulae, and using this transition function we then construct a transition function between ultrafilters and filters. Lastly, we use this transition function to construct the final transition function between ultrafilters and sets of ultrafilters.

The following lemma tells us how to construct the transition functions from an ultrafilter to a single formula.

Lemma 17. *For arbitrary ultrafilter $u \in \mathcal{U}$ and formula $\varphi \in \mathcal{L}$,*

$$L_0\varphi \in u \textit{ implies } \sup\{r \mid L_r\varphi \in u\} \leq \inf\{s \mid M_s\varphi \in u\}$$

Proof. We first argue that there exists some $r, s \in \mathbb{Q}_{\geq 0}$ such that $L_r\varphi \in u$ and $M_s\varphi \in u$, which implies that the sets we are considering are non-empty. By assumption $L_0\varphi \in u$. Suppose towards a contradiction that for all $s \in \mathbb{Q}_{\geq 0}$, $M_s\varphi \notin u$ implying, since u is an ultrafilter, that for all $s \in \mathbb{Q}_{\geq 0}$, $\neg M_s \in u$, which by axiom **R4** further implies that $\neg L_0\varphi \in u$, which contradicts our assumption that $L_0\varphi \in u$.

Let $x = \sup\{r \mid L_r\varphi \in u\}$ and $y = \inf\{s \mid M_s\varphi \in u\}$. Suppose towards a contradiction that $x > y$. Let $q \in \mathbb{Q}_{\geq 0}$ such that $x \geq q > y$, then by axiom **A2**, $L_q\varphi \in u$. Let $s \in \mathbb{Q}_{\geq 0}$ such that $x \geq q > s \geq y$, then by axiom **A2'**, $M_s\varphi \in u$. But by axiom **A8**, we have that $L_q\varphi \to \neg M_s\varphi$ as $q > s$, contradiction! ■

Next we define the transition function from ultrafilters to formulae $\theta_{\mathcal{L}}$.

Definition 18. For arbitrary $u \in \mathcal{U}$, $\varphi \in \mathcal{L}$ we define the transition function $\theta_{\mathcal{L}} : \mathcal{U} \to [\mathcal{L} \to \mathfrak{I}]$ as

$$\theta_{\mathcal{L}}(u)(\varphi) = \begin{cases} \emptyset & \text{if } L_0\varphi \notin u \\ \langle \sup\{r \mid L_r\varphi \in u\}, \inf\{s \mid M_s\varphi \in u\}\rangle & \text{otherwise.} \end{cases}$$

▲

We now prove that this function satisfies a finite version of conditions **I-III**.

Lemma 19. *For arbitrary $u \in \mathcal{U}$, $\varphi, \psi \in \mathcal{L}$ it holds that*

(i) $\theta_{\mathcal{L}}(u)(\bot) = \emptyset$,

(ii) $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \theta_{\mathcal{L}}(u)(\varphi) \uplus \theta_{\mathcal{L}}(u)(\psi)$, *and*

(iii) $L_0(\varphi \wedge \psi) \in u$ *implies* $\theta_{\mathcal{L}}(u)(\varphi \wedge \psi) = \theta_{\mathcal{L}}(u)(\varphi) \mathbin{\text{⩀}} \theta_{\mathcal{L}}(u)(\psi)$.

Proof. We shall now argue for *(i)*, *(ii)* and *(iii)* in turn. Let $u \in \mathcal{U}$ be an ultrafilter and $\varphi, \psi \in \mathcal{L}$ formulae.

(i) $\theta_{\mathcal{L}}(u)(\bot) = \emptyset$

By **A1**, $\neg L_0\bot \in u$, so $L_0\bot \notin u$, which by Definition 18 implies that $\theta^-_{\mathcal{L}}(u)(\bot) = \emptyset$.

(ii) $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi)$

Case $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \emptyset$ **iff** $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) = \emptyset$

$(\implies)$

Suppose that $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \emptyset$ and further towards a contradiction that $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$. Since $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \emptyset$ we have by Definition 18 that $L_0(\varphi \vee \psi) \notin u$ and because $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$ that $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ or $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$. Without loss of generality we assume that $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$, i.e. $L_0\varphi \in u$. Since $\vdash \varphi \to (\varphi \vee \psi)$ we have by axiom **R3** that $\vdash L_0\varphi \to L_0(\varphi \vee \psi)$ and thus $L_0(\varphi \vee \psi) \in u$ which contradicts our assumption that $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \emptyset$.

$(\impliedby)$

Suppose now that $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) = \emptyset$, i.e. $\theta_{\mathcal{L}}(u)(\varphi) = \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$. Suppose towards a contradiction that $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) \neq \emptyset$, i.e. $L_0(\varphi \vee \psi) \in u$. By axiom **A6** $\vdash L_0(\varphi \vee \psi) \to L_0\varphi \vee L_0\psi$ implying $L_0\varphi \vee L_0\psi \in u$ which contradicts our assumption that $\theta_{\mathcal{L}}(u)(\varphi) = \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$ and therefore $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) = \emptyset$ implies $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \emptyset$.

Case $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \langle x, y\rangle$ **iff** $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) = \langle x, y\rangle$

As shown in the previous case,

$$\theta_{\mathcal{L}}(u)(\varphi \vee \psi) \neq \emptyset \text{ iff } \theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) \neq \emptyset. \tag{1}$$

Suppose that $\theta_{\mathcal{L}}(u)(\varphi \vee \psi) = \langle x_\vee, y_\vee\rangle$, and $\theta_{\mathcal{L}}(u)(\varphi) \biguplus \theta_{\mathcal{L}}(u)(\psi) = \langle x_\uplus, y_\uplus\rangle$.

We shall now argue that $x_\vee = x_\uplus$. By Definition 18, $x_\vee = \sup\{r \mid L_r(\varphi \vee \psi) \in u\}$. By Definition 2 and Equation 1, either of the following cases holds

(i) $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$ or

(ii) $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$ or

(iii) $\theta_{\mathcal{L}}(u)(\varphi) = \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$.

Case (i) $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ **and** $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$

Assume towards a contradiction that $x_\vee < x_\uplus$. This means that there exists a $t \in \mathbb{Q}_{\geq 0}$ such that $x_\vee < t < x_\uplus$ which implies that there exist formulae $\varphi, \psi \in \mathcal{L}$ such that $L_r\varphi \in u$ and $L_s\psi \in u$ where $\min\{r, s\} > t$. By axiom **A3**, $L_r\varphi \in u$ and $L_s\psi \in u$ implies that $L_{\min\{r,s\}}(\varphi \vee \psi) \in u$, which by axiom **A2** implies that $L_t(\varphi \vee \psi) \in u$, but this contradicts that $\sup\{r \mid L_r(\varphi \vee \psi) \in u\} < t$.

Now assume towards a contradiction that $x_\vee > x_\uplus$. This means that there exists $t \in \mathbb{Q}_{\geq 0}$ such that $x_\vee > t > x_\uplus$. If $x_\vee$ is rational, then $L_{x_\vee}(\varphi \vee \psi) \in u$ because either it is part of the set which we take the supremum over, or it is the limit of a sequence of rational numbers, in which case axiom **R1** guarantees that the limit point is also in the set. If $x_\vee$ is irrational, then there must exist some $q \in \mathbb{Q}_{\geq 0}$ such that $L_q(\varphi \vee \psi) \in u$ and $x_\vee > q > t > x_\uplus$. In any case, we can use axiom **A2** to conclude that $L_t(\varphi \vee \psi) \in u$. Since $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$, we know that $L_0\varphi \in u$ and $L_0\psi \in u$. Since $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$, we can use axiom **R2** to get $L_t\varphi \in u$ and $L_t\psi \in u$, which contradicts our assumption that $\min\{\sup\{r \mid L_r\varphi \in u\}, \sup\{r \mid L_r\psi \in u\}\} < t$.

Hence $x_\vee = x_\uplus$.

Case (ii) $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ **and** $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$

As $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$, let $x_\varphi = \sup\{r \mid L_r\varphi \in u\}$. Given that $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$, $x_\varphi = x_\uplus = \sup\{r \mid L_r\varphi \in u\}$. Since $\theta_{\mathcal{L}}(u)(\psi) = \emptyset$, $\neg L_0\psi \in u$, and by axiom **A2** this implies $\neg L_r\psi \in u$ for any $r \in \mathbb{Q}_{\geq 0}$. By axiom **A6**, we know that $L_r(\varphi \vee \psi) \in u$ implies $L_r\varphi \vee L_r\psi \in u$, but since $L_r\psi \notin u$, this implies $L_r\varphi \in u$. By axiom **A7** we know that $L_r\varphi \in u$ implies $L_r(\varphi \vee \psi) \in u$, so $L_r(\varphi \vee \psi) \in u$ iff $L_r\varphi \in u$. Hence $x_{\uplus} = x_{\vee}$.

Case (iii) $\theta_{\mathcal{L}}(u)(\varphi) = \emptyset$ **and** $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$

Follows by similar arguments as in (ii).

We can argue that $y_{\vee} = y_{\uplus}$ in a similar fashion, using axioms **A2′**, **A3′**,**A6′**, **A7′**, **R1′**, and **R2′**.

(iii) $L_0(\varphi \wedge \psi) \in u$ **implies** $\theta_{\mathcal{L}}(u)(\varphi \wedge \psi) = \theta_{\mathcal{L}}(u)(\varphi) \Cap \theta_{\mathcal{L}}(u)(\psi)$

Suppose that $L_0(\varphi \wedge \psi) \in u$ which, by Definition 18, implies

$$\theta_{\mathcal{L}}(u)(\varphi \wedge \psi) = \langle \sup\{r \mid L_r(\varphi \wedge \psi) \in u\}, \inf\{s \mid M_s(\varphi \wedge \psi) \in u\}\rangle.$$

Since $L_0(\varphi \wedge \psi) \in u$ we have by axiom **R3** that $L_0\varphi \in u$ and $L_0\psi \in u$ implying, by Definition 18, that $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$ and $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$. Let

$$\langle x_\varphi, y_\varphi\rangle = \langle \sup\{r \mid L_r\varphi \in u\}, \inf\{s \mid M_s\varphi \in u\}\rangle = \theta_{\mathcal{L}}(u)(\varphi), \text{ and}$$
$$\langle x_\psi, y_\psi\rangle = \langle \sup\{r \mid L_r\psi \in u\}, \inf\{s \mid M_s\psi \in u\}\rangle = \theta_{\mathcal{L}}(u)(\psi).$$

We can now restate $\theta_{\mathcal{L}}(u)(\varphi \wedge \psi) = \theta_{\mathcal{L}}(u)(\varphi) \Cap \theta_{\mathcal{L}}(u)(\psi)$ as

$$\langle \sup\{r \mid L_r(\varphi \wedge \psi) \in u\}, \inf\{s \mid M_s(\varphi \wedge \psi) \in u\}\rangle = \langle \max\{x_\varphi, x_\psi\}, \min\{y_\varphi, y_\psi\}\rangle.$$

We now show that $\sup\{r \mid L_r(\varphi \wedge \psi) \in u\} = \max\{x_\varphi, x_\psi\}$.

By axiom **A4**, we know that if $L_r\varphi \in u$ and $L_s\psi \in u$, then $L_{\max\{r,s\}}(\varphi \wedge \psi) \in u$, so $\sup\{r \mid L_r(\varphi \wedge \psi) \in u\} \geq \max\{x_\varphi, x_\psi\}$. Assume towards a contradiction that $\sup\{r \mid L_r(\varphi \wedge \psi)\} > \max\{x_\varphi, x_\psi\}$. Then there exists some $t \in \mathbb{Q}_{\geq 0}$ such that

$$\sup\{r \mid L_r(\varphi \wedge \psi) \in u\} \geq t > \max\{x_\varphi, x_\psi\}.$$

By axiom **A2**, we know that $L_t(\varphi \wedge \psi) \in u$, and we know that $L_t\varphi \notin u$ and $L_t\psi \notin u$, so $\neg L_t\varphi \in u$ and $\neg L_t\psi \in u$. Hence axiom **A5** gives that $\neg L_t(\varphi \wedge \psi) \in u$, which is a contradiction. We conclude that $\sup\{r \mid L_r(\varphi \wedge \psi) \in u\} = \max\{x_\varphi, x_\psi\}$.

For $\inf\{s \mid M_s(\varphi \wedge \psi) \in u\} = \min\{y_\varphi, y_\psi\}$, symmetrical arguments can be made using axioms **A2′**, **A4′**, **A5′**, and **A9**.

■

Since we have infinitary rules in our logic, some infinite sets of formulae are equivalent to a finite formula. The following lemma tells us that when we have such a formula φ that can be described by an infinite set of formulae Φ, the transition function to φ is the same as the intersection of all the transition functions to $\psi \in \Phi$.

Lemma 20. *Let $\Phi \subseteq \mathcal{L}$ be an infinite set of formulae, $u \in \mathcal{U}$ an ultrafilter, and $\varphi \in \mathcal{L}$ a formula. Then if $\Phi \vdash \varphi$, $\varphi \vdash \Phi$, and $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$, it holds that*

$$\theta_{\mathcal{L}}(u)(\varphi) = \bigcap_{\psi \in \Phi} \theta_{\mathcal{L}}(u)(\psi).$$

Proof. Let $u \in \mathcal{U}$, $\varphi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$ such that $\Phi \vdash \varphi$, $\varphi \vdash \Phi$ and $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$. Because $\mathcal{L}$ is countable, we can enumerate the formulae of Φ such that $\Phi = \{\varphi_1, \ldots, \varphi_k \ldots\}$. We assume that Φ is ordered in such a way that $\vdash \varphi_{i+1} \to \varphi_i$ for any $i \in \mathbb{N}$, $i > 1$. Observe that we can impose such an ordering by letting for $i \in \mathbb{N}$: $\psi_1 = \varphi_1$ and $\psi_i = \psi_{i-1} \wedge \varphi_1$ for any

$i > 1$. Then $\Phi = \{\psi_1, \ldots, \psi_k \ldots\}$ and $\vdash \psi_{i+1} \to \psi_i$ for any $i > 1$. We have to show that $\theta_{\mathcal{L}}(u)(\varphi) = \bigcap\!\!\!\!+_i \theta_{\mathcal{L}}(u)(\varphi_i)$, i.e.

$$\sup\{x | L_x\varphi \in u\} = \sup_i \sup\{y \mid L_y\varphi_i \in u\}$$

and

$$\inf\{x \mid M_x\varphi \in u\} = \inf_i \inf\{y \mid M_y\varphi_i \in u\}.$$

We will first show that

$$\sup\{x | L_x\varphi \in u\} = \sup_i \sup\{y \mid L_y\varphi_i \in u\}.$$

Let $\sup\{x | L_x\varphi \in u\} = x_\varphi$ and $\sup_i \sup\{y \mid L_y\varphi_i \in u\} = y_\varphi$. Since for any i we have that $\vdash \varphi \to \varphi_i$, axiom **R2** yields that $\vdash L_y\varphi_i \wedge L_0\varphi \to L_y\varphi$ implying $x_\varphi \geq y_\varphi$. Suppose towards a contradiction that $x_\varphi > y_\varphi$ implying the existence of rationals $s, t \in \mathbb{Q}_{\geq 0}$ such that $x_\varphi > t + s > t > y_\varphi$. We must therefore have that $L_{t+s}\varphi \in u$ and for all i that $\neg L_t\varphi_i \in u$. By axiom **R5** we have that $\{\neg L_t\varphi_i\} \vdash \neg L_{t+s}\varphi$ implying $\neg L_{t+s}\varphi \in u$ which since $L_{t+s}\varphi \in u$ is a contradiction.

We now show that

$$\inf\{x \mid M_x\varphi \in u\} = \inf_i \inf\{y \mid M_y\varphi_i \in u\}.$$

Let $\inf\{x|M_x\varphi \in u\} = x_\varphi$ and $\inf_i \inf\{y \mid L_y\varphi_i \in u\} = y_\varphi$. Since for any i we have that $\vdash \varphi \to \varphi_i$, axiom **R2′** yields that $\vdash M_y\varphi_i \wedge L_0\varphi \to M_y\varphi$ implying $x_\varphi \leq y_\varphi$. Suppose towards a contradiction that $x_\varphi < y_\varphi$ implying the existence of rationals $s, t \in \mathbb{Q}_{\geq 0}$ such that $x_\varphi < t < t + s < y_\varphi$. We must therefore have that $M_t\varphi \in u$ and for all i that $\neg M_{t+s}\varphi_i \in u$. By axiom **R5′** we have that $\{\neg M_t\varphi_i\} \vdash \neg M_{t+s}\varphi$ implying $\neg M_{t+s}\varphi \in u$ which since $M_{t+s}\varphi \in u$ is a contradiction. ■

Now we extend the $\theta_{\mathcal{L}}$ function to filters by defining $\theta_{\mathcal{F}}$.

Definition 21. For arbitrary $u \in \mathcal{U}$, $F \in (\mathcal{F} \cup \{\emptyset\})$ we define the transition function $\theta_{\mathcal{F}} : \mathcal{U} \to [(\mathcal{F} \cup \{\emptyset\}) \to \mathfrak{I}]$ as

$$\theta_{\mathcal{F}}(u)(F) = \biguplus_{\varphi \in \llbracket F \rrbracket} \theta_{\mathcal{L}}(u)(\varphi).$$

▲

The following lemma shows that we can actually define $\theta_{\mathcal{F}}$ in two different, but equivalent ways. It will be used when showing that $\theta_{\mathcal{F}}$ satisfies condition **III**.

Lemma 22. *For arbitrary $u \in \mathcal{U}$, $F \in \{\mathcal{F} \cup \{\emptyset\}\}$ it holds that if $\theta_{\mathcal{F}}(u)(F) \neq \emptyset$, then*

$$\theta_{\mathcal{F}}(u)(F) = \bigcap\!\!\!\!\!+_{\psi \in \llbracket F \rrbracket} \theta_{\mathcal{L}}(u)(\psi).$$

Proof. Let $u \in \mathcal{U}$ and $F \in \mathcal{F} \cup \{\emptyset\}$. It can not be the case that $F = \emptyset$, because then $\theta_{\mathcal{F}}(u)(F) = \emptyset$, which contradicts our assumption that $\theta_{\mathcal{F}}(u)(F) \neq \emptyset$. Now,

$$\theta_{\mathcal{F}}(u)(F) = \biguplus_{\varphi \in \llbracket F \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset,$$

so there exists some $\varphi \in \llbracket F \rrbracket$ such that $\theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$, which implies that $L_0\varphi \in u$. Since $\vdash \varphi \to \psi$ for any $\psi \in \lceil\!\lceil F \rceil\!\rceil$, we have by axiom **R3** that $L_0\psi \in u$, so $\theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$ for any $\psi \in \lceil\!\lceil F \rceil\!\rceil$, and hence $\Cap_{\psi \in \lceil\!\lceil F \rceil\!\rceil} \theta_{\mathcal{L}}(u)(\psi) \neq \emptyset$.

Now assume $\theta_{\mathcal{F}}(u)(F) = \biguplus_{\varphi \in \llbracket F \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) = \langle x_\varphi, y_\varphi \rangle$ and $\Cap_{\psi \in \lceil\!\lceil F \rceil\!\rceil} \theta_{\mathcal{L}}(u)(\psi) = \langle x_\psi, y_\psi \rangle$. We first show that $x_\varphi = x_\psi$. Assume towards a contradiction that $x_\varphi < x_\psi$ which implies that there exists a $t \in \mathbb{Q}_{\geq 0}$ such that $x_\varphi < t < x_\psi$, $L_t\psi \in u$ for all $\psi \in \lceil\!\lceil F \rceil\!\rceil$, and $\neg L_t\varphi \in u$ for all $\varphi \in \llbracket F \rrbracket$. Since $\theta_{\mathcal{F}}(u)(F) = \biguplus_{\varphi \in \llbracket F \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \neq \emptyset$, there must exist some $\varphi' \in \llbracket F \rrbracket$ such that $\theta_{\mathcal{L}}(u)(\varphi') \neq \emptyset$, which implies that $L_0\varphi' \in u$. Since $\vdash \varphi' \to \psi$, we get by axiom **R2** that $L_t\varphi \in u$ which contradicts that $\neg L_t\varphi \in u$. Thus $x_\varphi \geq x_\psi$. Assume towards a contradiction that $x_\varphi > x_\psi$. Then there exists $t, s \in \mathbb{Q}_{\geq 0}$ where $s > 0$ such that $x_\varphi > t + s > t > x_\psi$ and $\neg L_t\psi \in u$ for all $\psi \in \lceil\!\lceil F \rceil\!\rceil$ and $L_{t+s}\varphi \in u$ for all $\varphi \in \llbracket F \rrbracket$. By axiom **R6**, this is a contradiction, so we can conclude that $x_\varphi = x_\psi$.

Assume towards a contradiction that $y_\varphi < y_\psi$. This implies that there exists $t, s \in \mathbb{Q}_{\geq 0}$ such that $y_\varphi < t < t + s < y_\psi$ and $M_{t+s}\varphi \in u$ for all $\varphi \in \llbracket F \rrbracket$ and $\neg M_t\psi \in u$ for all $\psi \in \lceil\!\lceil F \rceil\!\rceil$. By axiom **R6′**, this is a contradiction. Assume towards a contradiction that $y_\varphi > y_\psi$, which implies that there exists $t \in \mathbb{Q}_{\geq 0}$ such that $y_\varphi > t > y_\psi$ and $\neg M_t\varphi$ for all $\varphi \in \llbracket F \rrbracket$ and $M_t\psi \in u$ for all $\psi \in \lceil\!\lceil F \rceil\!\rceil$. Since we know that there exists some $\varphi' \in \llbracket F \rrbracket$ such that $L_0\varphi' \in u$, we can conclude by axiom **R2′** that $M_t\varphi' \in u$, which is a contradiction. Hence $y_\varphi = y_\psi$. ∎

From the previous lemmas we get the following corollary which gives a relation between formulae and their principal filters with the transition functions we have just defined.

Corollary 23. *For arbitrary ultrafilter $u \in \mathcal{U}$, if $F = \uparrow \varphi$ and $\theta_{\mathcal{F}}(u)(F) \neq \emptyset$, then $\theta_{\mathcal{F}}(u)(\uparrow \varphi) = \theta_{\mathcal{L}}(u)(\varphi)$.*

Proof. Suppose arbitrary formula $\varphi \in \mathcal{L}$ and let $F = \uparrow \varphi$ be its principal filter. Clearly, $\uparrow \varphi \vdash \varphi$ and $\varphi \vdash \uparrow \varphi$ and so we apply Lemma 20 and Lemma 22 to get

$$\theta_{\mathcal{F}}(u)(\uparrow \varphi) = \Cap_{\psi \in \lceil\!\lceil \uparrow\varphi \rceil\!\rceil} \theta_{\mathcal{L}}(u)(\psi) = \theta_{\mathcal{L}}(u)(\varphi).$$

∎

Next we prove that $\theta_{\mathcal{F}}$ satisfies the conditions **I-III**.

Lemma 24. *For any ultrafilter $u \in \mathcal{U}$,*

(I) $\theta_{\mathcal{F}}(u)(\emptyset) = \emptyset$,

(II) $\theta_{\mathcal{F}}(u)(\bigcup_i F_i) = \biguplus_i \theta_{\mathcal{F}}(u)(F_i)$, *and*

(III) $\theta_{\mathcal{F}}(u)(\bigcap_i F_i) \neq \emptyset$ *implies* $\theta_{\mathcal{F}}(u)(\bigcap_i F_i) = \Cap_i \theta_{\mathcal{F}}(u)(F_i)$.

Proof.

(I) $\theta_{\mathcal{F}}(u)(\emptyset) = \theta_{\mathcal{L}}(u)(\bot) = \emptyset$.

(II) Starting from the definition of $\theta_{\mathcal{F}}$, we have that

$$\begin{aligned} \theta_{\mathcal{F}}(u)\left(\bigcup_i F_i\right) &= \biguplus_{\varphi \in \llbracket \bigcup_i F_i \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \\ &= \biguplus_i \biguplus_{\varphi \in \llbracket F_i \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \\ &= \biguplus_i \theta_{\mathcal{F}}(u)(F_i). \end{aligned}$$

$$
\begin{tikzcd}
2^{\mathcal{U}} \arrow[r, "f"] \arrow[dr, "\theta(u) \circ f"'] & \mathcal{F} \arrow[d, "\theta(u)"] \\
& \mathfrak{I}
\end{tikzcd}
$$

Figure 5: Function composition of bijection $f : 2^{\mathcal{U}} \to \mathcal{F}$ and $\theta_{\mathcal{F}}$.

(III) Assume $\theta_{\mathcal{F}}(u)\left(\bigcap_i F_i\right) \neq \emptyset$. Then we can use Lemma 22 to obtain

$$
\begin{aligned}
\theta_{\mathcal{F}}(u)\left(\bigcap_i F_i\right) &= \mathop{\bigcap\hspace{-0.85em}+\hspace{0.35em}}_{\varphi \in \llbracket \bigcap_i F_i \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \\
&= \mathop{\bigcap\hspace{-0.85em}+\hspace{0.35em}}_{i} \mathop{\bigcap\hspace{-0.85em}+\hspace{0.35em}}_{\varphi \in \llbracket F_i \rrbracket} \theta_{\mathcal{L}}(u)(\varphi) \\
&= \mathop{\bigcap\hspace{-0.85em}+\hspace{0.35em}}_{i} \theta_{\mathcal{F}}(u)(F_i).
\end{aligned}
$$

■

For defining the transition function for sets of ultrafilters, we use the Stone duality from Boolean algebras to derive an isomorphism f between filters and sets of ultrafilters. Composing this isomorphism with $\theta_{\mathcal{F}}$ yields a function from the power set of ultrafilters into the set of intervals $\mathfrak{I}$. Figure 5 illustrates this composition. In what follows, let $f : 2^{\mathcal{U}} \to \mathcal{F}$ be the function defined for any $W \in 2^{\mathcal{U}}$ as

$$f(W) = \bigcap_{v \in W} v.$$

then $f^{-1} : \mathcal{F} \to 2^{\mathcal{U}}$ is defined for any $F \in \mathcal{F}$ as

$$f^{-1}(F) = \bigcup_{v \in \mathcal{U} : F \subseteq v} \{v\}.$$

The function f satisfies the following properties:

$$f(\bigcup_i W_i) = \bigcup_i f(W_i)$$

and

$$f(\bigcap_i W_i) = \bigcap_i f(W_i).$$

Definition 25. For arbitrary $u \in \mathcal{U}$, $W \in 2^{\mathcal{U}}$ we define the transition function $\theta_{\mathcal{U}} : \mathcal{U} \to \left[2^{\mathcal{U}} \to \mathfrak{I}\right]$ as

$$\theta_{\mathcal{U}}(u)(W) = \theta_{\mathcal{F}}(u)(f(W)).$$

▲

Lastly we define the labelling function by looking at the atomic propositions in the ultrafilter.

Definition 26. For any ultrafilter $u \in \mathcal{U}$ we define the labeling function $\boldsymbol{\ell}_{\mathcal{U}} : \mathcal{U} \to 2^{\mathcal{AP}}$ as

$$\boldsymbol{\ell}_{\mathcal{U}}(u) = \{p \in \mathcal{AP} \mid p \in u\}.$$

▲

We are now in a position to construct the canonical model which is a GTS with ultrafilters as states and prove that it is in fact a GTS.

Definition 27 (Canonical model)**.** $\mathcal{G}_{\mathcal{U}}$ is the *canonical model* defined as $\mathcal{G}_{\mathcal{U}} = (\mathcal{U}, \theta_{\mathcal{U}}, \boldsymbol{\ell}_{\mathcal{U}})$. ▲

Theorem 28. *$\mathcal{G}_{\mathcal{U}}$ is a GTS.*

Proof. We must verify that $\mathcal{G}_{\mathcal{U}}$ satisfies the properties (1)-(3) of Definition 8.

(1) $\mathcal{U}$ is a non-empty set of states.

(2) By Lemma 17, we know that if $L_0\varphi \notin u$, then $\theta^-_{\mathcal{L}}(u)(\varphi) \leq \theta^+_{\mathcal{L}}(u)(\llbracket\varphi\rrbracket)$. Since by definition,

$$\theta_{\mathcal{U}}(u)(W) = \theta_{\mathcal{F}}(u)\left(\bigcap_{w \in W} w\right) = \bigcap\!\!\!\!\!+_{w \in W} \ \biguplus_{\varphi \in \llbracket w \rrbracket} \theta_{\mathcal{L}}(u)(\varphi),$$

we thus have that $\theta^-_{\mathcal{U}}(u)(W) \leq \theta^+_{\mathcal{U}}(u)(W)$, so $\theta_{\mathcal{U}}$ does indeed map to $\mathcal{I}$.
We have that $\theta_{\mathcal{U}}(u)(\emptyset) = \theta_{\mathcal{F}}(u)(\emptyset) = \emptyset$ by Lemma 24, thus satisfying condition **I**.
Now observe that

$$\begin{aligned}\theta_{\mathcal{U}}(u)\left(\bigcup_i W_i\right) &= \theta_{\mathcal{F}}(u)\left(f\left(\bigcup_i W_i\right)\right) = \theta_{\mathcal{F}}(u)\left(\bigcup_i f(W_i)\right)\\ &= \biguplus_i \theta_{\mathcal{F}}(u)(f(W_i)) = \biguplus_i \theta_{\mathcal{U}}(u)(W_i)\end{aligned}$$

by Lemma 24, thus satisfying condition **II**.
Assume $\theta_{\mathcal{U}}(u)(\bigcap_i W_i) \neq \emptyset$. Then

$$\begin{aligned}\theta_{\mathcal{U}}(u)\left(\bigcap_i W_i\right) &= \theta_{\mathcal{F}}(u)\left(f\left(\bigcap_i W_i\right)\right) = \theta_{\mathcal{F}}(u)\left(\bigcap_i f(W_i)\right)\\ &= \bigcap\!\!\!\!\!+_i \theta_{\mathcal{F}}(u)(f(W_i) = \bigcap\!\!\!\!\!+_i \theta_{\mathcal{U}}(u)(W_i)\end{aligned}$$

by Lemma 24, thus satisfying condition **III**.

(3) By Definition 26, $\boldsymbol{\ell}_{\mathcal{U}}$ is a labeling function.

Hence we can conclude that $\mathcal{G}_{\mathcal{U}}$ is indeed a GTS. ■

6.3 Weak Completeness

Now that we have constructed the canonical model, we turn our attention to proving weak-completeness. To do this, we first prove the so-called Truth Lemma, from which weak-completeness follows easily.

For any ultrafilter v, we enumerate the formulae of v by $v = \{\psi_1^v, \psi_2^v, \psi_3^v, \dots\}$. Given $\varphi \in \mathcal{L}$, define $\mathcal{S}_\varphi$ as

$$\mathcal{S}_\varphi = \left\{ \bigcup_{v \in [\![\varphi]\!]} [\![\psi_{g(v)}^v]\!] \;\middle|\; g \in [[\![\varphi]\!] \to \mathbb{N}] \right\}.$$

Intuitively, each element of $\mathcal{S}_\varphi$ is constructed by choosing, through the choice function g, a formula from every ultrafilter in $[\![\varphi]\!]$. Then we take the union of the states that satisfy each of these formulae. From the definition of $\mathcal{S}_\varphi$, we conjecture that the set of ultrafilters obtained by taking the intersection of all elements in $\mathcal{S}_\varphi$ is the same as the set of ultrafilters obtained by taking the union of all ultrafilters which have $\uparrow \varphi$ as a subset.

Conjecture 29. $\displaystyle\bigcap_{S \in \mathcal{S}_\varphi} S = f^{-1} (\uparrow \varphi)$.

We now state a lemma that is essential for the proof of the Truth Lemma.

Lemma 30. *For any ultrafilter $u \in \mathcal{U}$ and formula $\varphi \in \mathcal{L}$, it holds that*

$$\theta_{\mathcal{U}} (u) ([\![\varphi]\!]) = \theta_{\mathcal{L}} (u) (\varphi) .$$

Proof. From the right-hand side of the equation we get,

$$\begin{aligned}
\theta_{\mathcal{L}} (u) (\varphi) &= \theta_{\mathcal{F}} (u) (\uparrow \varphi) && \text{Corollary 23} \\
&= \theta_{\mathcal{U}} (u) \left(f^{-1}(\uparrow \varphi)\right) && \text{Definition 25} \\
&= \theta_{\mathcal{U}} (u) \left(\bigcap_{S \in \mathcal{S}} S \right) && \text{Conjecture 29} \\
&= \biguplus_{S \in \mathcal{S}_\varphi} \theta_{\mathcal{U}} (u) (S) && \text{Definition 8, III}
\end{aligned}$$

From the left-hand side of the equation we get

$$
\begin{aligned}
\theta_{\mathcal{U}}(u)(\llbracket\varphi\rrbracket) &= \theta_{\mathcal{U}}(u)\left(\bigcup_{v\in\llbracket\varphi\rrbracket}\{v\}\right) \\
&= \biguplus_{v\in\llbracket\varphi\rrbracket} \theta_{\mathcal{U}}(u)(\{v\}) && \text{Definition 8, II} \\
&= \biguplus_{v\in\llbracket\varphi\rrbracket} \theta_{\mathcal{U}}(u)\left(\bigcap_{\psi\in v}\llbracket\psi\rrbracket\right) \\
&= \biguplus_{v\in\llbracket\varphi\rrbracket} \bigcap\!\!\!\!+_{\psi\in v} \theta_{\mathcal{U}}(u)(\llbracket\psi\rrbracket) && \text{Definition 8, III} \\
&= \biguplus_{v\in\llbracket\varphi\rrbracket} \bigcap\!\!\!\!+_{i\in\mathbb{N}} \theta_{\mathcal{U}}(u)(\llbracket\psi_i^v\rrbracket) && \text{Index } v \text{ as } v=\{\psi_0^v,\psi_1^v,\dots\} \\
&= \bigcap\!\!\!\!+_{g\in[\llbracket\varphi\rrbracket\to\mathbb{N}]} \biguplus_{v\in\llbracket\varphi\rrbracket} \theta_{\mathcal{U}}(u)\left(\llbracket\psi_{g(v)}^v\rrbracket\right) && \text{Result by p. 48-49 [3]} \\
&= \bigcap\!\!\!\!+_{g\in[\llbracket\varphi\rrbracket\to\mathbb{N}]} \theta_{\mathcal{U}}(u)\left(\bigcup_{v\in\llbracket\varphi\rrbracket}\llbracket\psi_{g(v)}^v\rrbracket\right) && \text{Definition 8, II} \\
&= \bigcap\!\!\!\!+_{S\in\mathcal{S}_\varphi} \theta_{\mathcal{U}}(u)(S) && \text{By definition of } \mathcal{S}_\varphi
\end{aligned}
$$

and hence the right-hand side and the left-hand side of the equation are equal. ■

We can now state and prove the Truth Lemma which states that each consistent formula φ in an ultrafilter u has a model where u is a state that satisfies φ and vice versa.

Lemma 31 (Truth Lemma)**.** *For arbitrary $u \in \mathcal{U}$ and consistent $\varphi \in \mathcal{L}$ it holds that*

$$\mathcal{G}_{\mathcal{U}}, u \models \varphi \textit{ iff } \varphi \in u.$$

Proof. Let $u \in \mathcal{U}$ be an ultrafilter and $\varphi \in \mathcal{L}$ a consistent formula. We will now show that

$$\mathcal{G}_{\mathcal{U}}, u \models \varphi \text{ iff } \varphi \in u$$

by structural induction on φ.

Case: $\varphi = p$

By the semantics of $\models$, $\mathcal{G}_{\mathcal{U}}, u \models p$ iff $p \in \boldsymbol{\ell}_{\mathcal{U}}$ and by Definition 26, $p \in \boldsymbol{\ell}_{\mathcal{U}}(u)$ iff $p \in u$ and thus $\mathcal{G}_{\mathcal{U}}, u \models p$ iff $p \in u$.

Case: $\varphi = \neg\psi$

$(\implies)$

We want to show that $\mathcal{G}_{\mathcal{U}}, u \models \neg\psi$ implies $\varphi \notin u$. Suppose that $\mathcal{G}_{\mathcal{U}}, u \models \neg\psi$ and suppose towards a contradiction that $\neg\psi \notin u$. Since u is an ultrafilter we must have that $\neg\neg\psi \in u$ and thus $\psi \in u$ which by induction implies $\mathcal{G}_{\mathcal{U}}, u \models \psi$ leading to a contradiction.

$(\impliedby)$

We want to show that $\neg\psi \in u$ implies $\mathcal{G}_\mathcal{U}, u \models \neg\psi$. Suppose that $\neg\psi \in u$ and suppose towards a contradiction that $\mathcal{U}, u \not\models \neg\psi$. By the semantics of $\models$, $\mathcal{U}, u \not\models \neg\psi$ implies $\mathcal{G}_\mathcal{U}, u \models \neg\neg\psi$ implying that $\mathcal{G}_\mathcal{U}, u \models \psi$ which, by induction, implies $\psi \in u$ leading to a contradiction.

Case: $\varphi = \psi \wedge \psi'$
By the semantics of $\models$, $\mathcal{G}_\mathcal{U}, u \models \psi \wedge \psi'$ iff $[\mathcal{G}_\mathcal{U}, u \models \psi$ and $\mathcal{G}_\mathcal{U}, u \models \psi']$ which is the case iff $[\psi \in u$ and $\psi' \in u]$ iff $\psi \wedge \psi' \in u$.

Case: $\varphi = L_r\psi$
By the semantics of $\models$, we have

$$\mathcal{G}_\mathcal{U}, u \models L_r\psi \text{ iff } \theta^-_\mathcal{U}(u)(\llbracket\psi\rrbracket) \geq r,$$

and by Lemma 30 we have

$$\theta^-_\mathcal{U}(u)(\llbracket\psi\rrbracket) \geq r \text{ iff } \theta^-_\mathcal{L}(u)(\psi) \geq r.$$

Now we wish to show that $\theta^-_\mathcal{L}(u)(\psi) \geq r$ iff $L_r\psi \in u$.
$(\implies)$
Let $x_\psi = \theta^-_\mathcal{L}(u)(\psi) = \sup\{x \mid L_x\psi \in u\} \geq r$. If x_ψ is rational, then either $L_{x_\psi}\psi \in u$ or x_ψ is the limit of some sequence of rational numbers, which by **R1** means that $L_{x_\psi}\psi \in u$. Either way, using axiom **A2**, this means that $L_r\psi \in u$. If x_ψ is irrational, then we know that there exists $q \in \mathbb{Q}_{\geq 0}$ such that $x_\psi > q > r$ and $L_q\psi \in u$. Again by using axiom **A2** we thus get $L_r\psi \in u$.
$(\impliedby)$
If $L_r\psi \in u$, then $\theta^-_\mathcal{L}(u)(\psi) = \sup\{x \mid L_x\psi \in u\} \geq r$, because if $\sup\{x \mid L_x\psi \in u\} < r$ then all $L_x\psi \in u$ must satisfy $x < r$, but we know that $L_r\psi \in u$ and $r \not< r$, so this is a contradiction.

We have thus shown that $\mathcal{G}_\mathcal{U}, u \models L_r\psi$ iff $L_r\psi \in u$.

Case: $\varphi = M_r\psi$
By the semantics of $\models$, we have

$$\mathcal{G}_\mathcal{U}, u \models M_r\psi \text{ iff } \theta^+_\mathcal{U}(u)(\llbracket\psi\rrbracket) \leq r$$

and by Lemma 30 we have

$$\theta^+_\mathcal{U}(u)(\llbracket\psi\rrbracket) \leq r \text{ iff } \theta^+_\mathcal{L}(u)(\psi) \leq r,$$

Next we show that $\theta^+_\mathcal{L}(u)(\psi) \leq r$ iff $M_r\psi \in u$.
$(\implies)$
Let $y_\psi = \theta^+_\mathcal{L}(u)(\psi) = \inf\{y \mid M_y\psi \in u\} \leq r$. If y_ψ is rational, then either $M_{y_\psi}\psi \in u$ or y_ψ is the limit of some sequence of rational numbers in which case **R1′** gives that $M_{y_\psi}\psi \in u$. This implies, by axiom **A2′**, that $M_r\psi \in u$. If y_ψ is irrational, then there exists $q \in \mathbb{Q}_{\geq 0}$ such that $y_\psi < q < r$ and $M_q\psi \in u$, and again **A2′** then implies $M_r\psi \in u$.
$(\impliedby)$
If $M_r\psi \in u$, then it must be the case that $\theta^+_\mathcal{L}(u)(\psi) = \inf\{y \mid M_y\psi \in u\} \leq r$, because otherwise $y > r$ for all $y \in \mathbb{Q}_{\geq 0}$ such that $M_y\psi \in u$, but we know that $M_r\psi \in u$ and $r \not> r$.

■

Theorem 32 (Weak Completeness). *The axiomatic system defined in Subsection 6.1 is weakly complete with respect to the semantics defined in Section 5, i.e.*

$$\models \varphi \quad \textit{implies} \quad \vdash \varphi.$$

Proof.

$$\models \varphi \quad \text{implies} \quad \vdash \varphi$$

is equivalent to

$$\nvdash \varphi \quad \text{implies} \quad \not\models \varphi$$

which is equivalent to

the consistency of $\neg\varphi$ implies the existence of a model for $\neg\varphi$,

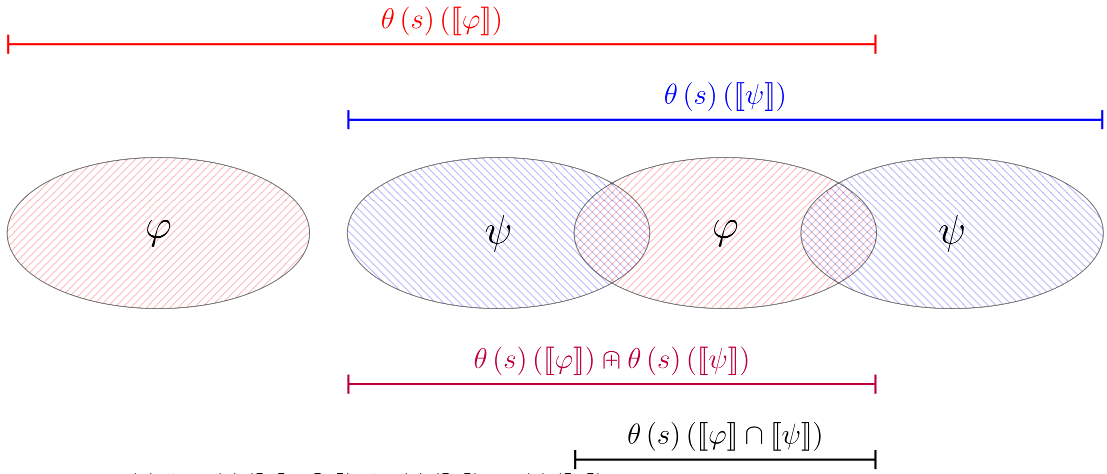
and this is guaranteed by the Truth Lemma. ■

7 Concluding Remarks

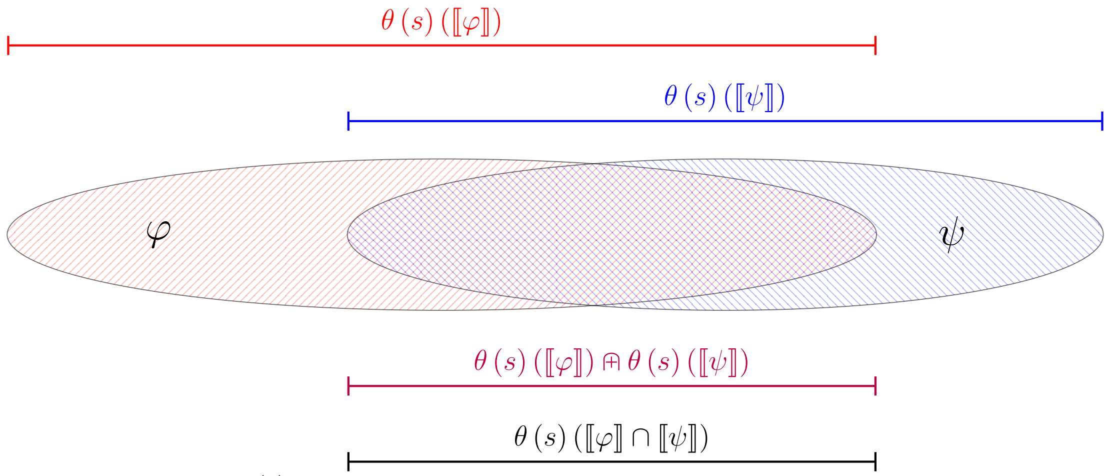
The first contribution of this paper is the, to the best of our knowledge, novel notion of generalized weighted transition systems that, together with the accompanying modal logic, can be used for approximate reasoning about weighted systems. Secondly, and mainly, we give an axiomatization, and show it to be weak-complete. To this end we have adopted an established method based on a canonical model representation, that has previously been used to show completeness of Markovian logics and completeness of weighted modal logic with relation to WTS semantics. Thirdly we have shown that the proposed logic is invariant under a suitable definition of bisimulation.

There are still many new things to explore regarding the proposed models and logic. In the proof theoretic direction, it would be interesting to explore a strong-complete axiomatization of the proposed logic, i.e. an axiomatization such that for any formula $\varphi \in \mathcal{L}$ and any set of formulae $\Phi \subseteq \mathcal{L}$, $\Phi \vdash \varphi$ iff $\Phi \models \varphi$. Since our logic is non-compact, strong-completeness does not follow directly from weak-completeness. As mentioned, we suspect that not all the presented axioms are independent, e.g. we suspect that the axioms **R5** and **R5**$'$ might be provable from **R6** and **R6'**. **R5** and **R5**$'$ have uncountably many instances, so if they could be removed as axioms, and instead be proved as results, this would leave us with an axiomatization with countably many instances, and we could then invoke the Rasiowa-Sikorski lemma to prove Lindenbaum's lemma.

Another direction for future work could be looking into the inherent limitations imposed by the transition function of our models. The conditions **II** and **III** poses some limitations on the models we can represent within our framework. Consider the situation depicted in Figure 6a, where the state spaces for the formulae φ and ψ are partitioned with regards to possible transition weights. As $\theta(s)(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket) \neq \theta(s)(\llbracket\varphi\rrbracket) \Cap \theta(s)(\llbracket\psi\rrbracket)$, we cannot model this system within our framework of GTSs. We can, however, model an approximation of systems, but as illustrated in Figure 6b the bounds derived from the approximate model is an over-approximation of the actual bounds. As such, the modeling formalism would have to be refined in order to give better approximations, i.e. tighter bounds. Somewhat related to this point, it would also be interesting to look into the precise relationship between WTSs and GTSs. Clearly, if we restrict intervals in a GTS to be degenerate and only consider transitions to singleton sets of states, then we get a WTS. Conversely, every WTS can be seen as a GTS by constructing intervals from the weights on transitions in the WTS. One could investigate whether such a construction is necessarily



(a) As $\theta(s)(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket) \neq \theta(s)(\llbracket\varphi\rrbracket) \Cap \theta(s)(\llbracket\psi\rrbracket)$, this is not a model in our formalism.



(b) We can, however, model an approximation of the system

Figure 6: An example of the limitations of our models.

unique, and whether, if we know some property holds for a GTS, we could then deduce that it must also hold for some class of WTSs through the relationship between WTSs and GTSs.

A third possibility for future work is to give an algorithm for finding suitable values such that a given property holds. More precisely, we would want an algorithm that given some property φ can give us suitable r and s such that $L_r\varphi$ and $M_s\varphi$ holds. Ideally, such an algorithm could give the maximum value of r such that $L_r\varphi$ holds and the minimum value s such that $M_s\varphi$ holds.

References

[1] P. Blackburn, J.F.A.K. van Benthem, and F. Wolter. *Handbook of Modal Logic*. Studies in Logic and Practical Reasoning. Elsevier Science, 2006.

[2] Ronald Fagin and Joseph Y. Halpern. Reasoning about knowledge and probability. *J. ACM*, 41(2):340–367, March 1994.

[3] S. Givant and P. Halmos. *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. Springer New York, 2008.

[4] Robert Goldblatt. An abstract setting for henkin proofs. *Topoi*, 3(1):37–41, 1984.

[5] Robert Goldblatt. Deduction systems for coalgebras over measurable spaces. *J. Log. Comput.*, 20(5):1069–1100, 2010.

[6] Aviad Heifetz and Philippe Mongin. Probability logic for type spaces. *Games and Economic Behavior*, 35(1–2):31 – 53, 2001.

[7] Line Juhl, Kim G. Larsen, and Jiri Srba. Modal transition systems with weight intervals. *The Journal of Logic and Algebraic Programming*, 81(4):408 – 421, 2012. Special Issue: 2009 Special Issue: {NWPT} 2009.

[8] Dexter Kozen, Radu Mardare, and Prakash Panangaden. Strong completeness for markovian logics. In Krishnendu Chatterjee and Jiri Sgall, editors, *Mathematical Foundations of Computer Science 2013 - 38th International Symposium, MFCS 2013, Klosterneuburg, Austria, August 26-30, 2013. Proceedings*, volume 8087 of *Lecture Notes in Computer Science*, pages 655–666. Springer, 2013.

[9] Kim G. Larsen and Radu Mardare. Complete proof systems for weighted modal logic. *Theor. Comput. Sci.*, 546:164–175, 2014.

[10] Kim Guldstrand Larsen, Radu Mardare, and Prakash Panangaden. Taking it to the limit: Approximate reasoning for markov processes. In Branislav Rovan, Vladimiro Sassone, and Peter Widmayer, editors, *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27-31, 2012. Proceedings*, volume 7464 of *Lecture Notes in Computer Science*, pages 681–692. Springer, 2012.

[11] Kim Guldstrand Larsen and Arne Skou. Bisimulation through probabilistic testing. In *Conference Record of the Sixteenth Annual ACM Symposium on Principles of Programming Languages, Austin, Texas, USA, January 11-13, 1989*, pages 344–352. ACM Press, 1989.

[12] Alfred Tarski. On some fundamental concepts of metamathematics. In John Corcoran, editor, *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, chapter 3, pages 30–37. Hackett Publishing, reprint edition, January 1983.

[13] Chunlai Zhou. A complete deductive system for probability logic. *J. Log. Comput.*, 19(6):1427–1454, 2009.